On geodesics in low regularity

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Non-regular spacetime geometry
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Overview

Theoretical outcome of a long term project on

Geodesics and geodesic completeness
of impulsive gravitational waves

jointly with

- Jiří Podolský, Robert Švarc (Relativity Group @ Prague)
- Clemens Sämann (Mathematics @ Vienna)

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Low regularity changes props. of geodesics

Riemannian counterexample [Hartman&Wintner, 1951]

$$\mathbf{g}_{ij}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |x|^\lambda \end{pmatrix} \qquad \text{on } (-1,1) \times \mathbb{R} \subseteq \mathbb{R}^2$$

- $\lambda \in (1,2) \implies \mathbf{g} \in \mathcal{C}^{1,\lambda-1}$ Hölder, slightly below $\mathcal{C}^{1,1}$
- (nevertheless) geodesic equation uniquely solvable

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- (nevertheless) geodesic equation uniquely solvable

BUT

- shortest curves from (0,0) to (0,y) are two symmetric arcs

  $\rightsquigarrow$ minimising curves not unique, even locally

- the y-axis is a geodesic which is

  non-minimising between any of its points

The Riemannian case

Added regularity of shortest curves

- $\mathbf{g} \in \mathcal{C}^0 \implies$ shortest (Lipschitz) curves exist [Hilbert, 1899]
- $\mathbf{g} \in \mathcal{C}^{0,\alpha} \implies$ all shortest curves are $\mathcal{C}^{1,\beta}$ with $\beta = \frac{\alpha}{2-\alpha}$ (optimal) [Calabi, Hartman, 70]
  in particular [Lytchak, Yaman, 06]
- $\mathbf{g} \in \mathcal{C}^{0,1} \implies$ all shortest curves are $\mathcal{C}^{1,1}$ and $\ddot{\gamma} = 0$ a.e.
- $\mathbf{g} \in \mathcal{C}^1 \implies$ all shortest curves satisfy $\ddot{\gamma} = 0$ and $\gamma \in \mathcal{C}^2$

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The Lorentzian case is different

- problems with length structure

  $\rightsquigarrow$ Lorentzian length spaces...talk by C. Sämann

  $\rightsquigarrow$ use geodesic equation
- added regularity of maximising curves ?

Lorentzian metrics & low regularity

Lorentzian counterexample [Chrusciel,Grant,12]

$$ds^2 = -(du + (1 - |u|^\lambda)dx)^2 + dx^2 \qquad \in \mathcal{C}^{0,\lambda} \quad (\lambda \in (0,1])$$

$\lambda < 1$: null geodesics branch $\rightsquigarrow$ causal bubble; no push up

General remarks

- $\mathbf{g} \in \mathcal{C}^{1,1}$: almost as good as the smooth case
- $g \in \mathcal{C}^{0,1} \Rightarrow$ causally plain (no bubbles)
- $g \in \mathcal{C}^0$: Cauchy time functions [CG,12, Fathi,Siconolfi,12]
  Avez-Seifert [Sämann, 15]

Focus on locally Lipschitz metrics: $\mathbf{g} \in \mathcal{C}^{0,1} \Rightarrow \Gamma \in L^\infty_{\text{loc}}$
Geodesic equations have locally bounded but discontinuous r.h.s.

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Filippov solutions I: Basic idea

- replace ODE with discont. r.h.s. by a **differential inclusion** relation

$$\dot{x}(t) = F(t, x(t)) \quad \rightsquigarrow \quad \dot{x}(t) \in \mathcal{F}[F](t, x(t))$$

where the **Filippov set-valued map** associated with F is

$$\mathcal{F}[F](t,x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \mathrm{co}\Big(F\big(B_\delta(t,x)\big) \setminus S\Big).$$

(non-empty, closed and convex set)

- A **Filippov solution** of the ODE is an absolutely continuous curve satisfying the inclusion relation almost everywhere.
- AC-functions can be reconstructed from their derivatives: $f \in AC(\mathbb{R}) \;\Leftrightarrow \exists f'$ a.e., L-integrable, and $f(x) = f(a) + \int_a^x f'(t)dt$.

Filippov solutions II: Existence

General existence theorem for differential inclusions

$$\dot{\xi}(s) \in A(s, \xi(s)) \text{ a.e.}, \quad \xi(t_0) = x_0 \qquad (t_0, x_0) \in J \times \mathbb{R}^n$$

has an AC-solution if the set valued map $(t, x) \to A(t, x)$ satisfies

1. $t \mapsto A(t, x)$ is Lebesgue measurable on J for all fixed x,
2. $x \mapsto A(t, x)$ is upper semi-continuous for almost all t, and
3. $\sup_{x \in \mathbb{R}^n} |A(t, x)| \leq \beta(t) \in L^1_{\text{loc}}(J)$ for almost all t.

Simple Corollary

If $F \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ then the ODE

$$\dot{x}(t) = F(x(t)), \quad x(t_0) = x_0$$

possesses Filippov solutions, that is AC-curves $x : J \to \mathbb{R}^n$ with

$$\dot{x}(t) \in \mathcal{F}(F)(x(t)), \qquad x(t_0) = x_0.$$

Existence of geodesics

Every Lipschitz metric has $\mathcal{C}^1$-geodesics

Let (M, g) be a $\mathcal{C}^\infty$-manifold with a $\mathcal{C}^{0,1}$-semi Riemannian metric. Then the geodesic equation has Filippov solutions, which are $\mathcal{C}^1$.

Rademacher: $\mathbf{g} \in \mathcal{C}^{0,1} \Rightarrow \Gamma \in L^\infty_{\text{loc}}$
Rewrite geodesic equation for in first order form:

$$\dot{z} = F(z(t)) \quad \text{where } z = (x, \dot{x}), \quad F(z) = (\dot{x}^i, -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k)$$

Corollary provides Filippov solutions which are AC.
Hence the geodesics are curves with AC-speeds.

Regularity almost matches up with simple LY-result in Riemannian case.

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Uniqueness: The general picture

- $g \in \mathcal{C}^{0,1}$ is much below classical threshold for uniqueness ($g \in \mathcal{C}^{1,1}$)
- One-sided Lipschitz conditions

$$(F(x) - F(x'))^T (x - x') \leq L \|x - x'\|_2^2$$

give one-sided uniqueness of F-solutions
without the need of continuous r.h.s. (here: Γ)
But: ill-suited for piecewise continuous r.h.s.

- more powerful results for piecewise continuous r.h.s.

Filippov solutions III: Uniqueness

Consider $D \subseteq \mathbb{R}^n$ connected

- split into two parts D^+, D^- by a $\mathcal{C}^2$-hypersurface $N = \partial D^+ = \partial D^-$
- $F \in \mathcal{C}^1(D^\pm)$ up to the boundary N
- $F^\pm :=$ extensions of $F|_{D^\pm}$ to the boundary N
- $F_N^\pm :=$ projections of $F^\pm$ on the unit normal $\vec{n}$ of N
  pointing from D^- to D^+

Uniqueness results

- All F-solutions are unique, unless $F_N^+ > 0$ and $F_N^- < 0$
  rules out repulsive trajectories
- If $F_N^\pm > 0$ all F-solutions are unique and pass from D^- to D^+.
  Analogously for $F_N^\pm < 0$ and passing from D^+ to D^-.
  rules out sliding motion

A Uniqueness result

Theorem (g smooth off a totally geodesic hypersurface)

Let $(M, \mathbf{g})$ be a $\mathcal{C}^\infty$-manifold with a $\mathcal{C}^{0,1}$-semi Riemannian metric. Assume that

- N is a totally geodesic $\mathcal{C}^2$-hypersurface, and
- $\mathbf{g} \in \mathcal{C}^2(M \setminus N)$.

Then all (Filippov) geodesics starting not on N are unique and those who hit N pass through.

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N is called totally geodesic, if
every (F-)geodesic starting tangentially in N stays (initially) in N.

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- Locally write $N = \{x^1 = 0\}$, $D^\pm = \{x^1 > \pm 0\}$ then $\vec{n} = e_1$ and for the geodesic $\gamma(t) = (x^1(t), \dots)$
- Rewrite geodesic equation as first order system $\rightsquigarrow$ $F_N^\pm = \dot{x}^1$
  $\rightsquigarrow$ only have to show that $\dot{x}^1 \neq 0$ if $x^1 = 0$
- But this follows for all geodesics starting off N and reaching it since N is totally geodesic.

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Impulsive gravitational waves

Exact radiative spacetimes that

- model short but strong pulses of gravitational radiation
- models of ultrarelativistic particle
- curvature concentrated on the null hypersurface $\{U = 0\}$
- continuous form $\mathbf{g} \in \mathcal{C}^{0,1}$ vs. 'distributional form' $\mathbf{g} \in \mathcal{D}'$

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Nonexpanding impulsive gravitational waves

on constant curvature background

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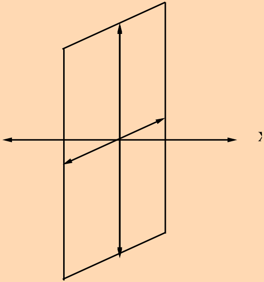
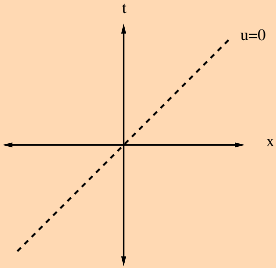
e background



ultrarelativistic particle



spacetime diagr. $(\Lambda = 0)$

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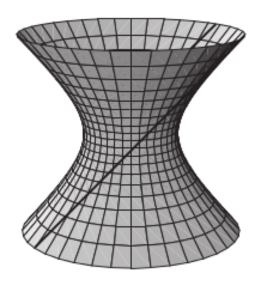
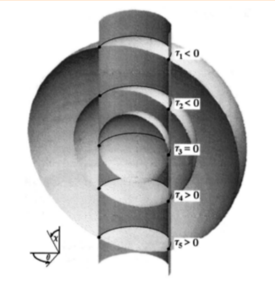


de Sitter universe



propagating wave

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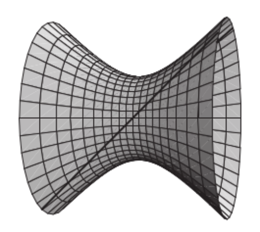
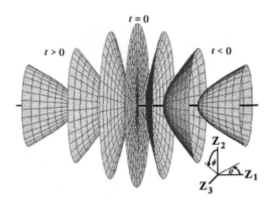


anti-de Sitter universe



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Nonexpanding impulsive gravitational waves

on constant curvature background

Goals

1. Geodesic completenss:
   analytically 'singular' vs. geometrically 'non-singular'
2. Explicitly calculate geodesics

Metric and geodesic equation

$$ds^2 = \frac{g_{ij}\,(U, X^k)\,\mathrm{d}X^i\mathrm{d}X^j - 2\,\mathrm{d}U\mathrm{d}V}{\left(1 + \frac{\Lambda}{12}(\delta_{ij}X^iX^j - 2UV - 2U_+G)\right)^2}$$

with

- $g_{ij} = \delta_{ij} + 2U_+\,H_{,ij} + U_+^2\,\delta^{kl}H_{,ik}H_{,jl}$, $G = H - X^iH_{,i}$
- H smooth fct of the spatial variables, and U_+ the kink-fct.

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$$\begin{aligned}
&\ddot{U} - 2\frac{\omega_{,U}}{\omega}\dot{U}^2 - g_{ij}\frac{\omega_{,V}}{\omega}\dot{X}^i\dot{X}^j - 2\frac{\omega_{,i}}{\omega}\dot{U}\dot{X}^i = 0\,,\\
&\ddot{V} - 2\frac{\omega_{,V}}{\omega}\dot{V}^2 - \left(g_{jk}\frac{\omega_{,U}}{\omega} - \frac{1}{2}g_{jk,U}\right)\dot{X}^j\dot{X}^k - 2\frac{\omega_{,i}}{\omega}\dot{V}\dot{X}^i = 0\,,\\
&\ddot{X}^i + \left[{}^s\Gamma^i_{kl} - \frac{1}{\omega}\left(\delta^i_k\omega_{,l} + \delta^i_l\omega_{,k} - g^{ij}g_{kl}\omega_{,j}\right)\right]\dot{X}^k\dot{X}^l - 2g^{ij}\frac{\omega_{,j}}{\omega}\dot{V}\dot{U}\\
&\qquad -2\delta^i_k\frac{\omega_{,V}}{\omega}\dot{V}\dot{X}^k + \left(g^{ij}g_{jk,U} - 2\delta^i_k\frac{\omega_{,U}}{\omega}\right)\dot{U}\dot{X}^k = 0\,.
\end{aligned}$$

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$\mathcal{C}^1$-matching of the geodesics

- Physicists like to derive the geodesics by matching geodesics of background across wave-surface.
- Only possible if geodesics — cross the wave-surface at all
  — are $\mathcal{C}^1$ across the wave-surface
  — are unique

Existence, regularity, uniqueness, completeness

- $\mathbf{g} \in \mathcal{C}^{0,1}$ $\Rightarrow$ For any initial condition Filippov solutions to the geodesic equation exist
  $\Rightarrow$ they are curves with AC velocities, in particular $\mathcal{C}^1$
- $\mathbf{g} \in \mathcal{C}^\infty$ off the wave surface $N := \{U = 0\}$
- The wave srfc. N is totally geodesic:
  - N totally geodesic in the background
  - Geodesics are $\mathcal{C}^1$-curves and background geodesics off N

  $\Rightarrow$ All geodesics with data given off N are unique and they cross N
  $\Rightarrow$ The $\mathcal{C}^1$-matching applies
- $\mathbf{g}$ is the background metric off N
  $\Rightarrow$ geodesic completeness

The explicit matching

For the geodesics in non-expanding impulsive gravitational waves on any constant curvature background we obtain

$$
\begin{aligned}
\mathcal{U}_{\rm i}^- &= 0 = \mathcal{U}_{\rm i}^+\,, & \dot{\mathcal{U}}_{\rm i}^- &= \dot{\mathcal{U}}_{\rm i}^+\,,\\
\mathcal{V}_{\rm i}^- &= \mathcal{V}_{\rm i}^+ - H_{\rm i}\,, & \dot{\mathcal{V}}_{\rm i}^- &= \dot{\mathcal{V}}_{\rm i}^+ - H_{{\rm i},X}\,\dot{x}_{\rm i}^+ - H_{{\rm i},Y}\,\dot{y}_{\rm i}^+\\
& & &\quad +\tfrac{1}{2}\big((H_{{\rm i},X})^2 + (H_{{\rm i},Y})^2\big)\,\dot{\mathcal{U}}_{\rm i}^+\,,\\
x_{\rm i}^- &= x_{\rm i}^+\,, & \dot{x}_{\rm i}^- &= \dot{x}_{\rm i}^+ - H_{{\rm i},X}\,\dot{\mathcal{U}}_{\rm i}^+\,,\\
y_{\rm i}^- &= y_{\rm i}^+\,, & \dot{y}_{\rm i}^- &= \dot{y}_{\rm i}^+ - H_{{\rm i},Y}\,\dot{\mathcal{U}}_{\rm i}^+\,.
\end{aligned}
$$

w.r.t. the conformally flat coordinates of the background

Related result: Expanding impulsive gravitational waves

$$\mathrm{d}s^2 = \frac{2\left|(V/p)\,\mathrm{d}Z + U_+\,p\,\bar{H}\,\mathrm{d}\bar{Z}\right|^2 + 2\,\mathrm{d}U\,\mathrm{d}V - 2\epsilon\,\mathrm{d}U^2}{\left[\,1 + \frac{1}{6}\Lambda U(V - \epsilon U)\,\right]^2},$$

with

- $p = 1 + \epsilon Z\bar{Z}$, $\epsilon = -1, 0, +1$,
- $H(Z) = \frac{1}{2}[h'''/h' - (3/2)(h''/h')^2]$

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$$\begin{aligned}
&\ddot{U} - \frac{2}{\omega}(\omega_{,U} + \epsilon\omega_{,V})\,\dot{U}^2 + \Big(\frac{\omega_{,V}}{\omega}\,g_{ij} - \frac{1}{2}g_{ij,V}\Big)\dot{X}^i\dot{X}^j = 0\,,\\
&\ddot{V} - 2\frac{\omega_{,V}}{\omega}\,\dot{V}^2 + 4\epsilon\frac{\omega_{,V}}{\omega}\,\dot{V}\dot{U} - \frac{2\epsilon}{\omega}(\omega_{,U} + 2\epsilon\omega_{,V})\dot{U}^2\\
&\quad + \Big(\frac{g_{ij}}{\omega}(\omega_{,U} + 2\epsilon\omega_{,V}) - (\epsilon g_{ij,V} + \frac{1}{2}g_{ij,U})\Big)\dot{X}^i\dot{X}^j = 0\,,\\
&\ddot{X}^i - \Big(2\delta^i_j\frac{\omega_{,V}}{\omega} - g^{il}g_{jl,V}\Big)\dot{V}\dot{X}^j - \Big(2\delta^i_j\frac{\omega_{,U}}{\omega} - g^{il}g_{jl,U}\Big)\dot{U}\dot{X}^j + {}^{(s)}\Gamma^i{}_{jk}\dot{X}^j\dot{X}^k = 0\,.
\end{aligned}$$

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$\mathcal{C}^1$-matching procedure

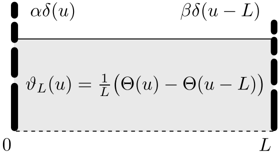
- N is a cone, hence not totally geodesic
- Uniqueness and crossing of wave surface by hand
- $\rightsquigarrow$ completeness

Less but still related result

Distributional full Brinkmann form of IGW—gyratons

$$ds^2 = h_{ij}dx^i dx^j - 2dudr + H(x)\delta_{\alpha,\beta}(u)du^2 + 2a_i(x)\vartheta_L(u)dudx^i$$

where $\delta_{\alpha,\beta}(u) = \alpha\delta(u) + \beta\delta(u-L)$
and $\vartheta_L(u) = \dfrac{1}{L}\big(\Theta(u) - \Theta(u-L)\big)$



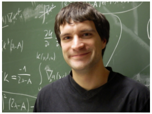
- regularisation of δ and Θ
- completeness of geodesics in the regularised smooth spacetime
  - via a fixed point argument
  - for small regularisation parameter, say $\varepsilon \le \varepsilon_0$
- BUT ε_0 depends on the initial values of the geodesic
- formulation in terms of nonlinear distributional geometry based on Colombeau algebras

List of completeness results

Prague Relativity Group

Jiří Podolský Robert Švarc

Diana Vienna



Clemens Sämann Alexander Lecke

- $\mathcal{C}^{0,1}$, $\Lambda = 0$, non-exp. [Lecke, S., Švarc, 14]
- $\mathcal{C}^{0,1}$, $\Lambda \neq 0$, non-exp. [Podolský, Sämann, S., Švarc, 15]
- $\mathcal{D}'$, $\Lambda \neq 0$, non-exp. [Sämann, S., Lecke, Podolský, 16 & 17]
- $\mathcal{C}^{0,1}$, $\Lambda \neq 0$, expanding [Podolský, Sämann, S., Švarc, 16]
- $\mathcal{D}'$, general non-flat wave-surface [Sämann, S., 12, 15]
- $\mathcal{D}'$, gyratons [Podolský, S., Švarc, 14] [Podolský, Sämann, S., Švarc, 16]

Outlook

- properties of F-geodesics
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- Ehlers-Kundt conjecture
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Thank you for your attention!

Some related Literature

[LSŠ,14] A. Lecke, R. Steinbauer, R. Švarc, *The regularity of geodesics in impulsive pp-waves.* GRG 46 (2014)

[PSS,14] J. Podolský, R. Steinbauer, R. Švarc, *Gyratonic pp-waves and their impulsive limit.* PRD 90 (2014)

[PSSŠ,15] J. Podolský, C. Sämann, R. Steinbauer, R. Švarc, *The global existence, uniqueness and $\mathcal{C}^1$-regularity of geodesics in nonexpanding impulsive gravitational waves.* CQG 32 (2015)

[PSSŠ,16] J. Podolský, C. Sämann, R. Steinbauer, R. Švarc, *The global uniqueness and $\mathcal{C}^1$-regularity of geodesics in expanding impulsive gravitational waves.* CQG 33 (2016)

[PŠSS,17] J. Podolský, R. Švarc, C. Sämann, R. Steinbauer, *Penrose junction conditions extended: impulsive waves with gyratons*, arXiv:1704.08570

[SS,12] C. Sämann, R. Steinbauer, *On the completeness of impulsive gravitational wave spacetimes.* CQG 29 (2012)

[SS,15] C. Sämann, R. Steinbauer, *Geodesic completeness of generalized space-times.* in Pseudo-differential operators and generalized functions. Pilipovic, S., Toft, J. (eds) Birkhäuser/Springer, 2015

[SS,17] C. Sämann, R. Steinbauer, *Geodesics in nonexpanding impulsive gravitational waves with Λ, part II*, arXiv:1704.05383

[SSLP,16] C. Sämann, R. Steinbauer, A. Lecke, J. Podolský, *Geodesics in nonexpanding impulsive gravitational waves with Λ, part I*, CQG 33 (2016)

[SSŠ,16] C. Sämann, R. Steinbauer, R. Švarc, *Completeness of general pp-wave spacetimes and their impulsive limit.*, CQG 33 (2016)

[S,14] R. Steinbauer, *Every Lipschitz metric has $\mathcal{C}^1$-geodesics.* CQG 31, 057001 (2014)

Thank you for your attention!