FWF-Research Project P12023-MAT

Distributional Methods in Einstein's Theory of Gravitation

Final Report

April 2002

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Summary

The research project P12023-MAT "Distributional Methods in Einsteins Theory of Gravitation" could be successfully finished in autumn 2001 after a life-span of almost four years.

All the goals of the proposal were reached and beyond these important new results have been obtained and new lines of research were initiated. Besides successfully applying nonlinear generalized functions in a number of problems arising in general relativity, important theoretical advances within the theory of generalized functions itself could be achieved; in particular, a *geometric* theory of generalized functions could be developped.

The publications originating form this project substantially exceed the scale initially hoped for, both in quality and in quantity.

Contents

I. GOALS

The project *"Distributional Methods in Einsteins Theory of Gravitation"* was aimed at adapting distributional methods for use in general relativity, with principal emphasis on the nonlinear theory of generalized functions as developed by J. F. Colombeau, M. Oberguggenberger, E. E. Rosinger and others.

The theory of distributions—brought in its final form by L. Schwartz in the mid-20th century—allows for a mathematically consistent description of very singular objects (Dirac measure, etc.) and is widely applied in the theory of linear PDEs and linear field theories of physics. However, it is a *genuinely linear* theory; the principal notion is that of a linear functional on certain function spaces and products cannot be defined in general without contradictions. Hence one encounters principal and conceptional problems on applying distribution theory in nonlinear situations.
In the case of the *genuinely nonlinear* theory of general relativity—its field equations, i.e., Einstein's equations, written in suitable coordinates take the form of a coupled system of quasi-linear partial differential equations of 2nd order for the coefficients of the space-time metric—R. Geroch and J. Traschen have analyzed the usefulness of linear distributional methods in great detail ([Ger87]). Their well-known "No-Go-Theorems" imply that applications of linear distribution theory in relativity have to be restricted to special cases and the use of less general ad-hoc methods (cf. eg. [Lic71], [Tau80], [Cho93], [Bal95]). A consistent description of singular space-time geometries—whose physical relevance has been emphasized by the singularity theorems of R. Penrose and S. Hawking (see eg. [Haw73], Chap. 7)—cannot be achieved with these methods. This applies in particular to cosmic strings and black hole geometries of the Kerr-Newman family (containing *the* prototype of a singular space-time, the Schwarzschild solution).

At this point the theory of algebras of generalized functions developed since the 1980ies by J. F. Colombeau ([Col84], [Col85], [Col90a], [Col92]), M. Oberguggenberger ([Obe92]), E. E. Rosinger ([Ros78], [Ros80]) and others enters the picture. Within this theory one constructs algebras of generalized functions containing the algebra of smooth functions as a subalgebra and the vector space of distributions as a subspace, while at the same time preserving maximal consistency with respect to classical analysis—in the precise sense of L. Schwartz' "impossibility result" ([Sch54]). In Colombeau's constructive approach these algebras are defined as quotient of the algebra $(\mathcal{C}^\infty)^I$ where I denotes the range of a suitable regularization parameter. The basic idea, i.e., *regularization by smooth functions* together with explicit *asymptotic estimates* in orders of the regularization parameter (which are used in the quotient construction) become a powerful analytical tool.
The theory of algebras of generalized functions quickly evolved into a useful tool to treat nonlinear PDEs where coefficients, data or prospective solutions are non-smooth (see e.g [Bia92a], [Bia92b], [Col90b], [Col93], [Col94a], [Key95], [Obe92], and for applications in numerics [Bia90], [Ber95], [Ber93]). In particular, this theory immediately provides a stable solution concept for such equations.

At this level the goals of the project may be formulated: the methods of the theory of algebras of generalized functions should be adapted for the needs of general relativity and in turn be applied to solve problems in the description of singular space-times.

II. ORGANIZATION OF THE PROJECT

The run-time of the project was November 1997 to August 2001; it was possible to span this unusually long period of almost four years by neutrally shifting the costs of the proposed Ph.D. position for two years.

Roland Steinbauer the coworker designated by the application was not at disposal at the beginning of the project since his scholarship by the Austrian Academy of science was extended. In his place *J. Mark Heinzle* entered the project and was financed from November 1997 to April 1998 by a "Forschungsbeihilfe für Diplomanden" (scholarship for diploma students).
Roland Steinbauer was assigned a Ph.D position from Mai 1998 to October 1999. However, his position was reduced to half-time from July 1998 onward since he also held a half-time teaching position at the Department of Mathematics of the University of Vienna during this period.
Moreover, starting from November 1999 (R. Steinbauer was fully employed by the University since then) again J. Mark Heinzle entered the project as a coworker. He held a half-time Ph.D. position until October 2000.
Then the project was paused until June 2001. In July and August 2001 *Michael Kunzinger* and again Roland Steinbauer were assigned full post-doc positions to support their research stay at Southampton (UK).

During the course of the project the following conference stays could be financed:
R. Steinbauer, “Spanish Relativity Meeting (ERE99)” (University of the Basque Country, Bilbao, Spain), September 1999.
M. Kunzinger and R. Steinbauer, “16th Conference on General Relativity and Gravitation (GR16)” (Durban, South Africa), July 2001.

III. LINES OF DEVELOPMENT

The first main goal of the project as formulated in the proposal was the adaption of nonlinear generalized function techniques to use them in *applications in general relativity.*

The first application dealt with was the *distributional description of impulsive pp-waves*, a class of singular space-time geometries where already the metric tensor contains the δ-distribution. The line element for these geometries which model a gravitational shock wave ([Pen72]) can explicitly be written as

$$ds^2 \;=\; f(x,y)\delta(u)du^2 - du dv + dx^2 + dy^2. \tag{1}$$

Impulsive pp-waves play a prominent role in the description of highly energetic scattering processes in the framework of (a yet-to-be-found theory of) quantum gravity.
In a first step (partly building upon earlier works, [Fer90], [Bal97]) in [J1] and [J2] a complete distributional treatment of geodesics in impulsive pp-waves was achieved. These results in turn provided the key for a mathematical clarification of a long standing question: beside the distributional form (1) of the metric impulsive pp-waves are often modeled using a continuous line element. Both forms of the metric—although physically equivalent (see eg. [S2])—can only be related to one another by a transformation which is distributionally ill-defined. In particular, it does not comprise an allowed change of coordinates in the sense of general relativity. In the paper [J3] which was elected one of the “Highlights 1999-2000” of the editors of the respective journal, it was shown that both forms of the metric can be related by a generalized coordinate transformation in the sense of the theory of nonlinear generalized functions. Moreover the methods applied could be given a sensible physical interpretation.

Continuing this line of research a distributional study of *spherical impulsive gravitational waves* was initiated in cooperation with Jirí Podolský from the Institute for Theoretical Physics of the University of Prague. These space-times model a spherical gravitational shock wave and have a much richer structure than pp-waves (cf. eg. [Pod99]) which can be seen immediately from the distributional form of the metric tensor, i.e.,

$$ds^2 \;=\; 2\,\frac{w^2}{\psi^2}\,|d\xi - f\,\delta(u)du\,|^2 + 2\,du\,dw - 2\epsilon\,du^2 + w\Big[(f_\xi + \bar f_{\bar\xi}) - \frac{2\epsilon}{\psi}(f\bar\xi + \bar f\xi)\Big]\delta(u)du^2. \tag{2}$$

Note that this line element already contains distributionally ill-defined squares of the Dirac-measure. It was first explicitly derived by J. Griffiths and J. Podolský ([Pod99]) and by a formal transformation related to the standard form of Robinson-Trautman solutions of type N. Since this situation is analogous to the pp-wave case the quest for a mathematically rigorous treatment arises immediately. A study of geodesics in these space-times is the subject of work [J4] which also aims at laying the foundations for a mathematically sound treatment of the above mentioned transformation.

Another issue processed in the course of the project is the distributional description of the most prominent singular solution of Einstein's equations—the *Schwarzschild space-time.* Using the geometric theory of generalized functions in [J5] a general framework was presented which allows to discuss all previous approaches given in the literature (eg. [Bal93], [Kaw97]) from a unified point of view. Moreover, using the Kerr-Schild structure of the space-time a first conceptionally satisfying derivation of the physically reasonable energy-momentum tensor was given.
In work [J6] the Schwarzschild metric is analyzed from the point of view of Hadamard regularizations. This theory of so-called pseudo-functions ([Bla00]) is a alternative theory of nonlinear distributions, which, in particular, is designed to treat point-singularities of the form present in the Schwarzschild metric. The results achieved in [J6] agree with the ones derived in [J5] (see above).

In addition to these applications in general relativity another (new and unforeseen) line of research could be initiated together with researchers form the Department of Mathematics of the University of Vienna (in part supported by FWF-Project P10472-MAT, “Nonlinear transformation groups and distributions”): the *geometric theory of generalized functions.*

At the heart of general relativity lies the general principle of covariance: there exists no preferred system of coordinates, or more mathematically speaking: general relativity is a diffeomorphism invariant theory—the stage of relativity is a (four-dimensional) pseudo-Riemannian manifold.
The theory of algebras of generalized functions on the other hand, was primarily developed as a tool in nonlinear PDEs (see above)—a field where diffeomorphism invariance in many respects is not an issue. Indeed the most widely used variant of the theory—the so-called *full* algebras distinguished by a canonical embedding of Schwartz distributions—is not diffeomorphism invariant; some of its basic building blocks are not invariant under the natural action of a coordinate transformation.
Under the influence of applications in a geometric setting and, in particular, in Lie group analysis of differential equations (see eg., [Kun00] and [Dap02]) and in general relativity (for an overview see see eg. [Vic99]) a certain shift of focus within the theory of nonlinear generalized functions itself had taken place; a geometric construction of an algebra of generalized function had become a central issue.

Using earlier works by J. F. Colombeau and A. Meril ([Col94b]) and J. Jelínek ([Jel99]) in paper [J7] (together with M. Grosser and E. Farkas, both Department of Mathematics, University of Vienna) the first construction of a *diffeomorphism invariant full Colombeau algebra* was given. The construction is done locally on open sets of Euclidean space and uses calculus in convenient vector spaces (put forward by A. Kriegl and P. Michor in [Kri97]) to deal with the unavoidable infinite dimensional analysis.
Applications in general relativity, however, put yet another demand on the construction of an algebra of generalized functions: all its ingredients should be defined intrinsically and globally on a manifold. Such a construction indeed was achieved in [J8] (with J. Vickers, Department of Mathematics, University of Southampton, UK). It provides a globally defined sheaf of differential algebras (with respect to the Lie derivative along smooth vector fields) of (scalar) generalized functions and a canonical embedding of distributions with the embedding commuting with Lie derivatives.

Another important variant of the theory of nonlinear generalized functions—the *special algebras*—although lacking a *canonical* embedding of distributions provide an especially flexible method of modeling singularities. Since these algebras are diffeomorphism invariant by definition they lend themselves to geometric applications in a very natural way.
Building upon earlier works by J. W. De Roever and M. Damsma ([De 91]) as well as by the group of S. Pilipović at the University of Novi Sad, Yugoslavia ([Dap96]) in publication [J9] a complete geometric theory of *generalized sections of vector bundles* could be developed. It provides maximal consistency with respect to the classical distributional theory of G. De Rham (De Rham currents, [De 84]), and J. E. Marsden ([Mar68]). Moreover the point value characterization of generalized functions on $\mathbb{R}^n$ ([Obe99]) could be generalized to the manifold level.
This construction was extended to a *generalized pseudo-Riemannian geometry* in [J10]. For the first time it provides a complete and consistent framework allowing to deal with singular space-time geometries in general relativity thereby significantly generalizing the classical distributional setting put forward by P. Parker ([Par79]).

In publication [J11] the study of generalized functions *taking values in a manifold* was initiated. A comparable concept is of course not available within distribution theory. However, the need for such a notion arises naturally upon considering the flow of a non-smooth vector field or the geodesics of a generalized space-time metric. The key idea is provided by a chart independent characterization of the asymptotic estimates in terms of the regularization parameter.
This local characterization could be proved to be equivalent to a simple global criterion in [J12] (with J. Vickers). This new characterization allows for the construction of a global and functorial theory of generalized functions valued in a manifold as well as of generalized vector bundle homomorphisms. In particular a point value description could be established in both cases.

Finally, current research ([J13] with M. Oberguggenberger, Department of Engineering Mathematics, Geometry and Computer Science, University Innsbruck and J. Vickers) deals with flows of non-smooth vector fields, in particular extending the theory of J. E. Marsden ([Mar68]).
Also [J14] (with J. Vickers) aims at initiating a study of generalized connections in principal fiber bundles extending the theory of linear connections introduced in [J10].

A comprehensive presentation of the geometric theory of generalized functions including its applications in Lie group analysis of differential equations as well as in general relativity was given in the research monograph [B1] (with M. Grosser and M. Oberguggenberger).

IV. COOPERATIONS

During the course of the project the coworkers could initiate and pursue a lot of interesting and stimulating national and international cooperations.

First of all the formation of the research group DiANA (Differential Algebras and Nonlinear Analysis; `http://www.mat.univie.ac.at/~diana`) by M. Grosser, G. Hörmann, M. Kunzinger, E. Farkas and R. Steinbauer (University of Vienna) and M. Oberguggenberger (University Innsbruck) has to be mentioned. Since its foundation in 1997/98, initiated by M. Kunzinger, this group is concerned with the further development of the theory of algebras of generalized functions and its applications in mathematical physics. The intense and manifold cooperation between the group members has led to a large number of high quality publications.

A close cooperation has been established with the group of C. J. S. Clarke, J. Vickers and J. Wilson at the Department of Mathematics at the University of Southampton, UK. The main focus of this collaboration lies in applications in general relativity as well as the geometric construction of spaces of generalized functions. During the course of the project several stays of J. Vickers and J. Wilson in Austria as well as a three-month-stay of M. Kunzinger and R. Steinbauer in Southampton could be realized.

Moreover a cooperation with J. Podolský (Institute of Theoretical Physics, University of Prague) focusing on the distributional treatment of impulsive gravitational waves could be established. M. Kunzinger, R. Steinbauer and H. Urbantke visited Prague several times, J. Podolský spent one research stay in Vienna.

During a research and conference stay of M. Kunzinger and R. Steinbauer in Durban (South Africa) the long standing contact of the DiANA group with one of the fathers of the theory of nonlinear generalized functions, E. E. Rosinger could be maintained. Moreover, contacts to several researchers working in general relativity could be established during the GR16 conference, most importantly with J. Griffiths (Loughborough University, UK) and B. Nolan (Dublin City University, Ireland). B. Nolan is going to visit Innsbruck and Vienna in May 2002.

Finally the long existing contact with the research group of S. Pilipović (Department of Mathematics, University of Novi Sad, Yugoslavia) could be intensified. The cornerstone of the construction of diffeomorphism invariant full Colombeau algebras was layed during a workshop in Novi Sad in July 1998.

V. RESULTS

To our great satisfaction we may say that not only all the goals of the project have completely been reached but also beyond these new lines of research have been opened and important results have been achieved.
On the one hand the project has succeeded in applying generalized function methods in general relativity thereby surpassing by far the initial objectives of the proposal; on the other hand it was one of the driving forces behind the development of the geometric theory of generalized functions. In particular, in this field important and difficult problems could be solved and the door to a wide range of new developments could be opened, both theoretical within the theory of nonlinear generalized functions as well as concerning applications in a geometric context.

The project resulted in altogether 9 publications in international highly ranked journals–some of them among the top-10 journals of mathematics. Five more papers are either submitted for publication, available as preprints or in preparation. Moreover, the book-project [B1]—which of course transcends the horizon of the project—could be completed successfully in 2001. A detailed exposition of the core of the project is contained in the Ph.D thesis of R. Steinbauer ([P1]).

Additionally, 7 more papers have been published in proceedings volumes and 23 talks have been given at international congresses or at renowned institutes.

List of Publications and Talks (updated March 2003)

1. Applications in General Relativity

A Publications in Refereed Journals

[J1] Steinbauer, R. Geodesics and geodesic deviation for impulsive gravitational waves. *J. Math. Phys.*, **39**:2201–2212, 1998.

[J2] Kunzinger, M., Steinbauer, R. A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves. *J. Math. Phys.*, **40**:1479–1489, 1999.

[J3] Kunzinger, M., Steinbauer, R. A note on the Penrose junction conditions. *Class. Quant. Grav.*, **16**:1255–1264, 1999.

[J4] Podolský, J., Steinbauer, R. Geodesics in spacetimes with expanding impulsive gravitational waves. *Phys. Rev. D.*, to appear 2003.

[J5] Heinzle, J. M., Steinbauer, R. Remarks on the distributional Schwarzschild geometry. *J. Math. Phys.*, **43**:1493–1508, 2002.

[J6] Heinzle, J. M. Hadamard regularization of the Schwarzschild geometry. *Preprint*, 2002.

B Proceedings

[S1] Steinbauer, R. Distributional description of impulsive gravitational waves. In Grosser, M., Hörmann, G., Kunzinger, M., Oberguggenberger, M., editor, *Nonlinear Theory of Generalized Functions*, volume 401 of *Chapman & Hall/CRC Research Notes in Mathematics*, pages 267–274, Boca Raton, 1999. CRC Press.

[S2] Steinbauer, R. On the geometry of impulsive gravitational waves. In Vulcanov, D., Cotaescu, I., editor, *Proceedings of the 8th Romanian Conference on General Relativity and Gravitation.* Mirton Publishing House, 1999.

[S3] Steinbauer, R. On the impulsive limit of gravitational waves. In Ibanez, J., editor, *Recent Developments in Gravitation, Proceedings of the Spanish Relativity Meeting. ERE-99*, pages 307–312. Universidad del Pais Vasco, 2000.

[S4] Steinbauer, R. Nonlinear distributional geometry and general relativity. Contribution to Proceedings of the International Conference on Generalized Functions (ICGF 2000, Guadeloupe). *Integral Transf. Special Funct.*, to appear, 2003.

C Further Publications

[P1] Steinbauer, R. Distributional Methods in General Relativity. *Ph.D. Thesis*, University of Vienna, 2000.

[P2] Steinbauer, R. Distributionelle Krümmung von Strings. *Seminar notes*, 1998.

D Talks

[V1] Steinbauer, R. Distributional Description of Impulsive Gravitational Waves. VIII Romanian Conference on General Relativity and Gravitation. Bistrita, Rumania, June 1998.

[V2] Steinbauer, R. Nonlinear Generalized Functions and General Relativity. Workshop: Nonlinear Theory of Generalized Functions. Department of Mathematics, University of Novi Sad, Yugoslavia, July 1998.

[V3] Steinbauer, R. ,,Discontinuous Diffeomorphisms". Workshop: Nonlinear Theory of Generalized Functions. Department of Mathematics, University of Novi Sad, Yugoslavia, July 1998.

[V4] Steinbauer, R. On the Geometry of Impulsive Gravitational Waves. 2nd Samos Meeting on Cosmology, Geometry and Relativity ,,Mathematical & Quantum Aspects of Relativity and Cosmology". Pythagoreon, Greece, August 1998.

[V5] Steinbauer, R. Impulsive Robinson-Trautman Lösungen. Department of Engineering Mathematics, Geometry and Computer Science, University Innsbruck, June 1999.

[V6] Steinbauer, R. On the impulsive limit of gravitational waves. Spanish Relativity Meeting (ERE99). University of the Basque Country, Bilbao, Spain, September 1999.

[V7] Steinbauer, R. Colombeau algebras and multiplication of distributions. Institute for Theoretical Physics, University Prague, November 1999.

[V8] Steinbauer, R. Colombeau algebras in general relativity: Impulsive waves and other topics. Institute for Theoretical Physics, University Prague, November 1999.

[V9] Steinbauer, R. Applications of generalized functions in general relativity. International Conference on Generalized Functions (ICGF 2000). Universite' des Antilles et de la Guyane, Guadeloupe, France, April 2000.

[V10] Kunzinger, M. Algebras of generalized functions and general relativity. Department of Mathematics, University of Southampton, UK, July 2001.

[V11] Steinbauer, R. Spherical impulsive gravitational waves. Department of Mathematics, University of Southampton, UK, July 2001.

[V12] Kunzinger, M. Nonlinear distributional geometry for General Relativity. 16th Conference on General Relativity and Gravitation (GR16). Durban, South Africa, July 2001.

[V13] Steinbauer, R. Geodesics in expanding impulsive gravitational waves. 16th Conference on General Relativity and Gravitation (GR16). Durban, South Africa, July 2001.

[V14] Heinzle, J. M. Distributional description of the Schwarzschild geometry. Department of Mathematics, University of Southampton, UK, August 2001.

[V15] Steinbauer, R. Generalized pseudo-Riemannian geometry for general relativity. Journees Relativistes 2001. Dublin, Ireland, September 2001.

2. Geometric Theory of Generalized Functions

A Book

[B1] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R. *Geometric Theory of Generalized Functions with Applications to Relativity*, volume **537** of *Mathematics and its Applications*. Kluwer, 2001.

B Publications in Refereed Journals

[J7] Grosser, M., Farkas, E., Kunzinger, M., Steinbauer, R. On the foundations of nonlinear generalized functions I, II. *Mem. Am. Math. Soc.*, **153**(729), 2001.

[J8] Grosser, M., Kunzinger, M., Steinbauer, R., Vickers, J. A global theory of algebras of generalized functions. *Adv. Math.*, **166**(1):179–206, 2002.

[J9] Kunzinger, M., Steinbauer, R. Foundations of a nonlinear distributional geometry. *Acta Appl. Math.*, **71**:179–206, 2002.

[J10] Kunzinger, M., Steinbauer, R. Generalized pseudo-Riemannian Geometry. *Trans. Amer. Math. Soc.*, **354** (10):4179-4199, 2002.

[J11] Kunzinger, M. Generalized functions valued in a smooth manifold. *Monatsh. Math.*, **137**:31-49, 2002.

[J12] Kunzinger, M., Steinbauer, R., Vickers, J. Intrinsic characterization of manifold-valued generalized functions. *Proc. London Math. Soc*, to appear, 2003.

[J13] Kunzinger, M., Oberguggenberger, M., Steinbauer, R., Vickers, J. Generalized flows and singular ODEs on differentiable manifolds. *Preprint*, 2003.

[J14] Kunzinger, M., Steinbauer, R., Vickers, J. Generalized connections and curvature. *Preprint*, 2003.

C Proceedings

[S5] Grosser, M., Kunzinger, M., Steinbauer, R., Urbantke, H., Vickers, J. Diffeomorphism invariant construction of nonlinear generalized functions. In Neugebauer, G., Collier, R., editor, *Proceedings of the International European Conference on Gravitation, Journées Relativistes 99*, volume **9** Spec. Iss. of *Annalen der Physik*. Wiley, 2000.

[S6] Steinbauer, R. Diffeomorphism invariant Colombeau algebras. Part I: Local theory. Contribution to Proceedings of the International Conference on Generalized Functions (ICGF 2000, Guadeloupe), *Integral Transf. Special Funct.*, to appear, 2003.

[S7] Kunzinger, M. Diffeomorphism invariant Colombeau algebras. Part III: Global theory. Contribution to Proceedings of the International Conference on Generalized Functions (ICGF 2000, Guadeloupe), *Integral Transf. Special Funct.*, to appear, 2003.

D Talks

[V16] Urbantke, H. (Poster presentation) A global theory of algebras of generalized functions. Journees Relativistes 1999. University Weimar, Germany, September 1999.

[V17] Grosser, M., Steinbauer, R. Algebren verallgemeinerter Funktionen auf Mannigfaltigkeiten. Mathematisches Kolloquium, University of Vienna, October 1999.

[V18] Steinbauer, R. Diffeomorphism invariant Colombeau algebras, Part 1: Local theory. International Conference on Generalized Functions (ICGF 2000). Universite' des Antilles et de la Guyane, Guadeloupe, France, April 2000.

[V19] Steinbauer, R. The geometry of nonlinear generalized functions. Department of Engineering Mathematics, Geometry and Computer Science, University Innsbruck, June 2001.

[V20] Kunzinger, M. Nonlinear distributional geometry. Department of Mathematics, University of Southampton, UK, July 2001.

[V21] Steinbauer, R. Generalized semi-Riemannian Geometry. Department of Mathematics, University of Southampton, UK, August 2001.

[V22] Steinbauer, R. Nonlinear distributional geometry, Part 1. 15. ÖMG-Kongress (15th Congress of the Austrian Mathematical Society). University of Vienna, September 2001.

[V23] Kunzinger, M. Nonlinear distributional geometry, Part 3. 15. ÖMG-Kongress (15th Congress of the Austrian Mathematical Society). University of Vienna, September 2001.

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