Wave equations on singular space-times: results and perspectives

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1. **Intro, motivation & aims**

2. **The classical theory of normally hyperbolic operators**
   - Normally hyperbolic operators
   - Causality theory: global hyperbolicity
   - Classical existence theory

3. **The case of low regularity metrics**
   - A local existence and uniqueness result
   - A global existence and uniqueness result
   - Comments and outlook

Outline

The theme

Solving the Cauchy problem for wave(-type) operators on Lorentzian manifolds with a metric of low regularity.

The ingredients

- M a smooth manifold with a weakly regular Lorentzian metric g
- $\Box_g$ the wave operator of g, i.e.,

$$\Box_g = g^{ij}\nabla_i\nabla_j = |\det g|^{-\frac{1}{2}}\,\partial_i\,(|\det g|^{\frac{1}{2}}\,g^{ij}\,\partial_j)$$

This is a (scalar) PDE on M with coefficients of low regularity.

The model (Generalised metrics [M.K.& R.S., 02])

A generalized L-metric is a symmetric section

$$g \in \mathcal{G}^0_2(M) \cong \mathcal{G}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^0_2(M)$$

(special Colombeau algebra with smooth ε-dependence) with

- *a representative $(g_\varepsilon)_\varepsilon$ consisting of smooth L-metrics, and*
- $\det(g)$ *invertible in* $\mathcal{G}(M)$

Results (Local Existence and uniqueness)

Local existence and uniqueness theorems for the Cauchy problem for the wave operator of weakly singular Lorentzian metrics in the Colombeau algebra.

- conical space times [J. Vickers & J. Wilson, 2000]
- generalisation to essentially locally bounded metrics [J. Grant, E. Mayerhofer & R.S., 2009]
- generalisation to tensors, refined regularity [C. Hanel, 2011]

Project (Global Existence and uniqueness)

Global existence and uniqueness for the Cauchy problem for

- *normally hyperbolic operators in*
- *globally hyperbolic space-times*

with metrics in the Colombeau algebra.

work in progress, jointly with G. Hörmann and M. Kunzinger

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Normally hyperbolic operators, 1

Definition

A 2nd order differential operator $P : \mathcal{C}^\infty(M, E) \to \mathcal{C}^\infty(M, E)$ acting on sections of a vector bundle (E, π, M) is called normally hyperbolic if its principal symbol is given by a Lorentzian metric g on M, i.e.,

$$\sigma(P)(x, \xi) = -g_x(\xi, \xi)\,\mathrm{Id}_E \qquad (x \in M,\ \xi \in T_x^*M \setminus \{0\}).$$

Locally: $P = -g^{ij}(x)\partial_i\partial_j + A^i(x)\partial_i + B(x)$

Examples

- wave operator or metric d'Alembertian $\Box_g$
- connection d'Alembertian: $\Box^\nabla := -\mathrm{tr}_g \otimes \mathrm{Id}_E(\nabla^{T^*M\otimes E} \circ \nabla)$
- Yamabe operator, squares of Dirac operators

Normally hyperbolic operators, 2

Facts

- *Weitzenböck formula: For every normally hyperbolic operator P there exists a unique connection ∇ on E and a unique homomorphism field $B_P \in \Gamma(\mathrm{Hom}(E, E))$ such that*

$$P = \Box^{\nabla} + B_P.$$

- *Huygens operators: subclass with sharp wave propagation*
  - *P. Günther, Hygens' principle and Hyperbolic Equations, Academic Press, Boston, 1988.*
  - *H. Baum, I. Kath, Ann. Glob. Anal. Geom,* **14**, *315-371, 1996.*
- *Local existence on small (RCCSV) domains using Riesz distributions and Hadamard's construction.*
- *Global existence and well-posedness on globally hyperbolic space-times.*

Causality: Global Hyperbolicity

Geometric key notion allowing to formulate Cauchy problems

Theorem (Characterising global hyperbolicity)

For a space-time (M, g) the following are equivalent:

(i) *M is globally hyperbolic, i.e.,*
- *M is strongly causal* *(no almost closed timelike curves)*
- *and the causal diamonds $J^-(p) \cap J^+(q)$ are all compact.*
  ($J^+(q), J^-(p)$, causal future and past)

(ii) *M has a Cauchy hypersurface S.*
(Every inextendible timelike curve meets S exactly once.)

(iii) *M is isometric to $\mathbb{R} \times S$ with metric* [A. Bernal, M. Sánchez, 05]

$$-\beta(t,x)\, dt^2 + h_t(x) \qquad \text{where}$$

- *β is a smooth and positive function, and*
- *h_t is a smooth one-parameter family of Riemannian metrics on S.*

Note: Each $\{t\} \times S$ is a spacelike Cauchy hypersurface in M.

Classical existence theory

Theorem (Global well-posedness [Bär, Ginoux, Pfäffle, 07])

Let
- *(M, g) be globally hyperbolic,*
- *S be a spacelike Cauchy hypersurface with future directed timelike unit nomal vector field n,*
- *P be normally hyperbolic acting on sections in E.*

Then

(i) *The Cauchy problem*

$$Pu = f, \qquad u|_S = u_0, \qquad \nabla_n u|_S = u_1.$$

has a unique solution $u \in \mathcal{C}^\infty(M, E)$ for each $u_0, u_1 \in \mathcal{D}(S, E)$ and each $f \in \mathcal{D}(M, E)$.

(ii) *In addition, $\operatorname{supp}(u) \subseteq J\Big(\operatorname{supp}(u_0) \cup \operatorname{supp}(u_1) \cup \operatorname{supp}(f)\Big)$.*

(causal propagation: $J(A) = J_+(A) \cup J_-(A)$, causal future and past)

(iii) *The mapping*

$$\mathcal{D}(S, E) \times \mathcal{D}(S, E) \times \mathcal{D}(M, E) \ni (u_0, u_1, f) \;\mapsto\; u \in \mathcal{C}^\infty(M, E)$$

is linear and continuous.

Distributional Data

Theorem (Global existence and uniqueness—$\mathcal{D}'$-data)

Let
- *(M, g) be globally hyperbolic,*
- *S be a spacelike Cauchy hypersurface with future directed timelike unit nomal vector field n,*
- *P be normally hyperbolic acting on sections in E.*

Then

(i) *The Cauchy problem*

$$Pu = f, \qquad u|_S = u_0, \qquad \nabla_n u|_S = u_1.$$

has a unique solution $u \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}'(S, E))$ for each $u_0, u_1 \in \mathcal{E}'(S, E)$ and each $f \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{E}'(S, E))$.

(ii) *In addition,* $\operatorname{supp}(u) \subseteq J\Big(\operatorname{supp}(u_0) \cup \operatorname{supp}(u_1) \cup \operatorname{supp}(f)\Big)$.

Key ideas: $M \cong \mathbb{R} \times S \rightsquigarrow Pu = f \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}'(S, E))$ possible
$\rightsquigarrow$ wave front set of f hence u avoids normal direction to S
$\rightsquigarrow u \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}'(S, E))$

Outline

Reminder: Solving PDEs in $\mathcal{G}$

To prove existence and uniqueness of solutions $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}$ of a PDE

$$P\,u \;=\; \sum_{|\alpha|\leq m} a^\alpha \partial^\alpha u \;=\; f \qquad (a^\alpha = [(a^\alpha_\varepsilon)_\varepsilon],\; f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}$$

proceed as follows:

(1) Solve $P_\varepsilon u_\varepsilon = f_\varepsilon$ in $\mathcal{C}^\infty$ for fixed ε on some common domain
obtaining a solution candidate $(u_\varepsilon)_\varepsilon$

(2) Show that $(u_\varepsilon)_\varepsilon$ is moderate
obtaining existence of solutions $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}$

(3) Show that disturbing $(f_\varepsilon)_\varepsilon$ and $(P_\varepsilon)_\varepsilon$ by elements of the ideal only changes $(u_\varepsilon)_\varepsilon$ by an element of the ideal
obtaining uniqueness of $u \in \mathcal{G}$

Local result: 3 conditions on the metric

Pick $p \in U \subseteq M$ relatively compact

(all norms derived from some smooth R-metric)

(A) $\forall K \subset\subset U \quad \forall k \quad \forall \eta_1, \ldots, \eta_k \in \mathfrak{X}(M)$

- $\sup_K ||L\eta_1 \ \ldots \ L\eta_k \ g_\varepsilon \ || = O(\varepsilon^{-k})$
- $\sup_K ||L\eta_1 \ldots L\eta_k \ g_\varepsilon^{-1}|| = O(\varepsilon^{-k})$

in particular g_ε, g_ε^{-1} locally uniformly bounded

$\Rightarrow$existence of a hypersurface $S \ni p$, uniformly spacelike with unit normal vector $n = [(n_\varepsilon)_\varepsilon]$

Hence we have an initial surface for the Cauchy problem.

(B) $\forall K \subset\subset U: \ \sup_K ||\nabla_{g^\varepsilon} n_\varepsilon|| = O(1)$ $\quad \Rightarrow \ ||L_n g_\varepsilon||_{e_\varepsilon} = O(1)$

(C) For each ε, S is a past compact, spacelike hypersurface and $\partial J_\varepsilon^+(S) = S$.
Moreover, $\bigcap_\varepsilon J_\varepsilon^+(S)$ contains some non-empty open set A.

$\Rightarrow$ existence of classical solutions on common domain

Hence we have a solution candidate.

A local result

Theorem (J. Grant, E. Mayerhofer & R.S., 09)

Let g be a generalised metric such that (A)–(C) holds.
Then there exists some open neighbourhood $V \subseteq U$ of p where the Cauchy problem

$$\Box_g u = 0, \qquad u|_S = u_0, \quad L_n u|_S = u_1$$

has a unique solution $u \in \mathcal{G}(V)$ for all $u_0, u_1 \in \mathcal{G}(S)$.

Key steps of the proof:

- (C) provides us with a solution candidate
- (A) & (B) allow to carry out higher order energy estimates which give existence and uniqueness in $\mathcal{G}$.

The global result: definitions

Definition (Generalising normal hyperbolicity)

A 2nd order PDO P with $\mathcal{G}$-coefficients is called normally hyperbolic if its principal symbol is given by a generalised L-metric.

Definition (Generalising global hyperbolicity)

There is a (classical) isometry taking M to $\mathbb{R} \times S$ and g to

$$-\beta(t,x)\, dt^2 + h(t,x) \qquad \text{where}$$

- $\beta \in \mathcal{G}(\mathbb{R} \times S)$ with $\beta_\varepsilon \geq C > 0$ on compact sets
- h is a $\mathcal{G}$-section (of $\mathrm{pr}_2^*(T^0_2 S)$ where $\mathrm{pr}_2 : \mathbb{R} \times S \to S$) s.t:

$$\forall K \subset\subset \mathbb{R} \times S \quad \exists q : \ |\det{}_3\, h(t,x)| > \varepsilon^q.$$

Consequences:

- Each $\{t\} \times S$ is a Cauchy hypersurface in (M, g_ε) for all ε.
- The Cauchy problem for P_ε has a global solution on M for all ε.

The global result

Theorem

Let P be a generalised normally hyperbolic operator on a generalised globally hyperbolic space-time (M, g) and suppose that conditions (A) and (B) hold. Then the Cauchy problem

$$Pu = f, \qquad u|_S = u_0, \quad L_n u|_S = u_1$$

has a unique solution $u \in \mathcal{G}(M, E)$ for all compactly supported $u_0, u_1 \in \mathcal{G}(S, E)$ and all $f \in \mathcal{G}(M, E)$.

Key steps of the proof:

- Classical theory of normally hyperbolic operators provides us with a solution candidate.
- (A) & (C) still allow us to do the energy estimates, which give existence and uniqueness.

Comments and outlook

- Variants of the result ([*C.Hanel*, 2010]):
  Condition (A) is not necessary to prove moderateness: either
  - replace (A) by $g_\varepsilon, g_\varepsilon^{-1} = O(1)$
    (i.e., no conditions on derivatives but moderateness)
    and still have the existence and uniqueness result, or
  - keep (A) and use it to calculate precise power of ε-asymptotics of (derivatives) of the solution.
- Connecting to the theory of first order systems
  see Christian Spreitzer's talk
- Perspectives, questions, projects?
  - Is condition (B) really necessary?
  - compatibility with $\mathcal{D}'$-result
  - connect to more classical approaches ($\mathcal{C}^{1,1}$ or GT space-times)
  - more general metrics: log-type growth in ε replacing $O(1)$
    (Hölder-Zygmund classes)
  - ...
  - go non-linear??? (Einstein equations)

Some references

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