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ORIGINAL ARTICLE

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Magnetic shape-memory alloys: thermomechanical modelling and analysis

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Abstract A phenomenological model for the coupled thermo-electro-magneto-mechanical and phase-transformation behaviour of magnetic shape-memory alloys is advanced in small strains and eddy current approximation. The corresponding system of strongly nonlinear relations is tackled via a suitable enthalpy-like transformation. A fully implicit regularized time-discretization scheme is devised and proved to be stable and convergent. In particular, the convergence proof for discrete solutions entails that a suitably weak, energy-conserving solution to the continuous nonlinear system exists. Moreover, several particular models as e.g. ferro/paramagnetic transformation in ferromagnetic materials, martensitic transformation in shape memory allows, or just a simple thermistor problem are covered just as special cases.

Keywords Magnetic shape-memory alloys · Martensitic phase transformation · Ferro/paramagnetic phase transformation · Eddy currents · Weak solutions · Existence · Time discretization

Mathematics Subject Classification (2010) 74C10 · 74N30 · 35K87 · 35Q60 · 35Q74 · 74F15 · 80A17

1 Introduction

Shape-memory alloys (SMAs) are metallic alloys presenting an amazing thermomechanical behaviour: comparably large strains can be induced by either thermal or mechanical stimuli [38]. This is the macroscopic effect of thermomechanically driven structural phase transitions in the material between different crystallographic variants: the austenite (symmetric, stable at high temperatures) and different martensites (less symmetric, stable at low temperatures). At suitably high temperatures, SMAs can recover strains up to 8 % during loading unloading cycles: this is the *superelastic* effect. At lower temperatures, deformations are permanent but can be recovered via a thermal treatment: this is the *shape memory* effect, [38–40,48]. The superelastic and the

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shape memory effect are at the basis of an amazing variety of innovative applications ranging from sensors and actuators to aerospace, biomedical, and seismic Engineering [30], just to mention some relevant examples of application fields. Correspondingly, the interest in a reliable phenomenological modelling of the complex thermomechanical SMA behaviour has nourished an intense research activity in the last decades [78]. Without any claim of completeness, we shall minimally refer the reader to [6,36,38,44,45,56,57,73,75–77,79,89] for reference SMA modelling results.

Some SMAs (including Ni_2MnGa, NiMnInCo, NiFeGaCo, FePt, FePd, among others) have been recently observed to show a remarkable magnetostrictive behaviour and are hence termed Magnetic SMAs (MSMAs). Indeed, the martensitic phase in MSMAs presents the classical ferromagnetic texture of magnetic domains. This mesostructure can be affected by a magnetic field and changes by magnetic domain wall motion, magnetization vector rotation, and magnetic-field-driven martensitic-variant reorientation. The first two effects above are present in all ferromagnetic materials, whereas martensitic-variant reorientation is specific of MSMAs and is usually referred to as the *ferromagnetic shape memory* effect. For instance, a Ni_2MnGa single crystal can develop up to a 10 % strain (at a 1–3 MPa activation stress under the effect of a 1 T magnetic field), whereas a *TerFeNOL-D* polycrystal, one of the most performing *giant* magnetostrictive materials to date, shows a maximal 0.2 % strain (at 60 MPa stress and 0.2 T field). The reader shall be referred, with no claim of completeness, to [29,48,49,58,66,90] as well as to the review in [50] for a minimal literature overview on MSMAs.

Our aim here is to present and analyse a phenomenological, internal-variable-type model for the coupled thermo-electro-magneto-mechanical and phase-change behaviour of MSMAs single crystals. At first, we introduce a thermodynamically consistent model of all the relevant phenomena by suitably coupling conservation and constitutive laws with Maxwell's relations. In particular, the model follows by specifying the free energy density of the medium along with the dissipation encountered during evolution and imposing classical constitutive choices (Section 2).

In this regard, the model may be considered as an extension of former ones in the direction of including additional effects. In particular, the mechanical and phase-change part of the model (that is, the SMA part) is directly constructed on the celebrated Souza-Auricchio modelling ansatz [8,84] which is, however, known to be not directly fitted to include thermal effects [53,54]. Indeed, the thermal behaviour in this model is somehow closer to the classical Frémond approach [38], extended here to a tensorial setting for the internal-phase variables. The ferromagnetic behaviour of the medium is described following ideas from [15,16], which, however, did not include the discussion of the Maxwell system but rather assumed the magnetic field as a datum. It is important to point out that our model features a parabolic (viscous plus rate-independent and diffusive) evolution of the internal-phase variable and of the magnetization.

The main focus of the paper is to prove the existence of a suitably weak solution to the system (Section 3). This is quite challenging as the resulting system of PDEs and evolutionary variational inequalities show strong nonlinearities and couplings. Our analytic strategy is twofold. At first, we perform a specific enthalpy-like change of variables in the heat transfer equation. This entails in particular a simplification of some terms in the internal energy expression at the expense of the appearance of additional nonlinear terms (Section 3). Secondly, we develop a time-discretization procedure. In particular, we consider a fully implicit time-discretization scheme which we prove to be weakly solvable (Section 4), conditionally stable (Section 5), and convergent (Section 6). The limit of time-discrete solutions is checked to be a weak solution of the original problem. Additionally, the analysis of the time discretization may be of some use in the direction of validating and applying the model.

To our knowledge, a macroscopic model encompassing the full variety of effects driving the evolution of MSMAs is unprecedented both from the modelling and the analytic viewpoint. Indeed, on the phenomenological level, a wealth of contributions have been addressed to the modelling of specific submodels including thermomechanics, SMAs (thermomechanics and phase-change), magneto-electro-mechanics, ferro-to-paramagnetic transitions (thermomagnetism), ferromagnetism in MSMA (magnetomechanics and phase-change), and so on. We shall review some of these, also with the aim of placing our contribution in a correct perspective, in the forthcoming Sect. 2.10.

2 The model

2.1 Tensors

We will denote by $\mathbb{R}^{3\times3}_{\rm sym}$ the space of symmetric 3×3 tensors endowed with the natural scalar product $\boldsymbol{a}{:}\boldsymbol{b} = \mathrm{tr}(ab) = a_{ij}b_{ij}$ (summation convention) and the corresponding norm $|\boldsymbol{a}|^2 = \boldsymbol{a}{:}\boldsymbol{a}$ for all $\boldsymbol{a},\ \boldsymbol{b} \in \mathbb{R}^{3\times3}_{\rm sym}$. The space $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ is orthogonally decomposed as $\mathbb{R}^{3\times3}_{\mathrm{sym}} = \mathbb{R}^{3\times3}_{\mathrm{dev}} \oplus \mathbb{R}\mathbf{1}_2$, where $\mathbb{R}\mathbf{1}_2$ is the subspace spanned by the identity 2-tensor $\mathbf{1}_2$, and $\mathbb{R}^{3\times3}_{\mathrm{dev}}$ is the subspace of deviatoric symmetric 3×3 tensors. For all $\boldsymbol{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$, we let $\boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^{3\times3}_{\mathrm{sym}})$ denote the standard symmetric gradient defined as $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^\top)/2$.

2.2 Reference configuration

We shall assume the reference configuration $\Omega \subset \mathbb{R}^3$ of the body to be nonempty, bounded, connected, and with Lipschitz continuous boundary $\Gamma = \partial\Omega$. We moreover require Ω to be bounded and either convex or smooth. The space dimension 3 plays essentially no role throughout the analysis and we would be in the position of reformulating our results in $\mathbb{R}^d$ with no particular intricacy. We assume that the boundary Γ is partitioned in two disjoint open sets Γ_{D} and Γ_{N} with $\partial\Gamma_{\mathrm{D}} = \partial\Gamma_{\mathrm{N}}$ (in $\partial\Omega$). We ask Γ_{D} to be such that there exists a positive constant c_{Korn} depending on Γ_{D} and Ω such that the Korn inequality

$$c_{\mathrm{Korn}} \|\boldsymbol{u}\|^2_{H^1(\Omega;\mathbb{R}^3)} \le \|\boldsymbol{u}\|^2_{L^2(\Gamma_{\mathrm{D}};\mathbb{R}^3)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})}, \tag{2.1}$$

holds true for all $\boldsymbol{u} \in H^1(\Omega; \mathbb{R}^3)$. It would indeed suffice to impose Γ_{D} to have a positive surface measure (see, e.g. [31, Thm. 3.1, p. 110]).

2.3 State variables

Moving within the small-strain realm, we shall specify the state variables of our model as θ, $\boldsymbol{\varepsilon}$, $\boldsymbol{m}$, $\boldsymbol{b}$, $\boldsymbol{e}$, and $\boldsymbol{z}$. Here, $\theta > 0$ represents the absolute temperature of the medium, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ is the symmetrized strain related to the displacement $\boldsymbol{u} : \Omega \to \mathbb{R}^3$, $\boldsymbol{m} \in \mathbb{R}^3$ is the magnetization, $\boldsymbol{b} \in \mathbb{R}^3$ is the magnetic induction, and $\boldsymbol{e} \in \mathbb{R}^3$ is the electric field.

The state variable $\boldsymbol{z} \in \mathbb{R}^N$ is the vectorial descriptor of the crystallographic phase distribution in the material. We shall keep some generality here in order to possibly incorporate in our discussion different modelling frames. However, let us mention that a reference choice for such an internal variable is

$$\boldsymbol{z} = (\boldsymbol{p}, \pi) \in \mathbb{R}^m \times \mathbb{R}. \tag{2.2}$$

In the latter, $\boldsymbol{p} \in \mathbb{R}^m$ describes the local proportion of each of the m martensitic variants, $m \in \mathbb{N}$. As such, it takes values in the simplex $A = \{p_i \ge 0,\ p_1+\cdots+p_m = 1\}$. We have specifically in mind the cases $m = 3$ and $m = 6$ which correspond to cubic-to-tetragonal (3 variants) and cubic-to-orthorhombic (6 variants) austenite–martensite systems, respectively. In particular, these are the active martensitic systems in the MSMAs as Ni_2MgGa, FePd, and FePt, among others. On the other hand, the scalar $\pi \in [0, 1]$ represents the local proportion of total martensitic phase. As our analysis does not directly rely on the choice (2.2), we keep generality in the following.

2.4 Free energy

The specific free energy ψ of the medium is additively decomposed as

$$\begin{aligned}\psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{z}, \boldsymbol{b}, \boldsymbol{m}, \nabla\boldsymbol{z}, \nabla\boldsymbol{m}, \theta) &= \psi_{\mathrm{THERM}}(\theta, \boldsymbol{z}, \boldsymbol{m}) + \psi_{\mathrm{MEC}}(\boldsymbol{\varepsilon}, \boldsymbol{z}) + \psi_{\mathrm{MAG}}(\boldsymbol{b}, \boldsymbol{m}) \\ &+\psi_{\mathrm{COUP}}(\boldsymbol{m}, \boldsymbol{z}) + \psi_{\mathrm{NL}}(\nabla\boldsymbol{z}, \nabla\boldsymbol{m}) + \psi_{\mathrm{CONST}}(\boldsymbol{z}, \boldsymbol{m})\end{aligned} \tag{2.3}$$

where we choose

$$\psi_{\mathrm{THERM}}(\theta, \boldsymbol{z}, \boldsymbol{m}) = \alpha_0(\theta) + \alpha_1(\theta)\gamma(\boldsymbol{z}) + \frac{a_0}{2}(\theta-\theta_{\mathrm{C}})|\boldsymbol{m}|^2, \tag{2.4a}$$

$$\psi_{\mathrm{MEC}}(\boldsymbol{\varepsilon}, \boldsymbol{z}) = \frac{1}{2}\mathbb{C}(\boldsymbol{\varepsilon}-E_{\mathrm{tr}}(\boldsymbol{z})){:}(\boldsymbol{\varepsilon}-E_{\mathrm{tr}}(\boldsymbol{z})) + \frac{1}{2}\mathbb{H}\boldsymbol{z}{\cdot}\boldsymbol{z}, \tag{2.4b}$$

$$\psi_{\mathrm{MAG}}(\boldsymbol{b}, \boldsymbol{m}) = \frac{1}{2\mu_0}(\boldsymbol{b}-\mu_0\boldsymbol{m})^2, \tag{2.4c}$$

$$\psi_{\mathrm{COUP}}(\boldsymbol{m}, \boldsymbol{z}) = \psi_{\mathrm{C}}(\boldsymbol{m}, \boldsymbol{z}) + \frac{a_0}{2}\theta_{\mathrm{C}}|\boldsymbol{m}|^2, \tag{2.4d}$$

$$\psi_{\mathrm{NL}}(\nabla z, \nabla \boldsymbol{m}) = \frac{\kappa_1}{2}|\nabla z|^2 + \frac{\kappa_2}{2}|\nabla \boldsymbol{m}|^2, \tag{2.4e}$$

$$\psi_{\mathrm{CONST}}(z, \boldsymbol{m}) = \delta_K(z) + \delta_S(\boldsymbol{m}). \tag{2.4f}$$

The term $\psi_{\mathrm{THERM}}(\theta, z, \boldsymbol{m})$ encodes the thermal response of the medium. More precisely, the contribution $\alpha_0(\theta)$ is purely caloric and represents the heat capacity of the body, $\alpha_1(\theta)\gamma(z)$ takes into account the latent heat associated with the phase change, in particular α_1 is the temperature-dependent latent heat density of the medium. In the frame of our reference choice (2.2), we can choose $\gamma(z) = \pi$ to represent the fact that one can associate a latent heat to the martensitic–austenitic transition, whereas the latent heat corresponding to martensitic reorientation can be assumed to be negligible. Note that, along with this choice, the function γ turns out to be smooth (and actually linear). This contrasts with the original formulation of the Souza-Auricchio model where a nonsmooth term arises causing indeed severe modelling inconsistencies and analytic drawbacks, see [53,54]. The last term $a_0(\theta-\theta_{\mathrm{C}})|\boldsymbol{m}|^2$ in ψ_{THERM} represents the thermomagnetic coupling. In particular, $\theta_{\mathrm{C}} > 0$ represents the Curie temperature and a_0 is positive [74].

The first term in the mechanic energy term $\psi_{\mathrm{MEC}}(\boldsymbol{\varepsilon}, z)$ translates the assumption of linear material response. In particular, $\mathbb{C} \in \mathbb{R}^{3^4}_{\mathrm{sym}}$ is the isotropic elasticity tensor (symmetric and positive definite) and the linear function $E_{\mathrm{tr}} : \mathbb{R}^N \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ represents the stress-free symmetrized strain of the crystal, given its phase z. In relation to position (2.2), a possible choice is

$$E_{\mathrm{tr}}(z) = \sum_{i=1}^{m} {E_{\mathrm{tr}}}^i \boldsymbol{p}_i \tag{2.5}$$

where ${E_{\mathrm{tr}}}^i = \epsilon_L(3\boldsymbol{\varepsilon}_i \otimes \boldsymbol{\varepsilon}_i - \mathbf{1}_2)/\sqrt{6}$ is the stress-free symmetrized strain corresponding to the pure i phase, and the parameter $\epsilon_L > 0$ measures the maximal strain obtainable by realignment of martensitic variants (typically around 5 %). Moreover, $\mathbb{H} \in \mathbb{R}^{3^4}_{\mathrm{sym}}$ is a positive-definite and symmetric hardening tensor. Again in the context of example (2.2), one could choose $\mathbb{H}z = h_1 {E_{\mathrm{tr}}}^{\top} \otimes E_{\mathrm{tr}}(z) + h_2\pi$ for some hardening parameters $h_1,\ h_2 > 0$.

The term $\psi_{\mathrm{MAG}}(\boldsymbol{b}, \boldsymbol{m})$ is the classical magnetic energy contribution, see below. In particular, the parameter μ_0 is the magnetic permeability of void. Some additional discussion on this term is in Sect. 2.8 below.

The coupling term $\psi_{\mathrm{COUP}}(\boldsymbol{m}, z)$ is responsible for the coupling between magnetism, and phase. Our analysis is independent of the explicit form of the coupling function ψ_{C}. Let us, however, mention that an example for such a coupling could be given by

$$\psi_{\mathrm{C}}(\boldsymbol{m}, z) = \frac{b_0}{4}|\boldsymbol{m}|^4 - K|\boldsymbol{m}\cdot\boldsymbol{A}(z))|^2 \tag{2.6}$$

where $K > 0$ measures the anisotropic magnetic response. The two positive parameters a_0, b_0 trigger the ferromagnetic–paramagnetic transition. In particular, along with this choice, the coupling energy ψ_{COUP} switches from being nonconvex for ferromagnets below the Curie temperature θ_{C} to convex for paramagnets above θ_{C} [74]. The last term in (2.6) is an anisotropic energy which favours the alignment of the magnetization $\boldsymbol{m}$ with the easy axis of magnetization $\boldsymbol{A}(z)$ corresponding to the phase z [15]. The function $\boldsymbol{A} : \mathbb{R}^N \to \mathbb{R}^3$ can be chosen to be linear. In particular, by assuming (2.2) for $m = 3$, one can take $\boldsymbol{A}(z) = \boldsymbol{p}$. As such, the material parameter $K > 0$ modulates the magnetic anisotropy of the martensitic phase of the material. Note that we are assuming here that the austenite is not ferromagnetic. This is indeed a simplification as for many MSMAs, austenite is known to have a rather complex magnetic behaviour. We, however, believe that this simplification does not jeopardize the performance of the model in the vast majority of applicative situations.

The gradient terms in ψ_{NL} encode the nonlocality of the behaviour of the internal variables $\boldsymbol{m}$ and z. As such, they introduce length scales in the model (described by the coefficients $\kappa_1,\ \kappa_2 > 0$) which are to be fitted with respect to the experimentally observable sizes of typical martensitic and ferromagnetic textures in the specimens. In particular, the term in $\nabla\boldsymbol{m}$ is the so-called exchange energy. From the mathematical viewpoint, these terms clearly bear a crucial compactifying effect.

Finally, the term ψ_{CONST} constraints $\boldsymbol{m}$ and z to take value in some admissible set only. In particular, δ stands for the indicator function of such a set. In view of the applications we have in mind, the sets $K \subset \mathbb{R}^N$ and $S \subset \mathbb{R}^3$ will be assumed to be convex, closed, and bounded. Typically, S is the ball of radius equal to the so-called saturation magnetization at zero temperature. On the other hand, for the choice (2.2), the constraint K reads as $K = A\times[0, 1]$.

Note that the evolution of temperature, phase, and magnetization is fully coupled through the energy. The term ψ_{THERM} takes into account temperature-phase and temperature-magnetization couplings whereas the term ψ_{COUP} describes the phase-magnetization coupling. Of course these energy contributions play a simultaneous role in the description of the process. Still, we believe that this distinction clarifies the respective roles of these different energetic sources.

2.5 Dissipation

In order to describe the evolution of the system, for all given components of the state (z, θ), we define the dissipation (pseudo-)potential $\phi(z, \theta; \cdot)$ as

$$\phi(z, \theta; \dot{z}, \dot{\boldsymbol{m}}, \nabla\theta, \boldsymbol{e}) = \delta_Z^*(\dot{z}) + \frac{1}{2}D\dot{z}\cdot\dot{z} + \delta_M^*(\dot{\boldsymbol{m}}) + \frac{\alpha}{2}|\dot{\boldsymbol{m}}|^2 + \frac{1}{2}\mathbb{K}(z, \theta)\nabla\theta\cdot\nabla\theta + \frac{1}{2}\mathbb{S}(z, \theta)\boldsymbol{e}\cdot\boldsymbol{e}. \tag{2.7}$$

Here, $Z \subset \mathbb{R}^N$ and $M \subset \mathbb{R}^3$ are nonempty, convex, and closed sets containing a neighbourhood of $\mathbf{0}$, and δ_Z^* and δ_M^* denote the conjugate functions to the indicator functions δ_Z and δ_M, respectively. These terms are positively 1-homogeneous and hence encode rate-independent dissipation effects. On the other hand, the viscous behaviour of the model is described by the viscosity matrix coefficient $D > 0$ and by $\alpha > 0$. Finally, the smooth functions $\mathbb{K}$ and $\mathbb{S}$ are assumed to take values on positive-definite tensors. In particular, $\mathbb{K}$ and $\mathbb{S}$ correspond to the thermal and electric conductivity, respectively. This combination of viscous and rate-independent dissipation for SMA has already been considered e.g. in [85, Fig.1].

In view of the applicability of the model, let us stress that the actual values of the above-mentioned material parameters may be obtained from the literature, see e.g. [49,51,58,91]. In particular, the thermomechanical response of the material (that is, parameters and functions $\mathbb{C}$, α_0, α_1, $\mathbb{H}$, $\mathbb{K}$, the two sets M and Z, and the viscosity coefficients D and α) can be fitted from a suite of ordinary loading experiments at different temperatures and frequencies. The conductivity $\mathbb{S}$ and the Curie temperature θ_{C} are also easily accessible to experiments, whereas a_0 and b_0 should be fitted on the actual ferromagnetic–paramagnetic behaviour of the material under thermal treatments. Finally, the two scale parameters κ_1, κ_2 are phenomenological and have to be tailored in order to reflect the relevant dimensions of the polycrystalline aggregate (κ_1) and its magnetic domain distribution (κ_2). These features are in principle accessible to transmission electron microscopy, see for instance [28].

We shall once again remark that the model takes anisotropic effects into account as for the magnetization-phase coupling (throught the function A) and martensitic reorientation (through E_{tr}, respectively). Some other possible source of anisotropy could be the different thermomechanical or electrical behaviour of distinguished martensitic variants. These are at present not directly considered in the model. Their inclusion would call for additional dependencies of parameters on the phase. We expect this modification to be mathematically amenable, although at the expense of an even heavier notation.

2.6 Constitutive relations and flow rules

Give the free energy ψ, we classically define the entropy η, the stress $\boldsymbol{\sigma}$ the magnetic field $\boldsymbol{h}$, and the internal energy e, as

$$\eta = -\partial_\theta\psi = -\alpha_0'(\theta) - \alpha_1'(\theta)\gamma(z) - \frac{a_0}{2}|\boldsymbol{m}|^2, \tag{2.8a}$$

$$\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}}\psi = \mathbb{C}(\boldsymbol{\varepsilon} - E_{\mathrm{tr}}(z)), \tag{2.8b}$$

$$\boldsymbol{h} = \partial_{\boldsymbol{b}}\psi = \frac{1}{\mu_0}\boldsymbol{b} - \boldsymbol{m}, \tag{2.8c}$$

$$e = \psi + \theta\eta = \psi_{\mathrm{MEC}}(\boldsymbol{\varepsilon}, z) + \psi_{\mathrm{MAG}}(\boldsymbol{b}, \boldsymbol{m}) + \psi_{\mathrm{C}}(\boldsymbol{m}, z) + \psi_{\mathrm{NL}}(\nabla z, \nabla\boldsymbol{m}) + \alpha_0(\theta) - \theta\alpha_0'(\theta) + (\alpha_1(\theta) - \theta\alpha_1'(\theta))\gamma(z). \tag{2.8d}$$

Note that $\boldsymbol{b}$ is given in terms of $\boldsymbol{m}$ and $\boldsymbol{h}$ only. As such, it will often be eliminated from the following being replaced by $\mu_0(\boldsymbol{h}+\boldsymbol{m})$; in particular, $\psi_{\mathrm{MAG}}(\boldsymbol{b}, \boldsymbol{m}) = \frac{1}{2}\mu_0|\boldsymbol{h}|^2$. The heat flux and electric current are given in terms of the dissipation potential as

$$(-\boldsymbol{q}, \boldsymbol{j}) = \partial_{(\nabla\theta, \boldsymbol{e})}\phi \tag{2.9}$$

which, in view of (2.7), gives the *Fourier* and the *Ohm laws* $\boldsymbol{q} = -\mathbb{K}(z,\theta)\nabla\theta$ and $\boldsymbol{j} = \mathbb{S}(z,\theta)\boldsymbol{e}$, respectively. In particular, note that we are neglecting the thermoelectric Seebeck and Peltier cross effects.

The evolution of the medium is described by the coupling of the entropy equation with the mechanical equilibrium for $\boldsymbol{u}$ and Biot-type flow relations for the internal variables $(z,\boldsymbol{m})$. As for the electromagnetic fields $(\boldsymbol{e},\boldsymbol{h})$, we assume that the alloy is highly electrically conductive so that we can neglect the so-called displacement current; in other word, we neglect the contribution of the electrical energy $\frac{1}{2}\varepsilon_0|\boldsymbol{e}|^2$ to the overall energy balance with ε_0 being the vacuum permittivity. This leads to the consideration of the so-called eddy current approximation of the Maxwell's system. In conclusion, we aim at analysing the following system of relations:

Entropy equation:

$$\theta\dot{\eta} + \operatorname{div}\boldsymbol{q} = \zeta(z,\theta;\dot{z},\dot{\boldsymbol{m}},\boldsymbol{e}) := \partial_{(\dot{z},\dot{\boldsymbol{m}},\boldsymbol{e})}\phi\cdot(\dot{z},\dot{\boldsymbol{m}},\boldsymbol{e}), \tag{2.10a}$$

Quasistatic mechanical equilibrium:

$$\operatorname{div}\partial_{\boldsymbol{\varepsilon}}\psi = \boldsymbol{0}, \tag{2.10b}$$

Biot-type flow rules for the internal variables:

$$\partial_{(\dot{z},\dot{\boldsymbol{m}})}\phi + \partial_{(z,\boldsymbol{m})}\psi \ni \boldsymbol{0}, \tag{2.10c}$$

Eddy-current Maxwell's system:

$$\dot{\boldsymbol{b}} + \operatorname{curl}\boldsymbol{e} = \boldsymbol{0} \qquad \text{and} \qquad \operatorname{curl}\boldsymbol{h} = \boldsymbol{j} \tag{2.10d}$$

with $\boldsymbol{q}$ and $\boldsymbol{j}$ from (2.9). In the entropy equation (2.10a), the right-hand side ζ represents the entropy dissipation rate. The differential inclusion (2.10c) corresponds to a system of relations in $\mathbb{R}^3\times\mathbb{R}^N\times\mathbb{R}^3$. Note that we are not including here any body force for the sake of notational simplicity. The inclusion of a nonzero body force would, however, be straightforward.

We are assuming here no external magnetic field contribution (again nonzero external contribution can be treated with no difficulties). Indeed, $\boldsymbol{h}$ corresponds here solely to the self-induced magnetization field. In order to avoid additional technical difficulties, which, however, would not alter the tenet of our analysis, we restrict ourselves to the consideration of fields in Ω only. We neglect the contribution to $\boldsymbol{h}$ given by the region which is external to Ω and prescribe suitable boundary conditions at Γ instead, see (2.12c) below.

2.7 PDE system

Let us now write system (2.10) in terms of the choices (2.3)–(2.4) and (2.7) and the constitutive relations (2.8)–(2.9) as

Heat-transfer equation:

$$\begin{aligned}\big(c_0(\theta)+c_1(\theta)\gamma(z)\big)\dot{\theta} - \operatorname{div}\big(\mathbb{K}(z,\theta)\nabla\theta\big) &= \delta^*_Z(\dot{z}) + D\dot{z}\cdot\dot{z} + \delta^*_M(\dot{\boldsymbol{m}})\\ &+ \alpha|\dot{\boldsymbol{m}}|^2 + \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e} + \theta\alpha_1'(\theta)\gamma'(z)\dot{z} + a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}},\end{aligned} \tag{2.11a}$$

Mechanical quasistatic equilibrium:

$$\operatorname{div}\big(\mathbb{C}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - E_{\rm tr}(z)\big)\big) = \boldsymbol{0}, \tag{2.11b}$$

Biot-type flow rules for the internal variables:

$$\begin{aligned}&\partial\delta^*_Z(\dot{z}) + D\dot{z} + E_{\rm tr}^\top\mathbb{C}(E_{\rm tr}(z)-\boldsymbol{\varepsilon}(\boldsymbol{u})) + \mathbb{H}z + \partial_z\psi_{\rm C}(\boldsymbol{m},z)\\ &\quad - \kappa_1\Delta z + N_K(z) \ni -\alpha_1(\theta)\gamma'(z),\end{aligned} \tag{2.11c}$$

$$\partial\delta^*_M(\dot{\boldsymbol{m}}) + \alpha\dot{\boldsymbol{m}} + \partial_{\boldsymbol{m}}\psi_{\rm C}(\boldsymbol{m},z) - \kappa_2\Delta\boldsymbol{m} + N_S(\boldsymbol{m}) \ni \mu_0\boldsymbol{h} - a_0\theta\boldsymbol{m}, \tag{2.11d}$$

Eddy-current Maxwell's system:

$$\mu_0(\dot{\boldsymbol{h}}+\dot{\boldsymbol{m}}) + \operatorname{curl}\boldsymbol{e} = \boldsymbol{0}, \tag{2.11e}$$

$$\operatorname{curl}\boldsymbol{h} = \mathbb{S}(z,\theta)\boldsymbol{e}. \tag{2.11f}$$

Here, we have used the notation $c_0(\theta) = -\theta\alpha_0''(\theta)$ and $c_1(\theta) = -\theta\alpha_1''(\theta)$ and we have indicated with $N_K(z)$ and $N_S(\boldsymbol{m})$ the normal cones to K and S at z and $\boldsymbol{m}$, respectively. In particular, $\boldsymbol{\xi} \in N_K(z)$ if and only if $z \in K$ and $\boldsymbol{\xi}\cdot(z-\tilde{z}) \geq 0$ for all $\tilde{z} \in K$. Analogously for N_S.

2.8 Boundary and initial conditions

We complete system (2.11) by prescribing initial and boundary conditions. In particular, we ask for

$$\theta(\cdot,0)=\theta_0,\qquad z(\cdot,0)=z_0,\qquad \boldsymbol{m}(\cdot,0)=\boldsymbol{m}_0,\qquad \boldsymbol{h}(\cdot,0)=\boldsymbol{h}_0\qquad \text{in } \Omega, \tag{2.12a}$$

$$\boldsymbol{u}=\boldsymbol{0}\quad \text{on } \Gamma_{\rm D},\qquad \boldsymbol{\sigma}\boldsymbol{\nu}=\boldsymbol{0}\quad \text{on } \Gamma_{\rm N},\qquad \text{and} \tag{2.12b}$$

$$\frac{\partial z}{\partial \boldsymbol{\nu}}=\boldsymbol{0},\qquad \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}}=\boldsymbol{0},\qquad \mathbb{K}(z,\theta)\nabla\theta\cdot\boldsymbol{\nu}=f_{\rm b},\qquad \boldsymbol{\nu}\times\boldsymbol{h}=\boldsymbol{j}_{\rm b}\qquad \text{on } \Gamma \tag{2.12c}$$

with $\boldsymbol{\nu}$ denoting the unit outward normal to Γ. Here, $f_{\rm b}$ is a prescribed heat flux and $\boldsymbol{j}_{\rm b}$ is some prescribed electric current as the boundary, assumed to be tangent to the boundary at all times. In order to better understand the role of the boundary condition on $\boldsymbol{h}$, one can compute, for all smooth tests v,

$$\begin{aligned}\int_\Gamma (\boldsymbol{j}\cdot\boldsymbol{\nu})v\,{\rm d}S &= \int_\Gamma (\operatorname{curl}\boldsymbol{h}\cdot\boldsymbol{\nu})v\,{\rm d}S = \int_\Omega \operatorname{curl}\boldsymbol{h}\cdot\nabla v\,{\rm d}x + \int_\Omega (\operatorname{div}\operatorname{curl}\boldsymbol{h})v\,{\rm d}x\\ &= \int_\Gamma (\boldsymbol{\nu}\times\boldsymbol{h})\cdot\nabla v\,{\rm d}S = \int_\Gamma \boldsymbol{j}_{\rm b}\cdot\nabla v\,{\rm d}S = -\int_\Gamma \operatorname{div}_{\rm S}\boldsymbol{j}_{\rm b}\,v\,{\rm d}S.\end{aligned}$$

In particular, by the arbitrariness of v one has that $\boldsymbol{j}\cdot\boldsymbol{\nu}=-\operatorname{div}_{\rm S}\boldsymbol{j}_{\rm b}$ (surface divergence) so that one can pump electric current in the conducting medium by merely specifying $\boldsymbol{j}_{\rm b}$; cf. also [83]. It should be realized that the boundary conditions for the electromagnetic fields $(\boldsymbol{e},\boldsymbol{h})$ neglect the electromagnetic field outside the body Ω. This is a legitimate simplification in a lot of applications and facilitates the analysis because otherwise the parabolic nonlinear system on Ω would be coupled with the hyperbolic system outside Ω where the eddy current approximation could not be used. This altogether would effectively yield a nonlinear hyperbolic system.

2.9 Energy balance

In order to illustrate the variational structure of the model, we shall now rewrite the energy balance for system (2.11) along with the boundary conditions (2.12b)–(2.12c). To this aim, let us define $\boldsymbol{h}_{\rm b}=-\boldsymbol{\nu}\times\boldsymbol{j}_{\rm b}$ on Γ and extend it arbitrarily in Ω. By testing the Maxwell system (2.11e)–(2.11f) by $(\boldsymbol{h},-\boldsymbol{e})$ we find that

$$\begin{aligned}\int_\Omega \Big(\frac{{\rm d}}{{\rm d}t}\frac{\mu_0}{2}|\boldsymbol{h}|^2+\mu_0\dot{\boldsymbol{m}}\cdot\boldsymbol{h}\Big)\,{\rm d}x + \int_\Omega \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\,{\rm d}x &= \int_\Omega \big(\operatorname{curl}\boldsymbol{h}\cdot\boldsymbol{e}-\operatorname{curl}\boldsymbol{e}\cdot\boldsymbol{h}\big){\rm d}x\\ = \int_\Omega \big(\operatorname{curl}\boldsymbol{h}_{\rm b}\cdot\boldsymbol{e}-\operatorname{curl}\boldsymbol{e}\cdot\boldsymbol{h}_{\rm b}\big){\rm d}x &= \int_\Gamma \boldsymbol{j}_{\rm b}\cdot\boldsymbol{e}\,{\rm d}S.\end{aligned} \tag{2.13}$$

Let us now formally test the momentum equation (2.11b) by $\dot{\boldsymbol{u}}$, the flow rule (2.11c) by $\dot{z}$, and the flow rule (2.11d) by $\dot{\boldsymbol{m}}$, and add the resulting relations getting

$$\begin{aligned}&\int_\Omega \big(\partial_{(\dot z,\dot{\boldsymbol{m}})}\phi\cdot(\dot z,\dot{\boldsymbol{m}})+\alpha_1(\theta)\gamma'(z)\dot z+a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}}\big){\rm d}x\\ &\quad+\int_\Omega \big(\partial_{\boldsymbol{\varepsilon}}\psi:\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})+\partial_{(z,\boldsymbol{m})}\psi\cdot(\dot z,\dot{\boldsymbol{m}})+\partial_{(\nabla z,\nabla\boldsymbol{m})}\psi\cdot(\nabla\dot z,\nabla\dot{\boldsymbol{m}})\big)\,{\rm d}x = \mu_0\int_\Omega \boldsymbol{h}\cdot\dot{\boldsymbol{m}}\,{\rm d}x\end{aligned} \tag{2.14}$$

Finally, we integrate the heat transfer equation (2.11a) on Ω (that is, we test it by 1) and use the boundary conditions in order to obtain that

$$\begin{aligned}\int_\Omega \frac{{\rm d}}{{\rm d}t}e\,{\rm d}x &= \int_\Gamma f_{\rm b}{\rm d}S+\int_\Omega \big(\zeta(z,\theta;\dot z,\dot{\boldsymbol{m}},\boldsymbol{e})+\alpha_1(\theta)\gamma'(z)\dot z+a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}}\big){\rm d}x\\ &\quad+\int_\Omega \big(\partial_{\boldsymbol{\varepsilon}}\psi:\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})+\partial_{(z,\boldsymbol{m})}\psi\cdot(\dot z,\dot{\boldsymbol{m}})+\partial_{(\nabla z,\nabla\boldsymbol{m})}\psi\cdot(\nabla\dot z,\nabla\dot{\boldsymbol{m}})\big){\rm d}x+\int_\Omega \frac{{\rm d}}{{\rm d}t}\frac{\mu_0}{2}|\boldsymbol{h}|^2{\rm d}x\end{aligned} \tag{2.15}$$

with e from (2.8d). Eventually, by summing up relations (2.13)–(2.15), we get the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} e\,\mathrm{d}x = \int_{\Gamma} \big(\boldsymbol{j}_{\mathrm{b}}\cdot\boldsymbol{e} + f_{\mathrm{b}}\big)\,\mathrm{d}S \tag{2.16}$$

where ζ is from (2.10a) and where the two terms on the right-hand side clearly corresponds to the contributed energy to the system in terms of heat and current flux supplied through the boundary Γ.

2.10 Relation with other models

We shall collect here some comments on the relations between our model and previous contributions. Let us start by describing how the present model builds upon and extends previous ones. As already mentioned in the Introduction, the purely SMA part of the model (mechanics and phase-change) corresponds to the celebrated *Souza-Auricchio model*, although in a single-crystal setting. The reader is referred to [8–10,84] for some basic information and comment on its validity and robustness, especially regarding approximations. The Souza-Auricchio model presents a sound variational formulation, and it is hence particularly amenable to be extended in order to encompass to additional effects. In particular, it has been formulated in the finite strain regime [34,35,41] and extended to nonsymmetric material behaviours [13] and to the description of residual plasticity [11–13,32], see also [43,55]. The Souza-Auricchio model has been investigated from the analytical viewpoint [7] and combined with space discretization in [62,63].

The reader is referred to [33,53,54,61,64] for the delicate extension of the Souza-Auricchio model in the direction of *thermal effects*. Indeed, the original Souza-Auricchio model suffers from not making a clear distinction between martensite fraction and martensite orientation: the whole inelastic effects are encoded into a single tensorial variable. Here, in the spirit of the decomposition (2.2), we are rather distinguishing these two concepts, with the aim of relating a latent heat production to the martensite–austenite transition only. This in turn allows to consider the thermal contributions of the energy to be smoothly depending on the internal variable z. The issue whether martensitic transformation is or not a rate-independent phenomenon is disputed. On the one hand, a number of contributions focus on rate-independent flow rules, see for instance all the above-mentioned contributions on the Souza-Auricchio model. On the other hand, modelling featuring viscous evolutions are also available, see [85] and all the analyses on the Frémond model. Here, we rather follow this second line but augment our dissipation by rate-independent terms for completeness.

By postponing the discussion on the magnetoelectric behaviour, we have to mention that thermomechanics in SMAs is a quite investigated topic. A widely studied SMA model is due to Frémond [38,39], see [18,23,24,87] for results and references, cf. also [78] for a survey. The main difference between our model and Frémond's relies in the description of the martensitic structure. Indeed, the Frémond model is grounded on a *mixing* ansatz on free energies. Although basically available for an arbitrary number of martensitic variants, the analysis of the Frémond model has been restricted to the consideration of two variants only. In particular, this entails the possibility of performing a suitable variable change and make the thermomechanical coupling term in the free energy bilinear. On the other hand, the phase descriptor is a scalar and this prevents the model from describing efficiently reorientation of martensites. The present model instead features a possibly *tensorial descriptor* of the solid phase. At the same time, the interpretation of the mixing of energy is still available. Indeed, assume that the mechanical energy density results from the mixture $\sum_i p_i \frac{1}{2}\mathbb{C}_i(\boldsymbol{\varepsilon}-{E_{\mathrm{tr}}}^i){:}(\boldsymbol{\varepsilon}-{E_{\mathrm{tr}}}^i)$ with $\boldsymbol{p}=(p_i)_i$ ranging over the Gibbs simplex. Up to a constant, for $\mathbb{C}_i\equiv\mathbb{C}$ this results to $\frac{1}{2}\mathbb{C}(\boldsymbol{\varepsilon}-E_{\mathrm{tr}}(\boldsymbol{p})){:}(\boldsymbol{\varepsilon}-E_{\mathrm{tr}}(\boldsymbol{p}))$ with $E_{\mathrm{tr}}=\sum_i {E_{\mathrm{tr}}}^i p_i$, namely the first term in ψ_{MEC} under assumption (2.2). By considering the thermal term in the i-phase to be given by $c_i\theta(1-\ln\theta)$ for some heat capacity c_i, the coupling term between phase and temperature reads $\sum_i c_i p_i\theta(1-\ln\theta)$. This correspond to the second term in ψ_{THERM} in (2.3) upon choosing $\alpha_1(\theta)=\theta(1-\ln\theta)$ and $\gamma(z)=\sum_i c_i p_i$, cf. also [78, Rem. 4.4].

We shall mention here the model [5,85,86] featuring a mechanical free energy term of the type

$$\frac{1}{2}\mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\pi\boldsymbol{e}){:}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\pi\boldsymbol{e}).$$

The tensor $\pi\boldsymbol{e}$ corresponds to the inelastic strain induced by the phase transition where the scalar π represents the size of the inelastic strain (or the proportion of martensites) and the tensor $\boldsymbol{e}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$ with $|\boldsymbol{e}|=1$ corresponds to its direction. This setting can be included in our frame by choosing $\boldsymbol{z} = (\boldsymbol{\pi}, \boldsymbol{e})$ and $E_{\rm tr}((\boldsymbol{z})) = \boldsymbol{\pi}\boldsymbol{e}$. Note that, in contrast to [85], plasticity is here neglected.

Thermoviscoelasticity in SMAs has been considered in a series of papers. In [72,92], the possibility of reproducing the shape memory behaviour is interpreted as a specific thermoelastic behaviour and no description of the solid phase is considered. This follows the tenet of the Falk [36] and Falk-Konopka [37] models where the complex SMA behaviour is described as an effect of the nonconvexity of the mechanical energy landscape. In this direction see also [14,27,71,93]. The situation is fairly different here. The mechanical part of the energy is indeed convex and the SMA behaviour stems from the interaction of energy and dissipation instead. This amounts to an enhanced robustness of the modelling with respect to approximations, see [7]. In [67–70], the thermomechanical coupling term in the free energy is assumed to be linear in the temperature θ. This results in the uncoupling of thermal and mechanical variables in the internal energy, a circumstance which appears to be not so well tailored for SMAs. Additionally, the model features viscous elastic terms which enhance the compactness frame and renders a Schauder-fixed-point analysis amenable. We have to mention that the consideration of viscoelasticity for SMA, albeit disputable form the modelling viewpoint, is widely considered in the SMA literature for three-dimensional problems, we refer without claiming completeness to [1,2,22,23,47] for a collection of classical existence results. Here, we do not assume viscoelasticity instead.

Magnetostriction and MSMAs have been discussed in [29,60] as well as in [25,26,48–51,59,86,90]. In particular, the Souza-Auricchio model has been extended to MSMA in [3,4,15,16,88]. The present modelling moves exactly in the direction of the latter by including also the thermal and the electric evolution. Other MSMA phenomenological models of internal variable type are those by Hirsinger & Lexcellent [46] and Kiefer & Lagoudas [52], later reconsidered by Wang & Steinmann [91]. These two models, albeit basically informed by our same principles, differ from ours in the choice of Gibbs energy, which is comparably more complex. In particular, anisotropy is directly built in by means of the choice of specific anisotropic energy contributions. Moreover, no thermal and electric behaviour is considered. By neglecting the phase transition descriptor $\boldsymbol{z}$, our model corresponds to the magnetostrictive model from [83] under the assumption of a convex underlying free energy containing no strain gradients if magnetization forcing in (2.11b) and velocity influence in (2.11e) and gyromagnetic effects in (2.11d) are neglected; in addition, our notion of solution here is stronger than the one in [83].

As for the *ferro/paramagnetic transition*, our model has to be compared with the analysis in [81,82]. In the latter, no quasistatic approximation is considered and the evolution of the magnetization is driven by the Landau–Lifshitz–Gilbert equation. Moreover, a specific right-hand side in the Faraday-Maxwell relation (2.11e) arises in connection with the fact that magnetization is to be measured in the deformed configuration. We neglect this last aspect here by assuming small deformations. In turn, the analysis in [81,82] exploits the compactifying effect of the extra viscoelastic term in the momentum equation (and in the heat equation as a source term) and does not include the description of the phase evolution.

Our model covers also a number of simplified situations. By considering only the variables θ, $\boldsymbol{e}$, and $\boldsymbol{h}$, the corresponding reduced system (2.11a,e,f) describes a *thermistor* with eddy currents, cf. [80, Sect.12.7]. By further neglecting $\mu_0\dot{\boldsymbol{h}}$ in (2.11e), one has $\boldsymbol{e} = \nabla\phi_{\rm el}$ for an electrostatic potential $\phi_{\rm el}$. This allows to replace (2.11e,f) by a single equation ${\rm div}(\mathbb{S}(\theta)\nabla\phi_{\rm el}) = \boldsymbol{0}$.

3 Weak formulation, data qualification, and main result

The aim of this section is to introduce a suitably weak notion of solution to system (2.11) and present the existence results to be proved in Sects. 4–6.

We abbreviate $I = (0,T)$, $Q = I\times\varOmega$, and $\Sigma = I\times\varGamma$. In the following, we shall use some classical notation for function spaces, namely the Lebesgue spaces L^p, the Sobolev spaces $W^{k,p}$ and, in particular, $H^k = W^{k,2}$, and vector-valued functions. In particular, we will use the Hilbert spaces

$$
\begin{aligned}
H^1_{\varGamma_{\rm D}}(\varOmega;\mathbb{R}^3) &= \big\{\boldsymbol{u}\in H^1(\varOmega;\mathbb{R}^3)\ :\ \boldsymbol{u}=\boldsymbol{0}\text{ on }\varGamma_{\rm D}\big\},\\
H^2_{\rm N}(\varOmega;\mathbb{R}^m) &= \big\{\boldsymbol{g}\in H^2(\varOmega;\mathbb{R}^m)\ :\ \partial\boldsymbol{g}/\partial\boldsymbol{\nu}=\boldsymbol{0}\text{ on }\varGamma\big\}\quad\text{for } m=N,\ 3,\\
L^2_{\rm curl}(\varOmega;\mathbb{R}^3) &= \big\{\boldsymbol{h}\in L^2(\varOmega;\mathbb{R}^3)\ :\ {\rm curl}\,\boldsymbol{h}\in L^2(\varOmega;\mathbb{R}^3)\big\},\\
L^2_{{\rm curl},0}(\varOmega;\mathbb{R}^3) &= \big\{\boldsymbol{h}\in L^2_{\rm curl}(\varOmega;\mathbb{R}^3)\ :\ \boldsymbol{h}\times\boldsymbol{\nu}|_{\varGamma}=\boldsymbol{0}\big\}.
\end{aligned}
$$

3.1 Weak formulation

A specific challenge for the treatment of the problem is that the time derivative $\dot{\theta}$ in (2.11a) does not seem to allow for a priori estimation. The weak formulation should naturally reflect this difficulty and will, beside θ, use also a *renormalized temperature* which enjoys and a priori bound on its time derivative. In the simple case when γ is constant, this substitution is usually referred to as *enthalpy transformation* in the mathematical literature, and the renormalized temperature is then called, perhaps disputably from the physical viewpoint, *enthalpy*. We will devise this transformation even for γ nonconstant and refer to it as an enthalpy-like transformation.

Let us firstly compute

$$\big(c_0(\theta)+c_1(\theta)\gamma(z)\big)\dot{\theta} = \frac{\partial}{\partial t}\big(\widehat{c}_0(\theta)+\widehat{c}_1(\theta)\gamma(z)\big) - \widehat{c}_1(\theta)\gamma'(z)\cdot\dot{\boldsymbol{z}},$$

where $\widehat{c_i}$ denotes a primitive function of c_i, $i = 0, 1$. We introduce the enthalpy-like variable

$$w = \omega(z,\theta) := \widehat{c}_0(\theta)+\widehat{c}_1(\theta)\gamma(z) \quad \text{with } \widehat{c_i}(\theta) := \int_0^\theta c_i(\vartheta)\,\mathrm{d}\vartheta, \quad i = 0, 1. \tag{3.1}$$

Introducing the abbreviation

$$\mathbb{A}(z,\theta) = \theta\alpha_1'(\theta)\gamma'(z) + \widehat{c}_1(\theta)\gamma'(z),$$

we can rewrite the heat equation (2.11a) into the form

$$\begin{aligned}\dot{w} - \mathrm{div}\big(\mathbb{K}(z,\theta)\nabla\theta\big) = \delta_Z^*(\dot{z}) + D\dot{z}\cdot\dot{z} + \delta_M^*(\dot{\boldsymbol{m}}) + \alpha|\dot{\boldsymbol{m}}|^2 + \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\\ +\mathbb{A}(z,\theta)\dot{z} + a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}}.\end{aligned} \tag{3.2}$$

Note that, due to the natural assumption about positive heat capacity (3.5i) below, $\omega(z,\cdot)$ is invertible and we can express θ in terms of w and z, thus eliminating it like in [83]. Here, however, e.g. in [19], we keep both variables w and θ in the formulation and mostly in the analysis of the problem. This turns out to simplify the formulation, the assumptions, and some arguments. By applying Green's formula, the curl-formula, and by-part integration in time, we obtain the following definition.

Definition 1 (*Weak solution*) We say that $(\theta, z, \boldsymbol{m}, \boldsymbol{u}, \boldsymbol{h}, \boldsymbol{e})$ is a *weak solution* to the initial boundary value problem for system (2.11) with the initial and boundary conditions (2.12), along with the selections $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$, if

$$\theta \in L^1(I; W^{1,1}(\Omega)) \cap C(\bar{I}; L^1(\Omega)), \tag{3.3a}$$

$$z \in H^1(I; L^2(\Omega;\mathbb{R}^N)) \cap L^\infty(I; H^1(\Omega;\mathbb{R}^N)) \cap L^2(I; H^2_{\mathrm{N}}(\Omega;\mathbb{R}^N)), \quad z \in K \text{ a.e. in } Q, \tag{3.3b}$$

$$\boldsymbol{m} \in H^1(I; L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(I; H^1(\Omega;\mathbb{R}^3)) \cap L^2(I; H^2_{\mathrm{N}}(\Omega;\mathbb{R}^3)), \quad \boldsymbol{m} \in S \text{ a.e. in } Q, \tag{3.3c}$$

$$\boldsymbol{u} \in H^1(I; H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)), \tag{3.3d}$$

$$\boldsymbol{h} \in L^\infty(I; L^2_{\mathrm{curl}}(\Omega;\mathbb{R}^3)) \text{ and } \boldsymbol{h}-\boldsymbol{h}_{\mathrm{b}} \in L^2(I; L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3)), \tag{3.3e}$$

$$\boldsymbol{e} \in L^2(Q;\mathbb{R}^3), \tag{3.3f}$$

$$\boldsymbol{\eta}_1 \in L^\infty(Q;\mathbb{R}^N), \qquad \boldsymbol{\eta}_1 \in \partial\delta_Z^*(\dot{z}) \quad \text{a.e. in } Q, \tag{3.3g}$$

$$\boldsymbol{\eta}_2 \in L^\infty(Q;\mathbb{R}^3), \qquad \boldsymbol{\eta}_2 \in \partial\delta_M^*(\dot{\boldsymbol{m}}) \quad \text{a.e. in } Q, \tag{3.3h}$$

$$\boldsymbol{\xi}_1 \in L^2(Q;\mathbb{R}^N), \qquad \boldsymbol{\xi}_1 \in N_K(z) \quad \text{a.e. in } Q, \tag{3.3i}$$

$$\boldsymbol{\xi}_2 \in L^2(Q;\mathbb{R}^3), \qquad \boldsymbol{\xi}_2 \in N_S(\boldsymbol{m}) \quad \text{a.e. in } Q, \tag{3.3j}$$

and, for ω from (3.1),

$$\omega(z,\theta) \in L^\infty(I; L^1(\Omega)) \cap L^r(I; W^{1,r}(\Omega)) \quad \text{for some } 1 \le r < 5/4, \tag{3.3k}$$

and, with $w_0 = \omega(z_0, \theta_0)$, the following hold

$$\int_Q \Big(\mathbb{K}(z,\theta)\nabla\theta\cdot\nabla\widetilde{w} - w\dot{\widetilde{w}}\Big)\mathrm{d}x\mathrm{d}t = \int_Q \Big(\delta_Z^*(\dot{z}) + D\dot{z}\cdot\dot{z} + \delta_M^*(\dot{\boldsymbol{m}}) + \alpha|\dot{\boldsymbol{m}}|^2 + \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e} + \mathbb{A}(z,\theta)\dot{z}$$

$$+ a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}}\Big)\widetilde{w}\,\mathrm{d}x\mathrm{d}t + \int_\Sigma f_\mathrm{b}\widetilde{w}\,\mathrm{d}S\mathrm{d}t + \int_\Omega w_0\widetilde{w}(0)\,\mathrm{d}x \forall\widetilde{w} \in W^{1,\infty}(Q) \text{ with } \widetilde{w}(\cdot,T) = 0, \tag{3.4a}$$

$$\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - E_\mathrm{tr}(z(t))):\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}})\,\mathrm{d}x = 0 \quad \forall\widetilde{\boldsymbol{u}} \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3), \quad t \in (0,T), \tag{3.4b}$$

$$\boldsymbol{\eta}_1 + D\dot{z} + E_\mathrm{tr}{}^\top\mathbb{C}(E_\mathrm{tr}(z) - \boldsymbol{\varepsilon}(\boldsymbol{u})) + \mathbb{H}z + \partial_z\psi_\mathrm{C}(z,\boldsymbol{m})$$
$$-\kappa_1\Delta z + \boldsymbol{\xi}_1 = -\alpha_1\theta\gamma'(z) \quad \text{a.e. in } Q, \tag{3.4c}$$

$$\boldsymbol{\eta}_2 + \alpha\dot{\boldsymbol{m}} + \partial_{\boldsymbol{m}}\psi_\mathrm{C}(z,\boldsymbol{m}) - \kappa_2\Delta\boldsymbol{m} + \boldsymbol{\xi}_2 = \mu_0\boldsymbol{h} - a_0\theta\boldsymbol{m} \quad \text{a.e. in } Q, \tag{3.4d}$$

$$\int_Q \Big(\boldsymbol{e}\cdot\mathrm{curl}\,\widetilde{\boldsymbol{v}} - \mu_0(\boldsymbol{h}+\boldsymbol{m})\cdot\dot{\widetilde{\boldsymbol{v}}}\Big)\mathrm{d}x\mathrm{d}t = \mu_0\int_\Omega (\boldsymbol{h}_0+\boldsymbol{m}_0)\cdot\widetilde{\boldsymbol{v}}(0,\cdot)\,\mathrm{d}x$$
$$\forall\widetilde{\boldsymbol{v}} \in H^1(I; L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3)) \text{ with } \widetilde{\boldsymbol{v}}(\cdot,T) = \boldsymbol{0}, \tag{3.4e}$$

$$\mathrm{curl}\,\boldsymbol{h} = \mathbb{S}(z,\theta)\boldsymbol{e} \quad \text{and} \quad w = \omega(z,\theta) \quad \text{a.e. in } Q, \tag{3.4f}$$

together with the initial conditions (2.12a) for z and $\boldsymbol{m}$, while the initial conditions for w and $\boldsymbol{h}$ are already included in relations (3.4a) and (3.4e).

Let us remark that all weak solutions to the initial boundary value problem for system (2.11) can be checked to preserve energy. In particular, one can reproduce the argument of Sect. 2.9 and obtain an analogous energy balance, written in terms of w instead of θ, cf. (3.14) and (3.15) below.

3.2 Data qualification

Before moving on, let us enlist here the assumptions on the external loading data and on nonlinearities (i.e. data determining the material properties) that are going to be used in the sequel of the paper. We shall ask for the following:

Initial data:

$$w_0 \in L^1(\Omega), \qquad w_0 \geq 0 \quad \text{a.e. in } \Omega, \tag{3.5a}$$
$$z_0 \in H^1(\Omega;\mathbb{R}^N), \qquad z_0 \in K \quad \text{a.e. in } \Omega, \tag{3.5b}$$
$$\boldsymbol{m}_0 \in H^1(\Omega;\mathbb{R}^3), \qquad \boldsymbol{m}_0 \in S \quad \text{a.e. in } \Omega, \tag{3.5c}$$
$$\boldsymbol{h}_0 \in L^2_\mathrm{curl}(\Omega;\mathbb{R}^3). \tag{3.5d}$$

Boundary data $\boldsymbol{h}_\mathrm{b}$ and $\boldsymbol{j}_\mathrm{b}$:

$$f_\mathrm{b} \in L^1(\Sigma), \quad f_\mathrm{b} \geq 0, \tag{3.5e}$$
$$\exists\boldsymbol{h}_\mathrm{b} \in L^2(I; L^2_\mathrm{curl}(\Omega;\mathbb{R}^3)) \cap W^{1,1}(I; L^2(\Omega;\mathbb{R}^3)) \quad \text{such that } -\boldsymbol{\nu}\times\boldsymbol{h}_\mathrm{b} = \boldsymbol{j}_\mathrm{b}. \tag{3.5f}$$

Nonlinearities:

$$c_0,\ c_1,\ \alpha_1,\ \psi_\mathrm{C},\ E_\mathrm{tr}, \mathbb{K},\ \mathbb{S} \text{ are Lipschitz continuous} \tag{3.5g}$$
$$\gamma \in C^{1,1}(K), \tag{3.5h}$$
$$\exists\, M,\ \kappa_0 > 0,\ \tfrac{3}{2} > \eta > 0\ \forall(z,\theta) \in K\times\mathbb{R},\ \boldsymbol{\xi} \in \mathbb{R}^3:$$
$$0 < \kappa_0(1+\theta^+) \leq c_0(\theta) + c_1(\theta)\gamma(z) \leq M, \tag{3.5i}$$
$$\mathbb{K}(z,\theta)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq \kappa_0|\boldsymbol{\xi}|^2 \quad \text{and} \quad \mathbb{S}(z,\theta)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq \kappa_0|\boldsymbol{\xi}|^2, \tag{3.5j}$$
$$|\mathbb{K}(z,\theta)| \leq M(1+|\theta|^\eta) \quad \text{and} \quad |\mathbb{S}(z,\theta)| \leq M. \tag{3.5k}$$

As already commented above, assumption (3.5i) guarantees that $\omega(z,\cdot)$ from (3.1) is invertible. Let us denote its inverse by $\mathscr{T}(z,\cdot) : w \mapsto \theta$ so that

$$\mathscr{T}(z,w) = \big[\omega(z,\cdot)\big]^{-1}(w).$$

By exploiting the smoothness of $c_0,\ c_1$, and γ, one readily checks that $\mathscr{T} \in C^{1,1}(K \times \mathbb{R})$. In particular, we have by (3.5i) that

$$0 < \frac{1}{M} \le \partial_w \mathscr{T}(z,w) = \frac{1}{c_0(\mathscr{T}(z,w)) + c_1(\mathscr{T}(z,w))\gamma(z)} \le \frac{1}{\kappa_0} \tag{3.6}$$

so that $\partial_z \mathscr{T}(z,w) = -\partial_w \mathscr{T}(z,w)\gamma'(z)$ is bounded as well. Moreover, by using $\mathscr{T}(z,0) = 0$ (recall that $\widehat{c_i}(0) = 0$) one has that

$$\mathscr{T}(z,w) \ge 0 \iff w \ge 0.$$

Moreover, by redefining $c_0 = c_1 = 0$ on $(-\infty, 0]$ one has that

$$\forall w \ge 0 : \quad \mathscr{T}(z,-w) = 0 \tag{3.7}$$

so that $\mathscr{T} \ge 0$. Hence, by using again (3.5i) one gets

$$\mathscr{T}(z,w) \le |w|\,|\partial_w \mathscr{T}(z,w)| \le \frac{|w|}{\kappa_0(1+\mathscr{T}(z,w))}$$

and the bound

$$\sqrt{\kappa_0}\mathscr{T}(z,w) \le |w|^{1/2} \tag{3.8}$$

ensues. In the following, we will occasionally use the shorthand notation

$$\mathscr{K}_0(z,w) = \mathbb{K}(z, \mathscr{T}(z,w))\,\partial_w \mathscr{T}(z,w), \tag{3.9a}$$
$$\mathscr{K}_1(z,w) = \mathbb{K}(z, \mathscr{T}(z,w))\,\partial_z \mathscr{T}(z,w), \quad \text{and} \tag{3.9b}$$
$$\mathscr{A}(z,w) = \mathbb{A}(z, \mathscr{T}(z,w)). \tag{3.9c}$$

Let us record here some properties of the latter nonlinearities to be used later on. First of all, let us check that $\mathscr{K}_0$ is coercive

$$\mathscr{K}_0(z,w)\boldsymbol{\xi}\cdot\boldsymbol{\xi} = \mathbb{K}(z, \mathscr{T}(z,w))\,\partial_w \mathscr{T}(z,w)\,\boldsymbol{\xi}\cdot\boldsymbol{\xi} \ge \frac{1}{M}\kappa_0|\boldsymbol{\xi}|^2 \tag{3.10}$$

where we have used (3.5j) and (3.6). Secondly, let us point out that

$$|\mathscr{K}_0(z,w)| + |\mathscr{K}_1(z,w)| \le C\big(1+|w|^\eta\big), \tag{3.11}$$

which follow from (3.5g)–(3.5h), (3.5k), and (3.6)–(3.8) for some suitable data-dependent constant C by arguing as

$$\begin{aligned}|\mathscr{K}_0(z,w)| = |\mathbb{K}(z, \mathscr{T}(z,w))\,\partial_w \mathscr{T}(z,w)| &\le M\big(1+|\mathscr{T}(z,w)|^\eta\big)\frac{1}{\kappa_0}\\ &\le M\big(1+|w|^\eta|\partial_w \mathscr{T}(z,w)|^\eta\big)\frac{1}{\kappa_0} \le C\big(1+|w|^\eta\big)\end{aligned}$$

Finally, one directly checks that, for all $w \ge 0$, one has

$$\mathscr{A}(z,-w), \qquad \mathscr{K}_1(z,-w) = \mathbf{0}. \tag{3.12}$$

3.3 Existence result

Let us now formulate our main analytical result:

Theorem 1 *(*Existence of weak solutions*) Under assumptions* (3.5) *there exists a weak solution* $(\theta, z, \boldsymbol{m}, \boldsymbol{u}, \boldsymbol{h}, \boldsymbol{e})$ *to the initial and boundary value problem* (2.11)–(2.12) *due to Definition 1. Moreover, we have that* $\theta \geq 0$ *and, with* w *from* (3.1),

$$\dot{w} \in L^1(I; H^3(\Omega)^*), \qquad \dot{\boldsymbol{h}} \in L^2(I; L^2_{\mathrm{curl},0}(\Omega; \mathbb{R}^3)^*), \tag{3.13}$$

and this solution fulfils the following balance for the magnetomechanical energy

$$\Psi_{\mathrm{MM}}(t) = \int_\Omega \Big(\psi_{\mathrm{MEC}}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), z(t)\big) + \psi_{\mathrm{MAG}}\big(\boldsymbol{b}(t), \boldsymbol{m}(t)\big) + \psi_{\mathrm{COUP}}\big(\boldsymbol{m}(t), z(t)\big) + \psi_{\mathrm{NL}}\big(\nabla z(t), \nabla \boldsymbol{m}(t)\big) + \psi_{\mathrm{CONST}}\big(z(t), \boldsymbol{m}(t)\big)\Big) \mathrm{d}x,$$

cf. also (2.14)*, as*

$$\Psi_{\mathrm{MM}}(t) + \int_0^t \int_\Omega \delta^*_Z(\dot{z}) + D\dot{z}\cdot\dot{z} + \delta^*_M(\dot{\boldsymbol{m}}) + \alpha|\dot{\boldsymbol{m}}|^2 + \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\,\mathrm{d}x\mathrm{d}t = \Psi_{\mathrm{MM}}(0) + \int_0^t \left(\int_\Gamma \boldsymbol{j}_{\mathrm{b}}\cdot\boldsymbol{e}\,\mathrm{d}S - \int_\Omega \alpha_1(\theta)\gamma'(z)\cdot\dot{z} + a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}}\mathrm{d}x\right)\mathrm{d}t \tag{3.14}$$

for any $t \in I$*, as well as the total energy conservation, cf. also* (2.16) *i.e. with* w_0 *from Definition 1,*

$$\Psi_{\mathrm{MM}}(t) + \int_\Omega w(t)\,\mathrm{d}x = \Psi_{\mathrm{MM}}(0) + \int_\Omega w_0\,\mathrm{d}x + \int_0^t\int_\Gamma \boldsymbol{j}_{\mathrm{b}}\cdot\boldsymbol{e} + f_{\mathrm{b}}\,\mathrm{d}S\mathrm{d}t. \tag{3.15}$$

Let us note that giving sense to the integrals $\int_0^t\int_\Gamma \boldsymbol{j}_{\mathrm{b}}\cdot\boldsymbol{e}\,\mathrm{d}S$ in (3.14) and in (3.15) is a bit tricky because $\boldsymbol{e}$ itself does not have well-defined traces on Γ. The first idea is to understand it rather in sense of the bulk integral $\int_0^t\int_\Omega(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b}}\cdot\boldsymbol{e} - \boldsymbol{h}_{\mathrm{b}}\cdot\mathrm{curl}\,\boldsymbol{e})\mathrm{d}x\mathrm{d}t$, cf. also (2.13). Even more, since $\mathrm{curl}\,\boldsymbol{e} = -\mu_0(\dot{\boldsymbol{h}}+\dot{\boldsymbol{m}})$ is only in $L^2(I; L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3)^*)$, cf. (3.13), this last integral is rather only a duality and, as $\boldsymbol{h}_{\mathrm{b}} \notin L^2(I; L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3))$, like in [83], we should still employ the by-part integration in time i.e. eventually

$$\int_0^t\int_\Gamma \boldsymbol{j}_{\mathrm{b}}\cdot\boldsymbol{e}\,\mathrm{d}S\mathrm{d}t := \int_0^t\int_\Omega \big(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b}}\cdot\boldsymbol{e} + \mu_0\boldsymbol{h}_{\mathrm{b}}\cdot\dot{\boldsymbol{m}} - \mu_0\dot{\boldsymbol{h}}_{\mathrm{b}}\cdot\boldsymbol{h}\big)\mathrm{d}x\mathrm{d}t + \mu_0\int_\Omega \big(\boldsymbol{h}_{\mathrm{b}}(t)\cdot\boldsymbol{h}(t) - \boldsymbol{h}_{\mathrm{b}}(0)\cdot\boldsymbol{h}_0\big)\mathrm{d}x, \tag{3.16}$$

relying on the qualification of $\boldsymbol{h}_{\mathrm{b}} \in W^{1,1}(I; L^2(\Omega;\mathbb{R}^3))$, cf. (3.5f) above.

This existence result will be proved in the rest of the paper by means of a regularization and time-discretization argument. In particular, we discuss the approximation of the problem in Sect. 4. Suitable a priori estimates are then established in Sect. 5. Eventually, the passage to the limit is performed in Sect. 6. Note that as a by-product of the existence analysis, we are advancing a stable and convergence discretization scheme which might be of some independent applicative interest: in particular, it suggests a conceptual numerical strategy when implementing an additional spatial discretization and an iterative procedure to solve approximately the nonconstructive fixed point problem arising, in fact, in the proof of Lemma 1.

4 Time discretization

As already mentioned, we tackle the existence of a weak solution to system (2.11) in the sense of Definition 1 by considering a suitable time discretization combined with a regularization. Let us introduce an equi-distant partition of the interval $[0, T]$ with time-step $\tau > 0$, assuming $T/\tau \in \mathbb{N}$. By making use of the notation $\mathrm{D}_\tau^k v = (v_\tau^k - v_\tau^{k-1})/\tau$ for the increment, we shall be considering the fully implicit discretized and regularized system:

$$\begin{aligned}
&\mathrm{D}_\tau^k w - \mathrm{div}\big(\mathbb{K}(z_\tau^k, \theta_\tau^k)\nabla\theta_\tau^k\big) = \delta_Z^*(\mathrm{D}_\tau^k z_\tau^k) + D\mathrm{D}_\tau^k z{\cdot}\mathrm{D}_\tau^k z + \delta_M^*(\mathrm{D}_\tau^k \boldsymbol{m}) + \alpha|\mathrm{D}_\tau^k \boldsymbol{m}|^2 \\
&\quad + \frac{\mathbb{S}(z_\tau^k, \theta_\tau^k)\boldsymbol{e}_\tau^k{\cdot}\boldsymbol{e}_\tau^k}{1{+}\tau|\boldsymbol{e}_\tau^k|^2} + \frac{\mathbb{A}(z_\tau^k, \theta_\tau^k)\mathrm{D}_\tau^k z}{1 + \tau|\mathbb{A}(z_\tau^k, \theta_\tau^k)|} + a_0\theta_\tau^k \boldsymbol{m}_\tau^k{\cdot}\mathrm{D}_\tau^k \boldsymbol{m} =: r_\tau^k, && \text{(4.1a)}\\
&\mathrm{div}\,\mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\tau^k){-}E_{\mathrm{tr}}(z_\tau^k)) = \boldsymbol{0}, && \text{(4.1b)}\\
&\partial\delta_Z^*(\mathrm{D}_\tau^k z){+}D\mathrm{D}_\tau^k z{+}E_{\mathrm{tr}}{}^{\top}\mathbb{C}\big(E_{\mathrm{tr}}(z_\tau^k){-}\boldsymbol{\varepsilon}(\boldsymbol{u}_\tau^k)\big){+}\mathbb{H}z_\tau^k{+}\partial_z\psi_{\mathrm{C}}(\boldsymbol{m}_\tau^k, z_\tau^k)\\
&\quad - \kappa_1\Delta z_\tau^k{+}N_K(z_\tau^k) \ni -\alpha_1(\theta_\tau^k)\gamma'(z_\tau^k), && \text{(4.1c)}\\
&\partial\delta_M^*(\mathrm{D}_\tau^k \boldsymbol{m}){+}\alpha\mathrm{D}_\tau^k \boldsymbol{m}{+}\partial_{\boldsymbol{m}}\psi_{\mathrm{C}}(\boldsymbol{m}_\tau^k, z_\tau^k){-}\kappa_2\Delta\boldsymbol{m}_\tau^k{+}N_S(\boldsymbol{m}_\tau^k) \ni \mu_0\boldsymbol{h}_\tau^k{-}a_0\theta_\tau^k\boldsymbol{m}_\tau^k, && \text{(4.1d)}\\
&\mu_0(\mathrm{D}_\tau^k \boldsymbol{h} + \mathrm{D}_\tau^k \boldsymbol{m}) + \mathrm{curl}\,\boldsymbol{e}_\tau^k = \boldsymbol{0}, && \text{(4.1e)}\\
&\mathrm{curl}\,\boldsymbol{h}_\tau^k = \mathbb{S}(z_\tau^k, \theta_\tau^k)\boldsymbol{e}_\tau^k, && \text{(4.1f)}\\
&w_\tau^k = \omega(z_\tau^k, \theta_\tau^k). && \text{(4.1g)}
\end{aligned}$$

The mentioned regularization consist in the approximation of the unbounded terms $\mathbb{S}(z, \theta)\boldsymbol{e}{\cdot}\boldsymbol{e}$ and $\mathbb{A}(z_\tau^k, \theta_\tau^k)$ by the bounded terms $\mathbb{S}(z, \theta)\boldsymbol{e}{\cdot}\boldsymbol{e}/(1{+}\tau|\boldsymbol{e}|^2)$ and $\mathbb{A}(z_\tau^k, \theta_\tau^k)/(1 + \tau|\mathbb{A}(z_\tau^k, \theta_\tau^k)|)$, respectively. Note that these regularizations depend on the time-step τ and vanish in the limit for $\tau \to 0$.

The initial and boundary conditions (2.12) are discretized by letting

$$\begin{aligned}
&z_\tau^0 = z_0, \quad w_\tau^0 = w_0 = \omega(z_0, \theta_0), \quad \boldsymbol{m}_\tau^0 = \boldsymbol{m}_0, \quad \boldsymbol{h}_\tau^0 = \boldsymbol{h}_0, && \text{(4.2a)}\\
&\boldsymbol{u}_\tau^k = \boldsymbol{0} \ \text{ on } \Gamma_{\mathrm{D}}, \qquad \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\tau^k){-}E_{\mathrm{tr}}(z_\tau^k))\boldsymbol{\nu} = \boldsymbol{0} \ \text{ on } \Gamma_{\mathrm{N}}, && \text{(4.2b)}\\
&\frac{\partial z_\tau^k}{\partial\boldsymbol{\nu}} = \boldsymbol{0}, \quad \frac{\partial\boldsymbol{m}_\tau^k}{\partial\boldsymbol{\nu}} = \boldsymbol{0}, \quad \big(\mathbb{K}(z_\tau^k, \theta_\tau^k)\nabla\theta_\tau^k\big){\cdot}\boldsymbol{\nu} = f_{\mathrm{b},\tau}^k, \quad \boldsymbol{\nu}{\times}\boldsymbol{h}_\tau^k = \boldsymbol{j}_{\mathrm{b},\tau}^k \ \text{ on } \Gamma, && \text{(4.2c)}
\end{aligned}$$

where we have used $f_{\mathrm{b},\tau}^k := (1/\tau)\int_{(k-1)\tau}^{k\tau} f_{\mathrm{b}}(t)\,\mathrm{d}t$ and $\boldsymbol{j}_{\mathrm{b},\tau}^k := (1/\tau)\int_{(k-1)\tau}^{k\tau} \boldsymbol{j}_{\mathrm{b}}(t)\,\mathrm{d}t$. Our first result concerns the solvability of the discrete system (4.1): We shall find a vector

$$\big(\theta_\tau^k, z_\tau^k, \boldsymbol{m}_\tau^k, \boldsymbol{u}_\tau^k, \boldsymbol{h}_\tau^k, \boldsymbol{e}_\tau^k\big) \in \mathscr{H} := H^1\big(\Omega; \mathbb{R}{\times}\mathbb{R}^N{\times}\mathbb{R}^3{\times}\mathbb{R}^3\big) \times L^2_{\mathrm{curl}}\big(\Omega; \mathbb{R}^3\big) \times L^2_{\mathrm{curl}}\big(\Omega; \mathbb{R}^3\big) \tag{4.3}$$

solving system (4.1) in a suitable weak sense, see (4.10) below. In particular, measurable selections of the involved multivalued mappings are to be found. We have the following.

Lemma 1 (Existence of discrete solutions) *Under assumptions* (3.5)*, the boundary value system* (4.1)–(4.2) *possesses at least one weak solution. Moreover, all such solutions fulfil* $w_\tau^k \geq 0$ *and* $\theta_\tau^k \geq 0$ *a.e. on* Ω*.*

Sketch of the Proof. We apply the abstract existence theorem for equations involving set-valued nonlinear pseudomonotone coercive operators from Hilbert spaces to the corresponding duals. In particular, we let the base space be $\mathscr{H}$ and consider the mapping defined by system (4.1) (along with the conditions (4.2)) from the six-tuple $(\theta, z, \boldsymbol{m}, \boldsymbol{u}, \boldsymbol{h}, \boldsymbol{e})$ to the dual of $\mathscr{H}$.

To prove the coercivity of the underlying nonlinear operator, the equations in (4.1) are to be tested, respectively, by θ_τ^k, z_τ^k, $\boldsymbol{m}_\tau^k, \boldsymbol{u}_\tau^k$, $\boldsymbol{h}_\tau^k$, and $-\boldsymbol{e}_\tau^k$. The nonmonotone terms are to be estimated by Hölder's and Young's inequalities by exploiting the L^∞-boundedness of z_τ^k and $\boldsymbol{m}_\tau^k$ (due to the boundedness of K and S, respectively). Moreover, we use the boundedness of the regularized Joule-heat term (i.e. the $\mathbb{S}$-term) in (4.1a). The remaining terms are quite easily controlled. In particular, the mentioned test of the Maxwell equations (4.1e) by $\boldsymbol{h}_\tau^k$ and (4.1f) by $-\boldsymbol{e}_\tau^k$ uses the property

$$\int_\Omega \Big(\mathrm{curl}\,\boldsymbol{h}_\tau^k{\cdot}\boldsymbol{e}_\tau^k - \boldsymbol{h}_\tau^k\cdot\mathrm{curl}\,\boldsymbol{e}_\tau^k\Big)\,\mathrm{d}x = \int_\Gamma (\boldsymbol{\nu}{\times}\boldsymbol{h}_\tau^k){\cdot}\boldsymbol{e}_\tau^k\,\mathrm{d}S = \int_\Gamma \boldsymbol{j}_{\mathrm{b},\tau}^k{\cdot}\boldsymbol{e}_\tau^k\,\mathrm{d}S = -\int_\Gamma (\boldsymbol{e}_\tau^k{\times}\boldsymbol{h}_{\mathrm{b},\tau}^k){\cdot}\boldsymbol{\nu}\,\mathrm{d}S \tag{4.4}$$

where the last equality follows from having defined $\boldsymbol{h}_{\mathrm{b},\tau}^k = -\boldsymbol{\nu}\times\boldsymbol{j}_{\mathrm{b},\tau}^k$ and by using the estimate

$$\int_\Gamma (\boldsymbol{\nu}\times\boldsymbol{h}_\tau^k)\cdot\boldsymbol{e}_\tau^k\,\mathrm{d}S \le C\big\|\boldsymbol{h}_{\mathrm{b},\tau}^k\big\|_{L^2_{\mathrm{curl}}(\Omega;\mathbb{R}^3)}\big\|\boldsymbol{e}_\tau^k\big\|_{L^2_{\mathrm{curl}}(\Omega;\mathbb{R}^3)},$$

cf. [83] for details. The term $\int_\Omega \mathrm{D}_\tau^k w\theta_\tau^k \mathrm{d}x$ gives rise in particular to the term $\frac{1}{\tau}\int_\Omega \omega(z_\tau^k,\theta_\tau^k)\theta_\tau^k\mathrm{d}x$ which can be estimated from below by $\frac{\epsilon}{\tau}\|\theta_\tau^k\|^2_{L^2(\Omega)}$ with ϵ from (3.5i), and ensures full coercivity (also with respect to constants). Altogether, by taking into account the positive-definiteness assumption (3.10), the coercive left-hand side terms control $\|\cdot\|^2_{\mathscr{H}}$ with the Hilbert space $\mathscr{H}$ from (4.3). This dominates the growth of the remaining nonmonotone terms on the right-hand side of the heat equation (4.1a) by using the mentioned boundedness of K and S. $\mathbb{A}(z,\cdot)$. Moreover, from a comparison in (4.1e) and (4.1f), we also obtain estimates on curl $\boldsymbol{e}_\tau^k$ in $L^2(\Omega;\mathbb{R}^3)$ and curl $\boldsymbol{h}_\tau^k$ in $L^2(\Omega;\mathbb{R}^3)$. Hence, coercivity on the spaces indicated in (4.3) and the validity of the formula (4.4) follow.

The pseudomonotonicity of the full operator follows because all nonmonotone terms are of lower order except the term $\mathbb{S}$ in (4.1a). We just need to show weak–strong continuity. To this aim, by letting $(\theta_i, z_i, \boldsymbol{m}_i, \boldsymbol{u}_i, \boldsymbol{h}_i, \boldsymbol{e}_i)$ denote some weakly $\mathscr{H}$-converging sequence to some limit $(\theta, z, \boldsymbol{m}, \boldsymbol{u}, \boldsymbol{h}, \boldsymbol{e})$ and using relations (4.1e)–(4.1f), it suffices to check that

$$\begin{aligned}
&\limsup_{i\to\infty}\int_\Omega \mathbb{S}(z_i,\theta_i)(\boldsymbol{e}_i-\boldsymbol{e})\cdot(\boldsymbol{e}_i-\boldsymbol{e})\,\mathrm{d}x\\
&= \limsup_{i\to\infty}\int_\Omega \mathbb{S}(z_i,\theta_i)\boldsymbol{e}_i\cdot\boldsymbol{e}_i\,\mathrm{d}x + \lim_{i\to\infty}\int_\Omega \mathbb{S}(z_i,\theta_i)(\boldsymbol{e}-2\boldsymbol{e}_i)\cdot\boldsymbol{e}\,\mathrm{d}x\\
&= \limsup_{i\to\infty}\int_\Omega \mathrm{curl}\,\boldsymbol{h}_i\cdot\boldsymbol{e}_i\,\mathrm{d}x + \int_\Omega \mathbb{S}(z,\theta)(\boldsymbol{e}-2\boldsymbol{e})\cdot\boldsymbol{e}\,\mathrm{d}x\\
&= \limsup_{i\to\infty}\int_\Omega \Big(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e}_i + \mathrm{curl}\,(\boldsymbol{h}_i-\boldsymbol{h}_{\mathrm{b},\tau}^k)\cdot\boldsymbol{e}_i\Big)\,\mathrm{d}x - \int_\Omega \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\,\mathrm{d}x\\
&= \limsup_{i\to\infty}\int_\Omega \Big(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e}_i + (\boldsymbol{h}_i-\boldsymbol{h}_{\mathrm{b},\tau}^k)\cdot\mathrm{curl}\,\boldsymbol{e}_i\Big)\,\mathrm{d}x - \int_\Omega \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\,\mathrm{d}x\\
&= \limsup_{i\to\infty}\int_\Omega \bigg(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e}_i - \mu_0(\boldsymbol{h}_i-\boldsymbol{h}_{\mathrm{b},\tau}^k)\cdot\bigg(\frac{\boldsymbol{h}_i+\boldsymbol{m}_i}{\tau} - \frac{\boldsymbol{h}_\tau^{k-1}+\boldsymbol{m}_\tau^{k-1}}{\tau}\bigg)\bigg)\mathrm{d}x - \int_\Omega \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\,\mathrm{d}x\\
&\le \int_\Omega \bigg(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e} - \mu_0(\boldsymbol{h}-\boldsymbol{h}_{\mathrm{b},\tau}^k)\cdot\bigg(\frac{\boldsymbol{h}+\boldsymbol{m}}{\tau} - \frac{\boldsymbol{h}_\tau^{k-1}+\boldsymbol{m}_\tau^{k-1}}{\tau}\bigg) - \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\bigg)\mathrm{d}x\\
&= \int_\Omega \Big(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e} + (\boldsymbol{h}-\boldsymbol{h}_{\mathrm{b},\tau}^k)\cdot\mathrm{curl}\,\boldsymbol{e} - \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\Big)\,\mathrm{d}x\\
&= \int_\Omega \Big(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e} + \mathrm{curl}\,(\boldsymbol{h}-\boldsymbol{h}_{\mathrm{b},\tau}^k)\cdot\boldsymbol{e} - \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\Big)\,\mathrm{d}x = 0
\end{aligned} \tag{4.5}$$

where we used $\boldsymbol{h}_i \to \boldsymbol{h}$ weakly in $L^2(\Omega;\mathbb{R}^3)$, $\boldsymbol{m}_i \to \boldsymbol{m}$ strongly in $L^2(\Omega;\mathbb{R}^3)$, and the weak upper semi-continuity of the functional $\boldsymbol{h} \mapsto \int_\Omega -\mu_0\boldsymbol{h}\cdot\boldsymbol{h}/\tau\,\mathrm{d}x$. Hence, $\boldsymbol{e}_i \to \boldsymbol{e}$ strongly in $L^2(\Omega;\mathbb{R}^3)$ and the weak–strong continuity of the $\mathbb{S}$-term in (4.1a) follows. Then, the claimed existence follows standardly by theory of pseudomonotone operators using the classical Brézis theorem [20], possibly generalized for set-valued mappings having a convex potential, cf. e.g. [80, Sect. 5.3]. Eventually, one can test (4.1a) by $-(\theta_\tau^k)^- = \min(\theta_\tau^k, 0)$ and exploit (3.12) and the sign of f_b from (3.5e) in order to prove that $\theta_\tau^k \ge 0$ almost everywhere. In view of (3.5i), the mapping $\omega(z,\cdot)$ from (3.1) is increasing and, as $\omega(z,0)=0$, we have also $w_\tau^k = \omega(z_\tau^k,\theta_\tau^k) \ge 0$. □

Our next aim is that of specifying the discrete analogue of relations (3.4) for the time-discrete weak solutions. In order to do so, we shall preliminarily observe that indeed the Biot-type relations (4.1c)–(4.1d) are here actually solved strongly. In particular, by letting $\boldsymbol{\eta}_{1,\tau}^k$ and $\boldsymbol{\eta}_{2,\tau}^k$ be selections in $\partial\delta_Z^*(\mathrm{D}_\tau^k \boldsymbol{z})$ and $\partial\delta_M^*(\mathrm{D}_\tau^k \boldsymbol{m})$, respectively, and observing that $\boldsymbol{\eta}_{i,\tau}^k \in L^\infty$, the bounds in (3.5g) and (3.11) entail that both $\boldsymbol{z}_\tau^k$ and $\boldsymbol{m}_\tau^k$ solve a relation of the form

$$-\kappa\Delta\boldsymbol{g} + N_C(\boldsymbol{g}) \ni \boldsymbol{f} \tag{4.6}$$

in the dual of some H^1-space, where $\boldsymbol{f} \in L^2$ and C is a nonempty, convex, and closed set. In particular, there exists a measurable selection $\boldsymbol{\xi}$ such that $\boldsymbol{\xi} \in N_C(\boldsymbol{g})$ almost everywhere. It is hence straightforward to check that both $\boldsymbol{\xi}$ and $-\Delta\boldsymbol{g}$ belong to L^2 as a consequence of the monotonicity of the normal cone. The rigorous proof needs a smoothening argument: one can use an exterior penalty $\delta_{C,\epsilon}(\boldsymbol{g}) := \epsilon^{-1}\min_{\widetilde{\boldsymbol{g}}\in C}|\boldsymbol{g}-\widetilde{\boldsymbol{g}}|^2$ i.e. the Yosida approximation of δ_C, and consider the Dirichlet boundary value problem $\kappa\Delta\boldsymbol{g}_\epsilon - \delta_{C,\epsilon}'(\boldsymbol{g}_\epsilon) = \boldsymbol{f}$ with the boundary condition $\boldsymbol{g}_\epsilon = \boldsymbol{g}$ on Γ with $\boldsymbol{g}$ solving (4.6), which ensures $\delta_{C,\epsilon}'(\boldsymbol{g}_\epsilon) = 0$ on Γ so that the boundary term arising by the test by $\Delta\boldsymbol{g}_\epsilon$ disappears, which allows for estimation

$$\begin{aligned}\kappa\big\|\Delta\boldsymbol{g}_\epsilon\big\|_{L^2(\Omega)}^2 &\le \int_\Omega \kappa|\Delta\boldsymbol{g}_\epsilon|^2\,\mathrm{d}x + \delta_{C,\epsilon}''(\boldsymbol{g}_\epsilon)\nabla\boldsymbol{g}_\epsilon{\cdot}\nabla\boldsymbol{g}_\epsilon\,\mathrm{d}x\\ &= \int_\Omega \kappa|\Delta\boldsymbol{g}_\epsilon|^2\,\mathrm{d}x + \delta_{C,\epsilon}''(\boldsymbol{g}_\epsilon)\nabla\boldsymbol{g}_\epsilon{\cdot}\nabla\boldsymbol{g}_\epsilon\,\mathrm{d}x - \int_\Gamma \delta_{C,\epsilon}'(\boldsymbol{g}_\epsilon)\nabla\boldsymbol{g}_\epsilon{\cdot}\boldsymbol{\nu}\,\mathrm{d}S\\ &= \int_\Omega \big(\kappa\Delta\boldsymbol{g}_\epsilon - \delta_{C,\epsilon}'(\boldsymbol{g}_\epsilon)\big)\Delta\boldsymbol{g}_\epsilon\,\mathrm{d}x = -\int_\Omega \boldsymbol{f}{\cdot}\Delta\boldsymbol{g}_\epsilon\,\mathrm{d}x \le \big\|\boldsymbol{f}\big\|_{L^2(\Omega)}\big\|\Delta\boldsymbol{g}_\epsilon\big\|_{L^2(\Omega)},\end{aligned} \tag{4.7}$$

which gives $\|\Delta\boldsymbol{g}_\epsilon\|_{L^2(\Omega)} \le \|\boldsymbol{f}\|_{L^2(\Omega)}/\kappa$ and then, for $\boldsymbol{\xi}_\epsilon = \delta_{C,\epsilon}'(\boldsymbol{g}_\epsilon)$, also

$$\|\boldsymbol{\xi}_\epsilon\|_{L^2(\Omega)} = \|\kappa\Delta\boldsymbol{g}_\epsilon + \boldsymbol{f}\|_{L^2(\Omega)} \le 2\|\boldsymbol{f}\|_{L^2(\Omega)},$$

and these estimates obviously pass to the limit as $\epsilon \to 0$. As such, the equation is solved strongly and $\boldsymbol{g}$ actually belongs to H_N^2. In particular, this observation entails that $\boldsymbol{z}_\tau^k \in H_\mathrm{N}^2(\Omega;\mathbb{R}^N)$ and $\boldsymbol{m}_\tau^k \in H_\mathrm{N}^2(\Omega;\mathbb{R}^3)$ and the corresponding inclusions (4.1c) and (4.1d) are solved strongly.

For the sake of rewriting the discrete system in a more compact form, let us define the piecewise affine, the left-continuous piecewise-constant, and the right-continuous piecewise-constant interpolants on the time partition. In particular, given any vectors $\{v^k\}_{k=0}^{T/\tau}$, we define

$$v_\tau(t) = \frac{t-(k-1)\tau}{\tau}v_\tau^k + \frac{k\tau - t}{\tau}v_\tau^{k-1} \quad \text{for } t\in[(k-1)\tau, k\tau],\quad k=1,\dots,T/\tau, \tag{4.8a}$$

$$\bar{v}_\tau(t) = v^k \quad \text{for } t\in((k-1)\tau, k\tau],\quad k=0,\dots,T/\tau, \tag{4.8b}$$

$$\underline{v}_\tau(t) := v_\tau^{k-1} \quad \text{for } t\in[(k-1)\tau, k\tau),\quad k=0,\dots,T/\tau. \tag{4.8c}$$

Given also $\{w^k\}_{k=1}^{T/\tau}$, we will also use the discrete by-part summation formula

$$\sum_{k=1}^{\ell}\tau(\mathrm{D}_\tau^k w)v^k = w_\tau^\ell v^\ell - w^0 v^0 - \sum_{k=1}^{\ell}\tau w_\tau^{k-1}\mathrm{D}_\tau^k v. \tag{4.9}$$

We can specify our notion of discrete weak solution as that of a vector $\{(\theta_\tau^k, \boldsymbol{z}_\tau^k, \boldsymbol{m}_\tau^k, \boldsymbol{h}_\tau^k, \boldsymbol{e}_\tau^k)\}_{k=1}^{T/\tau}$ in $H^1(\Omega)\times H_\mathrm{N}^2(\Omega;\mathbb{R}^N)\times H_\mathrm{N}^2(\Omega;\mathbb{R}^3)\times H_{\Gamma_\mathrm{D}}^1(\Omega;\mathbb{R}^3)\times L_\mathrm{curl}^2(\Omega;\mathbb{R}^3)\times L_\mathrm{curl}^2(\Omega;\mathbb{R}^3)$ along with the selections $\{(\boldsymbol{\eta}_{1,\tau}^k, \boldsymbol{\eta}_{2,\tau}^k, \boldsymbol{\xi}_{1,\tau}^k, \boldsymbol{\xi}_{1,\tau}^k,)\}_{k=1}^{T/\tau} \in L^\infty(\Omega;\mathbb{R}^N\times\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^N\times\mathbb{R}^3)$ such that, in terms of the interpolants defined in (4.8),

$$\int_\Omega \dot{\bar{w}}_\tau(t)\,\widetilde{v}\,\mathrm{d}x + \int_\Omega \mathbb{K}(\bar{z}_\tau(t),\bar{\theta}_\tau(t))\nabla\bar{\theta}_\tau(t)\cdot\nabla\widetilde{v}\,\mathrm{d}x + \int_\Gamma \bar{f}_{\mathrm{b},\tau}(t)\,\widetilde{v}\,\mathrm{d}S$$
$$= \int_\Omega \bar{r}_\tau(t)\,\widetilde{v}\,\mathrm{d}x \quad \forall\widetilde{v}\in H^1(\Omega),\ t\in(0,T), \tag{4.10a}$$

$$\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_\tau(t)) - E_{\mathrm{tr}}(\bar{z}_\tau(t))){:}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}})\,\mathrm{d}x = 0 \quad \forall\widetilde{\boldsymbol{v}}\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3),\ t\in(0,T), \tag{4.10b}$$

$$\bar{\boldsymbol{\eta}}_{1,\tau} + D\dot{z}_\tau + E_{\mathrm{tr}}{}^\top\mathbb{C}(E_{\mathrm{tr}}(\bar{z}_\tau) - \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_\tau)) + \mathbb{H}\bar{z}_\tau + \partial_z\psi_{\mathrm{C}}(\bar{z}_\tau,\bar{\boldsymbol{m}}_\tau)$$
$$-\kappa_1\Delta\bar{z}_\tau + \bar{\boldsymbol{\xi}}_{1,\tau} = -\alpha_1(\bar{\theta}_\tau)\gamma'(\bar{z}_\tau),\ \bar{\boldsymbol{\eta}}_{1,\tau}\in\partial\delta^*_Z(\dot{z}_\tau),\quad \bar{\boldsymbol{\xi}}_{1,\tau}\in N_K(\bar{z}_\tau)\ \text{ a.e. in } Q, \tag{4.10c}$$

$$\bar{\boldsymbol{\eta}}_{2,\tau} + \alpha\dot{\boldsymbol{m}}_\tau + \partial_{\boldsymbol{m}}\psi_{\mathrm{C}}(\bar{z}_\tau,\bar{\boldsymbol{m}}_\tau) - \kappa_2\Delta\bar{\boldsymbol{m}}_\tau + \bar{\boldsymbol{\xi}}_{2,\tau} = \mu_0\bar{\boldsymbol{h}}_\tau - a_0\bar{\theta}_\tau\bar{\boldsymbol{m}}_\tau,$$
$$\bar{\boldsymbol{\eta}}_{2,\tau}\in\partial\delta^*_M(\dot{\boldsymbol{m}}_\tau),\quad \bar{\boldsymbol{\xi}}_{2,\tau}\in N_S(\bar{\boldsymbol{m}}_\tau)\ \text{ a.e. in } Q, \tag{4.10d}$$

$$\int_Q \bar{\boldsymbol{e}}_\tau\cdot\mathrm{curl}\,\bar{\boldsymbol{v}}_\tau - \mu_0(\underline{\boldsymbol{h}}_\tau + \underline{\boldsymbol{m}}_\tau)\cdot\dot{\boldsymbol{v}}_\tau\,\mathrm{d}x\mathrm{d}t = \mu_0\int_\Omega(\boldsymbol{h}_0+\boldsymbol{m}_0)\cdot\boldsymbol{v}_\tau(\cdot,0)\,\mathrm{d}x$$
$$\forall\boldsymbol{v}_\tau\in H^1(I;L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3))\quad \text{with } \boldsymbol{v}_\tau(\cdot,T) = \boldsymbol{0}, \tag{4.10e}$$

$$\mathrm{curl}\,\bar{\boldsymbol{h}}_\tau = \mathbb{S}(\bar{z}_\tau,\bar{\theta}_\tau)\bar{\boldsymbol{e}}_\tau \ \text{ and } \ \bar{w}_\tau = \omega(\bar{z}_\tau,\bar{\theta}_\tau)\ \text{ a.e. in } Q,\ \text{ and}$$
$$\bar{\boldsymbol{h}}_\tau - \boldsymbol{h}_{\mathrm{b}}\in L^2(0,T;L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3)). \tag{4.10f}$$

where $\bar{r}_\tau$ comes from (4.1a) via (4.8b).

5 A priori estimates

The existence result of Theorem 1 follows from the passage to the limit as $\tau\to 0$ into relations (4.10). This in turn relies on weak compactness. As such, we shall establish here a suitable set of a priori estimates on the discrete solutions which are independent of the regularization-discretization parameter τ. In particular, in what follows, we use the symbol C in order to indicate any positive constant just depending on data and independent of τ. Note that the actual value of C may change from line to line. Occasionally, dependencies of the constants will be traced.

Lemma 2 *(*A priori estimates I*) Under assumptions* (3.5) *we have that*

$$\|w_\tau\|_{L^\infty(I;L^1(\Omega))} \le C, \tag{5.1a}$$
$$\|z_\tau\|_{L^2(I;H^2_{\mathrm{N}}(\Omega;\mathbb{R}^N))\cap L^\infty(I;H^1(\Omega;\mathbb{R}^N))\cap H^1(I;L^2(\Omega;\mathbb{R}^N))} \le C, \tag{5.1b}$$
$$\|\boldsymbol{m}_\tau\|_{L^2(I;H^2_{\mathrm{N}}(\Omega;\mathbb{R}^3))\cap L^\infty(I;H^1(\Omega;\mathbb{R}^3))\cap H^1(I;L^2(\Omega;\mathbb{R}^3))} \le C, \tag{5.1c}$$
$$\|\bar{\boldsymbol{u}}_\tau\|_{H^1(I;H^1(\Omega;\mathbb{R}^3))} \le C, \tag{5.1d}$$
$$\|\bar{\boldsymbol{h}}_\tau\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))\cap L^2(I;L^2_{\mathrm{curl}}(\Omega;\mathbb{R}^3))} \le C, \tag{5.1e}$$
$$\|\boldsymbol{e}_\tau\|_{L^2(Q;\mathbb{R}^3)} \le C, \tag{5.1f}$$
$$\|\bar{\boldsymbol{\eta}}_{1\tau}\|_{L^\infty(Q;\mathbb{R}^N)} \le C, \qquad \|\bar{\boldsymbol{\eta}}_{2\tau}\|_{L^\infty(Q;\mathbb{R}^3)} \le C, \tag{5.1g}$$
$$\|\bar{\boldsymbol{\xi}}_{1\tau}\|_{L^2(Q;\mathbb{R}^N)} \le C, \qquad \|\bar{\boldsymbol{\xi}}_{2\tau}\|_{L^2(Q;\mathbb{R}^3)} \le C. \tag{5.1h}$$

Sketch of the Proof. The boundedness of $\bar{\boldsymbol{\eta}}_{i,\tau}$ in (5.1g) is immediate. Indeed, note that both δ^*_Z and δ^*_M are bounded since Z and M contain a small ball centered in the origin. As such, the corresponding subdifferentials are bounded as well.

Let us now test (4.1b), (4.10c), (4.10d), (4.10e), and (4.10f) by $\mathrm{D}^k_\tau\boldsymbol{u}$, $\mathrm{D}^k_\tau z$, $\mathrm{D}^k_\tau\boldsymbol{m}$, $\bar{\boldsymbol{h}}_\tau$, and $-\bar{\boldsymbol{e}}_\tau$, respectively. We use the fact that ψ_{C} is semiconvex (namely, convex up to a quadratic correction) due to (3.5g). This allows for estimation when compensated by viscosity in the involved arguments i.e. z and $\boldsymbol{m}$. For $\tau > 0$ sufficiently small, a suitable discrete chain rule yields

$$
\begin{aligned}
&\big(\partial_z\psi_{\rm C}(z_\tau^k,\boldsymbol{m}_\tau^k)+D{\rm D}_\tau^k z\big)\cdot{\rm D}_\tau^k z+\big(\partial_{\boldsymbol{m}}\psi_{\rm C}(z_\tau^k,\boldsymbol{m}_\tau^k)+\alpha{\rm D}_\tau^k\boldsymbol{m}\big)\cdot{\rm D}_\tau^k\boldsymbol{m}\\
&\quad=\Big(\partial_z\psi_{\rm C}(z_\tau^k,\boldsymbol{m}_\tau^k)+\frac{D}{\sqrt{\tau}}z_\tau^k\Big)\cdot{\rm D}_\tau^k z+\Big(\partial_{\boldsymbol{m}}\psi_{\rm C}(z_\tau^k,\boldsymbol{m}_\tau^k)+\frac{\alpha}{\sqrt{\tau}}\boldsymbol{m}_\tau^k\Big)\cdot{\rm D}_\tau^k\boldsymbol{m}\\
&\qquad-\frac{D}{\sqrt{\tau}}z_\tau^k\cdot{\rm D}_\tau^k z-\frac{\alpha}{\sqrt{\tau}}\boldsymbol{m}_\tau^k\cdot{\rm D}_\tau^k\boldsymbol{m}+D{\rm D}_\tau^k z\cdot{\rm D}_\tau^k z+\alpha|{\rm D}_\tau^k\boldsymbol{m}|^2\\
&\quad\geq\frac{1}{\tau}\bigg(\psi_{\rm C}(z_\tau^k,\boldsymbol{m}_\tau^k)+\frac{Dz_\tau^k\cdot z_\tau^k+\alpha|\boldsymbol{m}_\tau^k|^2}{2\sqrt{\tau}}-\psi_{\rm C}(z_\tau^{k-1},\boldsymbol{m}_\tau^{k-1})-\frac{Dz_\tau^{k-1}\cdot z_\tau^{k-1}+\alpha|\boldsymbol{m}_\tau^{k-1}|^2}{2\sqrt{\tau}}\bigg)\\
&\qquad-\frac{D}{\sqrt{\tau}}z_\tau^k\cdot{\rm D}_\tau^k z-\frac{\alpha}{\sqrt{\tau}}\boldsymbol{m}_\tau^k\cdot{\rm D}_\tau^k\boldsymbol{m}+D{\rm D}_\tau^k z\cdot{\rm D}_\tau^k z+\alpha|{\rm D}_\tau^k\boldsymbol{m}|^2\\
&\quad=\frac{\psi_{\rm C}(z_\tau^k,\boldsymbol{m}_\tau^k)-\psi_{\rm C}(z_\tau^{k-1},\boldsymbol{m}_\tau^{k-1})}{\tau}+\Big(1-\frac{\sqrt{\tau}}{2}\Big)\big(D{\rm D}_\tau^k z\cdot{\rm D}_\tau^k z+\alpha|{\rm D}_\tau^k\boldsymbol{m}|^2\big).
\end{aligned}
\tag{5.2}
$$

On the other hand, the free energy is convex in terms of $\boldsymbol{u}$. This allows for estimation even without viscosity in terms of $\dot{\boldsymbol{\varepsilon}}$. By using (5.2), we can estimate the magneto-electro-mechanical energy of the system (2.16), integrated over the time interval $[0,\ell\tau]$ with $\ell=1,\ldots,T/\tau$, as

$$
\begin{aligned}
&\int_\Omega\Big(\psi_{\rm MEC}(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_\tau^\ell),z_\tau^\ell)+\psi_{\rm C}(z_\tau^\ell,\boldsymbol{m}_\tau^\ell)+\psi_{\rm NL}(\nabla z_\tau^\ell,\nabla\boldsymbol{m}_\tau^\ell)+\frac{\mu_0}{2}|\boldsymbol{h}_\tau^\ell|^2\Big)\,{\rm d}x\\
&\quad+\tau\sum_{k=1}^{\ell}\int_\Omega\Big(\delta_Z^*({\rm D}_\tau^k z)+\delta_M^*({\rm D}_\tau^k\boldsymbol{m})+\Big(1-\frac{\sqrt{\tau}}{2}\Big)D{\rm D}_\tau^k z\cdot{\rm D}_\tau^k z+\Big(1-\frac{\sqrt{\tau}}{2}\Big)\alpha|{\rm D}_\tau^k\boldsymbol{m}|^2\Big){\rm d}x\\
&\quad+\tau\sum_{k=1}^{\ell}\int_\Omega\Big(\mathbb{S}(z_\tau^k,\theta_\tau^k)\boldsymbol{e}_\tau^k\cdot\boldsymbol{e}_\tau^k+\alpha_1(\theta_\tau^k)\gamma'(z_\tau^k)\cdot{\rm D}_\tau^k z+a_0\theta_\tau^k\boldsymbol{m}_\tau^k\cdot{\rm D}_\tau^k\boldsymbol{m}\Big)\,{\rm d}x\\
&\quad\leq\int_\Omega\Big(\psi_{\rm MEC}(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_0),z_0)+\psi_{\rm C}(z_0,\boldsymbol{m}_0)+\psi_{\rm NL}(\nabla z_0,\nabla\boldsymbol{m}_0)+\frac{\mu_0}{2}|\boldsymbol{h}_0|^2\Big)\,{\rm d}x+\tau\sum_{k=1}^{\ell}\int_\Gamma\boldsymbol{j}_{{\rm b},\tau}^k\cdot\boldsymbol{e}_\tau^k\,{\rm d}S.
\end{aligned}
\tag{5.3}
$$

In order to estimate also the dissipative terms, we now add to the latter the discrete heat transfer equation (4.1a) tested by $1/2$. One gets

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega w_\tau^\ell{\rm d}x-\frac{1}{2}\int_\Omega w_0{\rm d}x=\frac{\tau}{2}\sum_{k=1}^{\ell}\int_\Omega\Big(\delta_Z^*({\rm D}_\tau^k z)+\delta_M^*({\rm D}_\tau^k\boldsymbol{m})+D{\rm D}_\tau^k z\cdot{\rm D}_\tau^k z+\alpha|{\rm D}_\tau^k\boldsymbol{m}|^2\Big)\,{\rm d}x\\
&\qquad+\frac{\tau}{2}\sum_{k=1}^{\ell}\int_\Omega\bigg(\frac{\mathbb{S}(z_\tau^k,\theta_\tau^k)\boldsymbol{e}_\tau^k\cdot\boldsymbol{e}_\tau^k}{1+\tau|\boldsymbol{e}_\tau^k|^2}+\frac{\mathbb{A}(z_\tau^k,\theta_\tau^k){\rm D}_\tau^k z}{1+\tau|\mathbb{A}(z_\tau^k,\theta_\tau^k)|}+a_0\theta_\tau^k\boldsymbol{m}_\tau^k\cdot{\rm D}_\tau^k\boldsymbol{m}\bigg)\,{\rm d}x+\frac{\tau}{2}\sum_{k=1}^{\ell}\int_\Gamma f_{{\rm b},\tau}^k{\rm d}S.
\end{aligned}
$$

By choosing τ small enough and adding the latter to (5.3) we get

$$
\begin{aligned}
&\int_\Omega\Big(\psi_{\rm MEC}(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_\tau^\ell),z_\tau^\ell)+\psi_{\rm C}(z_\tau^\ell,\boldsymbol{m}_\tau^\ell)+\psi_{\rm NL}(\nabla z_\tau^\ell,\nabla\boldsymbol{m}_\tau^\ell)+\frac{\mu_0}{2}|\boldsymbol{h}_\tau^\ell|^2+\frac{1}{2}w_\tau^\ell\Big)\,{\rm d}x\\
&\qquad+\frac{\tau}{4}\sum_{k=1}^{\ell}\int_\Omega\Big(\delta_Z^*({\rm D}_\tau^k z)+\delta_M^*({\rm D}_\tau^k\boldsymbol{m})+D{\rm D}_\tau^k z\cdot{\rm D}_\tau^k z+\alpha|{\rm D}_\tau^k\boldsymbol{m}|^2+\mathscr{S}(z_\tau^k,w_\tau^k)\boldsymbol{e}_\tau^k\cdot\boldsymbol{e}_\tau^k\Big)\,{\rm d}x\\
&\quad\leq\int_\Omega\Big(\psi_{\rm MEC}(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_0),z_0)+\psi_{\rm C}(z_0,\boldsymbol{m}_0)+\psi_{\rm NL}(\nabla z_0,\nabla\boldsymbol{m}_0)+\frac{\mu_0}{2}|\boldsymbol{h}_0|^2+\frac{1}{2}w_0\Big)\,{\rm d}x
\end{aligned}
$$

$$+\,\tau\sum_{k=1}^{\ell}\int_{\Omega}\bigg(\frac{\mathbb{A}(z_\tau^k,\theta_\tau^k)\mathrm{D}_\tau^k z}{1+\tau|\mathbb{A}(z_\tau^k,\theta_\tau^k)|}-\alpha_1(\theta_\tau^k)\gamma'(z_\tau^k)\cdot\mathrm{D}_\tau^k z-\frac{1}{2}a_0\theta_\tau^k\boldsymbol{m}_\tau^k\cdot\mathrm{D}_\tau^k\boldsymbol{m}\bigg)\,\mathrm{d}x$$
$$+\,\tau\sum_{k=1}^{\ell}\int_{\Gamma}\bigg(\boldsymbol{j}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e}_\tau^k+\frac{1}{2}f_{\mathrm{b},\tau}^k\bigg)\,\mathrm{d}S. \tag{5.4}$$

As for to control the above right-hand side, we observe that the initial terms are bounded due to the smoothness of ψ_{C} (3.5g) and the (3.5b)–(3.5d), the boundary term containing f_{b} is bounded due to (3.5e) and the remaining boundary term can be estimated by imitating the scenario (3.16) and using the by-part summation like (4.9) as follows:

$$\tau\sum_{k=1}^{\ell}\int_{\Gamma}\boldsymbol{j}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e}_\tau^k\mathrm{d}S=\tau\sum_{k=1}^{\ell}\int_{\Omega}\big(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e}_\tau^k-\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\mathrm{curl}\,\boldsymbol{e}_\tau^k\big)\mathrm{d}x$$
$$=\tau\sum_{k=1}^{\ell}\int_{\Omega}\big(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e}_\tau^k+\mu_0\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot(\mathrm{D}_\tau^k\boldsymbol{h}+\mathrm{D}_\tau^k\boldsymbol{m})\big)\mathrm{d}x$$
$$=\tau\sum_{k=1}^{\ell}\int_{\Omega}\big(\mathrm{curl}\,\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\boldsymbol{e}_\tau^k+\mu_0\boldsymbol{h}_{\mathrm{b},\tau}^k\cdot\mathrm{D}_\tau^k\boldsymbol{m}\big)\mathrm{d}x+\mu_0\int_{\Omega}\big(\boldsymbol{h}_{\mathrm{b},\tau}^\ell\cdot\boldsymbol{h}_\tau^\ell-\boldsymbol{h}_{\mathrm{b},\tau}^0\cdot\boldsymbol{h}_0\big)\mathrm{d}x-\tau\mu_0\sum_{k=1}^{\ell}\int_{\Omega}\mathrm{D}_\tau^k\boldsymbol{h}_{\mathrm{b}}\cdot\boldsymbol{h}_\tau^{k-1}\,\mathrm{d}x$$
$$\le\mu_0\|\boldsymbol{h}_{\mathrm{b},\tau}^\ell\|_{L^2(\Omega;\mathbb{R}^3)}\big\|\boldsymbol{h}_\tau^\ell\big\|_{L^2(\Omega;\mathbb{R}^3)}+\tau\sum_{k=1}^{\ell}\bigg(\frac{1}{2\epsilon}\big\|\boldsymbol{h}_{\mathrm{b},\tau}^k\big\|_{L^2_{\mathrm{curl}}(\Omega;\mathbb{R}^3)}^2+\epsilon\big\|\boldsymbol{e}_\tau^k\big\|_{L^2(\Omega;\mathbb{R}^3)}^2+\epsilon\big\|\mathrm{D}_\tau^k\boldsymbol{m}\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\bigg)$$
$$+\,\tau\mu_0\sum_{k=1}^{\ell}\big\|\mathrm{D}_\tau^k\boldsymbol{h}_{\mathrm{b}}\big\|_{L^2(\Omega;\mathbb{R}^3)}\Big(1+\big\|\boldsymbol{h}_\tau^{k-1}\big\|_{L^2(\Omega;\mathbb{R}^3)}^2\Big)+\mu_0\|\boldsymbol{h}_{\mathrm{b},\tau}^0\big\|_{L^2(\Omega;\mathbb{R}^3)}\big\|\boldsymbol{h}_0\big\|_{L^2(\Omega;\mathbb{R}^3)} \tag{5.5}$$

with $\epsilon>0$ to be chosen sufficiently small, namely $\epsilon<\min(\kappa_0,\alpha)$ with κ_0 from (3.10), and then to be treated by a discrete Gronwall inequality, exploiting also (3.5f). As for the other terms in the right-hand side of (5.4), by using (3.11) and letting $\epsilon>0$ be suitably small, we proceed as follows

$$\tau\sum_{k=1}^{\ell}\int_{\Omega}\mathbb{A}(z_\tau^k,\theta_\tau^k)\mathrm{D}_\tau^k z_\tau^k\mathrm{d}x\le C\int_0^{\ell\tau}\!\!\int_{\Omega}\Big(1+\sqrt{\bar{w}_\tau}\Big)\,|\dot{z}_\tau|\,\mathrm{d}x\mathrm{d}t\le\epsilon\int_0^{\ell\tau}\!\!\int_{\Omega}|\dot{z}_\tau|^2\mathrm{d}x\mathrm{d}t+\frac{C}{\epsilon}\int_0^{\ell\tau}\!\!\int_{\Omega}\bar{w}_\tau\mathrm{d}x\mathrm{d}t+C,$$
$$-\,\tau\sum_{k=1}^{\ell}\int_{\Omega}\alpha_1(\theta_\tau^k)\gamma'(z_\tau^k)\cdot\mathrm{D}_\tau^k z\,\mathrm{d}x\le C\int_0^{\ell\tau}\!\!\int_{\Omega}\Big(1+\sqrt{\bar{w}_\tau}\Big)\,|\dot{z}_\tau|\,\mathrm{d}x\mathrm{d}t\le\epsilon\int_0^{\ell\tau}\|\dot{z}_\tau\|_{L^2(\Omega;\mathbb{R}^3)}^2\mathrm{d}t$$
$$+\,\frac{C}{\epsilon}\int_0^{\ell\tau}\!\!\int_{\Omega}\bar{w}_\tau\mathrm{d}x\mathrm{d}t+C,\ -\frac{\tau}{2}\sum_{k=1}^{\ell}\int_{\Omega}a_0\theta_\tau^k\boldsymbol{m}_\tau^k\cdot\mathrm{D}_\tau^k\boldsymbol{m}\mathrm{d}x\le C\int_0^{\ell\tau}\!\!\int_{\Omega}\Big(1+\sqrt{\bar{w}_\tau}\Big)\,|\dot{\boldsymbol{m}}_\tau|\,\mathrm{d}x\mathrm{d}t$$
$$\le\epsilon\int_0^{\ell\tau}\|\dot{\boldsymbol{m}}_\tau\|_{L^2(\Omega;\mathbb{R}^3)}^2\mathrm{d}t+\frac{C}{\epsilon}\int_0^{\ell\tau}\!\!\int_{\Omega}\bar{w}_\tau\mathrm{d}x\mathrm{d}t+C.$$

Hence, by collecting the latter into (5.4), recalling that $w_\tau^k\ge0$, and applying the (discrete) Gronwall lemma (possibly taking a small time-step), we obtain estimates (5.1a)–(5.1c) as well as the L^2-estimate of $\boldsymbol{e}_\tau$. Estimate (5.1e) follows from the identity

$$\mathrm{curl}\,(\bar{\boldsymbol{h}}_\tau-\bar{\boldsymbol{h}}_{\mathrm{b},\tau})=\mathbb{S}(z_\tau^k,\theta_\tau^k)\bar{\boldsymbol{e}}_\tau-\mathrm{curl}\,\bar{\boldsymbol{h}}_{\mathrm{b},\tau} \tag{5.6}$$

and from the already established bounds and (3.5k). Estimate (5.1f) follows then from equation (4.1e). Additionally, we also estimate $\bar{\boldsymbol{u}}_\tau$ in $L^\infty(I;H^1(\Omega;\mathbb{R}^3))$. Moreover, by the linearity of the solution mapping

$(z, \boldsymbol{b}) \mapsto \boldsymbol{u}$ for relation (3.4b), the bound on $\dot{\boldsymbol{u}}$ follows from that of $\dot{z}_\tau$. In particular, estimate (5.1d) ensues. Eventually, by comparison in relations (4.10c) and (4.10d), we have that $-\kappa_1\Delta\bar{z}_\tau + \bar{\boldsymbol{\xi}}_{1,\tau}$ and $-\kappa_2\Delta\bar{\boldsymbol{m}}_\tau + \bar{\boldsymbol{\xi}}_{2,\tau}$ are bounded in L^2. This entails the boundedness in H^2 for $\bar{z}_\tau$ and $\bar{\boldsymbol{m}}_\tau$, see (5.1b)–(5.1c), as well as the bounds (5.1h) on $\boldsymbol{\xi}_{i,\tau}$. □

Let us now refine the energy estimate of Lemma 2 in order to obtain some control of time and space variations of w. This resides on the use of a by now classical argument from [17].

Lemma 3 *(*A priori estimates II*) Under assumptions* (3.5)*, we have that*

$$\big\|\nabla\bar{w}_\tau\big\|_{L^r(Q;\mathbb{R}^3)} \le C_r, \quad \big\|\bar{w}_\tau\big\|_{L^q(Q)} \le C_q, \quad \textit{and} \tag{5.7a}$$

$$\big\|\nabla\bar{\theta}_\tau\big\|_{L^r(Q;\mathbb{R}^3)} \le C_r, \quad \big\|\bar{w}_\tau\big\|_{L^q(Q)} \le C_q, \quad \textit{with } 1 \le r < 5/4,, \ \ 1 \le q < 5/3, \tag{5.7b}$$

$$\big\|\dot{w}_\tau\big\|_{L^1(I;H^3(\Omega)^*)} \le C \quad \textit{and} \tag{5.7c}$$

$$\big\|\dot{\boldsymbol{h}}_\tau\big\|_{L^2(I;L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3)^*)} \le C. \tag{5.7d}$$

Sketch of the Proof. The doubly nonlinear structure of the heat equation, involving both ω and $\mathbb{K}$, makes this estimate quite technical. In order to obtain the bound on the gradient of w, we test equation (4.1a) by $j_\eta(w_\tau^k)$ with $j_\eta : w \mapsto 1 - (1{+}w)^{-\eta}$ with $\eta > 0$; note that we cannot use the test by $j_\eta(\theta_\tau^k)$. Indeed, this would make treatment of the $\mathbb{K}$-term simpler but does not pair well with the $\dot{w}$-term. Note that $\bar{r}_\tau$ from (4.1a) is uniformly bounded in $L^1(Q)$ by virtue of the estimates of Lemma 2 and of the bounds (3.11) and (3.5k). Since $\theta = \mathscr{T}(z, w)$, one can express the heat flux as

$$\mathbb{K}(z,\theta)\nabla\theta = \mathbb{K}(z, \mathscr{T}(z, w))\nabla\mathscr{T}(z, w) = \mathscr{K}_0(z, w)\nabla w + \mathscr{K}_1(z, w)\nabla z \tag{5.8}$$

where $\mathscr{K}_0$ and $\mathscr{K}_1$ are given in positions (3.9). The above-mentioned test provides the following bound

$$\begin{aligned}
\kappa_0\eta\int_Q \frac{|\nabla\bar{w}_\tau|^2}{(1{+}\bar{w}_\tau)^{1+\eta}}\,\mathrm{d}x\mathrm{d}t &= \kappa_0\int_Q j_\eta'(\bar{w}_\tau)|\nabla\bar{w}_\tau|^2\,\mathrm{d}x\mathrm{d}t\\
&\le \int_Q j_\eta'(\bar{w}_\tau)\mathscr{K}_0(\bar{z}_\tau,\bar{w}_\tau)\nabla\bar{w}_\tau{\cdot}\nabla\bar{w}_\tau\,\mathrm{d}x\mathrm{d}t = \int_Q \mathscr{K}_0(\bar{z}_\tau,\bar{w}_\tau)\nabla\bar{w}_\tau{\cdot}\nabla j_\eta(\bar{w}_\tau)\,\mathrm{d}x\mathrm{d}t\\
&\le \int_Q \mathscr{K}_0(\bar{z}_\tau,\bar{w}_\tau)\nabla\bar{w}_\tau{\cdot}\nabla j_\eta(\bar{w}_\tau)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \widehat{j}_\eta(\bar{w}_\tau(T,\cdot))\,\mathrm{d}x\\
&\le \int_\Omega \widehat{j}_\eta(w_0)\,\mathrm{d}x + \int_\Sigma \bar{f}_{\mathrm{b},\tau}\, j_\eta(\bar{w}_\tau)\,\mathrm{d}S\mathrm{d}t + \int_Q \big(\bar{r}_\tau j_\eta(\bar{w}_\tau) - \mathscr{K}_1(\bar{z}_\tau,\bar{w}_\tau)\nabla\bar{z}_\tau{\cdot}\nabla j_\eta(\bar{w}_\tau)\big)\,\mathrm{d}x\mathrm{d}t\\
&\le \|w_0\|_{L^1(\Omega)} + \|f\|_{L^1(\Sigma)} + \|\bar{r}_\tau\|_{L^1(Q)} + \eta\int_Q \frac{\mathscr{K}_1(\bar{z}_\tau,\bar{w}_\tau)\nabla\bar{z}_\tau{\cdot}\nabla\bar{w}_\tau}{(1{+}\bar{w}_\tau)^{1+\eta}}\,\mathrm{d}x\mathrm{d}t\\
&\le C + \|\bar{r}_\tau\|_{L^1(Q)} + \eta\int_Q \bigg(\frac{1}{4\epsilon}|\nabla\bar{z}_\tau|^2 + \epsilon\frac{|\mathscr{K}_1(\bar{z}_\tau,\bar{w}_\tau)|^2}{(1{+}\bar{w}_\tau)}\,\frac{|\nabla\bar{w}_\tau|^2}{(1{+}\bar{w}_\tau)^{1+\eta}}\bigg)\mathrm{d}x\mathrm{d}t\\
&= C + \|\bar{r}_\tau\|_{L^1(Q)} + \eta\int_Q \bigg(\frac{1}{4\epsilon}|\nabla\bar{z}_\tau|^2 + \epsilon C\frac{|\nabla\bar{w}_\tau|^2}{(1{+}\bar{w}_\tau)^{1+\eta}}\bigg)\mathrm{d}x\mathrm{d}t
\end{aligned}\tag{5.9}$$

where $\kappa_0 > 0$ is the coercivity constant in (3.10), we have used the bound on $\mathscr{K}_1$ from assumption (3.11), and $\widehat{j}_\eta$ is a primitive function of j_η. In the latter computation, we have also used the fact that

$$j_\eta(w_\tau^k)\mathrm{D}_\tau^k w \ge \mathrm{D}_\tau^k\widehat{j}_\eta(w) \tag{5.10}$$

which is a consequence of the convexity of $\widehat{j}_\eta$. By choosing $\epsilon > 0$ small enough, we absorb the last term in the right-hand side of (5.9) to the left. In particular, we have proved that the term $\eta\int_Q |\nabla\bar{w}_\tau|^2(1{+}\bar{w}_\tau)^{-(1+\eta)}\mathrm{d}x\mathrm{d}t$

is uniformly bounded in $L^1(Q)$. Estimate (5.7a) now follows from a careful application of the Gagliardo-Nirenberg inequality as in [80, Sects. 12.1 and 12.8-9].

For (5.7b), we use the already obtained estimates (5.7a) and (5.1b) and the identity

$$\nabla\bar{\theta}_\tau = \frac{1}{c_0(\bar{\theta}_\tau)+c_1(\bar{\theta}_\tau)\gamma(\bar{z}_\tau)}\bar{\nabla}w_\tau - \frac{\widehat{c}_1(\bar{\theta}_\tau)\gamma'(\bar{z}_\tau)}{c_0(\bar{\theta}_\tau)+c_1(\bar{\theta}_\tau)\gamma(\bar{z}_\tau)}\bar{\nabla}z_\tau \tag{5.11}$$

obtained by applying the ∇-operator to the identity (3.1). Note that we use also the assumption (3.5i) here. By uniform at most linear growth of $\mathscr{T}(z,\cdot)$ due to (3.6), the L^q-estimate (5.7a) of $\bar{w}_\tau$ is inherited by $\bar{\theta}_\tau$ i.e. the L^q-estimate in (5.7b) is proved, too.

In order to estimate $\dot{w}_\tau$, we use comparison into relation (4.10a) and get that

$$\begin{aligned}
\|\dot{w}_\tau\|_{L^1(I;(H^3(\Omega))^*)} &= \sup_{\|v\|_{L^\infty(I;H^3(\Omega))}\le 1} \int_Q \dot{w}_\tau v\,\mathrm{d}x\mathrm{d}t \\
&= \sup_{\|v\|_{L^\infty(I;H^3(\Omega))}\le 1} \Bigg\{ \int_\Sigma \bar{f}_{\mathrm{b},\tau} v\,\mathrm{d}S\mathrm{d}t + \int_Q \Big(\delta^*_Z(\dot{z}_\tau)+D\dot{z}_\tau\cdot\dot{z}_\tau+\delta^*_M(\dot{\boldsymbol{m}}_\tau)+\alpha|\dot{\boldsymbol{m}}_\tau|^2\Big) v\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_Q \left(\frac{\mathbb{S}(\bar{z}_\tau,\bar{\theta}_\tau)}{1+\tau|\bar{\boldsymbol{e}}_\tau|^2}\bar{\boldsymbol{e}}_\tau\cdot\bar{\boldsymbol{e}}_\tau + \frac{\mathbb{A}(\bar{z}_\tau,\bar{\theta}_\tau)\dot{z}_\tau}{1+\tau|\mathbb{A}(\bar{z}_\tau,\bar{\theta}_\tau)|} + a_0\bar{\theta}_\tau\bar{\boldsymbol{m}}_\tau\cdot\dot{\boldsymbol{m}}_\tau \right) v - \mathbb{K}(\bar{z}_\tau,\bar{\theta}_\tau)\nabla\bar{\theta}_\tau\cdot\nabla v\,\mathrm{d}x\mathrm{d}t \Bigg\}.
\end{aligned} \tag{5.12}$$

Owing to the already established estimates (5.7b) and the bound (3.5k) with $\eta < 3/2$, we have that the term $\mathbb{K}(\bar{z}_\tau,\bar{\theta}_\tau)$ is bounded in $L^{5/2-\epsilon}(Q;\mathbb{R}^{3\times 3})$ so that $\mathbb{K}(\bar{z}_\tau,\bar{\theta}_\tau)\nabla\bar{\theta}_\tau$ is bounded in $L^1(Q;\mathbb{R}^3)$. In particular, estimate (5.7c) ensues.

As for the estimate on $\dot{\boldsymbol{h}}_\tau$, one uses relation (4.1e) in order to infer that

$$\begin{aligned}
\|\dot{\boldsymbol{h}}_\tau\|_{L^2(I;L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3)^*)} &= \sup_{\|\boldsymbol{v}\|_{L^2(I;L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3))}\le 1} \int_Q \dot{\boldsymbol{h}}_\tau\cdot\boldsymbol{v}\,\mathrm{d}x\mathrm{d}t \\
&= \sup_{\|\boldsymbol{v}\|_{L^2(I;L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3))}\le 1} \frac{1}{\mu_0}\int_Q \big(\mathrm{curl}\,\bar{\boldsymbol{e}}_\tau+\mu_0\dot{\boldsymbol{m}}_\tau\big)\cdot\boldsymbol{v}\,\mathrm{d}x\mathrm{d}t \\
&= \sup_{\|\boldsymbol{v}\|_{L^2(I;L^2_{\mathrm{curl},0}(\Omega;\mathbb{R}^3))}\le 1} \frac{1}{\mu_0}\int_Q \big(\bar{\boldsymbol{e}}_\tau\cdot\mathrm{curl}\,\boldsymbol{v}+\dot{\boldsymbol{m}}_\tau\cdot\boldsymbol{v}\big)\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{5.13}$$

In particular, estimate (5.7d) follows from the already obtained bounds (5.1c) and (5.1f). □

6 Passage to the limit

This section brings to the proof of the existence result of Theorem 1 via the passage to the limit in the (regularization and) discretization parameter τ. We formulate this convergence statement as follows.

Proposition 1 (Convergence for $\tau \to 0$) *Under assumptions (3.5), the sequences θ_τ, z_τ, $\boldsymbol{m}_\tau$, $\boldsymbol{u}_\tau$, $\boldsymbol{h}_\tau$, $\boldsymbol{e}_\tau$, $\boldsymbol{\eta}_{i,\tau}$, and $\boldsymbol{\xi}_{i,\tau}$, which exist by Lemma 1, possess weakly∗ converging subsequences in the topologies of estimates (5.1). Moreover, one also has that the strong convergences including also for $\bar{w}_\tau$, namely:*

$$\bar{\theta}_\tau \to \theta \quad\text{and}\quad \bar{w}_\tau \to w \qquad \text{strongly in } L^{5/3-\epsilon}(Q) \text{ with any } 0<\epsilon\le 2/3, \tag{6.1a}$$

$$(\bar{z}_\tau,\bar{\boldsymbol{m}}_\tau) \to (z,\boldsymbol{m}) \qquad \text{strongly in } W^{1,6-\epsilon}(Q;\mathbb{R}^N\times\mathbb{R}^3) \text{ with any } 0<\epsilon\le 5, \tag{6.1b}$$

$$\dot{z}_\tau \to \dot{z} \qquad \text{strongly in } L^2(Q;\mathbb{R}^N), \tag{6.1c}$$

$$\dot{\boldsymbol{u}}_\tau \to \dot{\boldsymbol{u}} \qquad \text{strongly in } L^2(I;H^1(\Omega;\mathbb{R}^3)), \tag{6.1d}$$

$$\dot{\boldsymbol{m}}_\tau \to \dot{\boldsymbol{m}} \qquad \text{strongly in } L^2(Q;\mathbb{R}^3), \tag{6.1e}$$

$$\bar{\boldsymbol{e}}_\tau \to \boldsymbol{e} \qquad \textit{strongly in } \ L^2(Q;\mathbb{R}^3), \tag{6.1f}$$

$$(\boldsymbol{\eta}_{1,\tau},\boldsymbol{\eta}_{2,\tau}) \to (\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) \qquad \textit{weakly* in } \ L^\infty(Q;\mathbb{R}^N\times\mathbb{R}^3), \tag{6.1g}$$

$$(\boldsymbol{\xi}_{1,\tau},\boldsymbol{\xi}_{2,\tau}) \to (\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \qquad \textit{weakly in } \ L^2(Q;\mathbb{R}^N\times\mathbb{R}^3). \tag{6.1h}$$

Each limit $(\theta_\tau, z, \boldsymbol{m}, \boldsymbol{u}, \boldsymbol{h}, \boldsymbol{e})$ obtained by this way, along with $(\boldsymbol{\eta}_i, \boldsymbol{\xi}_i)$, is a weak solution to the initial boundary value problem (2.11)–(2.12) in accord to Definition 1. Additionally, it fulfils (3.13), (3.14), and (3.15).

Proof Owing to the uniform estimates from Lemma 2 and standard weak* compactness arguments, we have that the sequences $(\theta_\tau, z_\tau, \boldsymbol{m}_\tau, \boldsymbol{u}_\tau, \boldsymbol{h}_\tau, \boldsymbol{e}_\tau)$ and $(\boldsymbol{\eta}_{i,\tau}, \boldsymbol{\xi}_{i,\tau})$ admit weakly* converging subsequences (not relabelled) to the limit $(\theta, z, \boldsymbol{m}, \boldsymbol{u}, \boldsymbol{h}, \boldsymbol{e})$ and $(\boldsymbol{\eta}_i, \boldsymbol{\xi}_i)$ in the topologies of the estimates (5.1). By the Aubin-Lions theorem, convergences (6.1a) and (6.1b) follow. Note that the proof of convergence (6.1a) requires some interpolation technique (as in [80, Cor.7.8]) and, for the first convergence in (6.1a), one still uses that $\bar{\theta}_\tau \to \theta$ weakly in $L^{5/3-\epsilon}(Q)$ due to the a priori estimates and simultaneously $\mathscr{T}(\bar{z}_\tau, \bar{w}_\tau) \to \mathscr{T}(z, w)$ by the continuity of the Nemytskiĭ operator, so that $\bar{\theta}_\tau = \mathscr{T}(\bar{z}_\tau, \bar{w}_\tau)$ converges even strongly to $\theta = \mathscr{T}(z, w)$; note that (6.1a) follows even without having any information about $\dot{\theta}$ at disposal. Thus, we also have $w = \omega(z, \theta)$ as needed in the definition (3.4f). Owing to these convergences and to the continuity of the other Nemytskiĭ mappings induced by $\psi_{\rm C}'$ and γ', one can pass to the limit in equations/inclusions (4.10b)–(4.10f) getting, respectively, (3.4b)–(3.4f). The almost everywhere inclusions $\boldsymbol{\xi}_1 \in N_K(z)$ and $\boldsymbol{\xi}_2 \in N_S(\boldsymbol{m})$ follow directly from the strong convergences (6.1b).

In order to check for the inclusions in (3.3g) and (3.3h), we shall argue by lower semicontinuity and test equations (4.10c) and (4.10d) by $\bar{z}_\tau$ and $\bar{\boldsymbol{m}}_\tau$, respectively. By letting $\widetilde{\psi}$ be defined as the lower semicontinuous and convex function

$$\widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), z, \nabla z, \nabla \boldsymbol{m}) := \psi_{\rm MEC}(\boldsymbol{\varepsilon}(\boldsymbol{u}), z) + \psi_{\rm C}(\boldsymbol{m}, z) + \psi_{\rm NL}(\nabla z, \nabla \boldsymbol{m}) + \psi_{\rm CONST}(z, \boldsymbol{m}), \tag{6.2}$$

we get that

$$\begin{aligned}
&\limsup_{\tau\to 0} \int_Q \Big(\bar{\boldsymbol{\eta}}_{1\tau}\cdot\dot{z}_\tau + \bar{\boldsymbol{\eta}}_{2\tau}\cdot\dot{\boldsymbol{m}}_\tau\Big)\,\mathrm{d}x\mathrm{d}t \\
&\quad = \limsup_{\tau\to 0} \Bigg[\int_Q \Big(-\partial_A\widetilde{\psi}(\bar{\boldsymbol{A}}_\tau)\cdot\dot{\boldsymbol{A}}_\tau - D\dot{z}_\tau\cdot\dot{z}_\tau - \alpha|\dot{\boldsymbol{m}}_\tau|^2\Big)\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \int_Q \Big(-\alpha_1(\bar{\theta}_\tau)\gamma'(\bar{z}_\tau)\cdot\dot{z}_\tau - a_0\bar{\theta}_\tau\bar{\boldsymbol{m}}_\tau\cdot\dot{\boldsymbol{m}}_\tau + \mu_0\bar{\boldsymbol{h}}_\tau\cdot\dot{\boldsymbol{m}}_\tau\Big)\,\mathrm{d}x\mathrm{d}t\Bigg] \\
&\quad \le \int_\Omega \widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_0), z_0, \nabla z_0, \nabla\boldsymbol{m}_0)\,\mathrm{d}x - \liminf_{\tau\to 0}\int_\Omega \widetilde{\psi}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_\tau(T)), z_\tau(T), \nabla z_\tau(T), \nabla\boldsymbol{m}_\tau(T)\big)\,\mathrm{d}x \\
&\qquad - \lim_{\tau\to 0}\int_Q \Big(\alpha_1(\bar{\theta}_\tau)\gamma'(\bar{z}_\tau)\cdot\dot{z}_\tau + a_0\bar{\theta}_\tau\bar{\boldsymbol{m}}_\tau\cdot\dot{\boldsymbol{m}}_\tau\Big)\,\mathrm{d}x\mathrm{d}t \\
&\qquad - \liminf_{\tau\to 0}\int_Q \Big(D\dot{z}_\tau\cdot\dot{z}_\tau + \alpha|\dot{\boldsymbol{m}}_\tau|^2\Big)\,\mathrm{d}x\mathrm{d}t + \limsup_{\tau\to 0}\mu_0\int_Q \bar{\boldsymbol{h}}_\tau\cdot\dot{\boldsymbol{m}}_\tau\,\mathrm{d}x\mathrm{d}t \\
&\quad \le \int_\Omega \widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_0), z_0, \nabla z_0, \nabla\boldsymbol{m}_0)\mathrm{d}x - \int_\Omega \widetilde{\psi}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}(T)), z(T), \nabla z(T), \nabla\boldsymbol{m}(T)\big)\mathrm{d}x \\
&\qquad - \int_Q \Big(\alpha_1(\theta)\gamma'(z)\cdot\dot{z} + a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}} + D\dot{z}\cdot\dot{z} + \alpha|\dot{\boldsymbol{m}}|^2\Big)\,\mathrm{d}x\mathrm{d}t \\
&\qquad - \left(\frac{\mu_0}{2}\int_\Omega \big(|\boldsymbol{h}(T)|^2 - |\boldsymbol{h}_0|^2\big)\,\mathrm{d}x + \int_Q \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\,\mathrm{d}x\mathrm{d}t + \int_\Sigma \boldsymbol{j}_{\rm b}\cdot\boldsymbol{e}\,\mathrm{d}S\mathrm{d}t\right)
\end{aligned}$$

$$\overset{(3.16)}{=} \int_Q \Big(-\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), z, \nabla z, \nabla \boldsymbol{m}) - D\dot{z}\cdot\dot{z} - \alpha|\dot{\boldsymbol{m}}|^2 \Big)\,\mathrm{d}x\mathrm{d}t$$
$$- \int_Q \Big(\alpha_1(\theta)\gamma'(z)\cdot\dot{z} + a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}} - \mu_0\boldsymbol{h}\cdot\dot{\boldsymbol{m}} \Big)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_Q \Big(\boldsymbol{\eta}_1\cdot\dot{z} + \boldsymbol{\eta}_2\cdot\dot{\boldsymbol{m}} \Big)\,\mathrm{d}x\mathrm{d}t, \tag{6.3}$$

where we have used the shorthand notation $\boldsymbol{A} = (\boldsymbol{\varepsilon}(\boldsymbol{u}), z, \nabla z, \nabla\boldsymbol{m})$ and

$$\mu_0\int_Q \bar{\boldsymbol{h}}_\tau\cdot\dot{\boldsymbol{m}}_\tau\,\mathrm{d}x\mathrm{d}t = -\int_Q \bar{\boldsymbol{h}}_\tau\cdot\big(\mu_0\dot{\boldsymbol{h}}_\tau + \mathrm{curl}\,\bar{\boldsymbol{e}}_\tau\big)\,\mathrm{d}x\mathrm{d}t \le \frac{\mu_0}{2}\int_\Omega \big(|\boldsymbol{h}_0|^2 - |\boldsymbol{h}_\tau(T)|^2\big)\,\mathrm{d}x - \int_Q \mathrm{curl}\,\bar{\boldsymbol{e}}_\tau\cdot\bar{\boldsymbol{h}}_\tau\,\mathrm{d}x\mathrm{d}t$$
$$= \frac{\mu_0}{2}\int_\Omega \big(|\boldsymbol{h}_0|^2 - |\boldsymbol{h}_\tau(T)|^2\big)\,\mathrm{d}x - \int_Q \big(\bar{\boldsymbol{e}}_\tau\cdot\mathrm{curl}\,(\bar{\boldsymbol{h}}_\tau - \bar{\boldsymbol{h}}_{\mathrm{b},\tau}) + \mathrm{curl}\,\bar{\boldsymbol{e}}_\tau\cdot\bar{\boldsymbol{h}}_{\mathrm{b},\tau}\big)\mathrm{d}x\mathrm{d}t$$
$$= \frac{\mu_0}{2}\int_\Omega \big(|\boldsymbol{h}_0|^2 - |\boldsymbol{h}_\tau(T)|^2\big)\,\mathrm{d}x - \int_Q \big(\mathbb{S}(\bar{z}_\tau, \bar{\theta}_\tau)\bar{\boldsymbol{e}}_\tau\cdot\bar{\boldsymbol{e}}_\tau - \bar{\boldsymbol{e}}_\tau\cdot\mathrm{curl}\,\bar{\boldsymbol{h}}_{\mathrm{b},\tau} + \mu_0(\dot{\boldsymbol{m}}_\tau + \dot{\boldsymbol{h}}_\tau)\cdot\bar{\boldsymbol{h}}_{\mathrm{b},\tau}\big)\mathrm{d}x\mathrm{d}t$$
$$= \frac{\mu_0}{2}\int_\Omega \big(|\boldsymbol{h}_0|^2 - |\boldsymbol{h}_\tau(T)|^2\big)\,\mathrm{d}x - \int_Q \big(\mathbb{S}(\bar{z}_\tau, \bar{\theta}_\tau)\bar{\boldsymbol{e}}_\tau\cdot\bar{\boldsymbol{e}}_\tau - \bar{\boldsymbol{e}}_\tau\cdot\mathrm{curl}\,\bar{\boldsymbol{h}}_{\mathrm{b},\tau} + \mu_0\dot{\boldsymbol{m}}_\tau\cdot\bar{\boldsymbol{h}}_{\mathrm{b},\tau}\big)\mathrm{d}x\mathrm{d}t$$
$$+ \mu_0\int_Q \dot{\boldsymbol{h}}_{\mathrm{b},\tau}\cdot\bar{\boldsymbol{h}}_\tau(\cdot - \tau)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \big(\boldsymbol{h}_{\mathrm{b},\tau}(0)\cdot\boldsymbol{h}_0 - \boldsymbol{h}_{\mathrm{b},\tau}(T)\cdot\boldsymbol{h}_\tau(T)\big)\mathrm{d}x, \tag{6.4}$$

cf. also the manipulations in (4.5) and (5.5), and further we have used the weak lower semicontinuity of $\boldsymbol{h} \mapsto \int_\Omega |\boldsymbol{h}(T)|^2\,\mathrm{d}x$ and of $(z, \theta, \boldsymbol{e}) \mapsto \int_Q \mathbb{S}(z, \theta)\boldsymbol{e}\cdot\boldsymbol{e}\,\mathrm{d}x\mathrm{d}t$ so that

$$\limsup_{\tau\to 0}\int_Q \bar{\boldsymbol{h}}_\tau\cdot\dot{\boldsymbol{m}}_\tau\,\mathrm{d}x\mathrm{d}t \le \frac{1}{2}\int_\Omega \big(|\boldsymbol{h}_0|^2 - |\boldsymbol{h}(T)|^2\big)\,\mathrm{d}x - \int_Q \Big(\frac{\mathbb{S}(z, \theta)\boldsymbol{e}\cdot\boldsymbol{e} - \boldsymbol{e}\cdot\mathrm{curl}\,\boldsymbol{h}_\mathrm{b}}{\mu_0} + \dot{\boldsymbol{m}}\cdot\bar{\boldsymbol{h}}_\mathrm{b} - \dot{\boldsymbol{h}}_\mathrm{b}\cdot\boldsymbol{h} \Big)\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \big(\boldsymbol{h}_\mathrm{b}(0)\cdot\boldsymbol{h}_0 - \boldsymbol{h}_\mathrm{b}(T)\cdot\boldsymbol{h}(T)\big)\,\mathrm{d}x = \int_Q \boldsymbol{h}\cdot\dot{\boldsymbol{m}}\,\mathrm{d}x\mathrm{d}t \tag{6.5}$$

where the last equality is due to analogous calculus as used already in (6.4) that can be made rigorous by limiting a mollification. The last equality in (6.3) follows from the already established relations (3.4c), (3.4d), and (3.4e)–(4.10f), and the inclusions in (3.3i) and (3.3j). Moreover, we have exploited the fact that $z_\tau(T) \to z(T)$, $\boldsymbol{m}_\tau(T) \to \boldsymbol{m}(T)$, and $\boldsymbol{u}_\tau(T) \to \boldsymbol{u}(T)$ weakly in H^1; note that $\boldsymbol{u}(T)$ is determined uniquely by $z(T)$ and $\boldsymbol{m}(T)$ for the latter limsup-estimate in (6.3). The last equality in (6.3) is due to the classical chain rule result [21, Lemme 3.3, p.73]. Standard maximal monotonicity arguments [21] entail that $\boldsymbol{\eta}_1 \in \partial\delta_Z^*(\dot{z})$ and $\boldsymbol{\eta}_2 \in \partial\delta_S^*(\dot{\boldsymbol{m}})$, that is, the two inclusions in (3.3g) and (3.3h). Putting this information into the last equality in (6.3) results into the energy balance:

$$\int_\Omega \widetilde{\psi}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}(T)), z(T), \nabla z(T), \nabla\boldsymbol{m}(T)\big)\mathrm{d}x + \int_Q \Big(D\dot{z}\cdot\dot{z} + \alpha|\dot{\boldsymbol{m}}|^2 + \delta_Z^*(\dot{z}) + \delta_S^*(\dot{\boldsymbol{m}}) \Big)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_Q \Big(\alpha_1(\theta)\gamma'(z)\cdot\dot{z} + a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}} - \mu_0\boldsymbol{h}\cdot\dot{\boldsymbol{m}} \Big)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_0), z_0, \nabla z_0, \nabla\boldsymbol{m}_0)\mathrm{d}x. \tag{6.6}$$

We are hence left with the limit passage in the heat equation (4.10a). The essential step is to show the convergence of the heat sources. In order to do so, we start by discussing the dissipative terms. In particular, referring to the abridged notation (6.2), we use the discrete magneto-electro-mechanical energy inequality (5.3) and lower semicontinuity in order to obtain that

$$
\begin{aligned}
&\int_Q \Big(\delta_Z^*(\dot{z})+D\dot{z}\cdot\dot{z}+\delta_M^*(\dot{\boldsymbol{m}})+\alpha|\dot{\boldsymbol{m}}|^2+\mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\Big)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \liminf_{\tau\to0}\int_Q \Big(\delta_Z^*(\dot{z}_\tau)+D\dot{z}_\tau\cdot\dot{z}_\tau+\delta_M^*(\dot{\boldsymbol{m}}_\tau)+\alpha|\dot{\boldsymbol{m}}_\tau|^2+\mathbb{S}(\bar{z}_\tau,\bar{\theta}_\tau)\bar{\boldsymbol{e}}_\tau\cdot\bar{\boldsymbol{e}}_\tau\Big)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \limsup_{\tau\to0}\int_Q \Big(\delta_Z^*(\dot{z}_\tau)+\Big(1-\frac{\sqrt{\tau}}{2}\Big)D\dot{z}_\tau\cdot\dot{z}_\tau+\delta_M^*(\dot{\boldsymbol{m}}_\tau)+\Big(1-\frac{\sqrt{\tau}}{2}\Big)\alpha|\dot{\boldsymbol{m}}_\tau|^2+\mathbb{S}(\bar{z}_\tau,\bar{\theta}_\tau)\bar{\boldsymbol{e}}_\tau\cdot\bar{\boldsymbol{e}}_\tau\Big)\mathrm{d}x\mathrm{d}t\\
&\quad\le \int_\Omega \Big(\widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_0),z_0,\nabla z_0,\nabla\boldsymbol{m}_0)+\frac{\mu_0}{2}|\boldsymbol{h}_0|^2\Big)\,\mathrm{d}x\\
&\qquad-\liminf_{\tau\to0}\int_\Omega \Big(\widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\tau(T)),z_\tau(T),\nabla z_\tau(T),\nabla\boldsymbol{m}_\tau(T))+\frac{\mu_0}{2}|\boldsymbol{h}_\tau(T)|^2\Big)\,\mathrm{d}x\\
&\qquad-\lim_{\tau\to0}\int_Q \Big(\alpha_1(\bar{\theta}_\tau)\gamma'(\bar{z}_\tau)\cdot\dot{z}_\tau+a_0\bar{\theta}_\tau\bar{\boldsymbol{m}}_\tau\cdot\dot{\boldsymbol{m}}_\tau\Big)\,\mathrm{d}x\mathrm{d}t+\lim_{\tau\to0}\int_\Sigma \bar{\boldsymbol{j}}_{\mathrm{b},\tau}\cdot\bar{\boldsymbol{e}}_\tau\mathrm{d}S\mathrm{d}t\\
&\quad\le \int_\Omega \Big(\widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_0),z_0,\nabla z_0,\nabla\boldsymbol{m}_0)+\frac{\mu_0}{2}|\boldsymbol{h}_0|^2\Big)\,\mathrm{d}x\\
&\qquad-\int_\Omega \Big(\widetilde{\psi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\tau(T)),z_\tau(T),\nabla z_\tau(T),\nabla\boldsymbol{m}_\tau(T))+\frac{\mu_0}{2}|\boldsymbol{h}(T)|^2\Big)\,\mathrm{d}x\\
&\qquad-\int_Q \Big(\alpha_1(\theta)\gamma'(z)\cdot\dot{z}+a_0\theta\boldsymbol{m}\cdot\dot{\boldsymbol{m}}\Big)\,\mathrm{d}x\mathrm{d}t+\int_\Sigma \boldsymbol{j}_{\mathrm{b}}\cdot\boldsymbol{e}\,\mathrm{d}S\mathrm{d}t\\
&\quad= \int_Q \Big(\delta_Z^*(\dot{z})+D\dot{z}\cdot\dot{z}+\delta_M^*(\dot{\boldsymbol{m}})+\alpha|\dot{\boldsymbol{m}}|^2+\mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\Big)\,\mathrm{d}x\mathrm{d}t, \qquad (6.7)
\end{aligned}
$$

where $\lim_{\tau\to0}\int_\Sigma \bar{\boldsymbol{j}}_{\mathrm{b},\tau}\cdot\bar{\boldsymbol{e}}_\tau\mathrm{d}S\mathrm{d}t = \int_\Sigma \boldsymbol{j}_{\mathrm{b}}\cdot\boldsymbol{e}\,\mathrm{d}S\mathrm{d}t$ is to be understood in the sense of (3.16). The first inequality in (6.7) uses in particular $\liminf_{\tau\to0}\int_Q \mathbb{S}(\bar{z}_\tau,\bar{\theta}_\tau)\bar{\boldsymbol{e}}_\tau\cdot\bar{\boldsymbol{e}}_\tau\mathrm{d}x\mathrm{d}t \ge \int_Q \mathbb{S}(z,\theta)\boldsymbol{e}\cdot\boldsymbol{e}\,\mathrm{d}x\mathrm{d}t$, which uses the already proved strong convergence $\bar{z}_\tau$ and $\bar{w}_\tau$ and the continuity and positive semidefiniteness of $\mathscr{S}$, cf. [42, Sect.4.3, Thm.4.4]. The last equality in (6.7) is due to (6.6) summed with the electromagnetic energy balance (2.13) integrated over $[0,T]$. In the present regularity setting, this needs

$$
\mu_0\int_\Omega |\boldsymbol{h}(T)|^2\mathrm{d}x-\mu_0\int_\Omega |\boldsymbol{h}_0|^2\mathrm{d}x+\mu_0\int_Q \dot{\boldsymbol{m}}\cdot\boldsymbol{h}\,\mathrm{d}x\mathrm{d}t+\int_Q \mathbb{S}(\bar{z}_\tau,\bar{\theta}_\tau)\boldsymbol{e}\cdot\boldsymbol{e}\,\mathrm{d}x\,\mathrm{d}t=\int_\Sigma \boldsymbol{j}_{\mathrm{b}}\cdot\boldsymbol{e}\,\mathrm{d}\Gamma\,\mathrm{d}t
$$

where, again, the latter term is interpreted as in (3.16). Note that this is exactly the desired magnetomechanical energy balance (3.14).

In particular, (6.7) shows that all inequalities in (6.7) are, in fact, equalities. On the other hand, the integrand on the left-hand side of (6.7) is uniformly convex. This entails the strong convergences for $\dot{z}_\tau$ and $\dot{\boldsymbol{m}}_\tau$ as in (6.1c) and (6.1e). We also get the strong convergence $\bar{z}_\tau$ as in (6.1f); we refer to [83, Step 4 in the proof of Prop. 1] for details. As for the strong convergence (6.1d) for $\dot{\boldsymbol{u}}_\tau$, one takes the time derivative of (4.10b) and test it on $\dot{\boldsymbol{u}}_\tau$ so that, by using (6.1c), one can check that $\lim_{\tau\to0}\int_Q \mathbb{C}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\tau){:}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\tau)\,\mathrm{d}x\mathrm{d}t = \lim_{\tau\to0}\int_Q \mathbb{C}E_{\mathrm{tr}}(\dot{z}_\tau){:}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\tau)\,\mathrm{d}x\mathrm{d}t = \int_Q \mathbb{C}E_{\mathrm{tr}}(\dot{z}){:}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\,\mathrm{d}x\mathrm{d}t = \int_Q \mathbb{C}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}){:}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\,\mathrm{d}x\mathrm{d}t$ and then (6.1d) follows from the weak convergence $\dot{\boldsymbol{u}}_\tau \to \dot{\boldsymbol{u}}$ which is obvious.

Having (6.1c), (6.1e), and (6.1f) at our disposal, we have that $\bar{r}_\tau$ converges in $L^2(Q)$ with $\bar{r}_\tau$ defined by (4.1a) and the limit passage in the discrete heat equation towards (3.4a) is simple. As $\dot{w} \in L^1(I;H^3(\Omega)^*)$, we can test (3.4a) by 1 and add it to (3.14) in order to deduce (3.15). ◻

7 Concluding remarks

Let us collect here some comments on some possible developments of the theory, both from the modelling and from the analytic viewpoint.

Remark 1 (Nonmagnetic limit.) One can consider some asymptotic analysis leading to the suppression of the magnetization in the medium. Indeed, by letting the convex domain $M \to \mathbb{R}^3$ (in the Hausdorff sense, for instance) one has that, in the limit, $\delta^*_{\mathbb{R}^3} = \delta_{\mathbf{0}}$. Hence, $\dot{\boldsymbol{m}}$ is forced to be $\mathbf{0}$ and no magnetization evolution is possible. This procedure can be made rigorous by means of an adaptation of the Γ-convergence analysis for rate-independent processes developed in [65].

Remark 2 (No martensitic transformation.) Similarly as above, one could consider the limit $Z \to \mathbb{R}^N$ which entails $\delta^*_Z \to \delta_{\mathbf{0}}$. In the limit one has that $\dot{z} = \mathbf{0}$ so that no martensitic transformation is possible and the systems reduces to a model for thermomechanics and magnetism. Again, the rigorous argument towards the limit can be grounded on the analysis in [65].

Remark 3 (Fully rate-independent ferro/paramagnetic and martensitic transformation.) We are presently not in the position of proving the existence of a weak solution to the system in the purely rate-independent case, namely for $D = 0$ and $\alpha = 0$. Still, also in this case, some a priori estimates can be deduced. Indeed, if Z and M are bounded, one can still check for the boundedness of $\dot{z}_\tau$ and $\dot{\boldsymbol{m}}_\tau$ in L^1 so that also the corresponding dissipative and adiabatic heat sources are bounded in $L^1(Q)$. Hence, the estimates for w are still valid. This entails the stability of the numerical scheme. On the other hand, the limit passage is still not obvious. The difficulty relies in controlling the product $\mathbb{A}(z,\theta)\dot{z}$. Indeed, $\dot{z}$ is a priori a measure on $\bar{Q}$ and $\mathbb{A}(z,\theta)$ is in $L^\infty(Q)$ but can hardly be expected to range over $C(\bar{Q})$ and even continuously dependent on z and w in such a space. In this case, one should resort in designing an even weaker notion of solution.

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