On Lorentzian metrics of low differentiability

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joint work with
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GR in low regularity situations

1. Why?
   - physically reasonable: "concentrated" but locally integrable energy
   - singularity theorems: not just failure of smoothness
2. The problem
   - concentrated sources of the gravitational field
     curvature from a metric of low regularity
   - analytical demand: low regularity vs.
     geometrical demand: nonlinearities
3. Different approaches
   (0) the classical C^∞-setting still fine down to $C^{1,1}$-metrics
   (1) linear distributional geometry (tensor distributions)
   restricted "maximal" distributional setting using Sobolev spaces
   (2) nonlinear distributional geometry (Colombeau algebras)
   unrestricted generalised setting
4. Compatibilty
   - Compatibility: in the range where (1) and (2) work, do they agree?

Distributional setting(s) for GR

- distributional metric [Marsden, 68], [Parker, 79]

$$g \in \mathcal{D}'^{0}_{2}(M) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty} \mathcal{T}^0_2(M) \cong L_{\mathcal{C}^\infty}(\mathfrak{X}(M), \mathfrak{X}(M); \mathcal{D}'(M))$$

symmetric and nondegenerate, i.e., $g(X, Y) = 0 \; \forall Y \Rightarrow X = 0$.

$\rightsquigarrow$ no way to define, inverse, curvature, . . .

- "maximal reasonable" setting: Geroch-Traschen class

$$g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$$

(gt-setting) [Geroch&Traschen, 87], [LeFloch&Mardare, 07]

Pro's: may define curvature $\mathrm{Riem}[g]$, $\mathrm{Ric}[g]$, $\mathrm{R}[g]$, $W[g]$ in distributions
consistent limits $\rightsquigarrow$ valid modelling

Con's: Bianchi identities fail $\rightsquigarrow$ energy conservation ?
$\dim(\mathrm{supp}(\mathrm{Riem}[g])) \geq 3 \rightsquigarrow$ thin shells yes, but strings no!

The (special) Colombeau algebra

- scalars: $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(M) := \dfrac{\mathcal{E}_M(M)}{\mathcal{N}(M)}$ [DeRoever&Damsma, 91]

$$\begin{aligned} \mathcal{E}_M(M) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^{\infty}{}^{(0,1]} : \ \forall K\, \forall P\, \exists\, l : \ \sup_{x\in K} |Pu_\varepsilon(x)| = O(\varepsilon^{-l})\} \\ \mathcal{N}(M) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M) : \ \forall K \quad \forall m : \ \sup_{x\in K} | \ u_\varepsilon(x)| = O(\varepsilon^{m})\} \end{aligned}$$

fine sheaf of differential algebras w.r.t. $L_X u := [(L_X u_\varepsilon)_\varepsilon]$

- tensor fields: $\mathcal{G}^r_s(M) := \mathcal{E}_M{}^r_s(M)/\mathcal{N}^r_s(M)$ [Kunzinger&S., 02]

$$\begin{aligned} \mathcal{G}^r_s(M) \cong \mathcal{G}(M) \otimes_{\mathcal{G}} \mathcal{T}^r_s(M) \quad &\cong \quad L_{\mathcal{C}^\infty(M)}\big(\Omega^1(M)^r, \mathfrak{X}(M)^s; \mathcal{G}(M)\big) \\ &\cong \quad L_{\mathcal{G}(M)}\big(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)\big) \end{aligned}$$

fine sheaf of finitely generated and projective $\mathcal{G}(M)$-modules

- Embeddings: $\exists$ injective sheaf morphisms (basically convolution)

$$\iota : \mathcal{T}^r_s(\_) \hookrightarrow \mathcal{D}'{}^r_s(\_) \hookrightarrow \mathcal{G}^r_s(\_).$$

- association: $\mathcal{G} \ni u \approx v \in \mathcal{D}' :\Leftrightarrow \int u_\varepsilon \omega \to \langle v, \omega \rangle$

Generalised setting for GR

- **generalised metric:** (technicalities on the index skipped)
  $g \in \mathcal{G}^0_2(M)$ symmetric and $\det(g)$ invertible in $\mathcal{G}$, i.e.,

  $$\forall K \text{ comp. } \exists m : \inf_{p\in K} |\det(g_\varepsilon(p))| \geq \varepsilon^m \qquad (N_\varepsilon)$$

  captures idea of smoothing: locally $\exists$ representative g_ε consisting of smooth metrics and $\det(g)$ invertible in $\mathcal{G}$
- **usual machinery works**, i.e., [Kunzinger&S., 02]
  - pointwise characterization of nondegeneracy
  - raise and lower indices: $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := g(X, \,.\,) \in \mathcal{G}^0_1(M)$
  - $\exists!$ generalised Levi-Civita connection for g
  - generalised curvature $\mathrm{Riem}[g], \mathrm{Ric}[g], \mathrm{R}[g]$ via usual formulae
  - basic $\mathcal{C}^2$-compatibility: $g_\varepsilon \to g$ in $\mathcal{C}^2$, g a vacuum solution of Einstein's equation $\Rightarrow \mathrm{Ric}[g_\varepsilon] \to 0$ in $\mathcal{D}'^1_3$.

The question of compatibility

- $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$ two ways to calculate the curvature
  (i) gt-setting: coordinate formulae in $\mathcal{D}'$ resp. $W^{m,p}_{\mathrm{loc}}$
  $\rightsquigarrow \mathrm{Riem}[g] \in \mathcal{D}'^1_3$
  (ii) $\mathcal{G}$-setting: embed g via convolution with a mollifier
  usual formulae for fixed ε $\rightsquigarrow \mathrm{Riem}[g_\varepsilon] \in \mathcal{G}^1_3$
- Do we get the same answer?

$$\begin{array}{ccc} H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \ni g & \xrightarrow{\ *\rho_\varepsilon\ } & [g_\varepsilon] \in \mathcal{G} \\ \text{gt-setting}\big\downarrow & & \big\downarrow \mathcal{G}\text{-setting} \\ \mathrm{Riem}[g] & \xleftarrow{\ \lim_{\varepsilon\to 0}\ } & \mathrm{Riem}[g_\varepsilon] \end{array}$$

Answer: Yes, but...

On the gt-class of metrics

- $H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ is an algebra
- $f \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ invertible :⇔ loc. unif. bounded away from 0, i.e.,

$$\forall K \text{ compact } \exists C : \; |f(x)| \geq \; C > 0 \text{ a.e. on } K$$

then f^{-1} is again loc. unif. bded away from 0

Definition (Nondegenerate gt-metrics [LeFM07], [SV09])

A gt-regular metric is a section $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$, which is a Semi-Riemannian metric almost everywhere.
It is called nondegenerate, if

$$\forall K \text{ compact } \exists C : \; \left|\det g(x)\right| \geq \; C > 0 \text{ a.e. on } K. \qquad (N)$$

$\Rightarrow$ $g^{-1} \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$ and nondegenerate, i.e.,
det(g^{-1}) loc. unif. bded away from 0

Smoothing gt-regular metrics

- chartwise convolution with strict δ-nets ψ_ε, which are cut-off versions of a standard mollifier with vanishing moments:

$$\rho \in \mathcal{S}(\mathbb{R}^n),\ \int \rho = 1,\ \int x^\alpha \rho(x)dx = 0\ \forall |\alpha| \geq 1$$

$$\psi_\varepsilon(x) := \chi\left(\frac{x}{\sqrt{\varepsilon}}\right)\rho_\varepsilon(x) := \chi\left(\frac{x}{\sqrt{\varepsilon}}\right)\frac{1}{\varepsilon^n}\,\rho\left(\frac{x}{\varepsilon}\right)$$

- $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$: $g^\varepsilon_{ij} := g_{ij} * \psi_\varepsilon$, $\rightsquigarrow$ metric g_ε, $\iota(g) = [(g_\varepsilon)_\varepsilon]$

Lemma (Stability of the determinant)

Let g be nondegenerate, gt-regular, then

$$\det(g_\varepsilon) \ \to\ \det g \quad \textit{in} \quad H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}} \ \textit{for all}\ p < \infty.$$

- But (N) for g does not imply (N_ε) for g_ε and $m = 0$, (N^0_ε)!

g nondegenerate gt-regular metric $\not\Rightarrow$ g_ε generalised metric

Preserving nondegeneracy (1)

problem (1): preserving positivity for scalars

- want: $0 \le f \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ & loc. unif. bounded away from 0

  $\Rightarrow \ \forall K$ compact $\exists C, \varepsilon_0 : \ f_\varepsilon(x) \ge C > 0 \quad \forall x \in K, \ \varepsilon \le \varepsilon_0 \qquad (N'_\varepsilon)$

  Then $1/f_\varepsilon$ smooth, locally uniformly bounded net, and
  $1/f_\varepsilon \to 1/f$ in $\quad H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ for all $p < \infty$.
- true if $\psi_\varepsilon \ge 0$, but ρ with vanishing moments $\Rightarrow \rho \not\ge 0 \Rightarrow \psi_\varepsilon \not\ge 0$

Lemma (Existence of admissible mollifiers)

There exist moderate strict delta nets ρ_ε with

(i) $\mathrm{supp}(\rho_\varepsilon) \subseteq B_\varepsilon(0)$ *(ii)* $\int \rho_\varepsilon(x)\,dx = 1$

(iii) $\forall j \in \mathbb{N}\ \exists \varepsilon_0 : \ \int x^\alpha \rho_\varepsilon(x)\,dx = 0 \quad$ *for all* $1 \le |\alpha| \le j$ *and all* $\varepsilon \le \varepsilon_0$

(iv) $\forall \eta > 0\ \exists \varepsilon_0 : \ \int |\rho_\varepsilon(x)|\,dx \le 1 + \eta \quad$ *for all* $\varepsilon \le \varepsilon_0$.

Convolution with ρ_ε provides an embedding ι_ρ into $\mathcal{G}$ with (N'_ε).

Preserving nondegeneracy (2)

problem (2): preserving nondegeneracy for metrics

- want: $\forall K$ cp. $\exists C, \varepsilon_0 : |\det(g_\varepsilon)| \geq C > 0\ \forall x \in K,\ \varepsilon \leq \varepsilon_0 \quad (N^0_\varepsilon)$

Definition (Stability condition)

Let g be a gt-regular metric and $\lambda_1, \ldots, \lambda_n$ its eigenvalues.

(i) For any compact K we set $\mu_K := \min_{1 \leq i \leq n} \operatorname{essinf}_{x \in K} |\lambda^i(x)|$.

(ii) We call g stable if on any compact K there is A^K continuous, s. t.

$$\max_{i,j} \operatorname{essup}_{x \in K} |g_{ij}(x) - A^K_{ij}(x)| \leq C < \tfrac{\mu_K}{2n}.$$

Lemma (Nondegeneracy of smoothed gt-regular metrics)

Let g be a nondegenerate, stable, and gt-regular metric.
Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.
Then (N^0_ε) holds, and the embedding $\iota_\rho(g)$ is a gen. metric.

Stability results

Lemma (Stability of the inverse and Christoffel symbols)

Let g be a nondegenerate, stable, and gt-regular metric. Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.

(i) *The inverse of the smoothing $(g_\varepsilon)^{-1}$ is a smooth and locally uniformly bounded net (on rel. cp. sets for ε small), and*

$$(g_\varepsilon)^{-1} \to g^{-1} \text{ in } H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}} \text{ for all } p < \infty.$$

In particular, for any embedding we have that $(\iota_\rho(g))^{-1} \approx g^{-1}$.

(ii) *The Christoffel symbols of the smoothing $\Gamma_{ijk}[g_\varepsilon]$, $\Gamma^i_{jk}[g_\varepsilon]$ are smooth and L^2_{loc}-bounded nets (on rel. cp. sets for ε small), and*

$$\Gamma_{ijk}[g_\varepsilon] \to \Gamma_{ijk} \text{ and } \Gamma^i_{jk}[g_\varepsilon] \to \Gamma^i_{jk} \text{ in } L^2_{\mathrm{loc}}$$

In particular, for any embedding $\Gamma_{ijk}[\iota_\rho(g)] \approx \Gamma_{ijk}[g]$ and $\Gamma^i_{jk}[\iota_\rho(g)] \approx \Gamma^i_{jk}[g]$.

Compatibility results

Theorem (Compatibility of the gt- with the $\mathcal{G}$-setting)

Let g be a nondegenerate, stable, and gt-regular metric, and denote its Riemann tensor by $\mathrm{Riem}[g]$.
Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.
Then we have for the Riemann tensor $\mathrm{Riem}[g_\varepsilon]$ of g_ε

$$\mathrm{Riem}[g_\varepsilon] \to \mathrm{Riem}[g] \text{ in } \mathcal{D}'^{1}_{3}.$$

Hence for any embedding ι_ρ we have $\mathrm{Riem}[\iota_\rho(g)] \approx \mathrm{Riem}[g]$.

$$\begin{tikzcd}
\begin{array}{c} H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \\ \textit{nondeg., stable} \end{array} \ni g \arrow[rr, "*\iota_\rho\ \textit{admissible}"] \arrow[d, "\text{gt-setting}"'] & & {[g_\varepsilon] \in \mathcal{G}} \arrow[d, "\mathcal{G}\text{-setting}"] \\
\mathrm{Riem}[g] & & \mathrm{Riem}[g_\varepsilon] \arrow[ll, "\approx"']
\end{tikzcd}$$

Discussion

Relation to older stability results: $(g_n)_n$ gt-regular sequence

- [LeFloch&Mardare, 07]
  $g_n \to g$ in H^1_{loc}, $g_n^{-1} \to g^{-1}$ in $L^\infty_{\mathrm{loc}} \Rightarrow \mathrm{Riem}[g_n] \to \mathrm{Riem}[g]$, in $\mathcal{D}'^1_3$.
  for smoothings via convolution $g_n^{-1} \not\to g^{-1}$ in L^∞_{loc}.
- [Geroch&Traschen, 87]
  $g_n \to g$ in H^1_{loc}, $g_n^{-1} \to g^{-1}$ in L^2_{loc}, g_n, g_n^{-1} bded in L^∞_{loc} $(*)$
  $$\Rightarrow \mathrm{Riem}[g_n] \to \mathrm{Riem}[g] \text{ in } \mathcal{D}'^1_3.$$

Existence of approximating sequences with $(*)$

- [Geroch&Traschen, 87]
  Only for continuous g, open for general g.
- Positive answer for general g by the above Theorem.

Further prospects

- Jump conditions along singular hypersurfaces in the spirit of [LeFloch&Mardare, 07], [Lichnerowicz, 55-79] in the generalised setting plus compatibility. Applications to gravitational shock waves.

  Diploma thesis of Nastasia Grubic.

- Regularity of generalised solutions to wave equations in singular space-times. [Grant, Mayerhofer, S., 09]
- Wave equation on gt-regular space-times.
- Compatibility for generalised connections in fibre bundles. [Kunzinger, Vickers, S., 05]

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