ENTROPY DISSIPATION AND WASSERSTEIN METRIC METHODS FOR THE VISCOUS BURGERS’ EQUATION: CONVERGENCE TO DIFFUSIVE WAVES.

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Abstract. In this paper we study the large time behavior for the viscous Burgers’ equation with initial data in $L^1(\mathbb{R})$. In particular, after a time dependent scaling, we provide the optimal rate of convergence in relative entropy and Wasserstein metric, towards an equilibrium state corresponding to a positive diffusive wave. The main tool in our analysis is the reduction of the rescaled Burgers’ equation to the linear Fokker–Planck equation by means of the Hopf–Cole transformation. We then employ well known results concerning the decay in relative entropy and in Wasserstein metric towards stationary solutions for the Fokker–Planck equation.

1. Introduction

This paper is devoted to the study of the large–time behavior of the one–dimensional viscous Burgers’ equation

$$u_t + uu_x = \mu u_{xx},$$

(1.1)

with $L^1$ initial data ($\mu > 0$). The study of this equation (introduced by Burgers in 1940), has been the key point for the development of the theory of shock waves and diffusion waves for viscous and non–viscous systems of conservation laws. The interest of (1.1) lays on the fact that it is the simplest model combining both nonlinear propagation of waves and diffusive behavior of solutions. A first detailed analysis of this phenomena was performed by Hopf ([Hop50]) and successively by Whitam ([Whi74]), who both constructed the typical intermediate asymptotic states for this equation (with summable initial data), namely, the diffusive N-waves and the nonnegative diffusive waves, and studied the trend towards these profiles for solutions having initial datum in $L^1$.

The study of the viscous Burgers’ equation is naturally related to that of the inviscid Burgers’ equation

$$u_t + uu_x = 0.$$

(1.2)

Indeed, it is well known that the most intuitive criterion for the selection of the unique entropy solution of the Cauchy problem for (1.2) is the $\mu \to 0$ limit of the solutions of (1.1) (see [Daf00]).
At the stage of time–asymptotics, it is also well–known that a system of
conservation laws has the same long–time behavior as its viscosity approxi-
mation at the level of shock waves, while the diffusion waves for the viscous
system are replaced by the \( N \)–waves in the inviscid one (see the important
paper by Tai Ping Liu [Liu85] and the paper by Liu and Pierre [LP84]). We
refer to the introduction to [Liu85] for a clear explanation of the asymptotic
stability of nonlinear waves for viscous conservation laws.

Besides the above mentioned literature, other authors (we mention Zuazua,
Vazquez and Escobedo among the others) have widely investigated the as-
ymptotic behavior of solutions of convection diffusion equations. The con-
struction of diffusive waves as exact solutions of these equations and the
study of the rate of convergence in \( L^p \)–norms towards these profiles have
been developed in many papers (see e.g. [EZ91, EVZ93]).

The main tool in the study of the asymptotic behavior of Burgers’ equa-
tion is the Hopf–Cole transformation
\[
\phi(x, t) = e^{-\frac{1}{2} \int_{-\infty}^{x} u(y, t) dy},
\]
which reduces (1.1) to the linear heat equation
\[
\phi_t = \phi_{xx}.
\]
In the recent years, the theory of linear and nonlinear diffusion equations
has been decisively improved by the use of entropy dissipation methods. We
mention the papers by Carrillo and Toscani [CT00] and Felix Otto [Ott01]
for the porous medium–fast diffusion equation. The main tool in this study
is the use of a time dependent scaling which reduces the PME to a nonlinear
Fokker–Planck equation. Hence, the exponential decay towards equilibrium
turns to be equivalent to the validity of some convex Sobolev inequalities
(which generalize the former Log–Sobolev inequality by Gross [Gro75]). We
mention the two papers by Arnold, Markowich, Toscani and Unterreiter
[AMTU00, AMTU01] and the review paper [MV00] for the theory of ex-
ponential decay towards equilibrium for the linear Fokker–Planck equation,
the study of Log–Sobolev type inequalities and the Csiszár–Kullback type
inequalities (which convert the entropy dissipation results into an \( L^1 \) decay
to equilibrium). The rate of convergence obtained in these papers is proven
to be sharp.

The entropy dissipation machinery developed in the above mentioned pa-
ers is closely related to the so–called Wasserstein distance. This distance is
defined on the space of probability measures with finite second moment and
comes as the minimal quadratic cost in the variational Monge–Kantorovich
problem of optimal mass trantonization (see the Section 4 for the details).
Its relation with the entropy functionals has been clarified by Felix Otto
([Ott01]) in the context of gradient flows. More precisely, the porous medium
equation (and the linear heat equation as a special case) can be viewed as
the gradient flow of an entropy–type functional with respect to a Dirichlet
integral type metric defined on the space of probability measures, which is
entailed by a Riemannian structure. The Wasserstein metric comes as the minimal distance induced by this metric on the infinite-dimensional Riemannian manifold of probability densities.

The analysis of the time-decay for the Wasserstein metric in the one-dimensional case is simplified by the representation of the optimal map (in the variational problem mentioned above) involving the distribution functions of the solutions. More precisely, one can estimate the Wasserstein distance by computing directly the equation for the pseudo-inverse of the distribution function. This technique has been recently applied to general diffusion equations (see the review paper [CT]).

The results on the present paper use the fact that the viscous Burgers' equation inherits the gradient flow structure of the linear heat equation via the Hopf–Cole transformation. Following the same ideas as in the above papers, we perform a time dependent scaling in order to convert the asymptotic states into stationary solutions. Hence, we prove the exponential decay of the entropy dissipation in Theorem 3.2 for nonnegative solutions and in Theorem 3.7 for general sign-changing data (using a quadratic entropy dissipation functional). Afterwards, in Theorem 4.2 we prove exponential decay for the Wasserstein distance in the rescaled variables.

In the next section we describe in details the construction of the asymptotic states via the Hopf–Cole transformation. This is done first in the classical way ([Hop50]), then via the time dependent scaling which converts equation (1.1) into the Burgers–Fokker–Planck equation. We also describe briefly the behavior of solutions as $\mu \to 0$ in order to help a general understanding of the interplay between time-asymptotics and vanishing viscosity limit. Section 3 is devoted to the study of the entropy functionals. Finally, in Section 4 we prove the estimate for the Wasserstein distance.

2. The Hopf–Cole Transformation

The time-asymptotic analysis for the viscous Burgers' equation

$$u_t + uu_x = u_{xx}, \quad (2.1)$$

is based on the classical Hopf–Cole transformation, which reduces (2.1) to the linear heat equation. In this section, we first explain the classical process used to obtain the typical intermediate-asymptotic states for this equation (see [Hop50]). Afterwards, we set up a different framework of notations, in order to convert (2.1) into the linear Fokker–Planck equation.

2.1. The classical setting. We consider the initial value problem

$$\begin{cases} u_t + \left( \frac{u^2}{2} \right)_x = u_{xx} \\ u(x, 0) = u_0(x) \end{cases} \quad (2.2)$$

where $x \in \mathbb{R}$, $t > 0$, $u \in \mathbb{R}$ and $u_0$ is a given function in $L^1(\mathbb{R})$. It is well known (see [Hop50]) that, if $u(\cdot, t)$ is the solution of (2.2) at a positive time
t, then the following conservation property holds
\[ \int_{-\infty}^{+\infty} u(x,t) \, dx = \int_{-\infty}^{+\infty} u_0(x) \, dx. \]

We denote in the following
\[ \int_{-\infty}^{+\infty} u_0(x) \, dx = M. \tag{2.3} \]

The unique solution of (2.2) can found explicitely by means of the Hopf–Cole tranformation
\[ \phi(x,t) = \exp \left( -\frac{1}{2} \int_{-\infty}^{x} u(y,t) \, dy \right), \]
\[ u(x,t) = -2 \phi_x(x,t) \phi(x,t), \tag{2.4} \]
which reduces (2.2) to the linear heat equation \( \phi_t = \phi_{xx} \). Indeed, via the convolution formula for the heat equation, one obtains
\[ u(x,t) = \frac{\int_{-\infty}^{+\infty} x-y \, e^{-\frac{|x-y|^2}{2t}} \, dx \, dy \, \int_{-\infty}^{+\infty} u_0(z) \, dz \, dy}{\int_{-\infty}^{+\infty} e^{-\frac{|x-y|^2}{2t}} \, dx \, dy}. \tag{2.5} \]

Moreover, one can construct a diffusion wave type solution \( U_M \) for (2.2) with mass \( M \), corresponding to the solution
\[ \Phi_M(x,t) = 1 - C_M \int_{-\infty}^{x(2t+1)^{-1/2}} e^{-\frac{\zeta^2}{2}} \, d\zeta \tag{2.6} \]
of the heat equation. The constant \( C_M \) in the above formula is determined in order to match condition (2.3).

To construct \( \Phi_M \), we observe that the spatial derivative \( z(x,t) = -\phi_x(x,t) \) (\( \phi \) given by (2.4)) satisfies again the heat equation \( z_t = z_{xx} \) and it has an initial datum \( z_0 \in L^1(\mathbb{R}) \). Hence, one can consider the gaussian solution of the heat equation with the same mass as \( z_0 \), namely
\[ Z_M(x,t) = C_M(2t + 1)^{-1/2} e^{-\frac{x^2}{2(2t+1)}}, \tag{2.7} \]
and write the corresponding \( \Phi_M \) by taking the spatial primitive of \( Z_M \) (the limiting conditions for \( \Phi \) are determined by the conservation of mass). Finally, by replacing \( \phi \) with \( \Phi_M \) into (2.4), one obtains the diffusion wave
\[ U_M(x,t) = 2C_M(2t + 1)^{-1/2} \frac{\exp \left( -\frac{x^2}{2(2t+1)} \right)}{1 - C_M \int_{-\infty}^{x(2t+1)^{-1/2}} e^{-\frac{d^2}{2}} \, d\zeta}. \tag{2.8} \]
2.2. Intermediate asymptotics and small–viscosity regime. Let us consider equation (2.1) with a small viscosity parameter $\mu > 0$, i.e.
\[
\begin{cases}
  u_t + \left( \frac{u^2}{2} \right)_x = \mu u_{xx} \\
  u(x, 0) = u_0(x),
\end{cases}
\]
with initial datum $u_0 \in L^1(\mathbb{R})$ eventually sign–changing. In the limit as $\mu \to 0$, one recovers the unique entropy solution of the Cauchy problem for the inviscid Burgers’ equation
\[
\begin{cases}
  u_t + \left( \frac{u^2}{2} \right)_x = 0 \\
  u(x, 0) = u_0(x).
\end{cases}
\]
It is well–known that the quantities
\[
p = -\inf_{x \in \mathbb{R}} \int_{-\infty}^x u_0(y)dy \quad q = \sup_{x \in \mathbb{R}} \int_x^{+\infty} u_0(y)dy
\]
are invariant for equation (2.10). Hence one can construct the $N$–wave type solution
\[
N_{p,q}(x,t) = \begin{cases} 
\frac{x}{t} & \text{if } -\sqrt{2}pt \leq x \leq \sqrt{2}qt \\
0 & \text{otherwise},
\end{cases}
\]
which turns out to be the an attractor in the $L^1$ norm for any solution of (2.10) with initial datum having compact support and negative and positive masses given by $p$ and $q$ respectively (see [Lax57, DP75]). In case of nonnegative data with mass $M$, (2.11) becomes
\[
N_M(x,t) = \begin{cases} 
\frac{x}{t} & \text{if } 0 \leq x \leq \sqrt{2Mt} \\
0 & \text{otherwise}.
\end{cases}
\]
As we pointed out before, in the case of a nonnegative initial datum $u_0$ for the viscous Burgers’ equation (2.2), the mass $M = \int u_0$ is the only information needed to construct the asymptotic diffusive wave $U_M$ given in (2.8). With the small viscosity parameter $\mu$ as in (2.9), this solution becomes
\[
U_{M,\mu}(x,t) = 2\mu C_{M,\mu}(2t + 1)^{-\frac{1}{2}} \frac{\exp \left( -\frac{x^2}{2(2t+1)} \right)}{1 - C_{M,\mu} \int_{-\infty}^{x(2t+1)^{-\frac{1}{2}}} e^{-\frac{\zeta^2}{2\mu}} d\zeta} ,
\]
with $C_{M,\mu} = 1 - e^{-\frac{1}{2\mu}}$. It can be easily checked that, for any fixed $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, $U_{M,\mu}(x,t) \to N_M(x,t + 1/2)$ as $\mu \to 0$, where $N_M$ is given by (2.12), that is, the diffusive wave $U_{M,\mu}$ approximates a positive $N$–wave in the zero–viscosity limit. As pointed out by T.P. Liu in his introduction to [Liu85], this situation is an example of how the behavior of a nonlinear conservation law, at the level of diffusion waves, changes considerably with the presence of the viscosity (even when this is small).
the case the of Burgers’ equation, an intermediate state for the viscous case (for general possibly sign-changing initial data) is provided by

\[ \tilde{U}(x,t) = -2\mu \frac{\tilde{\phi}(x,t)}{1 - \int_{-\infty}^{x} \phi(y,t) \, dy}, \tag{2.14} \]

where

\[ \tilde{\phi}(x,t) = \frac{a}{\sqrt{2\pi \mu t}} e^{-\frac{x^2}{2\mu t}} - \frac{b}{\sqrt{2\pi \mu t}} e^{-\frac{x^2}{2\mu t}} \]

and

\[ a = \frac{1}{2\mu} \int_{-\infty}^{+\infty} u_0(y) e^{-\frac{1}{2\mu} \int_{-\infty}^{y} u_0(z) \, dz} \, dy \quad b = \frac{1}{2\mu} \int_{-\infty}^{+\infty} u_0^+(y) e^{-\frac{1}{2\mu} \int_{-\infty}^{y} u_0(z) \, dz} \, dy. \]

For fixed \((x,t)\), the function \(\tilde{U}\) tends to the \(N\)-wave defined in (2.11) as \(\mu\) tends to zero. The very interesting paper by Kim and Tzavaras ([KT01]) provides a quantitative understanding of the long–time–small–viscosity interplay for the Burgers’ equation. It turns out that that, when the viscosity is fixed small, at a first asymptotic stage the solution tends to take the shape of an approximate \(N\)-wave of the type (2.14) (thus, its behavior is mainly governed by convection). Then, at a very long time stage, the diffusion produces an interaction between the positive and the negative masses, and the smallest between them is consumed, while the profile of the solution tends to that of the diffusive wave \(U_M\) defined in (2.13).

In the present paper we do not deal with the zero viscosity limit, even though an interesting problem could be the understanding of the limiting behavior of the estimates carried out in our work as \(\mu\) approaches to zero.

We also mention that the entropy approach for the inviscid Burgers’ equation has been recently used by Dolbeault and Escobedo in [DE], where the authors use a time–dependent scaling in order to view the \(N\)-wave type solution as a stationary state. The convergence towards equilibrium is then proven by means of an entropy functional which provides a decay in a weighted \(L^1\)-norm.

2.3. The time–dependent scaling. We are now interested in the study of the asymptotic convergence of the solution \(u\) of (2.2) towards \(U_M\) given by (2.8) as \(t \to \infty\). To perform this task, we consider a time-dependent scaling which transforms this problem of the study of the asymptotic stability of a stationary state. This idea has been frequently employed in the study of the time–asymptotics for nonlinear diffusion equations (see [Bar52, CT00]). More precisely, we set

\[ y = y(x,t) = xR(t)^{-1} \]
\[ s = s(t) = \log R(t) \]
\[ u(x,t) = R(t)^{-1} \rho(y(x,t), s(t)), \tag{2.15} \]

where

\[ R(t) = (2t + 1)^{1/2}. \]
With this notation, (2.2) turns into the following Cauchy problem for the Burgers–Fokker–Plank equation

\[
\begin{cases}
\frac{\partial \rho}{\partial s} = \frac{\partial}{\partial y} \left( \frac{\partial \rho}{\partial y} + y \rho - \frac{\rho^2}{2} \right) \\
\rho(y, 0) = \rho_0(y) = u_0(y).
\end{cases}
\] (2.16)

Obviously, we have again the conservation of the mass

\[
\int_{-\infty}^{+\infty} \rho(y, s) dy = \int_{-\infty}^{+\infty} \rho_0(y) dy = M.
\]

To recover the suitable stationary solution with mass \(M\) of (2.16), we employ once again the Hopf–Cole transformation

\[
\tau(y, s) = \exp \left( -\frac{1}{2} \int_{-\infty}^{y} \rho(\zeta, s) d\zeta \right),
\] (2.17)

which converts (2.16) into

\[
\frac{\partial \tau}{\partial s} = \frac{\partial^2 \tau}{\partial y^2} + y \frac{\partial \tau}{\partial y}.
\]

Hence, the spatial derivative

\[
\psi(y, s) = -\frac{\partial \tau}{\partial y}(y, s) = \frac{1}{2} \rho(y, s) \exp \left( -\frac{1}{2} \int_{-\infty}^{y} \rho(\zeta, s) d\zeta \right)
\] (2.18)

satisfies the Cauchy problem for the linear Fokker–Planck equation

\[
\begin{cases}
\frac{\partial \psi}{\partial s} = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} + y \psi \right) \\
\psi(y, 0) = \psi_0(y) = \frac{1}{2} \rho_0(y) \exp \left( -\frac{1}{2} \int_{-\infty}^{y} \rho_0(\zeta) d\zeta \right).
\end{cases}
\] (2.19)

The inverse transformation of (2.18) is given by

\[
\rho(y, s) = -2 \frac{\psi(y, s)}{1 - \int_{-\infty}^{y} \psi(\zeta, s) d\zeta}.
\] (2.20)

An easy computation gives

\[
\int_{-\infty}^{+\infty} \psi_0(y) dy = -\int_{-\infty}^{+\infty} \frac{\partial}{\partial y} \left( \exp \left( -\frac{1}{2} \int_{-\infty}^{y} \rho_0(\zeta) d\zeta \right) \right) dy = 1 - e^{-\frac{M}{2}} =: m.
\]

Therefore, conservation of the total mass for the Fokker–Planck equation (2.19), implies

\[
\int_{-\infty}^{+\infty} \psi(y, s) dy = m
\]

for any \(s > 0\). It is well known that (2.19) has the unique Gaussian equilibrium \(\Psi_m\) with mass \(m\), namely

\[
\Psi_m(y) = \frac{m}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.
\] (2.21)
By putting $\Psi_m$ into (2.20), we recover our steady state for the Burgers–Fokker–Planck equation (2.16)

$$\rho^\infty_M(y) = \frac{2m e^{-\frac{y^2}{2}}}{1 - 2m \int_{-\infty}^{y} e^{-\frac{\zeta^2}{2}} d\zeta}.$$  \hspace{1cm} (2.22)

By returning back to the original variables $x, t, u$, it turns out that the stationary state $\rho^\infty_M$ corresponds precisely to the diffusion wave $U_M$ defined in (2.8).

We close this section by recalling an easy estimate for the function $\tau(y, t)$, defined by the Hopf–Cole transformation (2.17), which will be useful in the sequel. Since the initial datum for the original Burgers’ equation $u_0$ belongs in $L^1(\mathbb{R})$, the quantities

$$p = -\inf_{x \in \mathbb{R}} \int_{-\infty}^{x} u_0(y) dy$$  \hspace{1cm} (2.23)

$$q = \sup_{x \in \mathbb{R}} \int_{x}^{+\infty} u_0(y) dy = M + p$$

are then finite. In terms of $\tau$, relations (2.23) provide the following property for $\tau(\cdot, 0)$

$$e^{-\frac{(M+p)}{2}} \leq \tau(y, 0) \leq e^{\frac{p}{2}},$$

for any $y \in \mathbb{R}$. Hence, by simple maximum principle, we obtain the estimate

$$e^{-\frac{(M+p)}{2}} \leq \tau(y, t) \leq e^{\frac{p}{2}},$$  \hspace{1cm} (2.24)

for all $(y, t) \in \mathbb{R} \times \mathbb{R}_+$. 

As it was first proved by Hopf ([Hopf50]), the large time behavior of solutions of the original problem (2.2) is described by the diffusion wave $U_M$ defined by (2.8). Our purpose here is to investigate the large time behavior of $u(x, t)$ in terms of solutions of equation (2.16). Hence, hereafter we shall discuss the asymptotic stability of the stationary solution $\rho^\infty_M$ of equation (2.16).

3. Trend to equilibrium in relative entropy

In this section we analyze the convergence towards the stationary profile $\rho^\infty_M$ defined by (2.22) for the solution $\rho$ of equation (2.16) with initial datum $\rho_0 \in L^1(\mathbb{R})$. We start by treating the case of non–negative initial data $u_0$. The general case will be covered later on. Our choice of the functionals used to control the distance between $u$ and $\tilde{u}$ are the relative entropy functionals

$$H_e(\rho(s)|\rho^\infty_M) = \int_{-\infty}^{+\infty} e \left( \frac{\rho(y, s)}{\rho^\infty_M(y)} \right) \rho^\infty_M(y) dy,$$  \hspace{1cm} (3.1)
where \( e : \mathbb{R}_+ \to \mathbb{R}_+ \) is a smooth function satisfying the following conditions
\[
e(1) = 0
\]
\[
e''(h) \geq 0 \quad \text{for any } h \in \mathbb{R}_+, \quad e'' \text{ not identically } 0
\]
\[
(e''')^2 \leq \frac{1}{2} e'' e'''
\] (3.2)

Such functions are called entropy generating functions, and the corresponding functionals \( H_e \) are called admissible relative entropies (see [AMTU01]). In particular, one can prove that functions \( e \) satisfying the above conditions are strictly convex. Moreover, since \( \rho \) and \( \rho^\infty \) have the same mass, we can obtain the same functional \( H_e \) by “normalizing” the generating function \( e \) in such a way that \( e'(1) = 0 \). As a consequence of the above conditions, each generating function satisfies (see [AMTU01], Lemma 2.6)
\[
\chi(h) \leq e(h) \leq \varphi(h),
\] (3.3)
where we have denoted
\[
\chi(h) = \alpha(h + \beta) \log \frac{h + \beta}{1 + \beta} - \alpha(h - 1), \quad \varphi(h) = \mu_2(h - 1)^2,
\]
with \( \mu_2, \alpha, \beta \) nonnegative constants depending on \( e''(1) \) and \( e'''(1) \), namely
\[
\mu_2 = e''(1), \quad \alpha = -\frac{e''(1)^2}{e'''(1)}, \quad \beta = -\frac{e''(1) + e'''(1)}{e'''(1)}.
\] (3.4)

The constants (3.4) are well defined if \( e'''(1) \neq 0 \). In the case \( e'''(1) = 0 \), one has to set \( \chi(h) = \mu_2 (h-1)^2 \). Hence, the admissible relative entropy approach allows us to cover a large range of functionals, including the physical relative entropy \( \int \rho \log \frac{\rho}{\rho^\infty} dx \). We refer to [AMTU01] for a detailed explanation of the mathematical properties of the relative entropies and their generating functions.

By means of so-called Csiszár–Kullback inequalities, one can control the \( L^1 \) norm of the difference \( \rho - \rho^\infty \) in terms of the relative entropy functionals defined in (3.1).

**Theorem 3.1** (Csiszár–Kullback). There exists a positive constant \( C \) such that, for all functions \( \rho_1, \rho_2 \in L^1_+(\mathbb{R}) \), with
\[
\int_\mathbb{R} \rho_1(x) dx = \int_\mathbb{R} \rho_2(x) dx
\] (3.5)
and for all admissible generating functions \( e \), we have
\[
\|\rho_1 - \rho_2\|_{L^1(\mathbb{R})} \leq C H_e(\rho_1|\rho_2).
\] (3.6)

Moreover, in case of the quadratic generating function
\[
e(h) = \varphi(h) = (h - 1)^2,
\]
the relation (3.6) holds for all \( \rho_1 \in L^1(\mathbb{R}) \) (eventually sign-changing and without requiring the integral condition (3.5)) and \( \rho_2 \in L^1_+(\mathbb{R}) \).
We refer to [AMTU00] for a detailed analysis of Csiszár–Kullback type inequalities.

Let us state our first result of convergence in relative entropy. In what follows we denote the primitives
\[ F(y, s) := \int_{-\infty}^{y} \rho(\zeta, s) d\zeta, \quad F_{\infty}(y, s) := \int_{-\infty}^{s} \rho_{M}^\infty(\zeta, s) d\zeta. \] (3.7)

**Theorem 3.2.** Let \( \rho \) be the solution of (2.16) with \( \rho_0 \in L^1_+ (\mathbb{R}) \). Let \( \rho_{M}^\infty \) be given by (2.22). Let \( H_e \) be the admissible relative entropy functional generated by the function \( e \). Then, there exists a positive constant \( C \) depending on the mass \( M \) such that the following estimate holds:
\[ H_e(\rho(s) | \rho_{M}^\infty) \leq Ce^{-2s} \left[ H_e(\rho_0 | \rho_{M}^\infty) + \| F(\cdot, 0) - F_{\infty}(\cdot, 0) \|_{L^\infty(\mathbb{R})}^2 \right. + \left. \| F(\cdot, 0) - F_{\infty}(\cdot, 0) \|_{L^1(\mathbb{R})}^2 \right]. \] (3.8)

**Remark 3.3.** An easy computation shows that the condition \( F(\cdot, 0) - F_{\infty}(\cdot, 0) \in L^1(\mathbb{R}) \), needed to control the bracket in the r.h.s. of (3.8), is equivalent to the conditions
\[ \int_0^{+\infty} \int_{-\infty}^{x} \rho_0(y) dy dx < +\infty \] (3.9)
and
\[ \int_{-\infty}^{+\infty} \int_x^{+\infty} \rho_0(y) dy dx < +\infty. \] (3.10)
The conditions (3.9)–(3.10) are satisfied, for instance, in the case of initial data with finite second moment, i.e.
\[ \int_{-\infty}^{+\infty} |y|^2 \rho_0(y) dy < +\infty. \] (3.11)

Now, it turns out that whenever \( \rho_0 \) has finite relative logarithmic entropy, then (3.11) holds (see [ABM96]). Hence, for all generating functions \( e \) we have
\[ H_e(\rho_0 | \rho_{M}^\infty) < +\infty \Rightarrow \int_{-\infty}^{+\infty} |y|^2 \rho_0(y) dy < +\infty. \]

Therefore, the decay rate statement of Theorem 3.2 is valid for initial data in \( L^1_+ (\mathbb{R}) \) with finite relative entropy, without further assumptions.

To prove theorem 3.2, we recall more concepts concerning relative entropies. Let \( \rho, \gamma \in L^1_+ (\mathbb{R}) \) such that \( \int_{-\infty}^{+\infty} \rho(x) dx = \int_{-\infty}^{+\infty} \gamma(x) dx \) and let us denote \( h(x) = \rho(x) / \gamma(x) \). The *entropy dissipation* generated by \( e \) is defined as
\[ I_e(\rho | \gamma) = \int_{-\infty}^{+\infty} e''(h(y, s))(h_y(y, s))^2 \rho_{M}^\infty(y) dy. \]
Then, we recall the following *generalized logarithmic Sobolev inequality* (see [AMTU01, MV00]).
Theorem 3.4. Let $H_e$ be an admissible relative entropy with generating function $e$. Let $\rho, \gamma \in L^1_+(\mathbb{R}^d)$ be such that
\[ \int \rho(x)dx = \int \gamma(x)ds = M. \]
Then, the following inequality holds
\[ H_e(\rho|\gamma) \leq \frac{1}{2} I_e(\rho|\gamma). \] (3.12)

In the quadratic case, inequality (3.12) becomes a generalized Poincaré–type inequality. More precisely, if we set $h = \rho/\gamma$, we obtain
\[ \int_{-\infty}^{+\infty} (h - 1)^2 \gamma dx \leq C \int_{-\infty}^{+\infty} |h_x|^2 \gamma dx. \] (3.13)

We remark that positivity of $\rho$ is not needed for (3.13), i.e. it holds for $h = \rho/\gamma$ with $\rho \in L^1(\mathbb{R})$, $\gamma \in L^1_+(\mathbb{R})$ and $\int \rho dx = \int \gamma dx$. Inequalities (3.12) and (3.13) are crucial in the proof of the following theorem, which is the main ingredient for our results. Again, we refer to [AMTU01] for the proof.

Theorem 3.5. Let $\psi$ be the solution to the Cauchy problem for the Fokker Planck equation (2.19) with $\psi_0 \in L^1_+(\mathbb{R})$. Let $\Psi_m$ be the stationary solution given in (2.21). Then, for any generating function $e$, the corresponding relative entropy functional $H_e(\psi(s)|\Psi_m)$ satisfies the following estimate
\[ H_e(\psi(s)|\Psi_m) \leq H_e(\psi_0|\Psi_m)e^{-2s}. \] (3.14)

Moreover, in the quadratic case $\psi(h) = (h - 1)^2$, (3.14) is also valid for an eventually sign–changing initial datum $\psi_0 \in L^1(\mathbb{R})$.

The following lemma is also proved in [AMTU01] (Lemma 2.9) and will be used in the sequel.

Lemma 3.6. The generator $e$ of any relative entropy functional $H_e$ satisfies
\[ a) \quad e(\sigma) \leq e(\sigma_0) \left( \frac{\sigma}{\sigma_0} \right)^2 + \mu \left( \frac{\sigma}{\sigma_0} - 1 \right) (\sigma - 1), \quad \sigma \geq \sigma_0 > 0 \]
\[ b) \quad e(\sigma) \leq e(\sigma_0) \left( \frac{\sigma}{\sigma_0} \right) + \mu \left( \frac{\sigma}{\sigma_0} - 1 \right) (\sigma - 1), \quad \sigma_0 \geq \sigma > 0, \]
where $\mu = e''(1)$.

Now we can provide the proof of Theorem 3.2.

Proof of theorem 3.2. We write inequality (3.14) in terms of $\rho$ and $\rho_M^\infty$ by means of identity (2.18). We observe that
\[ \rho(y,s) = \frac{2\psi(y,s)}{\tau(y,s)}, \]
\[ \rho_M^\infty(y) = \frac{2\Psi_m(y)}{\tau_m(y)}, \]
We observe that estimate (2.24) (p = 0 in case of positive solutions) implies

$$
\sup_{y,s} \left\{ \left( \frac{\sigma(x,t)}{\sigma_0(x,t)} \right), \left( \frac{\sigma(x,t)}{\sigma_0(x,t)} \right)^2 \right\} \leq e^M.
$$

Hence, we have

$$
H_e(\rho(s)|\rho_M^\infty(s)) = \int_{-\infty}^{+\infty} e \left( \frac{\rho(y,s)}{\rho_M^\infty(y)} \right) \rho_M^\infty(y) dy \\
\leq e^{\frac{3M}{2}} \int_{-\infty}^{+\infty} \psi \left( \frac{\rho(y,s)}{\rho_M^\infty(y)} \right) \rho_M^\infty(y) \tau_m(y) dy \\
+ \mu \int_{-\infty}^{+\infty} (\tau_m(y) - \tau(y,s)) (\rho(y,s) - \rho_M^\infty(y)) \tau(y,s)^{-1} dy := I_1 + I_2.
$$

We estimate the term $I_1$ by means of (3.15),

$$
I_1 \leq e^{2M} e^{-2s} \int_{-\infty}^{+\infty} e \left( \frac{\rho_0(y)\tau(y,0)}{\rho_M^\infty(y)\tau_m(y)} \right) \rho_M^\infty(y) \tau_m(y) dy.
$$

By using again Lemma 3.6 with $\sigma(y) = \frac{\rho_0(y)\tau(y,0)}{\rho_M^\infty(y)\tau_m(y)}$, $\sigma_0(y) = \frac{\rho_0(y)}{\rho_M^\infty(y)}$, we obtain

$$
I_1 \leq e^{2M} e^{-2s} \left[ \int_{-\infty}^{+\infty} e \left( \frac{\rho_0(y)}{\rho_M^\infty(y)} \right) \rho_M^\infty(y) dy \\
+ \mu \int_{-\infty}^{+\infty} \rho_M^\infty(y) \tau_m(y) (\tau(y,0) - \rho_0(y) - \rho_M^\infty(y)) dy \\
+ \mu \int_{-\infty}^{+\infty} \rho_M^\infty(y) \tau_m(y) (\tau(y,0) - \tau_m(y))^2 dy \right] \\
\leq e^{-2s} C(M,\mu) \left[ H_e(\rho_0|\rho_M^\infty) \\
+ \|\rho_0 - \rho_M^\infty\|_{L^1} \|\tau(\cdot,0) - \tau_m\|_{L^\infty} + \|\tau(\cdot,0) - \tau_m\|_{L^\infty}^2 \right] \\
\leq C(M,\mu)e^{-2s} \left[ H_e(\rho_0|\rho_M^\infty) + \|\tau(\cdot,0) - \tau_m\|_{L^\infty}^2 \right], \quad (3.16)
$$
where we have used the estimate (2.24) and the inequality (3.6). The constant $C(M, \mu)$ depends on $M$ and $\mu$. Let us estimate the integral term $I_2$ as follows,

\[
I_2 \leq \mu e^M \int_{-\infty}^{+\infty} |\tau_m(y) - \tau(y, s)| |\rho(y, s) - \rho_M^\infty(y)| \, dx
\]

\[
\leq \frac{\mu e^M}{2} \left[ \frac{1}{\varepsilon} \|\tau_m - \tau(s)\|^2_{L^\infty(\mathbb{R})} + \varepsilon \|\rho(s) - \rho_M^\infty\|^2_{L^1(\mathbb{R})} \right]
\]

\[
\leq \frac{\mu e^M}{2} \left[ \frac{1}{\varepsilon} \|\tau_m - \tau(s)\|^2_{L^\infty(\mathbb{R})} + C\varepsilon H(\rho(s)|\rho_M^\infty) \right], \tag{3.17}
\]

where we have used once again inequality (3.6) and $\varepsilon > 0$ shall be fixed later on. The first term in the bracket of (3.17) is to be treated as follows. Using the notation in (2.15), we observe that

\[
\tau(y, s) = e^{-\frac{1}{2} \int_{-\infty}^{y} \rho(\zeta, s) \, d\zeta} = e^{-R(t(s)) \frac{1}{2} \int_{-\infty}^{y} u(R(t(s))\zeta, t(s)) \, d\zeta}
\]

\[
= e^{-\frac{1}{2} \int_{-\infty}^{\rho_R(t)} u(\zeta, t(s)) \, d\zeta} =: \phi(x(y, s), t(s)).
\]

As we have seen in the previous section, the function $\phi(x, t)$ above satisfies the heat equation $\phi_t = \phi_{xx}$. We denote

\[
\~\phi(x, t) = e^{-\frac{1}{2} \int_{-\infty}^{x} U_M(\zeta, t) \, d\zeta} = \tau_m(y(x, t), s(t)),
\]

\[
\bar{\phi} = \phi - \~\phi.
\]

Hence, $\phi$ satisfies $\phi_t = \phi_{xx}$ with initial datum $\phi_0(x) = \tau(x, 0) - \tau_m(x)$. The representation formula for the solution of the one-dimensional heat equation yields

\[
|\bar{\phi}(x, t)| = C \left| \int_{-\infty}^{+\infty} \phi_0(y) e^{-\frac{(x-y)^2}{4t}} \, dy \right| \leq \frac{C}{\sqrt{t}} \|\phi_0\|_{L^1(\mathbb{R})}. \tag{3.18}
\]

Now, by (2.4) and since $0 \leq \int_{-\infty}^{x} u_0(y) \, dy \leq M$, there exists a constant $K_M$ depending only on $M$ such that

\[
|\phi_0(x)| \leq K_M |F(x, 0) - F^\infty(x, 0)|, \tag{3.19}
\]

where $F$ and $F^\infty$ are defined by (3.7). Hence, there exists a fixed constant $K'_M$ such that

\[
\|\phi(t) - \~\phi(t)\|^2_{L^\infty(\mathbb{R})} \leq \frac{K'_M}{t} \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^1(\mathbb{R})}.
\]

Moreover, since

\[
\|\phi(t) - \~\phi(t)\|^2_{L^\infty(\mathbb{R})} \leq \|\phi_0\|^2_{L^\infty(\mathbb{R})},
\]

we have, from (3.19),

\[
\|\phi(t) - \~\phi(t)\|^2_{L^\infty(\mathbb{R})} \leq \frac{\tilde{C}_M}{2t + 1} \left[ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^1(\mathbb{R})} + \|\phi_0\|^2_{L^\infty(\mathbb{R})} \right]
\]

\[
\leq \frac{\tilde{K}_M}{2t + 1} \left[ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^1(\mathbb{R})} + \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^\infty(\mathbb{R})} \right], \tag{3.20}
\]
for constants $\tilde{C}_M, \tilde{K}_M$ depending on $M$. In terms of the rescaled time $s$, (3.20) reads

$$
\|\tau_m - \tau(s)\|^2_{L^\infty(\mathbb{R})} \leq e^{-2s}\tilde{K}_M \left[ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^1(\mathbb{R})} 
+ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^\infty(\mathbb{R})} \right].
$$

(3.21)

Hence, by choosing a sufficiently small $\varepsilon = \varepsilon(\mu, M) > 0$ in (3.17) and in view of (3.16), we obtain

$$
H_e(\rho(s)|\rho^\infty_M) \leq Ce^{-2s} \left[ H_e(\rho_0|\rho^\infty_M) + \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^1(\mathbb{R})} 
+ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^\infty(\mathbb{R})} \right]
$$

and the proof is complete. $\square$

The result in Theorem 3.2 is restricted to the case of positive initial data $\rho_0$ because the relative entropy functionals $H_e(\rho|\rho^\infty_M)$ are not defined for negative values of $\rho$, except for the one generated by the quadratic function $\varphi(h) = (h - 1)^2$. It is possible to obtain a similar result for general sign-changing solutions and for the only case of quadratic entropy $H_\varphi(\rho|\rho^\infty_M) = \int_{-\infty}^{+\infty} \rho^\infty_M \left( \frac{\rho_0}{\rho^\infty_M} - 1 \right)^2 dx$. We observe that we cannot employ lemma 3.6 in the quadratic case, since it is valid only for positive values of $\rho$.

**Theorem 3.7.** Let $\rho(y, s)$ be the solution of the IVP (2.16) with

$$
\int \rho_0^2(y)e^{y^2/2} dy < \infty,
$$

(3.22)

($\rho_0$ eventually sign-changing). Let $\rho^\infty_M$ be given by (2.22). Then, the following estimate holds

$$
\int_{-\infty}^{+\infty} (\rho(y, s) - \rho^\infty_M(y))^2 e^{y^2/2} dy 
\leq Ce^{-2s} \left[ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^1(\mathbb{R})} 
+ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|^2_{L^\infty(\mathbb{R})} 
+ \int_{-\infty}^{+\infty} |\rho_0(y) - \rho^\infty_M(y)|^2 e^{y^2/2} dy \right],
$$

(3.23)

with a constant $C$ depending on the initial datum.
Proof. From the estimate (2.24) (we recall that $\tau$ is expressed by (2.17)), we obtain

$$
\int_{-\infty}^{+\infty} (\rho(y, s) - \rho_M^\infty(y))^2 e^{\frac{s^2}{2}} dy \\
\leq e^{M+p} \int_{-\infty}^{+\infty} (\rho(y, s)\tau_m(y) - \rho_M^\infty(y)\tau_m(y))^2 e^{\frac{s^2}{2}} dy \\
\leq e^{M+p} \int_{-\infty}^{+\infty} \rho(y, s)^2 (\tau(y, s) - \tau_m(y))^2 e^{\frac{s^2}{2}} dy \\
+ e^{M+p} \int_{-\infty}^{+\infty} (\rho(y, s)\tau(y, s) - \rho_M^\infty(y)\tau_m(y))^2 e^{\frac{s^2}{2}} dy \\
:= J_1 + J_2. 
$$

We now employ the same argument as in the proof of the previous theorem to recover estimate (3.21). Hence, we estimate the term $J_1$ as follows

$$
J_1 \leq C_0 e^{-2s} \left[ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|_{L^1(\mathbb{R})}^2 \\
+ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|_{L^\infty(\mathbb{R})}^2 \right] \int_{-\infty}^{+\infty} (\rho(y, s))^2 e^{\frac{s^2}{2}} dy.
$$

Then, inequality (3.14) in the case of quadratic entropy gives

$$
J_1 \leq C_1 e^{-2s} \left[ \|F(\cdot, 0) - F^\infty(\cdot, 0)\|_{L^1(\mathbb{R})}^2 + \|F(\cdot, 0) - F^\infty(\cdot, 0)\|_{L^\infty(\mathbb{R})}^2 \right],
$$

where $C_1$ depends on the initial datum. By writing again inequality (3.14) in terms of $\rho, \rho_M^\infty$ in the case of quadratic entropy, we obtain the estimate for the term $J_2$,

$$
J_2 \leq CH \varphi(\psi(s)|\Psi_m) \leq C e^{-2s} H \varphi(\psi_0|\Psi_m),
$$

for a constant $C$ depending on the mass $M$. Now, we have

$$
H \varphi(\psi_0|\Psi_m) = \int_{-\infty}^{+\infty} (\psi_0(y) - \Psi_m(y))^2 \Psi_m(y)^{-1} dy \\
\leq \frac{1}{4} \int_{-\infty}^{+\infty} |\rho_0(y)\tau(y, 0) - \rho_M^\infty(y)\tau_m(y)|^2 C_M e^{\frac{s^2}{2}} dy \\
\leq C^1_M \int_{-\infty}^{+\infty} |\rho_0(y) - \rho_M^\infty(y)|^2 e^{\frac{s^2}{2}} dy \\
+ C^2_M \int_{-\infty}^{+\infty} |\tau(y, 0) - \tau_m(y)|^2 \rho_M^\infty(y)^2 e^{\frac{s^2}{2}} dy \\
\leq C^0_M \left[ \int_{-\infty}^{+\infty} |\rho_0(y) - \rho_M^\infty(y)|^2 e^{\frac{s^2}{2}} dy + \|\tau(\cdot, 0) - \tau_m(\cdot, 0)\|_{L^\infty(\mathbb{R})}^2 \right],
$$

where $C^0_M$ depends on the initial datum.
where all the constants above depend on the mass $M$. Hence, because of (3.19), we have

$$J_2 \leq C e^{-2s} \left[ \int_{-\infty}^{+\infty} |\rho_0(y) - \rho_M^\infty(y)|^2 e^{y^2} dy + \| F(\cdot, 0) - F^\infty(\cdot, 0) \|_{L^\infty(\mathbb{R})}^2 \right].$$

Thus, by substituting the estimates for the terms $J_1$ and $J_2$ into (3.24), we obtain (3.23).

We remark that in the previous theorem condition (3.22) is sufficient to control the terms involving the primitive $F$ in (3.23). This assertion follows from the observations in remark 3.3 and from the obvious inequality

$$\int_{-\infty}^{+\infty} \rho_0(y) y^2 dy \leq C \left( \int_{-\infty}^{+\infty} \rho_0^2(y) e^{y^2} dy \right)^{\frac{1}{2}}.$$

The results in Theorems 3.2 and 3.7 can be easily converted in terms of $L^1$ decay for the solution $\rho$ to equation (2.16) towards the stationary solution $\rho_M^\infty$, by means of the Csiszár–Kullback inequality. Moreover, as usual in this framework, by returning to the original variable $u$ one gets a polynomial rate of convergence towards diffusion waves for the viscous Burgers’ equation. We collect all these results in the following two corollaries.

**Corollary 3.8.** Let $\rho(y, s)$ be the solution to the Cauchy problem (2.16) with $\rho_0 \in L^1(\mathbb{R})$. Let $\rho_M^\infty$ be given by (2.22). Suppose that one of the following two conditions is satisfied:

i) $\rho_0 \geq 0$, $\int_{\mathbb{R}} y^2 \rho_0(y) dy < +\infty$, and $\int_{\mathbb{R}} \rho_0(y) \log \rho_0(y) dy < +\infty$

ii) $\int_{\mathbb{R}} \rho_0(y)^2 e^{y^2} dy < +\infty$.

Then, the following estimate holds

$$\| \rho(s) - \rho_M^\infty \|_{L^1(\mathbb{R})} \leq C_M e^{-s},$$

where $C_M$ depends on the mass $M$ of the initial datum.

**Corollary 3.9.** Let $u$ be the solution to (2.2) with initial datum $u_0 \in L^1(\mathbb{R})$. Let $U_M$ given by (2.8). Suppose that one of the following two conditions is satisfied

i) $u_0 \geq 0$, $\int_{\mathbb{R}} x^2 u_0(x) dx < +\infty$, and $\int_{\mathbb{R}} u_0(x) \log u_0(x) dx < +\infty$

ii) $\int_{\mathbb{R}} u_0(x)^2 e^{x^2} dx < +\infty$

Then, for all $t \geq 0$, the following inequality holds

$$\| u(t) - U_M(t) \|_{L^1(\mathbb{R})} \leq C (t + 1)^{-\frac{1}{2}},$$

where $C$ depends on the initial datum.
Remark 3.10. We mention here an alternative entropy dissipation approach to the Burgers’–Fokker–Planck equation (see [Cav00])

\[
\frac{\partial \rho}{\partial s} = \frac{\partial}{\partial y} \left( \frac{\partial \rho}{\partial y} + y \rho - \frac{\rho^2}{2} \right).
\]

We denote

\[
\rho(y, s) = \tilde{\rho}(y, s) e^{-\frac{y^2}{2}},
\]

\[
\rho^\infty_M(y) = \tilde{\rho}^\infty_M(y) e^{-\frac{y^2}{2}},
\]

where \(\rho^\infty_M\) is given in (2.22), as usual. Hence, the above equation becomes

\[
e^{-\frac{y^2}{2}} \tilde{\rho}_s = \left( e^{-\frac{y^2}{2}} \tilde{\rho}_y - \frac{1}{2} e^{-y^2} \tilde{\rho}^2 \right) _y.
\]

Since

\[
-\frac{e^{-\frac{y^2}{2}}}{2} = \left( \frac{1}{\rho^\infty_M} \right)_y,
\]

we have

\[
e^{-\frac{y^2}{2}} \tilde{\rho}_s = \left( e^{-\frac{y^2}{2}} \tilde{\rho}^2 \left( \frac{1}{\rho^\infty_M} - \frac{1}{\rho} \right) _y \right) y.
\]

and finally we can rewrite the Burgers–Fokker–Planck equation in the following way

\[
\rho_s = \left( e^{\frac{y^2}{2}} \rho^2 \left( e^{-\frac{y^2}{2}} \left( \frac{1}{\rho^\infty_M} - \frac{1}{\rho} \right) \right) _y \right) y.
\]

This suggests the use of an alternative entropy, namely

\[
H(\rho|\rho^\infty_M) = \int_{-\infty}^{+\infty} \rho(y) e^{-\frac{y^2}{2}} \left[ \frac{\rho(y)}{\rho^\infty_M(y)} - 1 - \log \frac{\rho(y)}{\rho^\infty_M(y)} \right] dy.
\]

Indeed, the entropy production \(I = -\frac{d}{ds} H\) is given by

\[
I(\rho|\rho^\infty_M) = \int_{-\infty}^{+\infty} \rho(y)^2 e^{-\frac{y^2}{2}} \left[ \left( e^{-\frac{y^2}{2}} \left( \frac{1}{\rho^\infty_M(y)} - \frac{1}{\rho(y)} \right) \right) _y \right]^2 dy > 0.
\]

The use of the above entropy allows us to require less restrictive conditions on the initial data in order to obtain convergence to equilibrium. However, in this case no exponential decay is proven, and both functional \(H(\rho)\) and \(I(\rho)\) blow up if evaluated at a density \(\rho(y)\) with a much ‘faster’ behavior than \(\rho^\infty_M\) at \(|y| \to \infty\). Nevertheless, this approach can be generalized to convection diffusion equations with general nonlinear convection, since it does not require the use of the Hopf–Cole transformation.
4. Evolution of the Wasserstein metric

This section is devoted to the study of the Wasserstein metric of solutions of the Burgers–Fokker–Planck equation

\[
\begin{aligned}
\partial \rho / \partial s &= \partial \partial y (\partial \rho / \partial y + y \rho - \rho^2 / 2), \\
\rho(y, 0) &= \rho_0(y) = u_0(y),
\end{aligned}
\]  

(4.1)

with initial datum \( \rho_0 \in L^1_+(\mathbb{R}) \) with finite second moment. Let us recall briefly some concepts concerning the Wasserstein metric. We denote by \( \mathcal{M}_2 \) the space of all probability densities on \( \mathbb{R} \) with finite second moment, i. e.

\( \mathcal{M}_2 = \left\{ \rho \in L^1_+(\mathbb{R}), \int_{\mathbb{R}} \rho(y) dy = 1, \int_{\mathbb{R}} y^2 \rho(y) dy < \infty \right\} \).

The Wasserstein metric \( d^2_2(\cdot, \cdot) \) on the space \( \mathcal{M}_2 \) is defined as follows,

\[
d^2_2(\rho_1, \rho_2) = \inf_{\rho_2 = T_\# \rho_1} \left\{ \int_{\mathbb{R}} (y - T(y))^2 \rho_1(y) dy \right\},
\]  

(4.2)

where the notation \( \rho_2 = T_\# \rho_1 \) means that the admissible maps \( T \) are the push–forwards between the two densities \( \rho_1 \) and \( \rho_2 \), i. e. the \( T \)'s satisfy

\[
\int_{\mathbb{R}} \varphi(y) \rho_2(y) dy = \int_{\mathbb{R}} \varphi(T(y)) \rho_1(y) dy,
\]

for any \( \varphi \in C^0_0(\mathbb{R}) \). The precise definition of the Wasserstein metric comes from a relaxed variational problem. More precisely, the set of admissible maps is embedded into the set of all probability measures \( \mu \) on \( \mathbb{R}^2 \) with marginals given by \( \rho_1 \) and \( \rho_2 \). The quadratic cost defined above is converted into

\[
\int \int_{\mathbb{R}^2} (y_0 - y_1)^2 \mu(dy_0, dy_1).
\]

It turns out that the optimal measure \( \mu^* \), which minimizes the relaxed variational problem, is supported on the graph of a map \( T^* : \mathbb{R} \to \mathbb{R} \), which is exactly the optimal map of the original variational problem (4.2). The relaxed problem above is a version of the Monge–Kantorovich mass transfer problem.

In the one dimensional case, the optimal map \( T^* \) can be expressed in the following simple way. Let us define the distribution functions

\[
F_i(y) = \int_{-\infty}^{y} \rho_i(y) dy \quad i = 1, 2
\]

and their pseudo–inverses \( F_i^{-1} : (0, 1) \to \mathbb{R} \)

\[
F_i^{-1}(\eta) = \inf \{ \omega : F_i(\omega) > \eta \}.
\]

Then, it can be easily proven that the optimal map \( T^* \) between \( \rho_1 \) and \( \rho_2 \) is

\[
T = F_2^{-1} \circ F_1.
\]
Hence, by definition of Wasserstein metric, we have
\[ d_2^2(\rho_1, \rho_2) = \int_{-\infty}^{+\infty} \left( y - (F_2^{-1} \circ F_1)(y) \right)^2 \rho_1(y) dy = \int_{0}^{1} \left( F_1^{-1}(\eta) - F_2^{-1}(\eta) \right)^2 d\eta. \]

(4.3)

Let us now discuss the asymptotic behavior of the Wasserstein metric \( d_2 \) between a solution \( \rho(y, s) \) of equation (4.1) with initial datum \( \rho_0 \in M_2 \) and the stationary solution \( \rho^\infty \) defined in (2.22). To simplify the notation, we denote by \( \rho^\infty \) the stationary state with unit mass. We observe that all the computations below can be generalized to the case of \( \int \rho_0 = M \) for any positive \( M \). As in the previous section, we employ the Hopf–Cole transformation
\[ \psi(y, s) = \frac{1}{2} \rho(y, s) \exp \left( -\frac{1}{2} \int_{-\infty}^{y} \rho(\zeta, s) d\zeta \right), \]
which reduces (4.1) to the linear Fokker–Planck equation
\[
\begin{cases}
\frac{\partial \psi}{\partial s} = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} + y \psi \right) \\
\psi(y, 0) = \psi_0(y) = \frac{1}{2} \rho_0(y) \exp \left( -\frac{1}{2} \int_{-\infty}^{y} \rho_0(\zeta) d\zeta \right).
\end{cases}
\]
(4.5)

We recall that the initial datum \( \psi_0 \) has total mass equal to \( m = 1 - e^{-\frac{1}{2}} \).

Also, we recall the following theorem by Otto (see [Ott01]), giving a result of exponential decay of the Wasserstein metric for the linear Fokker–Planck equation. This result is a special case of a more general one for nonlinear diffusion equations contained in [Ott01].

**Theorem 4.1.** Let \( \psi(y, s) \) be the solution to (4.5) with initial datum \( \psi_0 \in L_1^1(\mathbb{R}), \int \psi_0(y) dy = m, m = 1 - e^{-1/2}. \) Let \( \Psi_m \) be the corresponding gaussian state defined in (2.21). Then the Wasserstein distance \( d_2(\psi, \Psi_m) \) satisfies
\[ d_2^2(\psi(s), \Psi_m) \leq e^{-2s} d_2^2(\psi_0, \Psi_m). \]

(4.6)

We now state our result for equation (4.1).

**Theorem 4.2.** Let \( \rho(y, s) \) be the solution to (4.1), with initial datum \( \rho_0 \in L_1^1(\mathbb{R}), \int \rho_0(y) dy = 1. \) Let \( \rho^\infty \) be the stationary solution
\[ \rho^\infty(y) = \frac{2m}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \]
with \( m = 1 - e^{-1/2}. \) Then the Wasserstein distance \( d_2(\rho(s), \rho^\infty) \) satisfies the exponential decay estimate
\[ d_2^2(\rho(s), \rho^\infty) \leq Ce^{-2s} d_2^2(\rho_0, \rho^\infty), \]
where \( C \) is a fixed constant.
Proof. Let $\rho$ be the solution to (4.1), we define the corresponding $\psi$ by means of the transformation (4.4), i.e.
\[
\psi(y,s) = \frac{1}{2} \rho(y,s) \exp \left( -\frac{1}{2} \int_{-\infty}^{y} \rho(\zeta, s) d\zeta \right).
\] (4.9)

We also recall that
\[
\Psi_m(y) = \frac{1}{2} \rho_\infty(y) \exp \left( -\frac{1}{2} \int_{-\infty}^{y} \rho_\infty(\zeta) d\zeta \right).
\]

Let us now define the distribution functions
\[
F(y,s) = \int_{-\infty}^{y} \rho(\zeta, s) d\zeta,
\]
\[
F_\infty(y) = \int_{-\infty}^{y} \rho_\infty(\zeta) d\zeta,
\]
\[
G(y,s) = \int_{-\infty}^{y} \psi(\zeta, s) d\zeta,
\]
\[
G_\infty(y) = \int_{-\infty}^{y} \Psi_m(\zeta) d\zeta.
\]

Hence, since $\psi$ and $\Psi_m$ satisfy the Fokker–Planck equation, Theorem 4.1 and the representation (4.3) of the Wasserstein metric in one space dimension imply
\[
\int_{0}^{m} (G^{-1}(\eta, s) - G^{-1}_\infty(\eta))^2 d\eta \leq e^{-2s} \int_{0}^{m} (G^{-1}(\eta, 0) - G^{-1}_\infty(\eta))^2 d\eta,
\] (4.10)

where the symbol $^{-1}$ stands for pseudo–inversion. Now, because of (4.9), it is possible to write $F$ in terms of $G$ (and similarly $F_\infty$ in terms of $G_\infty$). We have
\[
G(\eta, s) = \int_{-\infty}^{\eta} \psi(\zeta, s) d\zeta = \int_{-\infty}^{\eta} \frac{1}{2} \rho(\zeta, s) \exp \left( -\frac{1}{2} \int_{-\infty}^{\zeta} \rho(\xi, s) d\xi \right) d\zeta
\]
\[
= - \int_{-\infty}^{\eta} \frac{\partial}{\partial y} \left( \exp \left( -\frac{1}{2} \int_{-\infty}^{\zeta} \rho(\xi, s) d\xi \right) \right) d\zeta
\]
\[
= 1 - \exp \left( -\frac{1}{2} \int_{-\infty}^{\eta} \rho(\zeta, s) d\zeta \right) = 1 - \exp \left( -\frac{1}{2} F(y, s) \right).
\]

We denote by $\alpha : (0,1) \to (0,m)$ the bijective function
\[
\alpha(t) = 1 - e^{-t/2},
\]
which has inverse $\alpha^{-1}(\tau) = -2 \log(1 - \tau)$. Hence, we can write
\[
G = \alpha \circ F \quad G^{-1} \circ \alpha = F^{-1}.
\]
Therefore, by simple change of variable, we obtain

\[
\begin{align*}
\int_0^1 (F^{-1}(\eta, s) - F_{\infty}^{-1}(\eta))^2 \, d\eta &= \int_0^1 (G^{-1}(\alpha(\eta), s) - G_{\infty}^{-1}(\alpha(\eta)))^2 \, d\eta \\
&= \int_0^m (G^{-1}(\xi, s) - G_{\infty}^{-1}(\xi))^2 \frac{2}{1 - \xi} \, d\xi \\
&\leq 2\sqrt{e} \int_0^m (G^{-1}(\xi, s) - G_{\infty}^{-1}(\xi))^2 \, d\xi \\
&\leq 2\sqrt{e} e^{-2s} \int_0^m (G^{-1}(\xi, 0) - G_{\infty}^{-1}(\xi))^2 \, d\xi \\
&= 2\sqrt{e} e^{-2s} \int_0^m (F^{-1}(\alpha^{-1}(\xi), 0) - F_{\infty}^{-1}(\alpha^{-1}(\xi)))^2 \, d\xi \\
&= \sqrt{e} e^{-2s} \int_0^1 (F^{-1}(\eta, 0) - F_{\infty}^{-1}(\eta))^2 \, e^{-\frac{\eta}{2}} \, d\eta \\
&\leq \sqrt{e} e^{-2s} \int_0^1 (F^{-1}(\eta, 0) - F_{\infty}^{-1}(\eta))^2 \, d\eta, 
\end{align*}
\]

which concludes the proof.

\[\Box\]

In the original variables (2.2), the above theorem is converted into a stability result for the Wasserstein metric, as stated in the following corollary.

**Corollary 4.3.** Let \( u \) be the solution of (2.2) with nonnegative initial datum \( u_0 \in L^1(\mathbb{R}) \) with finite second moment. Let \( U_M \) given by (2.8). Hence, there exists a fixed positive constant \( C \) (depending on the mass) such that

\[
d_2(u(t), U_M) \leq Cd_2(u_0, U_M). \quad \text{(4.11)}
\]

We remark that such a result cannot be improved in order to obtain a decay at the level of the original variables, because of the translation–invariance and the representation (4.3) of the Wasserstein metric.

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