The wave equation on space-times of low regularity

Roland Steinbauer

Faculty of Mathematics
University of Vienna

Relativity Seminar
Department of Theoretical Physics
Charles University Prague
May 31, 2011

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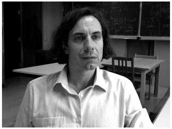
joint work with



Günther Hörmann

Relativity Seminar
Department of Theoretical Physics
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Michael Kunzinger

1. **Intro, motivation & aims**
2. **The classical existence theory**
   - Normally hyperbolic operators
   - Local existence theory
   - Global existence theory
3. **Interlude:**
   Yet another intro to nonlinear distributional geometry
4. **The case of low regularity metrics**
   - A local existence and uniqueness result
   - A global existence and uniqueness result
   - Comments and outlook

Outline

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   - Normally hyperbolic operators
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The theme

Solving the Cauchy problem for wave(-type) operators on Lorentzian manifolds with a metric of low regularity.

The ingredients

- M a smooth manifold with a weakly regular Lorentzian metric g
- $\Box$ the wave operator of g, i.e.,

$$\Box = g^{ij}\nabla_i\nabla_j = |\det g|^{-\frac{1}{2}}\,\partial_i\,(|\det g|^{\frac{1}{2}}\,g^{ij}\,\partial_j)$$

This is a (scalar) PDE on M with coefficients of low regularity.

The model (Generalised metrics [Kunzinger,S., 02])

A generalized L-metric is a symmetric section

$$g \in \mathcal{G}^0_2(M) \cong \mathcal{G}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^0_2(M)$$

(special Colombeau algebra) with

- *a representative* $(g_\varepsilon)_\varepsilon$ *consisting of smooth L-metrics, and*
- $\det(g)$ *invertible in* $\mathcal{G}(M)$

Results (Local Existence and uniqueness)

Local existence and uniqueness theorems for the Cauchy problem for the wave operator of weakly singular Lorentzian metrics in the Colombeau algebra.

- conical space times [J. Vickers & J. Wilson, 2000]
- generalisation to essentially locally bounded metrics [J. Grant, E. Mayerhofer & R.S., 2009]
- generalisation to tensors, refined regularity [C. Hanel, 2011]

Project (Global Existence and uniqueness)

Global existence and uniqueness for the Cauchy problem for

- *normally hyperbolic operators in*
- *globally hyperbolic space-times*

with metrics in the Colombeau algebra.

work in progress, jointly with G. Hörmann and M. Kunzinger

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Normally hyperbolic operators (1)

Definition

A 2nd order differential operator $P: \ \mathcal{C}^\infty(M, E) \to \mathcal{C}^\infty(M, E)$ acting on sections of a vector bundle (E, π, M) is called normally hyperbolic if its principal symbol is given by a Lorentzian metric g on M, i.e.,

$$\sigma(P)(x, \xi) = -g_x(\xi, \xi)\,\mathrm{Id}_E \qquad (x \in M,\ \xi \in T_x^* M \setminus \{0\}).$$

Locally: $P = -g^{ij}(x)\partial_i\partial_j + A^i(x)\partial_i + B(x)$

Examples

- wave operator or metric d'Alembertian $\Box$
- connection d'Alembertian: $\Box^\nabla := -\mathrm{tr}_g \otimes \mathrm{Id}_E(\nabla^{T^*M \otimes E} \circ \nabla)$
- Yamabe operator, squares of Dirac operators

Normally hyperbolic operators (2)

Facts

- *Weitzenböck formula: For every normally hyperbolic operator P there exists a unique connection ∇ on E and a unique homomorphism field $B_P \in \Gamma(\mathrm{Hom}(E, E))$ such that*

$$P = \Box^{\nabla} + B_P.$$

- *Huygens operators: subclass with sharp wave propagation*
  - *P. Günther, Hygens' principle and Hyperbolic Equations, Academic Press, Boston, 1988.*
  - *H. Baum, I. Kath, Ann. Glob. Anal. Geom,* **14***, 315-371, 1996.*
- *Local existence on small domains using Riesz distributions and Hadamard's construction.*
- *Global existence and well-posedness on globally hyperbolic space-times.*

The wave equation on Minkowski space

On n-dimensional Minkowski space set $\gamma(x) := -x_\mu x^\mu$

Definition (Riesz distributions)

For $\alpha \in \mathbb{C}$ with $\Re(\alpha) > n$ we define the continuous functions

$$R_\pm(\alpha)(x) := \begin{cases} \frac{2^{1-\alpha}\pi^{(2-n)/2}}{(\alpha/2-1)!((\alpha-n)/2)!}\gamma(x)^{(\alpha-n)/2} & x \in J_\pm(0) \\ 0 & \text{else.} \end{cases}$$

Theorem (Properties of the Riesz distributions)

(i) $\Box R_\pm(\alpha+2) = R_\pm(\alpha) \qquad (\Re(\alpha) < n+2)$ *(1)*

(ii) $\alpha \mapsto R_\pm(\alpha)$ *extends to a holomorphic family of distributions on* $\mathbb{C}$

(iii) $\mathrm{supp}(R_\pm(\alpha)) \subseteq J_\pm(0)$ $\quad$ *singsupp*$(R_\pm(\alpha)) \subseteq C_\pm(0)$

(iv) $R_\pm(0) = \delta$

So $R_\pm(2)$ is a fundamental solution of $\Box$ and we can solve the Cauchy problem.

Local existence on Lorentzian manifolds (1)

Let Ω be a normal neighbourhood of $x \in M$.

- Transport the Riesz distributions to Ω with the exponential map:

$$R^{\Omega}_{\pm}(\alpha, x) := \sqrt{|\det g_x|}\,(\exp_x)_* R_{\pm}(\alpha) \qquad \in \mathcal{D}'(\Omega)$$

We have analogous properties but (1) translates into

$$\Box R^{\Omega}_{\pm}(\alpha+2, x) = \Big(\frac{1}{2\alpha}(\Box(\gamma \circ \exp_x) - 2n) + 1\Big) R^{\Omega}_{\pm}(\alpha, x)$$

which makes it harder to find a fundamental solution.

- We make a formal ansatz

$$\mathcal{R}_{\pm}(x) := \sum_{k=0}^{\infty} V^k_x \, R^{\Omega}_{\pm}(2+2k, x) \qquad (2)$$

with the Hadamard coefficients V^k_x to be found.

Local existence on Lorentzian manifolds (2)

- Determine the Hadamard coefficients by differentiating (2) and demanding that $\mathcal{R}_\pm(x)$ is a fundamental solution of P at X. (Have to solve certain transport equations.)
- Introduce factors σ_j into (2) that enforce convergence to obtain an approximate fundamental solution $\tilde{\mathcal{R}}$:

$$P(.,D)\,\tilde{\mathcal{R}}_\pm(x) = \delta_x + K_\pm(x,.)$$

- $K_\pm \in \mathcal{C}^\infty$(!) define $(\mathcal{K}_\pm u)(x) := \int_{\bar{\Omega}} K_\pm(x,y)\,u(y)\,\sqrt{|\det g|}dy$.
- For Ω small, $id + \mathcal{K}_\pm$ is an isomorphism with bounded inverse given by the Neumann series

$$(id + \mathcal{K}_\pm)^{-1} = \sum_{j=0}^{\infty} (-\mathcal{K}_\pm)^j.$$

- Finally

$$F^\Omega_\pm(x) := (id + \mathcal{K}_\pm)^{-1} \circ \tilde{\mathcal{R}}_\pm(x)$$

is a fundamental solution of P at x; asymptotic expansion $\tilde{\mathcal{R}}_\pm(x)$.

Local existence on Lorentzian manifolds (3)

Theorem (Local existence)

Let

- (M, g) *be a time oriented Lorentzian manifold*
- P *be normally hyperbolic acting on sections in* E

Then each point $x \in M$ *possesses a relatively compact, causal neighborhood* Ω *such that for all (right hand sides)* $v \in \mathcal{D}(\Omega, E)$ *the distributions* $u_\pm$ *defined by*

$$\langle u_\pm, \varphi \rangle := \int_\Omega \langle F_\pm^\Omega(x), \varphi \rangle \, v(x) \, \sqrt{|\det g|} dx$$

is actually smooth and solves the PDE

$$P \, u_\pm \; = \; v.$$

Moreover, we have

$$\operatorname{supp}(u_\pm) \; \subseteq \; J_\pm^\Omega(\operatorname{supp}(v)).$$

Global Hyperbolicity

Pasting local solutions together needs good causality of M: geometric key notion allowing to formulate Cauchy problems

Theorem (Characterising global hyperbolicity)

For a space-time (M, g) the following are equivalent:

(i) *M is globally hyperbolic, i.e.,*
- *M is (strongly) causal* *(no (almost) closed causal curves)*
- *and the causal diamonds $J_-(p) \cap J_+(q)$ are all compact.*

(ii) *M has a Cauchy hypersurface S.*
(Every inextendible timelike curve meets S exactly once.)

(iii) *M is isometric to $\mathbb{R} \times S$ with metric* *[Bernal, Sánchez, 05]*

$$-\beta(t, x)\, dt^2 + h_t(x)$$ *where*

- *β is a smooth and positive function, and*
- *h_t is a smooth one-parameter family of Riemannian metrics on S.*

Note: Each $\{t\} \times S$ is a spacelike Cauchy hypersurface in M.

Global existence theory (1)

Lemma (Uniqueness of fundamental solutions)

Let M be a connected, time oriented L-manifold such that

(i) *the causality condition holds*

(ii) *the relation $\leq$ is closed*

(iii) *the time separation function is finite and continuous.*

Then for each point x in M there is at most one fundamental solution for P at x with past (future) compact support.

Now use this global uniqueness to patch together local fundamental solutions on RCCSV-domains to obtain global well posedness.

Global existence theory (2)

Theorem (Global well-posedness [Bär, Ginoux, Pfäffle, 07])

Let
- (M, g) *be globally hyperbolic,*
- S *be a spacelike Cauchy hypersurface with future directed timelike unit nomal vector field* n*,*
- P *be normally hyperbolic acting on sections in* E*.*

Then

(i) *The Cauchy problem*

$$Pu = f, \qquad u|_S = u_0, \qquad \nabla_n u|_S = u_1.$$

has a unique solution $u \in \mathcal{C}^\infty(M, E)$ *for each* $u_0, u_1 \in \mathcal{D}(S, E)$ *and each* $f \in \mathcal{D}(M, E)$.

(ii) *In addition,* $\mathrm{supp}(u) \subseteq J\Big(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)\Big)$.

(iii) *The mapping*

$$\mathcal{D}(S, E) \times \mathcal{D}(S, E) \times \mathcal{D}(M, E) \ni (u_0, u_1, f) \;\mapsto\; u \in \mathcal{C}^\infty(M, E)$$

is linear and continuous.

Distributional Data

Theorem (Global existence and uniqueness–$\mathcal{D}'$-data)

Let
- *(M, g) be globally hyperbolic,*
- *S be a spacelike Cauchy hypersurface with future directed timelike unit normal vector field n,*
- *P be normally hyperbolic acting on sections in E.*

Then

(i) *The Cauchy problem*

$$Pu = f, \qquad u|_S = u_0, \qquad \nabla_n u|_S = u_1.$$

has a unique solution $u \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}'(S, E))$ for each $u_0, u_1 \in \mathcal{E}'(S, E)$ and each $f \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{E}'(S, E))$.

(ii) *In addition, $\operatorname{supp}(u) \subseteq J\Big(\operatorname{supp}(u_0) \cup \operatorname{supp}(u_1) \cup \operatorname{supp}(f)\Big)$.*

Key ideas: $M \cong \mathbb{R} \times S \rightsquigarrow Pu = f \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}'(S, E))$ possible
$\rightsquigarrow$ wave front set of f hence u avoids normal direction to S
$\rightsquigarrow u \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}'(S, E))$

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Colombeau Algebras

Algebras of generalised functions in the sense of J.F. Colombeau [Colombeau 1984, 1985] are differential algebras

- that contain the vector space of distributions and
- display maximal consistency with classical analysis (in the light of L. Schwartz' impossibility result).
  In particular the construction preserves
  - the product of $\mathcal{C}^\infty$-functions
  - (Lie) derivatives of distributions.

Main ideas of the construction are

- regularisation of distributions by nets of $\mathcal{C}^\infty$-functions
- asymptotic estimates in terms of a regularisation parameter (quotient construction)

The (special) algebra on manifolds

- scalars: $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$

$$\begin{aligned} \mathcal{E}_M(M) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^{\infty}{}^{(0,1]} : \forall K\, \forall P\, \exists l : \sup_{x\in K} |Pu_\varepsilon(x)| = O(\varepsilon^{-l})\} \\ \mathcal{N}(M) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M) : \forall K \quad \forall m : \sup_{x\in K} | \ u_\varepsilon(x)| = O(\varepsilon^{m})\} \end{aligned}$$

  notation: $\mathcal{G} \ni u = [(u_\varepsilon)_\varepsilon]$
  fine sheaf of differential algebras w.r.t. $L_X u := [(L_X u_\varepsilon)_\varepsilon]$

- tensor fields: $\mathcal{G}^r_s(M) := \mathcal{E}_M{}^r_s(M)/\mathcal{N}^r_s(M)$

$$\begin{aligned} \mathcal{G}^r_s(M) \cong \mathcal{G}(M) \otimes_{\mathcal{G}} \mathcal{T}^r_s(M) &\cong L_{\mathcal{C}^\infty(M)}(\Omega^1(M)^r, \mathfrak{X}(M)^s; \mathcal{G}(M)) \\ &\cong L_{\mathcal{G}(M)}(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)) \end{aligned}$$

  fine sheaf of finitely generated and projective $\mathcal{G}(M)$-modules

- Embeddings: $\exists$ injective sheaf morphisms (basically convolution)

$$\iota : \mathcal{T}^r_s(\_) \hookrightarrow \mathcal{D}'^r_s(\_) \hookrightarrow \mathcal{G}^r_s(\_).$$

Generalised setting for GR

- generalised metric: (technicalities on the index skipped)
  $g \in \mathcal{G}^0_2(M)$ symmetric and $\det(g)$ invertible in $\mathcal{G}$, i.e.,

  $$\forall K \text{ comp. } \exists m : \inf_{p \in K} |\det(g_\varepsilon(p))| \geq \varepsilon^m \qquad (N_\varepsilon)$$

  - for all generalised points $g(\tilde{x})$ is non-degenerate as map $\tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}$ (pointwise generalised non-degeneracy)
  - there exists a representative g_ε consisting of smooth metrics and $\det(g)$ invertible in $\mathcal{G}$ (idea of smoothing)
  - g induces an isomorphism $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := g(X, \,.\,) \in \mathcal{G}^0_1(M)$
  - $\exists!$ generalised Levi-Civita connection for g
- generalised curvature $\mathrm{Riem}[g], \mathrm{Ric}[g], W[g], \mathrm{R}[g]$. defined via usual coordinate formulas for fixed ε
- basic $\mathcal{C}^2$-compatibility: $g_\varepsilon \to g$ in $\mathcal{C}^2$, g a vacuum solution of Einstein's equation $\Rightarrow \mathrm{Ric}[g_\varepsilon] \to 0$ in $\mathcal{D}'^1_3$.

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How to solve a PDE in $\mathcal{G}$

To prove existence and uniqueness of solutions $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}$ of a PDE

$$P\,u \;=\; \sum_{|\alpha|\leq m} a^\alpha \partial^\alpha u \;=\; f \qquad (a^\alpha = [(a^\alpha_\varepsilon)_\varepsilon],\; f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G})$$

proceed as follows:

(1) Solve $P_\varepsilon u_\varepsilon = f_\varepsilon$ in $\mathcal{C}^\infty$ for fixed ε on some common domain
obtaining a solution candidate $(u_\varepsilon)_\varepsilon$

(2) Show that $(u_\varepsilon)_\varepsilon$ is moderate
obtaining existence of solutions $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}$

(3) Show that disturbing $(f_\varepsilon)_\varepsilon$ and $(P_\varepsilon)_\varepsilon$ by elements of the ideal only changes $(u_\varepsilon)_\varepsilon$ by an element of the ideal
obtaining uniqueness of $u \in \mathcal{G}$

Local result: 3 conditions on the metric

Pick $p \in U \subseteq M$ relatively compact

(all norms derived from some smooth R-metric)

(A) $\forall K \subset\subset U \quad \forall k \quad \forall \eta_1, \ldots, \eta_k \in \mathfrak{X}(M)$

- $\sup_K ||L\eta_1 \ \ldots \ L\eta_k \ g_\varepsilon|| = O(\varepsilon^{-k})$
- $\sup_K ||L\eta_1 \ldots L\eta_k \ g_\varepsilon^{-1}|| = O(\varepsilon^{-k})$

in particular g_ε, g_ε^{-1} locally uniformly bounded

$\Rightarrow$existence of a hypersurface $S \ni p$, uniformly spacelike with unit normal vector $n = [(n_\varepsilon)_\varepsilon]$

Hence we have an initial surface for the Cauchy problem.

(B) $\forall K \subset\subset U: \ \sup_K ||\nabla_{g^\varepsilon} n_\varepsilon|| = O(1)$ $\qquad \Rightarrow \ ||L_n g_\varepsilon||_{e_\varepsilon} = O(1)$

(C) For each ε, S is a past compact, spacelike hypersurface and $\partial J_\varepsilon^+(S) = S$.
Moreover, $\bigcap_\varepsilon J_\varepsilon^+(S)$ contains some non-empty open set A.

$\Rightarrow$ existence of classical solutions on common domain

Hence we have a solution candidate.

The local result

Theorem ([Grant, Mayerhofer, S, 09])

Let g be a generalised metric such that (A)–(C) holds.
Then there exists some open neighbourhood $V \subseteq U$ of p where the Cauchy problem

$$\Box_g u = 0, \qquad u|_S = u_0, \quad L_n u|_S = u_1$$

has a unique solution $u \in \mathcal{G}(V)$ for all $u_0, u_1 \in \mathcal{G}(S)$.

Key steps of the proof:

- (C) provides us with a solution candidate
- (A) & (B) allow to carry out higher order energy estimates which give existence and uniqueness in $\mathcal{G}$.

Proof strategy: some details

key ingredient: higher order energy estimates

- use of ε-dependent Sobolev norms
- use of ε-dependent energy-momentum tensors
- switch back to sup-norms

step 1: existence of classical solutions u_ε for fixed ε (by (C))
$\rightsquigarrow$ candidate for $\mathcal{G}$-solution

step 2: existence of $\mathcal{G}$-solutions:
moderate data $\rightsquigarrow$ moderate initial energies
$\rightsquigarrow$ mod. energies for all times $\rightsquigarrow$ $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(V)$

step 3: uniqueness of $\mathcal{G}$-solutions:
independence of the choice of the representatives of v, w
negligible data $\rightsquigarrow$ [. . . . . .] $\rightsquigarrow$ $(u_\varepsilon)_\varepsilon \in \mathcal{N}(V)$

The energy momentum tensors

- From g_ε define a Riemannian metric:

$$e^\varepsilon_{ab} := g^\varepsilon_{ab} + 2n^\varepsilon_a n^\varepsilon_b$$

- Define the Energy momentum tensors:

$$\begin{aligned} T^{ab,0}_\varepsilon(u) &:= -\frac{1}{2} g^{ab}_\varepsilon u^2 \\ T^{ab,k}_\varepsilon(u) &:= \left(g^{ac}_\varepsilon g^{bd}_\varepsilon - \frac{1}{2} g^{ab}_\varepsilon g^{cd}_\varepsilon\right) e^{p_1 q_1}_\varepsilon \dots e^{p_{k-1} q_{k-1}}_\varepsilon \\ &\qquad (\nabla^\varepsilon_c \nabla^\varepsilon_{p_1} \dots \nabla^\varepsilon_{p_{k-1}} u)(\nabla^\varepsilon_d \nabla^\varepsilon_{q_1} \dots \nabla^\varepsilon_{q_{k-1}} u) \end{aligned}$$

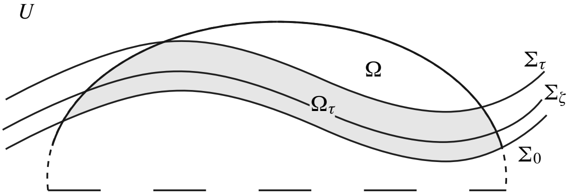
- $T^{ab,k}_\varepsilon(u)$ satisfies the dominant energy condition, i.e.

$$T^{ab,k}_\varepsilon(u)\, \eta^\varepsilon_a \eta^\varepsilon_b \geq 0 \quad \text{for any } \eta^\varepsilon_a \text{ timelike (w.r.t. } g_\varepsilon)$$

$$T^{ab,k}_\varepsilon(u)\, \eta^\varepsilon_a \quad \text{is non spacelike (w.r.t. } g_\varepsilon)$$

Higher order energies and the basic estimate

- Local foliation of space time



- Higher order Energies: $E^k_{\tau,\varepsilon}(u) := \sum_{j=0}^k \int_{\Sigma_\tau} T^{ab,j}_\varepsilon(u)\, n^\varepsilon_a\, n^\varepsilon_b\, \mu^\varepsilon_\tau$
- The dominant energy condition gives:

$$E^k_{\tau,\varepsilon}(u) \leq E^k_{\tau=0,\varepsilon}(u) + \sum_{j=0}^{k} \int_{\Omega_\tau} \left(n_b \nabla^\varepsilon_a T^{ab,j}_\varepsilon(u) + T^{ab,j}_\varepsilon(u) \nabla^\varepsilon_a n_b\right) \mu_\varepsilon$$

Energy estimates

Lemma (Energy estimates for the solution candidate)

Let u_ε be a solution candidate. Then for all orders of energies k there are constants C'_k, C''_k such that for all times τ $(0 \leq \tau \leq \gamma)$

$$\text{(i)} \quad E^k_{\tau,\varepsilon}(u_\varepsilon) \leq E^k_{0,\varepsilon}(u_\varepsilon) + C''_k \int_{\zeta=0}^{\tau} E^k_{\zeta,\varepsilon}(u_\varepsilon)d\zeta + C'_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int_{\zeta=0}^{\tau} E^j_{\zeta,\varepsilon}(u_\varepsilon)d\zeta$$

$$\text{(ii)} \quad E^k_{\tau,\varepsilon}(u_\varepsilon) \leq \left(E^k_{0,\varepsilon}(u_\varepsilon) + C'_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int_{\zeta=0}^{\tau} E^j_{\zeta,\varepsilon}(u_\varepsilon)d\zeta\right) e^{C''_k \tau}$$

Corollary (Moderateness of energies)

If the initial energy $(E^k_{0,\varepsilon}(u_\varepsilon))_\varepsilon$ is moderate [negligible] then

$$\sup_{0\leq\tau\leq\gamma} (E^k_{\tau,\varepsilon}(u_\varepsilon))_\varepsilon$$

is moderate [negligible].

The global result: definitions

Definition (Generalising normal hyperbolicity)

A 2nd order PDO P with $\mathcal{G}$-coefficients is called normally hyperbolic if its principal symbol is given by a generalised L-metric.

Definition (Generalising global hyperbolicity)

There is a (classical) isometry taking M to $\mathbb{R} \times S$ and g to

$$-\beta(t,x)\,dt^2 + h(t,x) \qquad \text{where}$$

- $\beta \in \mathcal{G}(\mathbb{R} \times S)$ with $\beta_\varepsilon \geq C > 0$ on compact sets
- h is a $\mathcal{G}$-section (of $\mathrm{pr}_2^*(T^0_2 S)$ where $\mathrm{pr}_2 : \mathbb{R} \times S \to S$) s.t:

$$\forall K \subset\subset \mathbb{R} \times S \quad \exists q : \ |\det{}_3\, h(t,x)| > \varepsilon^q.$$

Consequences:

- Each $\{t\} \times S$ is a Cauchy hypersurface in (M, g_ε) for all ε.
- The Cauchy problem for P_ε has a global solution on M for all ε.

The global result

Theorem ([HKS, 2011])

Let P be a generalised normally hyperbolic operator on a generalised globally hyperbolic space-time (M, g) and suppose that conditions (A) and (B) hold. Then the Cauchy problem

$$Pu = f, \qquad u|_S = u_0, \quad L_n u|_S = u_1$$

has a unique solution $u \in \mathcal{G}(M, E)$ for all compactly supported $u_0, u_1 \in \mathcal{G}(S, E)$ and $f \in \mathcal{G}(M, E)$.

Key steps of the proof:

- Classical theory of normally hyperbolic operators provides us with a solution candidate.
- (A) & (C) still allow us to do the energy estimates, which give existence and uniqueness.

Comments and outlook

- Variants of the result: [Hanel, 2011]
  Condition (A) is not necessary to prove moderateness: either
  - replace (A) by $g_\varepsilon, g_\varepsilon^{-1} = O(1)$
    (i.e., no conditions on derivatives but moderateness)
    and still have the existence and uniqueness result, or
  - keep (A) and use it to calculate precise power of ε-asymptotics of (derivatives) of the solution.
- Connecting to the theory of first order systems
  [Hanel, Hörmann, Spreitzer, S]
- Perspectives, questions, projects?
  - Is condition (B) really necessary?
  - compatibility with $\mathcal{D}'$-result
  - connect to more classical approaches ($\mathcal{C}^{1,1}$ or GT space-times)
  - more general metrics: log-type growth in ε replacing $O(1)$
    (Hölder-Zygmund classes)
  - ...
  - go non-linear??? (Einstein equations)

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Děkuji vám za pozornost