The wave equation on singular space-times

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Contents

Weakly singular space-times I

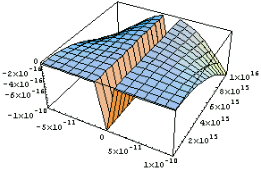
- General Relativity: space-time (M, g)

  Einstein equations $G_{ab}[g] = \kappa\, T_{ab}$

- classically: singularities defined geometrically

  i.e., via incomplete geodesics

- problem: many reasonable space-times are singular e.g.:



impulsive gravitational waves,
shell-crossing singularities,
cosmic strings, ...

- often (but not always) the metric is locally bounded but
  - below validity of standard differential geometry, i.e., $C^{1,1}$
  - below largest "reasonable" distributional class, i.e., $H^2_{loc} \cap L^\infty_{loc}$ [Geroch, Traschen, 87]

$\leadsto$ nonlinear distributional geometry (digression)

Weakly singular space-times II

- physically relevant: not geodesics but tidal forces on extended bodies near singularity   but unclear how to model!
- **idea:** study behavior of test fields instead [Clarke, 96] singularities as obstructions to the evolution of test fields rather than as obstructions to extending geodesics
- **so:** study Cauchy problem for the wave equation locally near singularity
- **problem:** $\Box_g u = \nabla^a(\nabla_a u) = |\det g|^{-\frac{1}{2}}\partial_a(|\det g|^{-\frac{1}{2}} g^{ab}\partial_b u)$ involves coefficients of low regularity
- **results:** local unique solvability
  - for shell-crossing singularities [Clarke, 98] (weak solution concept)
  - for cosmic strings [Vickers, Wilson, 00] (in $\mathcal{G}$)

$\rightsquigarrow$ generalization to "weakly singular" space-times (this talk)

Distributional differential geometry

[Schwartz, de Rham, Marsden, Parker, ...]

- distributions on manifolds: dual space of compactly supported n-forms
$$\mathcal{D}'(M) = (\Omega^n_c(M))'$$
- distributional sections in vector bundles
$$\begin{aligned} \mathcal{D}'(M,E) &= \Big(\Gamma_c(M, E^* \otimes \Omega^n_c(M)\Big)' \\ &\cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M,E) \\ &\cong L_{\mathcal{C}^\infty(M)}\Big(\Gamma(M,E^*), \mathcal{D}'(M)\Big) \end{aligned}$$
- extension of classical operations by continuity:
$$L_\xi,\ [\,,\,],\ \wedge,\ \iota_\xi,\ \{\,,\,\}\ \ldots$$

but with only one $\mathcal{D}'$-factor

Distributional semi-Riemannian geometry

- distributional **pseudo-Riemannian metric**

$$g \in \mathcal{D}'^0_2(M) = \{t : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'(M) : t\ \mathcal{C}^\infty(M) - \text{bilinear}\}$$

  symmetric and nondegenerate, i.e., $g(\xi, \eta) = 0\ \forall \eta \in \mathfrak{X}(M) \Rightarrow \xi = 0$.

- distributional **connection** $D : \mathfrak{X}(M) \times \mathcal{D}'^1_0(M) \to \mathcal{D}'^1_0(M)$
  $\mathcal{C}^\infty(M)$-linear in first, $\mathbb{R}$-linear in second slot, $D_\xi(f\eta) = fD_\xi\eta + \xi(f)\eta$
- **problems:**
  - **can't insert** $\mathcal{D}'$-vector fields into $\mathcal{D}'$-metric
  - **curvature** in general undefined: $R(\xi, \eta) = [D_\xi, D_\eta] - D_{[\xi,\eta]}$ (nonlin.!)
  - notion of **nondegeneracy** is too weak (no point-values $\rightsquigarrow$ non-local):
    $ds^2 = x^2\, dx^2$ (classically) degenerate in 0, but nondegenerate in $\mathcal{D}'$.
  - **index** of a distributional metric?
  - **geodesics** of a distributional metric?

Would need a notion of *manifold-valued* distributions.

Distributional General Relativity?

- differential geometry, Einstein equations:

  $g \in \mathcal{D}'(M, T^0_2(M))$ simply not possible!

- positive results in special cases
  - gravitational shock waves [Lichnerowicz, 1950-ies]
  - thin shells of matter, [Israel, 66]
  - ultrarelativistic black holes [Balasin, Nachbagauer, 1990-ies]
  - ...

  $$\dim \operatorname{supp}(T_{ab}) = \dim \operatorname{supp}(G_{ab}) = 3$$

- No-go results [Geroch-Traschen, 1987]
  - maximal "reasonable" class is $g_{ab} \in H^2_{loc} \cap L^\infty_{loc}$
    - curvature tensors, in particular G_{ab} defined in $\mathcal{D}'$
    - limit consistency, completeness
    - Bianchi identities ($\rightsquigarrow$ energy conservation) can't be formulated

    $$\Longrightarrow \ \dim \operatorname{supp}(G_{ab}) \geq 3$$
  - excludes many interesting examples, e.g. cosmic strings

Nonlinear distributional geometry

- Colombeau tensor fields (special version)
  usual quotient construction using:

$$(\mathcal{T}^r_s)_M : \ (u_\varepsilon) \in (\mathcal{T}^r_s)^{(0,1]} : \ \forall K \subset\subset M \ \forall P \ \exists N : \ \sup_{x \in K} \| P u_\varepsilon(x) \|_h = O(\varepsilon^{-N})$$

$$\mathcal{N}^r_s \quad : \ (u_\varepsilon) \in (\mathcal{T}^r_s)^{(0,1]} : \ \forall K \subset\subset M \qquad \forall\, m : \ \sup_{x \in K} \| \ u_\varepsilon(x) \|_h = O(\varepsilon^{m})$$

$$\boxed{\mathcal{G}^r_s(M) := (\mathcal{T}^r_s)_M / \mathcal{N}^r_s \qquad \mathcal{G}(M) := G^0_0(M)}$$

- $\mathcal{G}(\_)$ is a fine sheaf of differential algebras w.r.t. $L_X u := [(L_X u_\varepsilon)_\varepsilon]$
- $\mathcal{G}^r_s(\_)$ is a fine sheaf of $\mathcal{C}^\infty(M)$-modules and
  of $\mathcal{G}(M)$-modules; finitely gen., projective
- characterizations:

$$\begin{aligned} \mathcal{G}^r_s(M) \ &\cong \ \mathcal{G}(M) \otimes \mathcal{T}^r_s(M) \cong L_{\mathcal{C}^\infty(M)}\Big(\mathcal{T}^s_r(M), \mathcal{G}(M)\Big) \\ &\cong L_{\mathcal{G}(M)}\Big(\mathcal{G}^s_r(M), \mathcal{G}(M)\Big) \end{aligned}$$

Generalized pseudo-Riemannian geometry I

- For $g \in \mathcal{G}^0_2(M)$ TFAE
  - (i) $g : \mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M) \to \mathcal{G}(M)$ symm. & $\det g$ invertible in $\mathcal{G}(M)$
  - (ii) $\forall \tilde{x}$: $g(\tilde{x})$: $\widetilde{\mathbb{K}}^n \times \widetilde{\mathbb{K}}^n \to \widetilde{\mathbb{K}}$ symmetric and nondegenerate
  - (iii) On every rel. cp. chart there exists a representative that is a classical pseudo-Riemannian metric and $\det g$ is invertible in $\mathcal{G}(M)$

  If there exists a representative with constant index, g is called **generalized pseudo-Riemannian metric**.
- **results:**
  - index of is well-defined
    (proof by finite-dimensional perturbation theory)
  - inverse metric g^{-1} is a well-defined element of $\mathcal{G}^2_0(M)$
  - $\xi \mapsto g(\xi, .)$ induces a $\mathcal{G}(M)$-linear isomorphism: $\mathcal{G}^1_0(M) \to \mathcal{G}^0_1(M)$
    $\rightsquigarrow$ raising/lowering of indices

Generalized pseudo-Riemannian geometry II

- **Generalized connection:**

$$\hat{D} : \mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M) \to \mathcal{G}^1_0(M)$$

  $\mathcal{G}(M)$-linear in first, $\tilde{\mathbb{R}}$-linear in second slot, $\hat{D}_\xi(f\eta) = f\hat{D}_\xi\eta + \xi(f)\eta$

- **Fundamental Lemma:** For every generalized metric there exists a unique metric and torsion-free connection, the *generalized Levi-Civita connection*.

- **curvature** can be defined

$$\hat{R}_{\xi,\eta}\zeta \;:=\; \hat{D}_{[\xi,\eta]}\zeta - [\hat{D}_\xi, \hat{D}_\eta]\zeta$$

  Einstein equations can be formulated

- compatibility with
  - classical theory: $\hat{g} \approx_k g \Rightarrow$ gen. curvature $\approx_{k-2}$ class. curvature
  - Geroch-Traschen approach (recent work)

The wave equation on singular space-times: Overview

- **The Result:** Local existence and uniqueness Theorem for the Cauchy problem for the scalar wave equation for a large class of "weakly singular" (essentially locally bounded) space-time metrics in the (special) Colomebau algebra.

- **Class of metrics:**
  precise definition needs some geometric preparation
  - formulated entirely in $\mathcal{G}$
  - many weakly singular metrics can be modelled within this class
    - embedding via convolution with standard mollifier
    - physically motivated modelling
  - e.g.: impulsive pp-waves (in Rosen form), expanding spherical impulsive waves, cosmic strings, ...

- **the metric:** $g_{ab} \in \mathcal{G}^0_2(M)$ a generalized Lorentz metric
  $\Rightarrow \forall V \subseteq M$ rel. cp. $\exists$ repr. $(g^\varepsilon_{ab})_\varepsilon$ which is a classical L-metric and
  $\forall K \subset\subset U \exists N: \; inf_K || \det g_\varepsilon || \geq \varepsilon^m$
  use exclusively such representatives from now on
- **the geometry:** $p \in U \subseteq M$ open & rel. cp.
  $t : U \to \mathbb{R}$ smooth, $t(p) = 0$, $\sigma := dt \neq 0$ on U
  assume $\exists M_0 > 0: \; g_\varepsilon^{-1}(dt, dt) \leq 1/M_0^2$
  $\Longrightarrow \; \Sigma_\tau := \{q \in U : \; t(q) = \tau\}$ "uniformly spacelike" hypersurfaces
- **constructions:** normal vector $\xi^a_\varepsilon := g^{ab}_\varepsilon \sigma_b$; norm $V^2_\varepsilon := -g_\varepsilon(\xi_\varepsilon, \xi_\varepsilon)$
  corresponding normalized versions: $\hat{\xi}^a_\varepsilon = \xi^a_\varepsilon / V_\varepsilon$, $\hat{\sigma}^\varepsilon_a := \sigma_a / V_\varepsilon$
  note: only σ independent of ε!
  generalized Riemann metric: $e^\varepsilon_{ab} := g^\varepsilon_{ab} + 2\hat{\sigma}^\varepsilon_a \hat{\sigma}^\varepsilon_b$

The class of metrics

We define the following class of **weakly singular metrics**

(A) $\forall K \subset\subset U \;\forall k \;\forall \eta_1, \dots, \eta_k \in \mathfrak{X}(M)$

- $\sup_K ||L\eta_1 \dots L\eta_k \; g_\varepsilon || = O(\varepsilon^{-k})$
- $\sup_K ||L\eta_1 \dots L\eta_k \; g_\varepsilon^{-1}|| = O(\varepsilon^{-k})$

in particular g_ε, g_ε^{-1} locally uniformly bounded

$\Rightarrow \; 1/M_0 \leq \sqrt{-g_\varepsilon(\xi_\varepsilon, \xi_\varepsilon)} = V_\varepsilon \leq M_0$

(B) $\forall K \subset\subset U : \; \sup_K ||\nabla_{g^\varepsilon} \xi_\varepsilon|| = O(1)$

$\Rightarrow \; ||L_\xi g_\varepsilon||_{e_\varepsilon} = O(1), \quad ||\nabla_{g^\varepsilon} e_\varepsilon|| = O(1) = ||\nabla_{g^\varepsilon} e_\varepsilon^{-1}||_{e_\varepsilon}$

(C) for each ε : $\Sigma := \Sigma_0$ past cp., spacelike hypersurface s.t. $\partial J_\varepsilon^+(\Sigma) = \Sigma$ and $\exists A \neq \emptyset$, open $A \subseteq \bigcap_\varepsilon J_\varepsilon^+(\Sigma)$

$\Rightarrow$ existence of classical solutions on common domain

all norms $\| \;\|$ derived from some classical Riemannian background metric

Formulation of the result

- For given weakly singular g and $v, w \in \mathcal{G}(\Sigma)$ we consider the i.v.p.

$$\Box_g u = 0, \qquad u|_\Sigma = v, \quad L_{\hat{\xi}}|_\Sigma u = w \qquad (\Box)$$

  and ask for solutions in $\mathcal{G}(U)$

Theorem: Let (M, g) be a generalized space-time with a weakly singular metric (i.e., (A)–(C) hold) and let $v, w \in \mathcal{G}(\Sigma)$. Then for any $p \in \Sigma$ there exists an open neighborhood V where $(\Box)$ has a unique solution in $\mathcal{G}(V)$.

- In coordinates $(U, \{t, x^1, x^2, x^3\})$

$$g_\varepsilon = -V_\varepsilon^2 dt^2 + h_{ij}^\varepsilon \left(dx^i - N_\varepsilon^i dt\right) \otimes \left(dx^j - N_\varepsilon^j dt\right)$$

$$V_\varepsilon,\ \partial V_\varepsilon,\ h_{ij}^\varepsilon = O(1), \qquad \partial^\alpha V_\varepsilon,\ \partial^\alpha h_{ij}^\varepsilon,\ \partial^\alpha N_\varepsilon^i = O(\varepsilon^{|\alpha|})$$

$$\begin{aligned} \Box_{g^\varepsilon} u_\varepsilon &= |\det g_\varepsilon|^{-\frac{1}{2}} \partial_a (|\det g_\varepsilon|^{-\frac{1}{2}} g_\varepsilon^{ab} \partial_b) u_\varepsilon = f_\varepsilon \in \mathcal{N}(U) \\ u_\varepsilon(t = 0, x^i) &= v_\varepsilon(x^i), \quad L_{\hat{\xi}_\varepsilon} u_\varepsilon(t = 0, x^i) = w_\varepsilon(x^i) \qquad (\Box_\varepsilon) \end{aligned}$$

General strategy of proof

key ingredient: higher order energy estimates

- use of ε-dependent Sobolev norms
- use of ε-dependent energy-momentum tensors
- switch back to sup-norms

step 1: existence of classical solutions u_ε for fixed ε (by (C))
$\rightsquigarrow$ candidate for $\mathcal{G}$-solution

step 2: existence of $\mathcal{G}$-solutions:
moderate data $\rightsquigarrow$ moderate initial energies
$\rightsquigarrow$ moderate energies for all times $\rightsquigarrow (u_\varepsilon)_\varepsilon \in \mathcal{E}_M(V)$

step 3: uniqueness—part 1:
independence of the choice of the representatives of v, w
negligible data $\rightsquigarrow$ [......] $\rightsquigarrow (u_\varepsilon)_\varepsilon \in \mathcal{N}(V)$

step 2: uniqueness—part 2:
independence of the choice of the representative of g

Sobolev norms and energies I

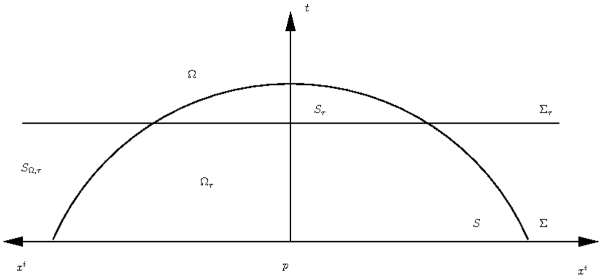
- use the R-metric $e^{\varepsilon}_{ab} := g^{\varepsilon}_{ab} + 2\hat{\sigma}^{\varepsilon}_{a}\hat{\sigma}^{\varepsilon}_{b}$ to define "pointwise" norm

$$|\nabla^{(j)}_{\varepsilon} u|^2 := ||\nabla^{\varepsilon}_{p_1} \dots \nabla^{\varepsilon}_{p_j} u||^2_{e_\varepsilon}$$
$$= e^{p_1 q_1}_{\varepsilon} \dots e^{p_j q_j}_{\varepsilon} (\nabla^{\varepsilon}_{p_1} \dots \nabla^{\varepsilon}_{p_j} u)\, (\nabla^{\varepsilon}_{q_1} \dots \nabla^{\varepsilon}_{q_j} u)$$

- 4D-Sobolev norms on Ω_τ w.r.t. ∇_{g^ε} resp. ∂

$$^{\nabla}\|u\|^{k}_{\Omega_\tau,\varepsilon} := \left(\sum_{j=0}^{k} \int_{\Omega_\tau} |\nabla^{(j)}_{\varepsilon}(u)|^2 \mu^\varepsilon \right)^{\frac{1}{2}}$$

$$^{\partial}\|u\|^{k}_{\Omega_\tau,\varepsilon} := \left(\sum_{\substack{p_1,\dots,p_j \\ 0\le j\le k}} \int_{\Omega_\tau} |\partial_{p_1} \dots \partial_{p_j} u|^2 \mu^\varepsilon \right)^{\frac{1}{2}}$$



Sobolev norms and Energies II

- 3D-Sobolev norms on S_τ w.r.t. ∇_{g^ε} resp. ∂

$$^{\nabla}\|u\|^k_{S_\tau,\varepsilon} := \Big(\sum_{j=0}^k \int_{S_\tau} |\nabla^{(j)}_\varepsilon(u)|^2 \mu^\varepsilon_\tau \Big)^{\frac{1}{2}}$$

$$^{\partial}\|u\|^k_{S_\tau,\varepsilon} := \Big(\sum_{\substack{p_1,\dots,p_j \\ 0\le j\le k}} \int_{S_\tau} |\partial_{p_1}\dots\partial_{p_j} u|^2 \mu^\varepsilon_\tau \Big)^{\frac{1}{2}}$$

- Energy momentum tensors ($k > 0$)

$$T^{ab,0}_\varepsilon(u) := -\frac{1}{2} g^{ab}_\varepsilon u^2$$

$$T^{ab,k}_\varepsilon(u) := \big(g^{ac}_\varepsilon g^{bd}_\varepsilon - \frac{1}{2} g^{ab}_\varepsilon g^{cd}_\varepsilon\big) e^{p_1 q_1}_\varepsilon \dots e^{p_{k-1} q_{k-1}}_\varepsilon (\nabla^\varepsilon_c \nabla^\varepsilon_{p_1} \dots \nabla^\varepsilon_{p_{k-1}} u)(\nabla^\varepsilon_d \nabla^\varepsilon_{q_1} \dots \nabla^\varepsilon_{q_{k-1}} u)$$

- Energies ($k \ge 0$)

$$E^k_{\tau,\varepsilon}(u) := \sum_{j=0}^k \int_{S_\tau} T^{ab,j}_\varepsilon(u) \widehat{\xi}^\varepsilon_a \widehat{\xi}^\varepsilon_b \widehat{\mu}^\varepsilon_\tau$$

Lemma 1: Equivalency of norms and energies

(i) $({}^{\nabla}\| \ \|_{S_{\tau,\varepsilon}}$ vs. energies)
There exist constants A, A' such that

$$A'({}^{\nabla}\|u\|^{k}_{S_{\tau,\varepsilon}})^2 \leq E^{k}_{\tau,\varepsilon}(u) \leq A({}^{\nabla}\|u\|^{k}_{S_{\tau,\varepsilon}})^2$$

(ii) $({}^{\nabla}\| \ \|_{S_{\tau,\varepsilon}}$ vs. ${}^{\partial}\| \ \|_{S_{\tau,\varepsilon}})$
For each k there exist positive constants B_k, B'_k such that

$$\begin{aligned} ({}^{\nabla}\|u\|^{k}_{S_{\tau,\varepsilon}})^2 &\leq B'_k \sum_{j=1}^{k} \frac{1}{\varepsilon^{2(k-j)}} ({}^{\partial}\|u\|^{j}_{S_{\tau,\varepsilon}})^2 \\ ({}^{\partial}\|u\|^{k}_{S_{\tau,\varepsilon}})^2 &\leq B_k \sum_{j=1}^{k} \frac{1}{\varepsilon^{2(k-j)}} ({}^{\nabla}\|u\|^{j}_{S_{\tau,\varepsilon}})^2 \end{aligned}$$

(iii) (basic energy estimate) $\forall \varepsilon, \tau$ the dominant energy condition gives:

$$E^{k}_{\tau,\varepsilon}(u) \leq E^{k}_{\tau=0,\varepsilon}(u) + \sum_{j=0}^{k} \int_{\Omega_\tau} \left(\xi_b \nabla^{\varepsilon}_{a} T^{ab,j}_{\varepsilon}(u) + T^{ab,j}_{\varepsilon}(u) \nabla^{\varepsilon}_{a} \xi_b \right) \mu_\varepsilon .$$

Proposition 1: Energy estimates

Let u_ε be a solution of $(\square_\varepsilon)$. Then $\forall k\ \exists C'_k, C''_k, C'''_k$ s.t. $\forall 0 \le \tau \le \gamma$

(i) $$E^k_{\tau,\varepsilon}(u_\varepsilon) \ \le\ E^k_{0,\varepsilon}(u_\varepsilon) + C'_k({}^\nabla\|f_\varepsilon\|^{k-1}_{\Omega_\tau,\varepsilon})^2 + C'''_k \int_{\zeta=0}^{\tau} E^k_{\zeta,\varepsilon}(u_\varepsilon)d\zeta + C''_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int_{\zeta=0}^{\tau} E^j_{\zeta,\varepsilon}(u_\varepsilon)d\zeta$$

(ii) $$E^k_{\tau,\varepsilon}(u_\varepsilon) \ \le\ \Big(E^k_{0,\varepsilon}(u_\varepsilon) + C'_k({}^\nabla\|f_\varepsilon\|^{k-1}_{\Omega_\tau,\varepsilon})^2 + C''_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int_{\zeta=0}^{\tau} E^j_{\zeta,\varepsilon}(u_\varepsilon)d\zeta\Big) e^{C'''_k\tau}$$

(iii) If the initial energy $(E^k_{0,\varepsilon}(u_\varepsilon))_\varepsilon$ is moderate [negligible] then

$$\sup_{0\le\tau\le\gamma} (E^k_{\tau,\varepsilon}(u_\varepsilon))_\varepsilon$$

is moderate [negligible].

Lemma 2: Auxiliary estimates

(i) (Bounds on initial energies from initial data)
Let u_ε be a solution of $(\Box_\varepsilon)$. If $(v_\varepsilon)_\varepsilon$, $(w_\varepsilon)_\varepsilon$ are moderate resp. negligible, then the initial energies $(E^k_{0,\varepsilon}(u_\varepsilon))_\varepsilon$, for each $k \geq 0$, are moderate resp. negligible nets of real numbers.
[estimate t-derivatives of the data using the wave equation
$\partial_t^2 u_\varepsilon =$
$-V_\varepsilon^2 \left(f_\varepsilon + \frac{2}{V_\varepsilon^2} N^i \partial_t \partial_i u_\varepsilon - \left(h_\varepsilon^{ij} - \frac{1}{V_\varepsilon^2} N_\varepsilon^i N_\varepsilon^j \right) \partial_i \partial_j u_\varepsilon + g_\varepsilon^{ab} \, \Gamma[\mathbf{g}_\varepsilon]^c{}_{ab} \frac{\partial u_\varepsilon}{\partial x^c} \right)$]

(ii) (Bounds on solutions from bounds on energies)
For $m > 3/2$ an integer, there exists a constant K and number N such that for all $u \in \mathcal{C}^\infty(\Omega_\tau)$ and for all $\zeta \in [0, \tau]$ we have

$$\sup_{x \in \Omega_\tau} |\partial_{x^{i_1}} \cdots \partial_{x^{i_l}} u(x)| \leq K \varepsilon^{-N} \sup_{0 \leq \zeta \leq \tau} E^{m+l}_{\zeta,\varepsilon}(u).$$

[uses Sobolev embedding plus Lemma 1]

Independence of the representative of the metric

(i) Let $\widehat{g}_\varepsilon$ be another representative and consider

$$\begin{aligned} \Box_{\hat{g}^\varepsilon} \hat{u}_\varepsilon &= f_\varepsilon \\ \hat{u}_\varepsilon(t=0, x^i) &= v_\varepsilon(x^i) \\ \partial_t \hat{u}_\varepsilon(t=0, x^i) &= w_\varepsilon(x^i) \end{aligned}$$

(ii) The difference $\widetilde{u}_\varepsilon := u_\varepsilon - \widehat{u}_\varepsilon$ solves

$$\Box_{\hat{g}^\varepsilon} \widetilde{u}_\varepsilon = f_\varepsilon - \Box_{\hat{g}^\varepsilon} u_\varepsilon, \text{ vanishing data}$$

(iii) $f_\varepsilon - \Box_{\hat{g}^\varepsilon} u_\varepsilon = (f_\varepsilon - \Box_{g^\varepsilon} u_\varepsilon) + (\Box_{g^\varepsilon} u_\varepsilon - \Box_{\hat{g}^\varepsilon} u_\varepsilon) = \Box_{g^\varepsilon} u_\varepsilon - \Box_{\hat{g}^\varepsilon} u_\varepsilon$ is negligible by well-definedness of $\Box_g$.

Outlook & Perspectives

- more general classes of metrics: log-type growth in ε replacing $O(1)$...Hölder-Zygmund classes
- generalization of Hadamard parametrix constructions
- generalized singularity theorems
  (extending geodesics in generalized sense)
- go non-linear: Einstein equations
- connections to cosmic censorship hypothesis

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