Linear and nonlinear distributional Lorentzian geometry

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Intro: The use of distributions

Distributions are used to mathematically describe a wide range of idealized configurations in physics.

Ex: replace extended source of a field by an idealized one
(point charges, thin shells or layers of matter, . . .)

This description works very well in linear field theories.

Two reasons:

1. mathematically sound formulation, rooted in functional analysis
2. physically reasonable since we have 'limit consistency'.
   (Ex: charges close to a point charge produce fields close to the Coulomb field.)

The general theme

Distribution theory is deeply routed in linear functional analysis.

- Existence of Green functions (Malgrange-Ehrenpreis)
- Schwartz kernel theorem (general integral operators)

However, distribution theory is a inherently linear theory!

- Non-existence of distributional solutions for PDEs with non-constant coefficients (Lewy example)
- No general product for arbitrary pairs of distributions

Quest

Mathematical description of concentrated sources in GR.

Challenges: nonlinearities and the geometric nature of GR

Need a nonlinear distributional geometry!

Outline

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Distributions: basic ideas and definitions

- *Key idea:* a function is a map
  taking points $x \in \mathbb{R}^n$ to complex numbers but also
  taking 'test functions' to numbers

$$f: \mathcal{C}_c^\infty \ni \varphi \mapsto \int f(x)\varphi(x)\, dx \quad \textit{instead of} \quad f: \mathbb{R}^n \ni x \mapsto f(x) \in \mathbb{C}$$

- take all such maps $\rightsquigarrow$ *dual space:* ($U \subseteq \mathbb{R}^n$ open)

$$\mathcal{D}'(U) := \{u: \mathcal{C}_c^\infty(U) \to \mathbb{C} \mid \textit{linear and continuous}\}$$

This is a very big space: not just (locally integrable) functions!

- *Scalar distributions on manifolds:* here functions
  take 'test n-forms' to numbers

$$\mathcal{D}'(M) := \{u: \Omega_c^n(M) \to \mathbb{C} \mid \textit{linear and continuous}\}$$

No metric structure needed!

Basic Distributional Geometry

[Schwartz, de Rham, ...]

- Scalar distributions on manifolds: $\mathcal{D}'(M) = \left[\Omega^n_c(M)\right]'$
  Distributions localize over open sets but not over points!
- Distributional tensor fields (dual space as well but easier):

$$\mathcal{D}'{}^r_s(M) = \mathcal{D}'(M) \otimes \mathcal{T}^r_s(M) = L_{\mathcal{C}^\infty(M)}(\Omega^1(M)^r, \mathfrak{X}(M)^s; \mathcal{D}'(M))$$

  - The usual local formulas work but components are distributions!

    e.g. $\mathcal{D}'{}^1_0 \ni X = \sum X^i \partial_i$ with $X^i \in \mathcal{D}'(M)$
  - Extend usual operations by continuity: $L_X,\ [\,,\,],\ \wedge,\ \iota_X,\ \{\,,\,\}, \dots$
    but with only one $\mathcal{D}'$-factor!
  - compare smooth case: $\mathcal{T}^r_s(M) = L_{\mathcal{C}^\infty(M)}(\Omega^1(M)^r, \mathfrak{X}(M)^s; \mathcal{C}^\infty(M))$
- More general, distributional sections of a vector bundle $E \to M$

$$\mathcal{D}'(M, E) = \mathcal{D}'(M) \otimes \Gamma(M, E) = L_{\mathcal{C}^\infty(M)}(\Gamma(M, E^*), \mathcal{D}'(M))$$

Outline

Distributional Lorentzian Metrics

Definition ([Marsden 69], [Parker 79])

A distributional metric is a element

$$g \in \mathcal{D}'^0_2(M) \quad \text{i.e.,} \quad g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'(M), \quad \mathcal{C}^\infty\text{-bilinear}$$

that is symmetric, and non-degenerate.

Problems:

- Non-degeneracy cannot be defined pointwise! Replacements:
  1. $g(X, Y) = 0$ for all **smooth** vector fields $Y \Rightarrow X = 0$
     $\rightsquigarrow \quad ds^2 = x^2\, dx^2$ is non-degenerate!
  2. g is a smooth Lorentzian metric off its singular support.

  Best choice: demand 1 and 2
- can't insert $\mathcal{D}'$-vector fields into $\mathcal{D}'$-metric
- g gives no isomorphism $\mathcal{D}'^1_0 \ni X \mapsto X^\flat := g(X, .) \in \mathcal{D}'^0_1$
- index, geodesics, etc. of a distributional metric?

Distributional Connections

Definition ([Marsden 69]?, [LeFloch, Mardare 07])

A distr. connection is a map

$$\nabla: \ \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'^1_0(M)$$

with the usual properties.

- extends to entire (smooth!) tensor-algebra
- standard formulas hold, e.g.
$$\nabla_{\partial_i}(Y^j\partial_j) = \big(\partial_i Y^k + \Gamma^k_{ij} Y^j\big)\partial_k \quad (\Gamma^k_{ij} \in \mathcal{D}')$$
- Fundamental lemma in a weak form [LeFM, 07]:
Every distributional metric has a unique 'Levi-Civita connection'
$$\begin{aligned} 2\nabla^\flat_X Y(Z) \quad &:= \quad X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(X,Y)\big) \\ &- \quad g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \end{aligned}$$
Only if g more regular (e.g. L^2_{loc}): $\Rightarrow \nabla^\flat_X Y = (\nabla_X Y)^\flat$

Curvature from a Distributional Connection?

- Coordinate-free analysis [LeFM, 07]
- Coordinate approach [Geroch, Traschen, 87]

$$R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}$$

  Look for special distributional connections for which the formula makes sense in $\mathcal{D}'$.

- A good choice is $\Gamma^i_{jk} \in L^2_{\text{loc}}$:

$$\Gamma^i_{lj} \in L^2_{loc} \subseteq L^1_{loc} \subseteq \mathcal{D}' \quad \Rightarrow \quad \Gamma^i_{lj,k} \in \mathcal{D}'$$

$$\Gamma^i_{km} \in L^2_{loc} \quad \Rightarrow \quad \Gamma^i_{km}\Gamma^m_{lj} \in L^1_{loc} \subseteq \mathcal{D}'$$

- Observe:

$$\Gamma^i_{jk} \in L^2_{\text{loc}} \Leftrightarrow \nabla_X Y \in L^2_{\text{loc}} \text{ for all smooth vector fields } X, Y$$

Curvature from a Distributional Connection

Definition ([LeFM, 07], [GT, 87])

(i) A distr. connection ∇ is called L^2_{loc}-connection if $\nabla_X Y \in L^2_{\mathrm{loc}}$.

(ii) The Riemannian curvature tensor $R \in \mathcal{D}'^1_3(M)$ for an L^2_{loc}-connection is defined by the usual formula, i.e.,
$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

(iii) Ricci, Weyl, and scalar curvature defined as usual.

Observation (L^2_{loc}-stability)

Assume for a sequence of L^2_{loc}-connections
$$\nabla^{(n)}_X Y \to \nabla_X Y \qquad \text{in } L^2_{loc}.$$
Then
$$R^{(n)}_{XY}Z \to R_{XY}Z \qquad \text{in } \mathcal{D}'^1_0(M).$$
The analogue holds true for the Ricci, Weyl, and scalar curvature.

Curvature from a Distributional Metric

- by the above: want induced connection to be in L^2_{loc}
- in local coordinates: $2\Gamma^i_{jk} = g^{il}\left(g_{lj,k} + g_{kl,j} - g_{jk,l}\right)$

Observation ([GT, 87])

The curvature is defined if the metric g belongs to $H^1_{loc} \cap L^\infty_{loc}$.

Recall: $H^1_{\mathrm{loc}} := \{u \in \mathcal{D}' : \; u, \partial_j u \in L^2_{\mathrm{loc}}\}$

Indeed:

$$g_{ij} \in H^1_{\mathrm{loc}} \;\Rightarrow g_{ij,l} \in L^2_{\mathrm{loc}}; \; g_{ij} \in L^\infty_{\mathrm{loc}} \;\Rightarrow \Gamma^i_{jk} \in L^2_{\mathrm{loc}}$$

$H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ is an algebra with a good notion of invertibility.

$f \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ invertible $:\Leftrightarrow$ loc. uniformly bounded away from 0,

$$\forall K \text{ compact } \exists C : \; |f(x)| \geq \; C > 0 \text{ a.e. on } K$$

then f^{-1} is again loc. unif. bded away from 0

The gt-class of metrics

Definition ([GT, 87], add on by [LeFM, 07], [S,Vickers, 09])

A distributional metric g is called gt-regular if

- $g \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$,
- g is a Lorentzian metric almost everywhere, and
- g is non-degenerate, i.e., $\det(g)$ is invertible in $H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$.

Theorem (Properties of gt-regular metrics, [G&T,87])

(i) *The Levi-Civita connection of g is an L^2_{loc}-connection.*

(ii) *The Riemann, Ricci, Weyl, and scalar curvature of g is defined.*

(iii) *We have H^1_{loc}-stability.*

(iv) *The Bianchi identities cannot be formulated in $\mathcal{D}'$.*

(v) $\dim(\mathrm{supp}(\mathrm{Riem}[g])) \geq 3$

Summary: Linear distributions for GR

The 'maximally reasonable' distributional setting essentially due to Geroch and Traschen uses Sobolev regularity

$$g \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}.$$

Pros:

- Allows to define curvature $\mathrm{Riem}[g]$, $\mathrm{Ric}[g]$, $W[g]$, $\mathrm{R}[g]$ in distributions.
- Has 'limit consistency' in H^1_{loc}.

Cons:

- Energy conservation cannot be formulated.
- Allows for only mild concentration of sources: thin shells of matter are okay but strings are out.

Outline

Warning: It really can go wrong! [GT, 87]

Example (Regularising a string (1))

$$ds_l^2 \;=\; -dt^2 + dz^2 + dr^2 + \beta_l(r)^2 d\phi^2$$

$$\beta_l(r) \;=\; \begin{cases} \frac{l}{\gamma}\,\sin(\frac{\gamma r}{l}) & (r \le l) \\ (r - l + \frac{l}{\gamma}\tan\gamma)\cos\gamma & (r > l) \end{cases} \quad (l > 0, \gamma \in (0, \frac{\pi}{2}]\,)$$

Outside ($r > l$): standard cone via $R = r - l + (l/\gamma)\tan\gamma$
Inside ($r < l$): mass density $\mu = 2\pi(1 - \cos\gamma) = \Delta$
Limit ($l \to 0$):

$$ds^2 \;=\; -dt^2 + dz^2 + dr^2 + r^2\cos^2\gamma\, d\phi^2, \quad \mu = \Delta$$

It is tempting to assign to the string the mass density

$$\rho_s = \mu\delta^{(2)}(r) = 2\pi(1 - \cos\gamma)\,\delta^{(2)}(r) = \Delta\delta^{(2)}(r).$$

Warning: It really can go wrong! [GT, 87]

Example (Regularising a string (2))

$$\bar{ds_l}^2 = e^{2\lambda f(r/l)} ds_l^2, \qquad \lambda > 0,\ f \geq 0,\ \mathrm{supp}(f) \subset\subset [1/2, 1]$$

$$\bar{\mu} = 2\pi(1 - \cos\gamma) - 2\pi \int_0^1 \frac{e^{2\lambda f(x)}}{\gamma} \left(\lambda^2 f'^2(x) \sin(\gamma x)\right) dx < \mu$$

Limit ($l \to 0$):

$$ds^2 = -dt^2 + dz^2 + dr^2 + r^2 \cos^2\gamma\, d\phi^2, \quad \bar{\mu} \neq \Delta$$

So we run into an inconsistency:

$$\bar{\rho}_s = \bar{\mu}\,\delta^{(2)}(r) \neq 2\pi(1 - \cos\gamma)\,\delta^{(2)}(r) = \rho_s$$

Outline

Products of distributions?

A multitude of examples show that there is no reasonable product

$$\mathcal{D}' \times \mathcal{D}' \to \mathcal{D}'.$$

Alternatives

1. *Regular intrinsic products*
   Classical algebras L^∞, H^s $(s > n/2)$, but not for all of $\mathcal{D}'$
2. *Irregular intrinsic products*
   Products of singular distributions by ad hoc-methods (e.g. special regularisation,...). This leads to ambiguities, e.g.
   $$\delta^2 = 0,\ c\delta,\ c\delta + c'\frac{1}{2\pi i}\delta,\ c\delta + c'\delta',\ \dots$$
3. *Extrinsic products, algebras containing $\mathcal{D}'$*
   Consistently assign a product to each pair of distributions, which no longer is a distribution.
   Only limited consistency with classical analysis possible.

What can be done

Reasonable requirements for an algebra $\mathcal{A}(+,\circ)$ containing $\mathcal{D}'$:

(i) $\mathcal{D}' \hookrightarrow \mathcal{A}$ linear, $f \equiv 1$ unity in $\mathcal{A}$

(ii) derivations $\partial_i : \mathcal{A} \to \mathcal{A}$ ($i = 1, \dots, n$), linear and Leibniz rule

(iii) $\partial_i|_{\mathcal{D}'}$ is the usual partial derivative ($i = 1, \dots, n$).

(iv) $\circ|_{\mathcal{C}\times\mathcal{C}}$ is the usual product

Theorem (Impossibility Result, [Schwartz, 54])

There is no associative, commutative algebra satisfying (i)–(iv).

(What is possible, [Colombeau, 84])

Construction of associative and commutative algebras satisfying (i)–(iii) and

(iv') $\circ|_{\mathcal{C}^\infty\times\mathcal{C}^\infty}$ *is the usual product.*

Outline

Colombeau Algebras: an overview

Algebras of generalised functions in the sense of J.F. Colombeau [Colombeau 84, 85] are differential algebras

- that contain the vector space of distributions and
- display maximal consistency with classical analysis (in the light of L. Schwartz' impossibility result).
  In particular the construction preserves
  - the product of $\mathcal{C}^\infty$-functions
  - (Lie) derivatives of distributions.

Main ideas of the construction are

- regularisation of distributions by nets of $\mathcal{C}^\infty$-functions
- asymptotic estimates in terms of a regularisation parameter (via a quotient construction)

Colombeau algebra on manifolds (1)

[Damsma, deRoever 91]

- $\mathcal{E}(M)$, the basic space: all sequences of smooth functions on M

$$\mathcal{E}(M) := \{(u_\varepsilon)_{\varepsilon\in(0,1]} : \ u_\varepsilon \in \mathcal{C}^\infty(M)\}$$

ε as regularisation parameter, $\lim_{\varepsilon\to 0} u_\varepsilon$ may or may not exist!

- $\mathcal{E}_M(M)$, moderate sequences: slow growth of all derivatives in ε

For all compact sets K, for any number of vector fields $X_1, \ldots, X_l$

$$\sup_{x\in K} |L_{X_1} \ldots L_{X_l} u_\varepsilon(x)| = O(\varepsilon^{-p}) \quad \text{for some } p \text{, as } \varepsilon \to 0$$

- $\mathcal{N}(M)$, negligible sequences: fast vanishing of all derivatives in ε

For all compact sets K, for any number of vector fields $X_1, \ldots, X_l$

$$\sup_{x\in K} |L_{X_1} \ldots L_{X_l} u_\varepsilon(x)| = O(\varepsilon^{q}) \quad \text{for all } q \text{, as } \varepsilon \to 0$$

Definition (The (special) Colombeau algebra)

$$\mathcal{G}(M) := \mathcal{E}_M(M) \,/\, \mathcal{N}(M)$$

Colombeau algebra on manifolds (2)

Sequences of smooth functions of moderate growth in ε, identify those that only differ by a fast vanishing sequence.

$$\mathcal{G}(M) := \mathcal{E}_M(M) \,/\, \mathcal{N}(M) \;\ni u = [(u_\varepsilon)_\varepsilon]$$

- Componentwise operations
  - products $uv = [(u_\varepsilon v_\varepsilon)_\varepsilon]$
  - Lie derivatives $L_X u = [(L_X u_\varepsilon)_\varepsilon]$
- Localises to open sets and to points
  in fact, u is determined by its (generalised) point values
- Embedding of $\mathcal{C}^\infty$-functions: as trivial sequence

$$\sigma : \; \mathcal{C}^\infty(M) \ni u \mapsto (u)_\varepsilon \in \mathcal{G}(M)$$

- The product of smooth functions is preserved:

$$\sigma(uv) = \sigma(u)\sigma(v)$$

Embedding of distributions

In essence, locally via convolution

in a chart $\iota: \mathcal{D}'(U) \ni u \mapsto [(u * \rho_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ *for a mollifier* ρ

$\rho \in \mathcal{S}(\mathbb{R}^n)$, $\int \rho = 1$, $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$, $\int x^\alpha \rho(x)dx = 0$ for all $\alpha > 0$

$$\Rightarrow (\iota - \sigma)(\mathcal{C}^\infty) \subseteq \mathcal{N}$$

Warning: This depends on choice of mollifier and chart!

Alternatives:

- use a geometrically preferred embedding (geometric flows [Dave and coworkers 07–])
- use an embedding suitable to the application (special modelling)
- use a more involved construction of the algebra, which allows for a canonically embedding of distributions $\rightsquigarrow$ full algebras

In any case: $L_X \circ \iota = \iota \circ L_X$

Association: Connecting back to $\mathcal{D}'$

coarse-grain the algebra to implement distributional equality

- $u, v \in \mathcal{G}$ are associated with each other, $u \approx v$, if for all $\omega \in \Omega^n_c$

$$\lim_{\varepsilon \to} \int (u_\varepsilon - v_\varepsilon)\,\omega = 0 \qquad \text{(for one/any pair of representatives)}$$

- $u \in \mathcal{G}$ is associated to $v \in \mathcal{D}'$, $u \approx v$, if for all $\omega \in \Omega^n_c$

$$\lim_{\varepsilon \to 0} \int u_\varepsilon\,\omega = \langle v, \omega \rangle \qquad \textit{(for one/any representative)}$$

Then v is called the distributional shadow of u.
If a shadow exists at all it is unique.

Example

- $\theta^2\,(= \iota(\theta)^2) \;\approx\; \theta$
- $\delta^2\,(= \iota(\delta)^2)$ has no distributional shadow

Colombeau tensor fields

Definition (Generalised sections)

$$\mathcal{G}^r_s(M) := \mathcal{E}_M{}^r_s(M) \,/\, \mathcal{N}^r_s(M),$$

where moderateness and negligibility are defined analogously.

Theorem (Characterising sections, [Kunzinger, S, 01])

$\mathcal{G}^r_s(M)$ is a finitely generated, projective $\mathcal{G}(M)$-module and

$$\begin{aligned}\mathcal{G}^r_s(M) \;=\; \mathcal{G}(M)\otimes\mathcal{T}^r_s(M) \;&=\; L_{\mathcal{C}^\infty(M)}\big(\Omega^1(M)^r, \mathfrak{X}(M)^s; \mathcal{G}(M)\big)\\ &=\; L_{\mathcal{G}(M)}\big(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)\big).\end{aligned}$$

- compare to $\mathcal{D}'{}^r_s(M) = \mathcal{D}'(M)\otimes\mathcal{T}^r_s(M) = L_{\mathcal{C}^\infty(M)}(\Omega_1(M)^r, \mathfrak{X}(M)^s; \mathcal{D}'(M))$
- Similarly for sections of a vector bundle $E \to M$: $\Gamma_{\mathcal{G}}(M, E)$

Outline

Generalised metrics

Definition (Generalised metrics [KS, 02])

We define a symmetric $g \in \mathcal{G}^0_2(M)$ to be a generalised metric by one of the following equivalent non-degeneracy conditions

(i) $\det(g)$ is invertible in $\mathcal{G}(M)$ (generalised non-degeneracy)

(ii) for all generalised points $g(\tilde{x})$ is nondegenerate as map $\tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}$ (pointwise generalised non-degeneracy)

(iii) there exists a representative g_ε consisting of smooth metrics and $\det(g)$ invertible in $\mathcal{G}(M)$ (idea of smoothing)

- technicalities on the index skipped
- g induces an isomorphism $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := g(X, \,.\,) \in \mathcal{G}^0_1(M)$
- g has a unique (generalised) Levi-Civita connection

Levi-Civita connection

Definition

A gen. connection is a map $\nabla: \mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M) \to \mathcal{G}^1_0(M)$ satisfying the usual conditions.

- extends to entire generalised(!) tensor-algebra
- standard formulas hold, e.g.

$$\nabla_{\partial_i}(Y^j\partial_j) = \big(\partial_i Y^k + \Gamma^k_{ij} Y^j\big)\partial_k \quad (Y^j, \Gamma^k_{ij} \in \mathcal{G}).$$

Theorem (Fundamental Lemma, [KS, 02])

For any generalised metric g there exists a unique generalised connection ∇ that is $(X, Y, Z \in \mathcal{G}^1_0(M))$

(∇_3) *torsion-free i.e.,* $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$

(∇_4) *metric, i.e.,* $\nabla_X g = 0 \Leftrightarrow X\big(g(X,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

It is called Levi-Civita connection and given by Koszul's formula.

Curvature from a generalised connection

Definition (Generalised curvature)

(i) Let g be a generalised metric with generalised Levi-Civita connection ∇. We define the generalised Riemannian curvature tensor $R \in \mathcal{G}^1_3(M)$ of g by the usual formula, i.e.,

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

(ii) Ricci, Weyl, and scalar curvature defined as usual.

Observation (Basic compatibility)

Let g be a generalised metric with representative g_ε such that

$$g_\varepsilon \to \tilde{g} \text{ locally in } \mathcal{C}^k.$$

Then any representative R_ε of the Riemann tensor R of g converges to the Riemann tensor $\tilde{R}$ of $\tilde{g}$ locally in $\mathcal{C}^{k-2}$.
The analogue holds true for the Ricci, Weyl, scalar curvature.

Outline

Applications in GR: an overview

- Curvature of cosmic strings
  [Clarke, Vickers, Wilson, 96], [Vickers and Coworkers, 99–01]
- Geometry of impulsive pp-waves
  [Balasin, 96], [KS, 98–04]
- (Ultrarelativistic) Kerr-Newman geometries
  [Balasin, 96–03], [S, 98], [Heinzle, S, 02]
- Singular Yang-Mills theory [KSV, 05]
- Linear distributional geometry renewed [LeFM, 07],
  applications [LeFloch and Coworkers 07–]
- Compatibility of $\mathcal{G}$- with the gt-setting [SV, 09]
- Wave equations in singular space times
  [VW, 00], [Grant, Mayerhofer, S, 09], [Hanel, 11]
  $\rightsquigarrow$ afternoon talk

Geometry of impulsive pp-waves (1)

Penrose cut&paste approach $\rightsquigarrow$ distributional form of the metric

$$ds^2 = f(x^A)\delta(u)du^2 + dudv - \delta_{AB}dx^A dx^B$$

Model in $\mathcal{G}$:

$$\hat{d}s^2 = f(x^A)D(u)du^2 + dudv - \delta_{AB}dx^A dx^B$$

with $D = [(\rho_\varepsilon)_\varepsilon]$ and ρ_ε a strict-δ-net:

$\mathrm{supp}(\rho_\varepsilon) \to \{0\}$, $\int \rho_\varepsilon \to 1$ and $||\rho_\varepsilon||_{L^1} \leq C$

Results (Geodesics)

- *The geodesic equation is uniquely solvable in $\mathcal{G}(\mathbb{R}; M)$.*
- *These generalised geodesic have associated distributions which are refracted, broken straight lines.*
- *The geodesic deviation equation is uniquely solvable in $\mathcal{G}^1_0(M)$.*

Geometry of impulsive pp-waves (2)

Rosen form of the metric

$$ds^2 = dudV - (\delta_{AB} + \frac{1}{2} u\theta(u)\partial_A\partial_B f)^2 dX^A dX^B$$

continuous! achieved via a discontinuous transformation.

Results (Interpretation in $\mathcal{G}$)

There is a generalised coordinate transform $[(T_\varepsilon)_\varepsilon]$ (implicitly defined using the generalised geodesics) such that:

$$\begin{tikzcd}
\textit{distr. metric} \arrow[r, "\delta \mapsto D"] \arrow[d, "?"'] & f(x^A)\rho_\varepsilon(u)du^2 + dudv - \delta_{AB}dx^A dx^B \arrow[d, "T_\varepsilon"] \\
\textit{cont. metric} & \begin{array}{r} dudV + 2(\dot{x}_\varepsilon^A \partial_B x_\varepsilon^A - \partial_B v_\varepsilon)dudX^B \\ +(\partial_B(x_\varepsilon^1 + x_\varepsilon^2)(dX^B))^2 \end{array} \arrow[l, "\lim_{\varepsilon\to 0}"']
\end{tikzcd}$$

Further theoretical work by [Erlacher, Grosser 08–10]

Outline

Compatibility with the gt-setting

- $g \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \rightsquigarrow$ two ways to calculate the curvature
  - (i) gt-setting: coordinate formulas in $\mathcal{D}'$ resp. $W^{m,p}_{\mathrm{loc}}$ $\rightsquigarrow \mathrm{Riem}[g] \in {\mathcal{D}'}^1_3$
  - (ii) $\mathcal{G}$-setting: embed g via convolution with a mollifier usual formulas for fixed ε $\rightsquigarrow \mathrm{Riem}[(g_\varepsilon)_\varepsilon] \in \mathcal{G}^1_3$
- Do we get the same answer?

$$\begin{tikzcd} H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \ni g \arrow[r, "*\rho_\varepsilon"] \arrow[d, "\text{gt-setting}"'] & {[(g_\varepsilon)_\varepsilon] \in \mathcal{G}} \arrow[d, "\mathcal{G}\text{-setting}"] \\ \mathrm{Riem}[g] & \mathrm{Riem}[(g_\varepsilon)_\varepsilon] \arrow[l, "\lim_{\varepsilon\to 0}"'] \end{tikzcd}$$

Regularising gt-metrics

- To preserve positivity need admissible mollifiers:
  special strict δ-nets, which are moderate and have
  - (i) asymptotically vanishing moments:
    $\forall j \in \mathbb{N}\ \exists \varepsilon_0 :\ \int x^\alpha \rho_\varepsilon(x)\, dx = 0 \quad$ for all $1 \leq |\alpha| \leq j$ and all $\varepsilon \leq \varepsilon_0$
  - (ii) arbitrarily small L^1-norms of their negative parts:
    $\forall \eta > 0\ \exists \varepsilon_0 :\ \int |\rho_\varepsilon(x)|\, dx \leq 1 + \eta \quad$ for all $\varepsilon \leq \varepsilon_0$
- $g \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} :\ g^\varepsilon_{ij} := g_{ij} * \rho_\varepsilon, \rightsquigarrow$ metric g_ε, $\iota(g) = [(g_\varepsilon)_\varepsilon]$
- Basic convergence result:
  $$f \in W^{m,p}_{\mathrm{loc}} \ \Rightarrow\ f_\varepsilon := f * \rho_\varepsilon \to f \quad \text{in} \quad W^{m,p}_{\mathrm{loc}}\ (m \geq 0,\ 1 \leq p < \infty)$$

Lemma (Stability of the determinant)

For gt-regular g: $\det(g_\varepsilon) \to \det g$ *in* $H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ *for all* $p < \infty$.

But non-degeneracy of g $\quad (|\det(g)| \geq C > 0$ a.e. on cp. sets)

$\not\Rightarrow$ non-degeneracy of g_ε $\quad (|\det(g_\varepsilon)| \geq C > 0$ on cp. sets).

Preserving non-degeneracy

problem: preserving non-degeneracy for gt-metrics

- want: $\forall K$ compact, $\exists C$: $|\det(g_\varepsilon)| \geq C_K > 0$ on K $\qquad (N_\varepsilon)$

Definition (Stability condition)

Let g be a gt-regular metric and $\lambda_i, \dots, \lambda_n$ its eigenvalues.

(i) For any compact K we set $\mu_K := \min_{1\leq i\leq n} \mathit{esinf}_{x\in K} |\lambda^i(x)|$.

(ii) We call g stable if on K there is A^K continuous, such that

$$\max_{i,j} \mathit{essup}_{x\in K} |g_{ij}(x) - A^K_{ij}(x)| \leq C < \frac{\mu_K}{2n}.$$

Lemma (Non-degeneracy of smoothed gt-regular metrics)

Let g be a stable gt-regular metric and let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.
Then (N_ε) holds, and the embedding $\iota_\rho(g)$ is a gen. metric.

Compatibility results

Theorem (Compatibility of the gt- with the $\mathcal{G}$-setting)

Let g be stable, gt-regular with Riemann tensor $\mathrm{Riem}[g]$*.*
Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$*.*
Then we have for the Riemann tensor $\mathrm{Riem}[(g_\varepsilon)_\varepsilon]$ *of* $[(g_\varepsilon)_\varepsilon]$

$$\mathrm{Riem}[(g_\varepsilon)_\varepsilon] \to \mathrm{Riem}[g] \text{ in } \mathcal{D}'^1_3 \qquad (\varepsilon \to 0).$$

Hence for any embedding ι_ρ *we have* $\mathrm{Riem}[\iota_\rho(g)] \approx \mathrm{Riem}[g]$*.*

$$\begin{array}{ccc}
\underset{\color{red}{\textit{stable}}}{H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}} \ni g & \xrightarrow{\;*\iota_\rho \;\; \color{red}{\textit{admissible}}\;} & [(g_\varepsilon)_\varepsilon] \in \mathcal{G} \\
\text{gt-setting} \downarrow & & \downarrow \mathcal{G}\text{-setting} \\
\mathrm{Riem}[g] & \xleftarrow{\;\approx\;} & \mathrm{Riem}[(g_\varepsilon)_\varepsilon]
\end{array}$$

Similar results hold for other curvature quantities.

Děkuji

vám za pozornost

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