The group $ISO(1,1)$

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Abstract

Some generalalities on $ISO(1,1)$ are discussed.

The group $ISO(1,1)$ is the group of affine metric-preserving and orientation preserving transformations of the plane $\mathbb{R}^2$, with the hyperbolic metric η of signature $+-$ given by

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Explicitly, we get a faithfull (defining) representation of the form

$$ISO(1,1) = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \;\middle|\; A \in SO(1,1)\,,\; b \in \mathbb{R}^2 \right\}\,.$$

Thus, we see that topologically we have $ISO(1,1) = SO(1,1) \times \mathbb{R}^2$. Since $\mathbb{R}^2$ is simply connected, we can answer the question about the kind of connectedness of $ISO(1,1)$ by looking at $SO(1,1)$ solely. But the group $SO(1,1)$ is the group of real 2×2 matrices

$$\begin{pmatrix} u & v \\ s & t \end{pmatrix}$$

with

$$u^2 - s^2 = 1\,, \quad uv - st = 0\,, \quad tu - sv = 1\,, \quad v^2 - t^2 = -1\,. \qquad (1)$$

Solving eqns.(1) directly in $\mathbb{R}$

Wo we may take $u = \sigma \cosh\alpha$ and $s = \sinh\alpha$, where $\sigma = \pm 1$ and $\alpha \in \mathbb{R}$. Similarly, we can take $t = \tau \cosh\beta$ and $v = \sinh\beta$. But then from $uv - st = 0$ we obtain

$$\cosh\alpha \sinh\beta - \sigma\tau \sinh\alpha \cosh\beta = \sinh(\beta - \sigma\tau\alpha) = 0$$

which implies $\alpha = \sigma\tau\beta$. And then from $ut - sv = 1$ we get

$$\cosh\alpha\cosh(\tau\sigma\alpha) - \tau\sigma\sinh\alpha\sinh(\tau\sigma\alpha) = \cosh(\alpha - \alpha\tau\sigma) = 1\,,$$

which implies $\tau\sigma = 1$ and thus all $SO(1,1)$ matrices are of the form:

$$\begin{pmatrix} \sigma\cosh\alpha & \sinh\alpha \\ \sinh\alpha & \sigma\cosh\alpha \end{pmatrix} = \sigma \begin{pmatrix} \cosh(\sigma\alpha) & \sinh(\sigma\alpha) \\ \sinh(\sigma\alpha) & \cosh(\sigma\alpha) \end{pmatrix}\,, \qquad \sigma = \pm 1.$$

The topology

Topologically, we have $SO(1,1) \cong \mathbb{R} \sqcup \mathbb{R}$ and thus $ISO(1,1) \cong \mathbb{R}^3 \sqcup \mathbb{R}^3$ (disjoint union) and thus we have that $ISO(1,1)$ **is not simply connected**. It is not even connected. Note that indeed, if a matrix of the given parametrization (with only $\cosh\alpha$ and not $-\cosh\alpha$ and thus the $SO(1,1)$ matrix has 11-component > 0) is in $ISO(1,1)$ then also minus that matrix is in $ISO(1,1)$, but we can not reach it with the parametrization, unless we use two real lines... and thus the sign σ. And thus $ISO(1,1)$ is not connected. However, the connected component containing the identity is simply connected. We can thus write $ISO(1,1) = ISO(1,1)^+ \sqcup ISO(1,1)^-$, where $ISO(1,1)^+$ is the identity component and $ISO(1,1)^-$ is reached by applying the matrix $-\mathbb{1}_{2\times 2}$ to $ISO(1,1)^+$.

Some more algebra

Introducing the variables $X = u - s$, $Y = u + s$, $Z = t - v$ and $W = t + v$ we can rewrite (1) as

$$XY = 1\,, \quad ZW = 1\,, \quad XW = YZ\,, \quad WX + YZ = 2\,. \tag{2}$$

We can now look for solutions in any commutative (and associative) ring with 1 and with characteristic not equal to 2. Since X, Y, W, Z all have to be invertible we get from the first two equations of (2) that $X = Y^{-1}$ and $W = Z^{-1}$. Using this in the third equation of (2) we obtain $X^2 = Z^2$, and hence $Z = WX^2$. Note that concluding $X = \pm Z$ might be wrong as in general $X + Z$ can be a zerodivisor of $X - Z$. Multiplying the fourth equation of (2) with X we obtain

$$2X = WX^2 + XYZ = Z + Z = 2Z$$

and thus $X = Z$. Since X is invertible, we immediately obtain $W = Y$. Hence the solutions of (2) are parameterized by the hyperbola $XY = 1$, which is the affine line without the zero.

Now we can look again at the original problem and see that $SO(1,1)$ is as a real manifold nothing more than the hyperbola, which is disconnected. Looking at the complexification something happens, as then the algebraic variety $XY = 1$ in $\mathbb{C}$ becomes connected. Furthermore, compactifying by adding the point in infinity, we get the variety $XY = Z^2$, which is noting more that the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding.