Ultrarelativistic Black Holes

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Abstract

In this brief note we explain in rather general terms the idea of ultrarelativistic limits of black hole space-times in general relativity.

Prelude: Ultrarelativistic limits. We prepare the grounds for understanding the concept of ultrarelativistic limits of black hole geometries by considering *simpler analogies.*

To begin with think of a ball made of *foam rubber.* If you shoot it hard enough it will be deformed: it will be squeezed in the direction of the motion and elongated in the perpendicular directions.

A somewhat more complicated example is the *electric field* of a charged body. Classical electrostatics tells us that a small ball with charge Q (which for simplicity we assume to be located at the origin of our coordinate system) creates an electric field according to Coulomb's law,

$$E(x) = \frac{Q}{4\pi}\,\frac{\hat{x}}{\|x\|^2}\,. \tag{1}$$

Here x denotes an arbitrary point in space, $\hat{x}$ denotes its direction and $\|x\|$ its distance to the origin. This means that a body with charge q located at the point x will feel a force F according to

$$F(x) = q\,E(x) = \frac{q\,Q}{4\pi}\,\frac{\hat{x}}{\|x\|^2}\,. \tag{2}$$

One way to illustrate the electric field E is to draw its *field lines* and its *equipotential surfaces*, see Figure 1. The field lines give the directions in which the electric force acts; in our case they are rays going straight out from the origin. The equipotential surfaces connect all points where the force has the same magnitude; in our case these are concentric spheres which appear as circles in Figure 1 since we clearly have to suppress one dimension in our drawing.

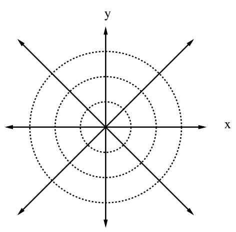

Figure 1: Electric field of a charge at rest

Now if we start to *move* the charged ball along, say, the x-axis with speed v its field changes much in the same way as the ball of foam rubber changes its shape: the equipotential surfaces get squeezed in the direction of the motion and elongated in the perpendicular directions, see Figure 2. This results in the circles of Figure 1 being deformed into the ellipses of Figure 2. Moreover, the faster the charge moves the stronger the field gets squeezed and the ellipses become narrower and narrower.

Finally we want to consider the case when the speed of the charge approaches the *speed of light* $c = 299792458\,m/s$—which according to Einstein's special theory of relativity is the maximum possible speed. This situation is called the *ultrarelativistic limit* and physicists speak of the ultrarelativistic electric field of the charge Q. What actually happens is that the ellipses get all squeezed together to a single line—or remembering that we have suppressed one dimension the ellipsoids get squeezed to a single plane. In other words the entire electric field is now concentrated on a single plane (see Figure 3), where it is, of course, very strong.

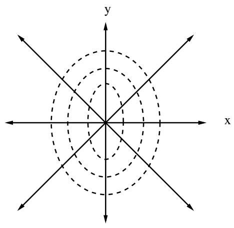

Figure 2: Electric field of a charge moving in the x-direction

This situation is precisely analogous to the situation of a jet plane flying with the speed of sound. All the acoustic energy is concentrated on a plane perpendicular to the direction of the motion which is carried along with the jet plane.

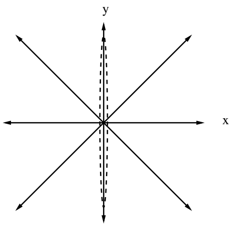

Figure 3: Ultrarelativistic electric field

The precise formula for the ultrarelativistic electromagnetic field is given by

$$F^{ik}(u,v,y,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -z \\ 0 & y & 0 & 0 \\ 0 & z & 0 & 0 \end{pmatrix} \frac{4Q}{\rho^2}\,\delta(u). \quad (3)$$

A vivid way to picture this situation is by using a *space-time diagram*. That is, we draw the time axis (labeled by t) vertically and—again restricted by the 2 dimensions of our drawing—one spatial direction, say x, horizontally. In this diagram the path taken by the charge moving with the speed of light in the x-direction is given by a 45 degree line, see Figure 4. The angle of 45 degrees arises because we stick to the commonly used practice to set the speed of light equal to one, $c = 1$.

Now the ultrarelativistic field, which is carried along with the charge, at each instant of time t' is *concentrated* on a plane perpendicular to the x-direction and containing the current position of the charge (t', x'). Since we had to suppress the perpendicular directions, the filed at each t' is pictured solely by the single point (t', x'). In total the entire history of the ultrarelativistic field is given by the collection of all such points, hence by the line $x = t$. It is useful to label this line by a new coordinate $u = x - t$ such that the ultrarelativistic field is concentrated on the 3-dimensional plane given by $u = 0$. This is also explicitly expressed by the term $\delta(u)$ in formula (3). Here δ denotes the Dirac function which is zero for all $u \neq 0$ and—loosely speaking—infinitely large at $u = 0$ such that its integral gives one, $\int_{-\infty}^{\infty} \delta(t)dt = 1$.

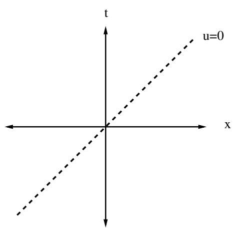

Figure 4: Space-time diagram of the ultrarelativistic field

Now if we want to apply the same procedure to the gravitational field of a body we have to use the adequate physical theory which is general relativity.

Interlude: General relativity, Albert Einstein's theory of gravity, created 99 years ago is the current description of gravitation in modern physics. It is an inherently *geometric theory* in the following sense: By Galileo's principle of equivalence all bodies fall the same in a gravitational field. The fact that *all* bodies are affected in precisely the same way, allows one to think of the gravitational field as being a property of the surroundig space itself. More precisely, the gravitational field manifests itself in the properties of space and time and, in particular, in its curvature. In fact, a massive body like e.g. the earth curves its surrounding space, see Figure 5.

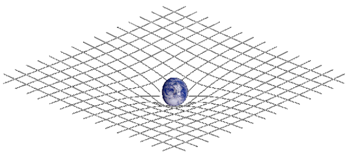

Figure 5: Space-time curvature

Slightly more technical the geometry of a (vector) space is mathematically described by an inner product, which determines how to measure lengths and angles. In general relativity one uses as fundamental mathematical entity the so-called *space-time metric* $\mathbf{g}_{ij}$ which provides such an inner product at each point of space-time. From $\mathbf{g}_{ij}$ one derives the curvature encoded in the so-called Riemann and Ricci curvature tensors $\mathbf{R}^i_{jkl}$ and $\mathbf{R}_{ij}$. Now the fundamental equation of general relativity— *Einstein's equation*—says that the curvature of space-time is determined by its mass-energy content encoded in the energy-momentum tensor $\mathbf{T}_{ab}$ and explicitly reads

$$\mathbf{R}_{ij} - \frac{1}{2}\mathbf{g}_{ij}\,\mathbf{g}_{lm}\mathbf{R}^{lm} = \frac{8\pi G}{c^4}\,\mathbf{T}_{ij}, \tag{4}$$

where $G = 6.67\,10^{-11}\,N(m/kg)^2$ is Newton's constant and c again is the speed of light. In what follows we shall also set $G = 1$ which amounts to use so-called geometric units.

In many ways the simplest (nontrivial) solution of equation (4) is the *Schwarzschild metric* which describes the gravitational field outside a spherically symmetric body, e.g. a star of total mass M. In the usual Schwarzschild coordinates $x^j = (t, r, \theta, \varphi)$ it explicitly reads

$$\begin{aligned} ds^2 &= \mathbf{g}_{ij}dx^i dx^j \qquad (5)\\ &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \end{aligned}$$

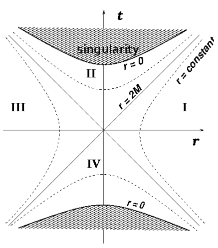

Figure 6: Extended Schwarzschild sol.

This form of the metric becomes singular on the so-called *Schwarzschild radius* $r = 2M$ which usually lies within the star. But for very dense bodies, e.g. very massive stars which undergo gravitational collapse the region $r < 2M$ becomes relevant. By changing to more suitable coordinates one finds that the range of the radial coordinate can be extended to all values $r > 0$. In this way one obtains the *maximally extended* Schwarzschild solution which contains the exterior regions I, III as well as the black and white hole regions II, IV, see Figure 6. In particular, the diagram illustrates the following scenario, notorious in science fiction: If an observer, e.g. an unlucky space ship starts in the exterior region I and enters region II by crossing the Schwarzschild radius $r = 2M$ it can never leave it again—recall that objects moving at the speed of light follow 45-degree lines. More dramatically the doomed observer, after crossing the Schwarzschild radius cannot even send any signal (e.g. a warning) to the outside world (region I) and will necessarily end its existence in the curvature singularity located at $r = 0$, depicted by the upper gray area in Figure 6.

The main part: Ultrarelativistic black holes. We now come to the main part of this note and explain the notion of ultrarelativistic black hole geometries, thereby taking up the line of ideas form the Prelude. There we discussed the ultrarelativistic electromagnetic field of a charge, now we want to look at the ultrarelativistic gravitational field of some ball of mass M. Our starting point is the exterior Schwarzschild solution discussed above, which models the gravitational field outside a spherically symmetric body and which we now write in so-called isotropic coordinates,

$$ds^2 = -\frac{(1-A)^2}{(1+A)^2}dt^2 + (1+A)^4(dx^2 + dy^2 + dz^2). \tag{6}$$

Here A is given by $A = 2M/r$, where (t, x, y, z) are 4-dimensional coordinates and $r = \sqrt{x^2 + y^2 + z^2}$ is the 3-dimensional radius.

We now outline the *ultrarelativistic boost* of the Schwarzschild space-time invented by P. Aichelburg and R. Sexl in the by now classical article [1]. The first step is to set the mass in motion, or equivalently to look at the massive ball if we move by it, say along the x-axis with constant speed v. Technically this is achieved by applying a boost, i.e., a velocity-depending Lorentz transformation

$$\bar{t} = \frac{t + xv}{\sqrt{1-v^2}}, \quad \bar{x} = \frac{x + tv}{\sqrt{1-v^2}}, \quad \bar{y} = y, \quad \bar{z} = z$$

to (6) which then takes the form

$$ds^2 = (1+A)^2(-d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) - \Big((1+A)^4 - \Big(\frac{1-A}{1+A}\Big)^4\Big)\frac{(d\bar{t} - vd\bar{x})^2}{1-v^2}. \tag{7}$$

Moreover, we relate the rest mass M of the body to its total energy p via $M = p\sqrt{1-v^2}$ and hence A changes to

$$A = \frac{2M}{r} = \frac{p(1-v^2)}{2\sqrt{(\bar{x} - v\bar{t})^2 + (1-v^2)(\bar{y}^2 + \bar{z}^2)}}.$$

In a second step we now want to take the speed v to its ultrarelativistic limit, $v \to c = 1$. Here certain unpleasant subtleties arise. Even if we scale the rest mass to zero, $M \to 0$, to keep the total energy p constant, which seems only reasonable, a limit of (7) simply does not exist. Only after performing another peculiar change of coordinates,

$$x' - vt' = \bar{x} - v\bar{t}, \quad x' + vt' = \bar{x} + v\bar{t} + 4p\log\Big(\sqrt{(\bar{x} - v\bar{t})^2 + (1-v^2)} - (\bar{x} - \bar{t})\Big)$$

and allowing for generalized functions[1] as coefficients of the metric we arrive at the result

$$ds^2 = -2du dv + dy'^2 + dz'^2 + 8p\,\delta(u)\log(\rho)du^2, \tag{8}$$

[1] This is mathematically somewhat delicate, see the Appendix.

where for convenience we have set $u = t' - x'$, $v = t' + x'$ and $\rho = \sqrt{y'^2 + z'^2}$.

From the form of the metric (8) we see the following: The term $2dudv + dy'^2 + dz'^2$ is just flat space, so no gravitational effect results from it. The final term $8p\,\delta(u)\log(\rho)du^2$ contains the Dirac function and hence is only non-zero if $u = 0$. So the only effect occurring is a very strong curvature on the 3-dimensional hyperplane given by $u = 0$. This hyperplane contains the entire history of the mass M which is located at $u = 0$, $y' = 0 = z'$ and we recover a picture similar to the ultrarelativistic electromagnetic field, see Figure 7. Again the perpendicular directions y', z' are suppressed such that each point on the line $u = 0$ represents a 2-dimensional plane, together making up the entire 3-plane where the field is concentrated.

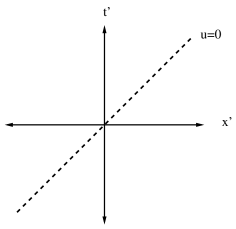

Figure 7: ultrarel. Schwarzschild field

Moreover, one finds that the metric (8) represents a gravitational wave and since the curvature is concentrated on a small part of space-time it actually is a so-called *impulsive gravitational wave*, a notion which we shall discuss in the final section.

Postlude: Impulsive gravitational waves.

We start briefly explaining the notion of *gravitational waves* by looking at a more familiar but analogous example. Electrodynamics allows for wave like solutions, that is electromagnetic waves propagating in vacuum. Depending on their frequency we distinguish e.g. visible light, ultraviolet radiation, X-rays or γ-rays. Similarly general relativity predicts wave like solutions, that is small changes of curvature traveling at the speed of light through space-time. The easiest way to picture gravitational waves is to think of them as small ripples of curvature traveling through the universe just like water waves travel on a pond after you drop a stone into it, see Figure 8.

Figure 8: Gravitational waves as ripples in space-time

Gravitational waves so far have only been *observed* indirectly. Indeed, in 1974 R. Hulse and J. Taylor studied the binary pulsar PSR 1913+16 whose precise observations provided firm evidence that the system is actually emitting gravitational radiation—a discovery which was awarded with the 1993 Nobel Prize in physics. With the highly sensitive large-scale gravitational wave interferometers such as LIGO or Virgo, a direct observation of gravitational waves will probably be feasible within the next couple of years.

Impulsive gravitational waves are geometries that model gravitational waves with very short duration but with a big amplitude. They have been introduced by R. Penrose (see e.g. [3]) who derived corresponding solutions of the Einstein equation using an elegant geometric construction called *scissors and paste approach*. In the easiest case, which leads to so called impulsive pp-waves one takes flat Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -2dudv + dy^2 + dz^2 \tag{9}$$

and cuts it into two halves along the 3-plane N given by $u = 0$. Then one

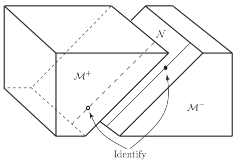

Figure 9: The scissors and paste method

reattaches the halves with a suitable warp, see Figure 9. More precisely, points with coordinates $(u = 0, v, y, z)$ on the boundary of the left half $\mathcal{M}_+$ of Minkowski space are matched to points on the boundary of the right half $\mathcal{M}_-$ according to the rule

$$(u = 0, v, y, z) \;\rightarrow\; (u = 0, v + H(y, z), y, z), \qquad (10)$$

where H is called the warping function. The resulting space-time metric takes the form

$$ds^2 = -2dudv + dy^2 + dz^2 + H(y, z)\delta(u)du^2 \qquad (11)$$

and the Einstein equations reduce to the Laplace equation for H,

$$\triangle H(y, z) = \left(\partial_y^2 + \partial_z^2\right) H(y, z) = 0.$$

It is now easy to see that the ultrarelativistic Schwarzschild solution (8) is precisely of the form (11) with $H(y, z) = 8p\log(\sqrt{y^2 + z^2})$, hence it is an *impulsive pp-wave.*

Finally we discuss the *effect* that an impulsive pp-wave has *on an observer*, i.e., what happens e.g. to a space ship that is hit by such a wave which—according to Aichelburg and Sexl—may be caused by a massive body flying by at the speed of light. In fact when the observer is hit by the wave at the point $(u = 0, v, y, z)$, i.e., when it passes the $u = 0$-plane at (y, z), then (the v-coordinate of) its position instantaneously changes according to the warp $v \rightarrow v + H(y, z)$, cf. Figure 9 and equation (10). Moreover, the v-direction of its path also changes in a way again determined by the (derivatives of the) warping function H. In the transverse directions y and z the path of the observer also instantaneously changes the direction. In total we have that any observer hit by the impulsive wave feels the following violent effect: it jumps in the v-direction and instantaneously changes the direction of its flight in the v, y and z-directions.

Appendix: The wider scientific area. So far we only have discussed the most simple case of an ultrarelativistic black hole space-time and its corresponding impulsive gravitational wave. In this appendix we briefly put the topic into its wider scientific context and supplement some main references.

The Aichelburg-Sexl procedure has been generalized by a number of authors to different black hole geometries of the Kerr-Newman family and also to the case of a non-vanishing cosmological constant, see [4, Sec. 3.5] and [2, Ch. 20] for an overview and the bulk of references collected there. In all cases the resulting ultrarelativistic space-times turn out to be impulsive gravitational waves of various kinds which can serve as simple exact models for violent gravitational wave bursts.

These geometries have been used extensively as simple test cases in particle physics, in particular, the scattering of various particles in these backgrounds has been studied from the 1990-ies on. These studies have some relevance in the quest for a quantum theory of gravitation which, however, is still (completely) open.

In a more theoretical context impulsive gravitational waves are mathematically delicate to handle. Due to the occurrence of the Dirac-delta in the metric (cf. equation (11)) one has to deal with very rough geometric objects while the underlying mathematical theory of Lorentzian geometry usually is only formulated in terms of very smooth functions. Hence these geometries have also been key eamples in the development of theories of (nonlinear) distributional geometry, see [5] for an overview.

Impulsive gravitational waves still are an area of active research, on the one hand within the field of exact radiative solutions of Einstein equations and, on the other hand in the context of low regularity Lorentzian geometry.

References

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