A Note on Distributional (Pseudo-)Riemannian Geometry

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The Theme: Basics of (Pseudo-)Riemannian Geometry for Metrics of Low Regularity

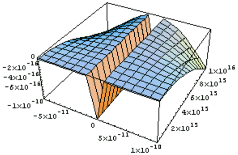
- Focus on
  1. The Levi-Civita connection for a metric of low regularity
  2. Curvature from a connection or metric of low regularity

- Settings
  - The classical C^∞-setting — still fine down to $C^{1,1}$-metrics
  - The distributional setting — all linear!
  - Nonlinear distributional geometry — using algebras of generalized functions (Colombeau algebras)

Outline

Why is it needed?

- General Relativity
  - space-time (M, g) Einstein equations $G_{ab}[g] = \kappa\, T_{ab}$
  - problem: many reasonable space-times are singular e.g.:



impulsive gravitational waves, shell-crossing singularities, cosmic strings, ...

  - often the metric is below $C^{1,1}$

T_{ab} "concentrated" but locally integrable

- Singular Yang-Mills Theory
  - fractionally charged instantons

$\rightsquigarrow$ singular connections in fiber bundles

- renewed interest [LeFloch&Mardare, 2007]

Basic Distributional Geometry

[Schwartz, de Rham, Marsden, Parker, ...]

- distributions on manifolds: $\mathcal{D}'(M) = \left[\Omega^n_c(M)\right]'$
- distributional sections of a vector bundle $E \to M$
  (see M.Grosser's talk)

$$\begin{aligned}\mathcal{D}'(M,E) &:= \left[\Gamma(M,E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)\right]' \\ &\cong L_{\mathcal{C}^\infty(M)}\big(\Gamma(M,E^*), \mathcal{D}'(M)\big) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M,E)\end{aligned}$$

- in particular, distributional tensor fields

$$\begin{aligned}{\mathcal{D}'}^r_s(M) &:= \left[\mathcal{T}^s_r(M) \otimes \Omega^n_c(M)\right]' \\ &\cong L_{\mathcal{C}^\infty(M)}\big(\mathcal{T}^s_r, \mathcal{D}'(M)\big) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M)\end{aligned}$$

- extend operations by continuity: L_X, $[\,,\,]$, $\wedge$, ι_X, $\{\,,\,\}$,...
  but with only one $\mathcal{D}'$-factor!

Distributional (Pseudo-)Riemannian Geometry

- distributional metric:

$$g \in \mathcal{D}'^0_2(M) \text{ i.e., } g : \mathfrak{X} \times \mathfrak{X} \to \mathcal{D}'(M) : \ \mathcal{C}^\infty(M) - \text{bilinear}$$

  - symmetric, and
  - nondegenerate, i.e., $g(X, Y) = 0 \ \forall Y \Rightarrow X = 0$

- Problems:
  - notion of nondegeneracy is rather weak (non-local):

$$ds^2 = x^2\, dx^2$$

    (classically) degenerate in 0, but nondegenerate in $\mathcal{D}'$
  - can't insert $\mathcal{D}'$-vector fields into $\mathcal{D}'$-metric
  - g gives no isomorphism $\mathcal{D}'^1_0 \ni X \mapsto X^\flat := g(X, .) \in \mathcal{D}'^0_1$
  - index, geodesics, etc. of a distributional metric?

Reminder: Levi-Civita Connection ($\mathcal{C}^\infty$-case)

Definition (Linear Connection)

A connection on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ s.t.

(∇_1) $\nabla_{fX+X'} Y = f\nabla_X Y + \nabla_{X'} Y$ ($\mathcal{C}^\infty$-linear)

(∇_2) $\nabla_X(fY + Y') = f\nabla_X Y + X(f)Y + \nabla_X Y'$ ($\mathbb{R}$-linear, Leibniz)

Theorem (Fundamental Lemma)

On (M, g) there exists a unique connection ∇ that is

(∇_3) *torsion-free i.e.,* $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$

(∇_4) *metric, i.e.,* $\nabla_X g = 0 \Leftrightarrow X\big(g(X, Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

called Levi-Civita connection of g, given by Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X\big(g(Y, Z)\big) + Y\big(g(Z, X)\big) - Z\big(g(X, Y)\big) \\ &- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Distributional Connections

Definition (Marsden, 1969)

A distr. connection is a map $\nabla : \mathfrak{X}(M) \times \mathcal{D}'^1_0(M) \to \mathcal{D}'^1_0(M)$ satisfying (∇_1), (∇_2) (appropriately read).

- not very flexible

Definition (LeFloch&Mardare, 2007)

A distr. connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'^1_0(M)$ satisfying (∇_1), (∇_2) (appropriately read).

- used from now on
- extends to entire (smooth!) tensor-algebra
- standard formulas hold, e.g.

$$\nabla_{\partial_i}(Y^j\partial_j) = \left(\partial_i Y^k + \Gamma^k_{ij} Y^j\right)\partial_k \quad (\Gamma^k_{ij} \in \mathcal{D}')$$

Levi-Civita connection ($\mathcal{D}'$-case)

- problem: $\nabla_X g$, in particular $g(\nabla_X Y, Z)$ cannot be defined
- workaround [LeF&M]: r.h.s. of Koszul's formula ($X, Y, Z \in \mathfrak{X}(M)$)

$$\begin{aligned} F(X,Y,Z) \quad := \quad & X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(X,Y)\big) \\ - \quad & g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \end{aligned}$$

• defined for a $\mathcal{D}'$-metric g
• C^∞-linear in Z hence defines a $\mathcal{D}'$-one-form $\nabla^\flat_X Y$

Definition ("Distributional Levi-Civita Connection")

Given a distributional metric g we define $\nabla^\flat_X Y(Z) = (1/2)\,F(X,Y,Z)$.

- properties:

$$\begin{array}{ll} (\nabla_3)^\flat & \nabla^\flat_X Y - \nabla^\flat_Y X - [X,Y]^\flat = 0 \\ (\nabla_4)^\flat & X\big(g(Y,Z)\big) - \nabla^\flat_X Y(Z) - \nabla^\flat_X Z(Y) = 0 \end{array}$$

- but no $\mathcal{D}'$-vector field, since no iso $\mathcal{D}'^1_0 \to \mathcal{D}'^0_1$
so no distributional connection as defined before
- only if g more reg. (e.g. L^2_{loc}): $\Rightarrow \nabla^\flat_X Y = (\nabla_X Y)^\flat \Rightarrow (\nabla_3), (\nabla_4)$

Curvature from a Distributional Connection?

- $\mathcal{C}^\infty$-case: $R: \ \mathfrak{X}(M)^3 \to \mathfrak{X}(M), \qquad R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$
- $\mathcal{D}'$-case, i.e., use distributional curvature
  problem: $\nabla_Y Z \in \mathcal{D}'^1_0 \ \rightsquigarrow \ \nabla_X \nabla_Y Z$ not defined
- workaround: look for special distributional connections with

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{A}(M) \subseteq \mathcal{D}'^1_0(M)$$

  such that ∇ can be extended ($X \in \mathfrak{X}, Y \in \mathcal{A}, \theta \in \Omega^1$)

$$\begin{aligned} \nabla : \ \mathfrak{X}(M) \times \mathcal{A}(M) \quad &\to \quad \mathcal{D}'^1_0(M) \\ \underbrace{\nabla_X Y}_{\in \mathcal{D}'^1_0}(\theta) \quad &:= \quad X\Big(\underbrace{Y(\theta)}_{\in \mathcal{D}'}\Big) - \underbrace{\nabla_X \theta}_{\in \mathcal{A}}(\underbrace{Y}_{\in \mathcal{A}}) \qquad \in \mathcal{D}'(M) \end{aligned}$$

- obvious choice $\mathcal{A} = (L^2_{\mathrm{loc}})^1_0$

Curvature from a Distributional Connection

Definition (LF&M, 2007, Geroch&Traschen 1987)

(i) A distr. connection ∇ is called L^2_{loc}-connection if $\nabla_X Y \in (L^2_{\mathrm{loc}})^1_0$.

(ii) The Riemannian curvature tensor $R \in \mathcal{D}'^1_3(M)$ for an L^2_{loc}-connection is defined by the usual formula, i.e.,
$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

(iii) Ricci and scalar curvature defined as usual.

Observation (L^2_{loc}-stability)

Assume for a sequence of L^2_{loc}-connections
$$\nabla^{(n)}_X Y \to \nabla_X Y \qquad in\ L^2_{loc}.$$
Then
$$R^{(n)}_{XY}Z \to R_{XY}Z \qquad in\ \mathcal{D}'^1_0(M).$$
The analogue holds true for the Ricci and scalar curvature.

Curvature from a Distributional Metric

- by the above: want induced connection to be in L^2_{loc}
- locally in the $\mathcal{C}^\infty$- setting for a frame E_α

$$\nabla_X Y = g^{\alpha\beta} g(\nabla_X Y, E_\beta) E_\alpha \qquad g_x^{\alpha\beta} := \big(g(E_\alpha, E_\beta)_x\big)^{-1}$$

Definition (esentially (G&T,1987))

A distributional metric g is called gt-regular if $g \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ and uniformly nondegenerate, i.e., $|\det\big(g(E_\alpha, E_\beta)\big)| \geq C$

Theorem (G&T,1987)

Let g be gt-regular. Then

(i) *The Levi-Civita connection of g is an L^2_{loc}-connection.*

(ii) *The Riemann, Ricci and scalar curvature of g is defined.*

(iii) *H^1_{loc}-stability* (iv) *No-go:* $\operatorname{codim}\operatorname{supp}(Riem) \leq 1$

Colombeau Algebras

Algebras of generalized functions in the sense of J.F. Colombeau [Colombeau 1984, 1985] are differential algebras

- that contain the vector space of distributions and
- display maximal consistency (in the light of L. Schwartz' impossibility result) w.r.t. classical analysis.
  In particular they, preserve
  - the product of $\mathcal{C}^\infty$ functions
  - Lie derivatives of distributions.

Main ideas of the construction are

- regularization of distributions by nets of $\mathcal{C}^\infty$-functions
- asymptotic estimates in terms of a regularization parameter (quotient construction)

The (Special) Algebra on Manifolds

- scalars: $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$

$$\begin{aligned}\mathcal{E}_M(M) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^{\infty(0,1]} :\ \forall K\ \forall P\ \exists l :\ \sup_{x\in K}|Pu_\varepsilon(x)| = O(\varepsilon^{-l})\}\\ \mathcal{N}(M) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M) :\ \forall K\quad \forall m :\ \sup_{x\in K}|\ u_\varepsilon(x)| = O(\varepsilon^{m})\}\end{aligned}$$

  fine sheaf of differential algebras w.r.t. $L_X u := [(L_X u_\varepsilon)_\varepsilon]$

- sections of $E \to M$: ($\Gamma(M,E)$ ... $\mathcal{C}^\infty$-sections)

$$\begin{aligned}\Gamma_{\mathcal{G}}(M,E) &:= \Gamma_{\mathcal{M}}(M,E)/\Gamma_{\mathcal{N}}(M,E)\\ &\cong L_{\mathcal{C}^\infty(M)}\big(\Gamma(M,E^*),\mathcal{G}(M)\big) \cong \mathcal{G}(M)\otimes_{\mathcal{C}^\infty}\Gamma(M,E)\\ &\cong L_{\mathcal{G}(M)}\big(\Gamma_{\mathcal{G}}(M,E),\mathcal{G}(M)\big)\end{aligned}$$

  in particular, generalized tensor fields $\mathcal{G}^r_s(M)$
  fine sheaf of finitely generated and projective $\mathcal{G}(M)$-modules

- Embeddings: $\exists$ injective sheaf morphisms (basically convolution)

$$\iota : \Gamma(\_\,,E) \hookrightarrow \mathcal{D}'(\_\,,E) \hookrightarrow \Gamma_{\mathcal{G}}(\_\,,E).$$

Generalized Metrics

Definition (Kunzinger&S., 2002)

We define a symmetric $g \in \mathcal{G}^0_2(M)$ to be a generalized metric by one of the following equivalent conditions

(i) $\det(g)$ invertible in $\mathcal{G}(M)$
(generalized nondegeneracy)

(ii) for all generalized points $g(\tilde{x})$ is nondegenerate as map $\tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}$ (see H. Vernaeve's talk)
(pointwise generalized nondegeneracy)

(iii) locally there exists a representative g_ε consisting of smooth metrics and $\det(g)$ invertible in $\mathcal{G}(M)$
(idea of smoothing)

- technicalities on the index skipped
- g induces an isomorphism $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := g(X,\,.\,) \in \mathcal{G}^0_1(M)$

Levi-Civita Connection

Definition

A gen. connection is a map $\nabla : \mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M) \to \mathcal{G}^1_0(M)$ satisfying (∇_1), (∇_2) (appropriately read).

- extends to entire generalized(!) tensor-algebra
- standard formulas hold, e.g.

$$\nabla_{\partial_i}(Y^j\partial_j) = \big(\partial_i Y^k + \Gamma^k_{ij} Y^j\big)\partial_k \quad (Y^j, \Gamma^k_{ij} \in \mathcal{G}).$$

Theorem (Fundamental Lemma)

For any generalized metric g there exists a unique generalized connection ∇ that is ($X, Y, Z \in \mathcal{G}^1_0(M)$)

(∇_3) *torsion-free i.e.,* $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$

(∇_4) *metric, i.e.,* $\nabla_X g = 0 \Leftrightarrow X\big(g(X,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

It is called Levi-Civita connection and given by Koszul's formula.

Curvature from a Generalized Connection

Definition (Generalized Curvature)

(i) Let g be a generalized metric with generalized Levi-Civita connection ∇. We define the generalized Riemann curvature tensor $R \in \mathcal{G}^1_3(M)$ of g by the usual formula, i.e.,

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

(ii) Ricci and scalar curvature defined as usual.

Observation (Basic Compatibility)

Let g be a generalized metric with representative g_ε such that

$$g_\varepsilon \to \tilde{g} \text{ locally in } \mathcal{C}^k.$$

Then any representative R_ε of the Riemann tensor R of g converges to the Riemann tensor $\tilde{R}$ of $\tilde{g}$ locally in $\mathcal{C}^{k-2}$. The analogue holds true for the Ricci and scalar curvature.

Compatibility with Geroch&Traschen

- Question: For a gt-regular metric g we have to ways to define the curvature
  1. (1) proceed in the distributional way
  2. (2) embed g into $\mathcal{G}^0_2(M)$ and proceed in the $\mathcal{G}$-setting

  Do we get the same answer?

Theorem (Vickers&S., 2007)

Let g be a gt-regular metric with distributional curvature $R \in \mathcal{D}'^1_3(M)$. Denote by $\hat{g} \in \mathcal{G}^0_2(M)$ the generalized metric obtained by convolution of g with a [...] mollifier.

Then any representative $\hat{R}_\varepsilon$ of the generalized curvature $\hat{R}$ of $\hat{g}$ converges to R in distributions, i.e.,

$$\hat{R}_\varepsilon \to R \quad \text{in}\, \mathcal{D}'^1_3(M).$$

Some References

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