The Haar measure on $SU(2)$

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March 14, 2008

Abstract

We give a way to find the Haar measure on $SU(2)$.

1 Introduction

From some abstract mathematics we know that one a compact Lie group $G$ there exists an up to scaling unique left-invariant integration measure $d\mu$:

$$\int_G f(gh)d\mu(h) = \int_G f(h)d\mu(h).$$

The measure $d\mu$ is called the Haar measure.

Knowing existence is of course nice, but in practice (as in physics and chemistry) one needs a way to construct the Haar measure. For $SU(2)$ there are enough nice properties that make it tractable to find the Haar measure without too much effort. We will use a number of facts:

- We assume the reader knows what the Lie group $SU(2)$ is; for example see the text “Notes on SU(2) and SO(3)” on my website http://www.mat.univie.at.at/westra/physmath.html.
- Any element $g$ of $SU(2)$ can be written as
  $$g = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$  
  In this way the underlying manifold of $SU(2)$ is the three-sphere $S^3$. See the same text of my website for a derivation.
- $SU(2)$ has a natural representation on $\mathbb{C}^2$ where it leaves the inner product $\langle z, w \rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2$ invariant. Note that we write $z, w, \ldots$ for elements of $\mathbb{C}^2$.

2 Representations on homogeneous polynomials

Consider the linear complex vector space $V_j$ of dimension $j + 1$ of polynomials of homogeneous degree $j$ in the coordinates $z_1$ and $z_2$. We get a representation of $SU(2)$ on $V_j$ by defining the action

$$(U_g p)(z) = p(g^{-1}z), \quad p \in V_j, g \in SU(2).$$

We thus have a map $SU(2) \rightarrow GL(j, \mathbb{C})$ given by $g \mapsto U_g$. It is easy to see that $U_{gh} = U_h U_g$ and $U_e = id_{V_j}$.

We equip $V_j$ with the following inner product: Let $D$ be the solid three-sphere embedded in $\mathbb{C}^2$

$$D = \{ z \in \mathbb{C}^2 \mid \langle z, z \rangle \leq 1 \}.$$

Then for $p, q \in V_j$ we define

$$\langle p, q \rangle = \int_D \overline{p(z)} q(z),$$

where $\overline{p(z)}$ is the complex conjugate of $p(z)$.
where \( Dz = dz_1 dz_2 dz_3 dz_4 \). The integration measure \( Dz \) satisfies

\[
\int_D Dz = 4\pi^3 \int_{s^2+y^2 \geq 1} xy \, dx \, dy = \frac{\pi^2}{2}.
\]

For further reference it is useful to note the following

\[
\int_{s^2+y^2 \geq 1} x^{2n+1} y^{2m+1} \, dx \, dy = \frac{1}{4} \int_{0 \leq s \leq 1} s^n t^m \, ds \, dt = \frac{1}{4} \int_0^1 s^n (1-s)^m \, ds = \frac{n! m!}{(n + m + 1)!}.
\]

We then find

\[
\int_D Dz z_1^a z_2^b z_3^c z_4^d = \frac{\pi^2}{2} \frac{a! b! c! d!}{(a + b + 1)!} \delta_{ab} \delta_{cd},
\]

so that the monomials \( z_1^a z_2^b \) form a basis for \( V_j \) where \( 0 \leq a \leq j \). We can thus define \( p^j_a = z_1^a z_2^{-j} \) and find

\[
(p^j_a, p^j_b) = \frac{\pi^2}{j+1} \frac{1}{a^j} \delta_{ab}.
\]

Although this representation is nice already in itself, it is not the object of study. The reason we expanded was to give the reader some preparation of how to think of what follows.

The measure \( Dz \) is invariant under \( SU(2) \)-transformations. Indeed, under \( z \mapsto g \cdot z \) we have \( dz_1 \, dz_2 = \det gdz_1 \, dz_2 = dz_1 \, dz_2 \).

The group \( SU(2) \) is inside \( \mathbb{C}^2 \) as the subset \( S^3 = \{ z \in \mathbb{C}^2 \mid \langle z, z \rangle = 1 \} \), which is just the boundary of \( D \). In mathematical words: \( SU(2) = \partial D \). In \( \mathbb{C}^2 \) we can use spherical coordinates \( r, \theta, \varphi, \chi \) and then \( SU(2) \) corresponds to the subset where \( r = 1 \). Given a smooth function, we can restrict it to a smooth function on \( SU(2) \). In fact, we may parameterize \( z_1 = r \cos \theta e^{i \varphi}, \ z_2 = r \sin \theta e^{i \chi} \) with \( 0 \leq \theta \leq \pi \) and \( 0 \leq \varphi, \chi \leq 2\pi \). Then the polynomials \( p^j_a \) restrict to the functions \( P^j_a \) on \( SU(2) \) defined by

\[
P^j_a(\theta, \varphi, \chi) = (\cos \theta)^a (\sin \theta)^{j-a} e^{ia\varphi} e^{i\chi}.
\]

Hence for each \( j \) we get a class of functions on \( SU(2) \); the restriction of \( V_j \) to \( SU(2) \). In fact, it is a theorem of Hermann Weyl and his student Peter that says that every integrable function on \( SU(2) \) can be approximated by the functions \( P^j_a \), where integrable is with respect to the Haar measure. Now, ... what can this Haar measure be???

## 3 The Haar measure

The functions on \( SU(2) \) can be extended to functions on \( D \) by decomposing a function into its \( V_j \) components and then multiplying with the radial coordinate \( r \). For functions on \( D \) we have an \( SU(2) \)-invariant measure. Hence we obtain an invariant measure on \( SU(2) \) by restricting the one on \( D \) to its boundary. Note that an \( SU(2) \) transformation only changes the angles in a function and not its \( r \)-dependence. Hence when we split off the radial part of the measure

\[
Dz = \mu(\theta, \varphi, \chi) d\theta d\varphi d\chi,
\]

we have that

\[
\int_0^1 r^3 \int_{S^3} \mu(\theta, \varphi, \chi) \, d\theta d\varphi d\chi \, f(r, \theta, \varphi, \chi),
\]

is invariant under rotating the angles by means of an \( SU(2) \)-transformation. The \( SU(2) \)-invariance means

\[
\int_D Dz(U_h f)(z_1, z_2) = \int_D Dz f(z_1, z_2).
\]

Since \( SU(2) \) only acts on the radial part of the functions, we must have that

\[
\int_{S^3} \mu(\theta, \varphi, \chi) \, d\theta d\varphi d\chi \, f(r, \theta, \varphi, \chi)
\]

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is invariant under $SU(2)$-transformations for each $r$. Hence also for $r = 1$, which gives back the original
function $f$ on $SU(2)$.

We find $\mu(\theta, \varphi, \chi)$ by writing out

$$dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 = r^3 |\det J| \, d\theta d\varphi d\chi,$$  \hspace{1cm} (15)

where

$$r^3 J = \begin{pmatrix} 2\alpha & 2\alpha & 2\alpha & 2\alpha \\ \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial \chi} \\ \frac{\partial}{\partial \bar{z}_1} & \frac{\partial}{\partial \bar{z}_2} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial \chi} \\ \frac{\partial}{\partial z_2} & \frac{\partial}{\partial \bar{z}_2} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial \chi} \end{pmatrix}$$ \hspace{1cm} (16)

A small calculation reveals

$$\mu(\theta, \varphi, \chi) d\theta d\varphi d\chi = \sin 2\theta \, d\theta d\varphi d\chi.$$ \hspace{1cm} (17)