The Penrose and Hawking Singularity Theorems revisited

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Overview

Long-term project on

Lorentzian geometry and general relativity

with metrics of low regularity

jointly with

- ‘theoretical branch’ (Vienna & U.K.):
  Melanie Graf, James Grant, Günther Hörmann, Mike Kunzinger, Clemens Sämann, James Vickers
- ‘exact solutions branch’ (Vienna & Prague):
  Jiří Podolský, Clemens Sämann, Robert Švarc

Contents

Table of Contents

Singularities in GR

- singularities occur in exact solutions; high degree of symmetries
- singularities as obstruction to extend causal geodesics [Penrose, 65]

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- (iii) initial condition $\rightsquigarrow$ causal geodesics start focussing
- (i) energy condition $\rightsquigarrow$ focussing goes on $\rightsquigarrow$ focal point
- (ii) causality condition $\rightsquigarrow$ no focal points
- way out: one causal geodesic has to be incomplete, i.e., $\neg$ (iv)

The classical theorems

Theorem ([Penrose, 1965] Gravitational collapse)

A spacetime is future null geodesically incomplete, if

(i) $Ric(X, X) \geq 0$ *for every null vector* X

(ii) *There exists a non-compact Cauchy hypersurface* S *in* M

(iii) *There exists a trapped surface*
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Theorem ([Hawking, 1967] Big Bang)

A spacetime is future timelike geodesically incomplete, if

(i) $Ric(X, X) \geq 0$ *for every timelike vector* X

(ii) *There exists a compact space-like hypersurface* S *in* M

(iii) *The unit normals to* S *are everywhere converging,* $\theta := -tr\mathbf{K} < 0$.

Hawking's Thm: proof strategy ($\mathcal{C}^2$-case)

- **Analysis**: θ evolves along the normal geodesic congruence of S by *Raychaudhury's* equation

$$\theta' + \frac{\theta^2}{3} + \mathrm{Ric}(\dot\gamma, \dot\gamma) + \mathrm{tr}(\sigma^2) = 0$$

  - (i) $\implies \theta' + (1/3)\theta^2 \leq 0 \implies (\theta^{-1})' \geq 1/3$
  - (iii) $\implies \theta(0) < 0 \implies \theta \to \infty$ in finite time $\implies$ focal point

- **Causality theory**: $\exists$ longest curves in the Cauchy development
  $\implies$ no focal points in the Cauchy development

- **completeness** $\implies \overline{D^+(S)} \subseteq exp([0, T] \cdot \mathbf{n}_S) \ldots$ compact
  $\implies$ horizon $H^+(M)$ compact, $\rightsquigarrow$ 2 possibilities

  (1) $H^+(M) = \emptyset$. Then $I^+(S) \subseteq D^+(S) \implies$ timlike incomplete $\lightning$

  (2) $H^+(M) \neq \emptyset$ compact $\implies p \mapsto d(S, p)$ has min on $H^+(S)$
  But from every point in $H^+(M)$ there starts a past null generator γ (inextendible past directed null geodesic contained in $H^+(S)$)
  and $p \mapsto d(S, p)$ strictly decreasing along $\gamma \implies$ unbounded $\lightning$

Regularity for the singularity theorems of GR

Pattern singularity theorem **[Senovilla, 98]**

In a $\mathcal{C}^2$-spacetime the following are incompatible

(i) Energy condition
(ii) Causality condition
(iii) Initial or boundary condition
(iv) Causal geodesic completeness

Theorem allows (i)–(iv) and $g \in \mathcal{C}^{1,1} \equiv \mathcal{C}^{2-}$. But $\mathcal{C}^{1,1}$-spacetimes

- are physically reasonable models
- are not *really* singular (curvature bounded)
- still allow unique solutions of geodesic eq. $\rightsquigarrow$ formulation sensible

Moreover below $\mathcal{C}^{1,1}$ we have

- unbded curv., non-unique geos, no convexity $\rightsquigarrow$ 'really singular'

Hence $\mathcal{C}^{1,1}$ is the natural regularity class for singularity theorems!

Table of Contents

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What is is?

GR and Lorentzian geometry on spacetime manifolds $(M, \mathbf{g})$, where M is smoot **but** g **is non-smooth** (below $\mathcal{C}^2$)

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Why is it needed?

1. Physics: Realistic matter models $\rightsquigarrow \mathbf{g} \notin \mathcal{C}^2$
2. Analysis: ivp solved in Sobolev spaces $\rightsquigarrow \mathbf{g} \in H^{5/2}(M)$

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Physics and Analysis	vs.	**Lorentzian geometry**
want/need low regularity		needs high regularity

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But isn't it just a silly game for mathematicians?

NO! Low regularity really changes the geometry!

Why Low Regularity?

(1) Realistic matter—Physics

- want discontinuous matter configurations $\rightsquigarrow \mathbf{T} \notin \mathcal{C}^0 \implies \mathbf{g} \notin \mathcal{C}^2$
- finite jumps in $\mathbf{T} \rightsquigarrow \mathbf{g} \in \mathcal{C}^{1,1}$ (derivatives locally Lipschitz)
- more extreme situations (impulsive waves): $\mathbf{g}$ piecew. $\mathcal{C}^3$, globally $\mathcal{C}^0$

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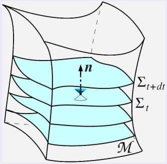
(2) Initial value problem—Analysis

Local existence and uniqueness Thms.
for Einstein eqs. in terms of Sobolev spaces



- classical [CB,HKM]: $\mathbf{g} \in H^{5/2} \implies \mathcal{C}^1(\Sigma)$
- recent big improvements [K,R,M,S]: $\mathbf{g} \in \mathcal{C}^0(\Sigma)$

Low regularity changes the geometry

Riemannian counterexample [Hartman&Wintner, 51]

$$\mathbf{g}_{ij}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |x|^\lambda \end{pmatrix} \qquad \text{on } (-1,1)\times\mathbb{R} \subseteq \mathbb{R}^2$$

- $\lambda \in (1,2) \implies \mathbf{g} \in \mathcal{C}^{1,\lambda-1}$ Hölder, slightly below $\mathcal{C}^{1,1}$
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BUT

- shortest curves from (0,0) to (0,y) are two symmetric arcs

  $\rightsquigarrow$ minimising curves not unique, even locally

- the y-axis is a geodesic which is

  non-minimising between any of its points

GR and low regularity

The challenge

Physics and Analysis	vs.	**Lorentzian geometry**
want/need low regularity		needs high regularity

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Lorentzian geometry and regularity

- classically $\mathbf{g} \in \mathcal{C}^\infty$, for all practical purposes $\mathbf{g} \in \mathcal{C}^2$
- things go wrong below $\mathcal{C}^2$
  - convexity goes wrong for $\mathbf{g} \in \mathcal{C}^{1,\alpha}$ $(\alpha < 1)$ [HW, 51]
  - causality goes wrong, light cones "bubble up" for $\mathbf{g} \in \mathcal{C}^0$ [CG, 12]

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Things that can be done

- impulisve grav. waves $g \in \mathrm{Lip}$, $\mathcal{D}'$ [J.P., R.Š., C.S., R.S., A.L.]
- causality theory for continuous metrics [CG, 12], [Sämann, 16]
- singularity theorems in $\mathcal{C}^{1,1}$ [KSSV, 15], [KSV, 15]

Table of Contents

Again: Why go to $\mathcal{C}^{1,1}$?

Recall:

Theorem (Pattern singularity theorem [Senovilla, 98])

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Theorem allows (i)–(iv) and $g \in \mathcal{C}^{1,1}$. But $\mathcal{C}^{1,1}$-spacetimes

- are physically okay/not singular
- allow to formulate the theorems

$\mathcal{C}^{1,1}$ is the natural regularity class for the singularity theorems.

The classical Theorems

Theorem **[Hawking, 1967]**

A $\mathcal{C}^2$ -spacetime is future timelike geodesically incomplete, if

(i) $\text{Ric}\,(X,X) \geq 0$ for every timelike vector X

(ii) There exists a compact space-like hypersurface S in M

(iii) The unit normals to S are everywhere converging

Theorem **[Penrose, 1965]**

A $\mathcal{C}^2$ -spacetime is future null geodesically incomplete, if

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(cp. achronal spacelike 2-srf. w. past-pt. timelike mean curvature)

The $\mathcal{C}^{1,1}$-Theorems

Theorem **[Kunzinger, S., Stojković, Vickers, 2015]**

A $\mathcal{C}^{1,1}$-spacetime is future timelike geodesically incomplete, if

(i) $\mathrm{Ric}\,(X,X) \geq 0$ for every smooth timelike local vector field X

(ii) There exists a compact space-like hypersurface S in M

(iii) The unit normals to S are everywhere converging

Theorem **[Kunzinger, S., Vickers, 2015]**

A $\mathcal{C}^{1,1}$-spacetime is future null geodesically incomplete, if

(i) $\mathrm{Ric}\,(X,X) \geq 0$ for every Lip-cont. local null vector field X

(ii) There exists a non-compact Cauchy hypersurface S in M

(iii) There exists a trapped surface $\mathcal{T}$
(cp. achronal spacelike 2-srf. w. past-pt. timelike mean curvature)

Obstacles in the $\mathcal{C}^{1,1}$-case

- No appropriate version of calculus of variations available (second variation, maximizing curves, focal points, index form, ...)
- $\mathcal{C}^2$-causality theory rests on local equivalence with Minkowski space. This requires good properties of exponential map.

$\rightsquigarrow$ big parts of causality theory have to be redone

- Ricci tensors is only L^∞

$\rightsquigarrow$ problems with energy conditions

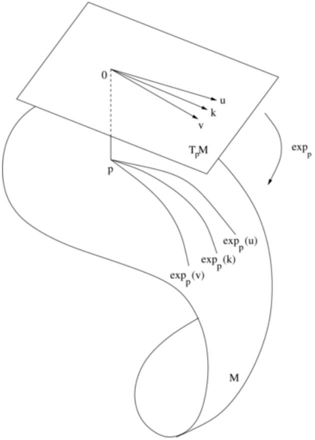
strategy:

- Proof that the exponential map is a bi-Lipschitz homeo
- Re-build causality theory for $\mathcal{C}^{1,1}$-metrics regularisation adapted to causal structure replacing calculus of var.
- use surrogate energy condition

Table of Contents

The exponential map in low regularity

- $exp_p : \; T_pM \ni v \mapsto \gamma_v(1) \in M$, where γ_v is the (unique) geodesic starting at p in direction of v
- $\mathbf{g} \in \mathcal{C}^2 \; \Rightarrow exp_p$ local diffeo
- $\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ loc. homeo [W,32]



Optimal regularity

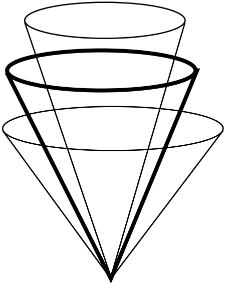
$\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ bi-Lipschitz homeo

- [KSS,14]: regularisation & comparison geometry
- [Minguzzi,15]: refined ODE methods

$\rightsquigarrow$ bulk of causality theory remains true in $\mathcal{C}^{1,1}$ [CG,12, KSSV,14, Ming.,15]

Chrusciel-Grant regularization of the metric

Regularisation adapted to the causal structure [CG,12], [KSSV, 14]



Sandwich null cones of $\mathbf{g}$ between null cones of two approximating families of smooth metrics so that

$$\check{\mathbf{g}}_\varepsilon \prec \mathbf{g} \prec \hat{\mathbf{g}}_\varepsilon.$$

- applies to continuous metrics
- local convolution plus small shift

Properties of the approximations for $\mathbf{g} \in \mathcal{C}^{1,1}$

(i) $\check{\mathbf{g}}_\varepsilon$, $\hat{\mathbf{g}}_\varepsilon \to \mathbf{g}$ locally in C^1

(ii) $D^2\check{\mathbf{g}}_\varepsilon$, $D^2\hat{\mathbf{g}}_\varepsilon$ locally uniformly bded. in ε, but $\mathrm{Ric}[\mathbf{g}_\varepsilon] \not\to \mathrm{Ric}[\mathbf{g}]$

Surrogate energy condition (Hawking case)

Lemma **[KSSV, 15]**

Let $(M, \mathbf{g})$ be a $\mathcal{C}^{1,1}$-spacetime satisfying the energy condition

$$\mathrm{Ric}\,[\mathbf{g}]\,(X, X) \geq 0 \quad \text{for all timelike local } \mathcal{C}^\infty\text{-vector fields } X.$$

Then for all $K \subset\subset M \;\; \forall C > 0 \;\; \forall \delta > 0 \;\; \forall \kappa < 0 \;\; \forall \varepsilon$ small

$$\mathrm{Ric}\,[\check{\mathbf{g}}_\varepsilon](X, X) > -\delta \quad \forall X \in TM|_K : \; \check{\mathbf{g}}_\varepsilon(X, X) \leq \kappa, \; \|X\|_h \leq C.$$

Proof.

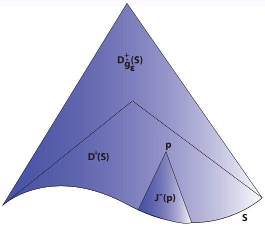
- $\check{g}_\varepsilon - g * \rho_\varepsilon \to 0$ in $\mathcal{C}^2 \rightsquigarrow$ only consider $g_\varepsilon := g * \rho_\varepsilon$
- $R_{jk} = R^i_{jki} = \partial_{x^i}\Gamma^i_{kj} - \partial_{x^k}\Gamma^i_{ij} + \Gamma^i_{im}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{ij}$
- Blue terms$|_\varepsilon$ converge uniformly
- For red terms use variant of Friedrich's Lemma:

$$\rho_\varepsilon \geq 0 \implies \big(\mathrm{Ric}[\mathbf{g}](X, X)\big) * \rho_\varepsilon \geq 0$$

$$\big(\mathrm{Ric}[\mathbf{g}](X, X)\big) * \rho_\varepsilon - \mathrm{Ric}[\mathbf{g}_\varepsilon](X, X) \to 0 \text{ unif.}$$

The $\mathcal{C}^{1,1}$-proof (Hawking case)

- $D^+(S) \subseteq D^+_{\check{\mathbf{g}}_\varepsilon}(S)$:



- Limiting argument $\Rightarrow \exists$ maximising $\mathbf{g}$-geodesic γ for all $p \in D^+(S)$ and $\gamma = \lim \gamma_{\check{\mathbf{g}}_\varepsilon}$ in $\mathcal{C}^1$
- Surrogate energy condition for $\check{\mathbf{g}}_\varepsilon$ and Raychaudhury equation
  $\Rightarrow D^+(S)$ relatively compact
  otherwise $\exists$ $\check{\mathbf{g}}_\varepsilon$-focal pt. too early
  $\Rightarrow H^+(S) \subseteq \overline{D^+(S)}$ compact
- Derive a contradiction as in the $\mathcal{C}^\infty$-case using $\mathcal{C}^{1,1}$-causality

Surrogate energy condition (Penrose case)

Lemma **[KSV, 15]**

Let $(M, \mathbf{g})$ be a $\mathcal{C}^{1,1}$-spacetime satisfying the energy condition

$$\mathrm{Ric}\,[\mathbf{g}]\,(X, X) \geq 0 \quad \text{for every local Lip. null vector field } X.$$

Then for all $K \subset\subset M$ $\forall C > 0$ $\forall \delta > 0$ $\exists \eta > 0$ s.t. we have

$$\mathrm{Ric}\,[\hat{\mathbf{g}}_\varepsilon](X, X) > -\delta$$

for all $p \in K$ and all $X \in T_pM$ with $\|X\|_h \leq C$ which are close to a $\mathbf{g}$-null vector in the sense that

$$\exists Y_0 \in TM|_K \quad \mathbf{g}\text{-null}, \quad \|Y_0\|_h \leq C, \quad d_h(X, Y_0) \leq \eta.$$

The $\mathcal{C}^{1,1}$-proof (Penrose case)

- Choose $\hat{\mathbf{g}}_\varepsilon$ globally hyperbolic (stability [NM,11], [S,15])
- Surrogate energy condition is strong enough to guarantee that

$$E^+_\varepsilon(\mathcal{T}) = J^+_\varepsilon(\mathcal{T}) \setminus I^+_\varepsilon(\mathcal{T}) \quad \text{is relatively compact}$$

  in case of null geodesic completeness
- $\hat{\mathbf{g}}_\varepsilon$ globally hyperbolic $\Rightarrow$

$$E^+_\varepsilon(\mathcal{T}) = \partial J^+_\varepsilon(\mathcal{T}) \text{ is a } \hat{\mathbf{g}}_\varepsilon\text{-achronal, compact } \mathcal{C}^0\text{-hypersrf.}$$

- $\mathbf{g} < \hat{\mathbf{g}}_\varepsilon \Rightarrow E^+_\varepsilon(\mathcal{T})$ is $\mathbf{g}$-achronal
- derive usual (topological) contradiction

Table of Contents

Lemma [Hawking and Penrose, 1967]

In a causally complete $\mathcal{C}^2$-spacetime, the following cannot all hold:

1. Every inextendible causal geodesic has a pair of conjugate points
2. M contains no closed timelike curves and
3. there is a future or past trapped achronal set S

Theorem

A $\mathcal{C}^2$-spacetime M is causally incomplete if Einstein's eqs. hold and

1. M contains no closed timelike curves
2. M satisfies an energy condition
3. *Genericity*: nontrivial curvature at some pt. of any causal geodesic
4. M contains either
    - a trapped surface
    - some p s.t. convergence of all null geodesics changes sign in the past
    - a compact spacelike hypersurface

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Some related Literature

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Thank you for your attention!