An application of the Segre embedding

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Abstract

We pose and answer a question concerning rational functions on a surface in \( \mathbb{P}^3 \) once raised during an algebra class.

1 The question

Consider the surface \( X \) defined in \( \mathbb{P}^3 \), where we use coordinates \((z_0 : z_1 : z_2 : z_3)\), defined as the zero locus of the following equation:

\[
X : \ z_0 z_3 - z_1 z_2 = 0 .
\]

(1)

Define the following open sets in \( \mathbb{P}^3 \):

\[
U_i = \{(z_0 : z_1 : z_2 : z_3) \in \mathbb{P}^3 | z_i \neq 0 \}, \quad i = 0, 1, 2, 3 .
\]

(2)

We define the following open sets on \( X \):

\[
X_i = X \cap U_i, \quad i = 0, 1, 2, 3.
\]

(3)

We denote \( \mathcal{O}_X \) the structure sheaf on \( X \), that is, \( \mathcal{O}_X(V) \) is the algebra of regular functions on \( V \), where \( V \) is an open set on \( X \).

Consider the following regular functions:

\[
f \in \mathcal{O}_X(X_1), \quad f = \frac{z_0}{z_1}, \quad \text{and} \quad g \in \mathcal{O}_X(X_3), \quad g = \frac{z_2}{z_3} .
\]

(4)

On the intersection \( X_1 \cap X_3 \) they define the same element:

\[
f|_{X_1 \cap X_3} = g|_{X_1 \cap X_3} \in \mathcal{O}_X(X_1 \cap X_3) .
\]

(5)

Hence \( f \) and \( g \) define a single regular function \( F \) on \( X_1 \cup X_3 \). The question is whether there exists a rational function of the \( z_i \) that equals \( f \) on \( X_1 \) and \( g \) on \( X_3 \). In other words, can’t we get a single expression for \( F \)?
2 The answer

We define \( V = X_1 \cup X_3 \) and we will look for a description of the regular functions on \( V \).

Consider the Segre map \( \varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \) defined by

\[
\varphi : (x_0 : x_1) \times (y_0 : y_1) \mapsto (x_0y_0 : x_1y_0 : x_0y_1 : x_1y_1).
\]

Then clearly the image of \( \varphi \) is contained in \( X \). But in fact, the image of \( \varphi \) is precisely \( X \). We can set up a biregular equivalence between \( X \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \) by means of \( \varphi \); the inverse of \( \varphi \) is on \( X_0 \) given by:

\[
\varphi^{-1}|_{X_0} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : z_1) \times (z_0 : z_2)
\]
on \( X_1 \) the inverse is given by

\[
\varphi^{-1}|_{X_1} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : z_1) \times (z_1 : z_3)
\]
on \( X_2 \) the inverse is given by

\[
\varphi^{-1}|_{X_2} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_2 : z_3) \times (z_0 : z_2)
\]
and on \( X_3 \) the inverse is given by

\[
\varphi^{-1}|_{X_3} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_2 : z_3) \times (z_1 : z_3).
\]

Clearly \( X \) is the union of all the \( X_i \) and the complement of \( X_1 \cup X_3 \) consists of all the set where \( z_1 = z_3 = 0 \), the inverse image of which is the set where \( x_1 = 0 \). Hence \( V \) is biregular to \( \mathbb{C}^1 \times \mathbb{P}^1 \). A regular function on \( V \) thus pulls back to a regular function on \( \mathbb{C}^1 \times \mathbb{P}^1 \). The set of regular functions on \( \mathbb{C}^1 \times \mathbb{P}^1 \) are in the given setting the elements of \( \mathbb{C}[\frac{x_0}{x_1}] \). Using the inverse maps of \( \varphi \) we see that any polynomial in \( \frac{x_0}{x_1} \) pulls back to a polynomial function of \( \frac{z_0}{z_1} \) on \( X_1 \) and to a polynomial function of \( \frac{z_2}{z_3} \) on \( X_3 \). Hence we cannot meet both requirements of having neither powers of \( z_1 \) nor powers of \( z_3 \) in the denominator either of the two subsets \( X_1 \) or \( X_3 \). The answer to the question thus is: NO.