A Regularisation Approach to Causality Theory for Non-smooth Lorentzian metrics

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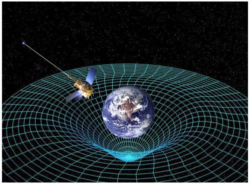
General Topic

General Relativity and semi-Riemannian geometry with metrics of low regularity

Prelude: The basic setup of General Relativity

- Albert Einstein's theory of gravity created exactly 99 years ago
- current description of gravitation in physics
- **geometric theory** due to Galileo's principle of equivalence: all bodies fall the same in a gravitational field
  $\rightsquigarrow$ gravitational field as property of the surrounding space
- Gravitational field influences how we measure lengths and angles hence the **curvature of space and time**



The mathematical setup of GR

Lorentzian geometry (basic geometric setup)

- smooth 4-dimensional space-time manifold M
- smooth space-time metric $\mathbf{g} \in \Gamma^0_2(M)$: at any T_pM symmetric, non-degenerate scalar product with signature $(-,+,+,+)$

Field equations (basic physical/analytical setup)

- Einstein Equations $\boxed{\mathbf{G}_{ij}[\mathbf{g}] := \mathbf{R}_{ij}[\mathbf{g}] - \frac{1}{2}\mathbf{R}[\mathbf{g}]\,\mathbf{g}_{ij} = 8\pi\mathbf{T}_{ij}}$
- Ricci-tensor $\mathbf{R}_{ij}$, curvature scalar $\mathbf{R}$ built from **Riemann tensor** $R^m{}_{ikp} = \partial_k\Gamma^m{}_{ip} - \partial_p\Gamma^m{}_{ik} + \Gamma^a{}_{ip}\Gamma^m{}_{ak} - \Gamma^a{}_{ik}\Gamma^m{}_{ap}$ and Christoffel symbols $\Gamma^i_{jk} = \mathbf{g}^{il}\Gamma_{ljk} = \frac{1}{2}\mathbf{g}^{il}(\partial_k\mathbf{g}_{lj} + \partial_j\mathbf{g}_{kl} - \partial_l\mathbf{g}jk)$

$$\Rightarrow \mathbf{R}_{ij}, \mathbf{R} \;\sim\; \partial^2\mathbf{g} + (\partial\mathbf{g})^2$$

- coupled system of 10 quasi-linear PDEs of 2^{nd} order for $\mathbf{g}$

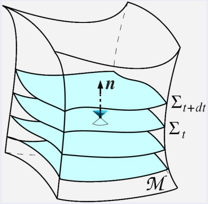
Why Low Regularity?

(1) Physics

- want discontinuous matter configurations $\rightsquigarrow \mathbf{T} \notin \mathcal{C}^0 \implies \mathbf{g} \notin \mathcal{C}^2$
- finite jumps in $\mathbf{T} \rightsquigarrow \mathbf{g} \in \mathcal{C}^{1,1}$
- standard approach: $\mathbf{g}$ piecewise $\mathcal{C}^3$, globally only $\mathcal{C}^1$
- more extreme situations (impulsive waves): $\mathbf{g}$ piecew. $\mathcal{C}^3$, globally $\mathcal{C}^0$

(2) Initial value problem—Analysis

- $3+1$-split: $M = \Sigma \times \{t\}$; C. data $(\Sigma_0, \mathbf{g}_0, \mathbf{k})$ with $\Sigma_0 = \{t = 0\}$, $\mathbf{g}(.,0) = \mathbf{g}_0$, $\partial_t \mathbf{g}(.,0) = \mathbf{k}$
- Local existence and uniqueness Thms. $(\mathbf{g}_0, \mathbf{k}) \in H^s \times H^{s-1}(\Sigma_0) \implies \mathbf{g} \in H^s(\Sigma)$
  - classical [CB,HKM]: $s > 5/2 \implies \mathbf{g} \in \mathcal{C}^1(\Sigma)$
  - recent big improvements [K,R,M,S]: $\mathbf{g} \in \mathcal{C}^0(\Sigma)$



GR and low regularity

The big quest

Physics and Analysis	vs.	**Lorentzian geometry**
want/need low regularity		needs high regularity
		to maintain standard results

Lorentzian geometry and regularity

- classically $\mathbf{g} \in \mathcal{C}^\infty$, for all practical purposes $\mathbf{g} \in \mathcal{C}^2$
  - exponential map works
  - existence of totally normal ngbhds. $\Rightarrow$ geodesically convex
  - causality theory works [C]
  - needed for singularity thms. [S]
- things go wrong below $\mathcal{C}^2$
  - causality goes wrong, light cones "bubble up" for $\mathbf{g} \in \mathcal{C}^0$ [CG12]
  - convexity goes wrong for $\mathbf{g} \in \mathcal{C}^{1,\alpha}$ $(\alpha < 1)$ [HW], see M.K.'s talk
- treshold $\mathbf{g} \in \mathcal{C}^{1,1}$: Unique solvability of geodesics eq. suffices ???

Causality theory

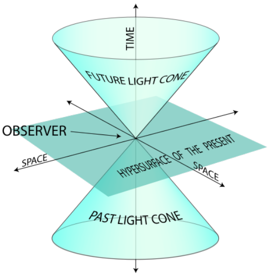
What is CT?

- essentially the theory of future & past
- tells how signals propagate, in particular how fields propagate $\rightsquigarrow$ PDE, see talks of G.H. and C.S.

Simplest ex: Minkowski space

$(M, \mathbf{g}) = (\mathbb{R}^4, \eta)$,
where $\eta = \mathrm{diag}(-1, 1, 1, 1)$

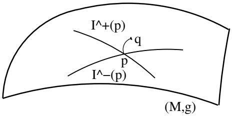
- $\eta(X, X) < 0$: timelike
- $\eta(X, X) = 0$: null (lightlike)
- $\eta(X, X) \leq 0$: causality
- $\eta(X, X) > 0$: spacelike



Local causality in a general space-time

Definitions

Timelike (causal) curve: $\gamma \in \mathcal{C}^{0,1}$
with $\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) < 0 \ (\leq 0)$ a.e.
Timelike/causal future $I^+(p)/J^+(p)$:
points reachable by future directed timelike (causal) curve



Expectation and classically true:

Locally the causality in any space-time is Minkowskian, in part.

- the local causal structure is given by the image of the lightcone under the exponential map
- the push up principles hold, in particular: any curve from p to $\partial I^+(p)$ is a null geodesic

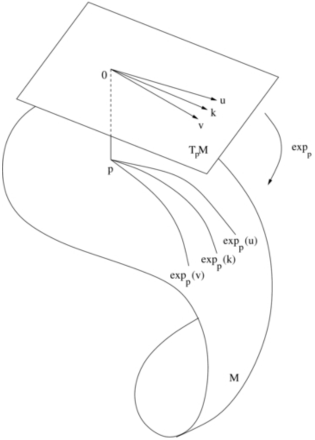
The exponential map in low regularity

The exponential map

- $exp_p : \ T_pM \ni v \mapsto \gamma_v(1) \in M$, where γ_v is the (unique) geodesic starting at p in direction of v
- maps rays through $0 \in T_pM$ to geodesics through $p \in M$

Regularity

- $\mathbf{g} \in \mathcal{C}^2 \Rightarrow exp_p$ local diffeo
- $\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ loc. homeo [W32]
- $\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ bi-Lipschitz homeo [KSS14],[M13]



Tools for causality theory with $\mathcal{C}^{1,1}$-metrics

Theorem (Maximal regularity of *exp* [KSS14])

If $\mathbf{g} \in \mathcal{C}^{1,1}$ then $\forall p \in M$ there exist open ngbhds. $\tilde{U}$ of $0 \in T_pM$ and U of p in M such that $\exp_p : \tilde{U} \to U$ is a bi-Lipschitz homeo.

Theorem (Existence of totally normal ngbhds. [KSS14])

If $\mathbf{g} \in \mathcal{C}^{1,1}$ then all $p \in M$ possess a basis of totally normal ngbhds.

Theorem (The Gauss Lemma [KSSV14])

If $\mathbf{g} \in \mathcal{C}^{1,1}$ then all $p \in M$ possess a basis of normal ngbhds. U with $\exp_p : \tilde{U} \to U$ a bi-Lipschitz homeo. and for almost all $x \in \tilde{U}$, if $v_x, w_x \in T_x(T_pM)$ and v_x is radial, then

$$\langle T_x \exp_p(v_x), T_x \exp_p(w_x) \rangle = \langle v_x, w_x \rangle.$$

Method of proof (details: M. Stojković's poster)

[KSS14], [KSSV14] use

- **regularisation technique**
  approximate $g \in \mathcal{C}^{1,1}$ by smooth $\mathbf{g}_\varepsilon$ gained via convolution
  $\Rightarrow \mathbf{g}_\varepsilon \to \mathbf{g} \in \mathcal{C}^1$ and $\mathbf{Riem}[\mathbf{g}_\varepsilon]$ locally uniformly bounded
  Beware: $\mathbf{Riem}[\mathbf{g}_\varepsilon] \not\to \mathbf{Riem}[\mathbf{g}]$

- **comparison geometry**
  new methods from Lorentzian comparison geometry [LeFC,08]

Alternative approach by **E. Minguzzi** [M13] uses

- careful ODE-analysis based on Picard-Lindelöf approximations
- inverse function theorem for Lipschitz maps

Merrits: [M13] gives somewhat stronger results but techniques do not extend below $\mathcal{C}^{1,1}$.

Causality for $\mathcal{C}^{1,1}$-metrics

Theorem (Local causality [KSSV14])

If $\mathbf{g} \in \mathcal{C}^{1,1}$ *then all* $p \in M$ *possess a basis of normal ngbhds.* $exp_p : \tilde{U} \to U$ *a bi-Lipschitz homeomorphism and*

$$I^+(p,U) = exp_p(I^+(0) \cap \tilde{U}), \qquad J^+(p,U) = exp_p(J^+(0) \cap \tilde{U})$$
$$\partial I^+(p,U) = \partial J^+(p,U) = exp_p(\partial I^+(0) \cap \tilde{U}).$$

Theorem (Push up principles [CG12])

If $\mathbf{g} \in \mathcal{C}^{0,1}$ *then we have*

- *If there is a timelike curve from* p *to* q *and a causal curve from* q *to* r *then there is a timelike curve from* p *to* r.
- *If a causal curve* α *from* p *to* q *has is timelike piece then there exists a timelike curve from* p *to* q *arbitrarily close to* α.

Main technique

Regularisations of the metric adapted to the causal structure
[CG12],[KSSV14]

If $g \in \mathcal{C}^0$ then for any $\varepsilon > 0$ there exist smooth metrics $\check{\mathbf{g}}_\varepsilon$ and $\hat{\mathbf{g}}_\varepsilon$ with

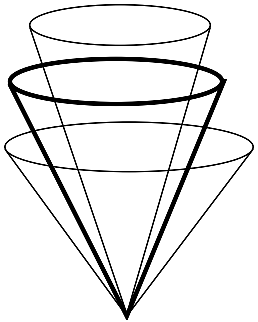
$$\check{\mathbf{g}}_\varepsilon \prec \mathbf{g} \prec \hat{\mathbf{g}}_\varepsilon,$$

$$d_h(\check{\mathbf{g}}_\varepsilon, \mathbf{g}) + d_h(\hat{\mathbf{g}}_\varepsilon, g) < \varepsilon$$

where $d_h(g_1, g_2) :=$

$$\sup_{0 \neq X, Y \in TM} \frac{|g_1(X,Y) - g_2(X,Y)|}{\|X\|_h \|Y\|_h}$$

and h is some Riem. backgrd metr.



$\mathbf{g} \prec \mathbf{h} \quad :\Leftrightarrow$
$\mathbf{g}(X,X) \leq 0 \Rightarrow \mathbf{h}(X,X) < 0$

Further results and outlook

$\mathcal{C}^{1,1}$-causality theory works!

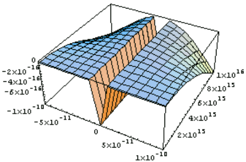
- Fundamental constructions (local causality, push up principles) of causality theory remain valid for $\mathbf{g} \in \mathcal{C}^{1,1}$.
- Accumulation curves of causal curves are causal. [CG12]
- This allows to obtain all of standard causality theory for $\mathbf{g} \in \mathcal{C}^{1,1}$ following the classical proofs. [KSSV14]

Outlook:
This (finally) puts us into a position to try to prove singularity theorems for $g \in \mathcal{C}^{1,1}$.

Geodescis in impulsive gravitational waves

Impulsive gravitational waves

- model short but strong pulses of gravitational radiation propagating in constant curvature backgrounds
- frequently described by a $\mathbf{g} \in \mathcal{C}^{0,1}$



Line element in the **non-expanding** case (coords $(U, V, Z, \bar{Z})$)

$$ds^2 = \frac{2\,|dZ + U_+(H_{,Z\bar{Z}}dZ + H_{,\bar{Z}\bar{Z}}d\bar{Z})|^2 - 2\,dUdV}{[\,1 + \frac{1}{6}\Lambda(Z\bar{Z} - UV - U_+G)\,]^2} \qquad (1)$$

where $G(Z, \bar{Z}) \equiv H - ZH_{,Z} - \bar{Z}H_{,\bar{Z}}$.

- curvature concentrated on the null hypersurface $\{U = 0\}$
- relevant models of ultrarelativistic particles

Geodesics: regularity, matching, completeness (details: A. Lecke's poster)

$\mathcal{C}^1$-matching of the geodesics in impulsive grav. waves

- Physicists like to derive the geodesics by matching the geodesics of the background across the wave-surface.
- This is only possible if the geodesics
  — cross the wave-surface at all, and
  — are $\mathcal{C}^1$ across the wave-surface

Task: Prove that these space-times are geodesically complete with $\mathcal{C}^1$-geodesics.

Problem: Geodesic eqs. are ODEs with discontinuous r.h.s.

$$\ddot{\gamma}^j(t) + \Gamma^j_{kl}(\gamma(t))\, \dot{\gamma}^k(t)\, \dot{\gamma}^l(t) = 0$$

$\mathbf{g}_{ij} \in \mathcal{C}^{0,1} \Rightarrow \Gamma^j_{kl} \in L^\infty_{\mathrm{loc}}$

Regularity of geodesics in imp. grav. waves

The case $\Lambda = 0$ [LSŠ14]

- simple structure of the metric $\rightsquigarrow$ equations can be written as non-autonomous system with U as "time"-parameter
- Geodesic equations possess unique globally defined solutions in the sense of Carathéodory and the solutions are $\mathcal{C}^1$-curves.
  $\Rightarrow$ geodesic completeness and $\mathcal{C}^1$-matching is ok!

The case $\Lambda \neq 0$

- U is **not** a parameter $\rightsquigarrow$ use Filippov's solution concept.
- Observation [S14]: $\mathbf{g} \in \mathcal{C}^{1,0} \Rightarrow$ geodesic equations possess solutions in the sense of Filippov which are $\mathcal{C}^1$-curves.
- [PSSŠ14]: In case of (1) solutions are unique and globally defined.
  $\Rightarrow$ geodesic completeness and $\mathcal{C}^1$-matching is ok!

Work in progress: expanding impulsive waves

Literature

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HVALA!