Geometric desingularization in slow-fast systems: non-uniform collapse of a folded critical manifold

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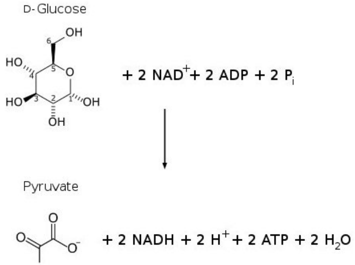
WK Summer Camp, Weissensee 2009

Segel and Goldbeter, 1994

Glycolytic Oscillations Model (GOM)

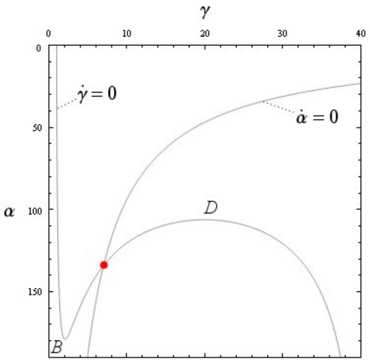
$$\begin{aligned}\dot{\alpha} &= \mu\rho^{-1} - \rho^{-1}\phi(\alpha,\gamma)\\ \dot{\gamma} &= \lambda\phi(\alpha,\gamma) - \gamma\end{aligned}$$

$$\phi(\alpha,\gamma) = \frac{\alpha^2(\gamma+1)^2}{L+\alpha^2(\gamma+1)^2}$$



L. Segel, A. Goldbeter, *Scaling in biochemical kinetics: dissection of a relaxation oscillator*, J. Math. Biol.(1994) 32: 147-160.

Limit cycle of relaxation type



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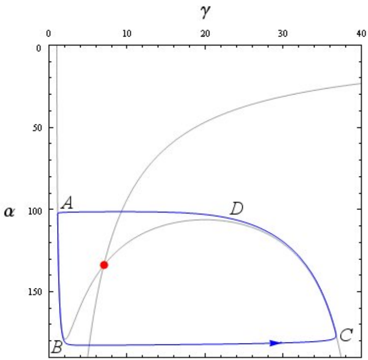

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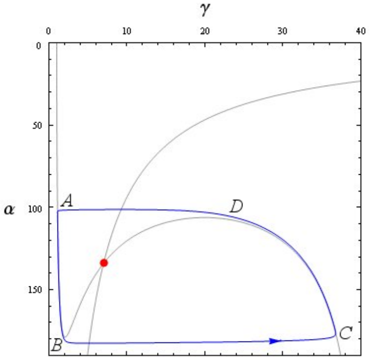


Method of Scaling

Region	α scale	γ scale	t scale
AB	$\sqrt{L/\lambda}$	1	ε^{-1}
BC	$\sqrt{L/\lambda}$	λ	1
CD	$\sqrt{L/\lambda}$	λ	$\mu\varepsilon^{-1}$
DA	$\sqrt{L}/\lambda$	λ	1

Problem

"... scaling approach can be said to have the disadvantage that it relies on an overall intuitive understanding of the underlying phenomenology.

What of more conventional approaches, which are based on the mathematical structure of the equation? "

L. Segel, A. Goldbeter, *Scaling in biochemical kinetics: dissection of a relaxation oscillator*, J. Math. Biol.(1994) 32: 147-160.

Geometric Singular Perturbation Approach

(GOM) in standard form of slow-fast systems

$$\begin{aligned} \dot{a} &= a^2b^2(\mu-1)+\mu\delta^2 \\ \varepsilon\dot{b} &= a^2b^2(1-b)+\delta^2(a^2b^2-b+\delta^2) \end{aligned} \qquad (1)$$

$$(\text{slow time } t, \qquad \dot{} = \frac{d}{dt}, \qquad 0<\varepsilon\ll 1)$$

Fast system equivalent for $\varepsilon > 0$

$$\begin{aligned} a' &= \varepsilon[a^2b^2(\mu-1)+\mu\delta^2] \\ b' &= a^2b^2(1-b)+\delta^2(a^2b^2-b+\delta^2) \end{aligned} \qquad (2)$$

$$(\text{fast time } \tau, \qquad ' = \frac{d}{dt}, \qquad \tau := t/\varepsilon)$$

Slow-fast subsystems for $\varepsilon = 0$

- **the reduced problem** (obtained from (1))

$$\begin{aligned} \dot{a} &= a^2b^2(\mu - 1) + \mu\delta^2 \\ 0 &= a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) \end{aligned} \tag{3}$$

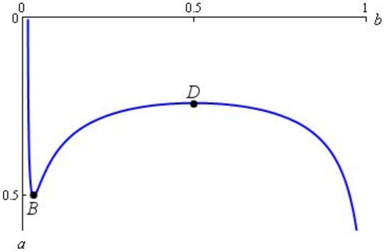
- **the layer problem** (obtained from (2))

$$\begin{aligned} a' &= 0 \\ b' &= a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) \end{aligned} \tag{4}$$

(3) defined on critical manifold S, (4) parametrized by a with equilibria on S

δ fixed: classical relaxation oscillations

$$\varepsilon = 0$$



Critical manifold S

$$a^2b^2(1-b)+\delta^2(a^2b^2-b+\delta^2) = 0$$

δ fixed: classical relaxation oscillations

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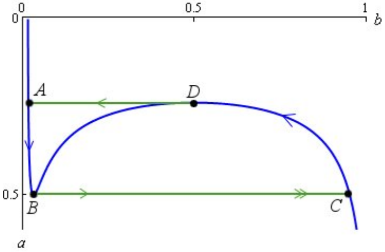

Idea: combine (3) with (4) to understand (1) and (2)

δ fixed: classical relaxation oscillations

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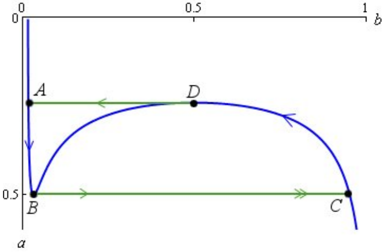
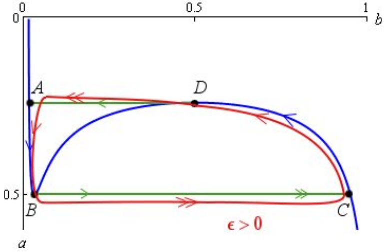


Idea: combine (3) with (4) to understand (1) and (2)

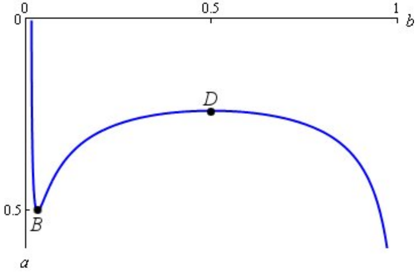
$\varepsilon > 0$



S perturbs to S_ε by Fenichel theory

Critical manifold S^δ

$$S^\delta = \{(a,b): \quad a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) = 0\}$$



$$\delta = 1/8$$

Critical manifold S^δ

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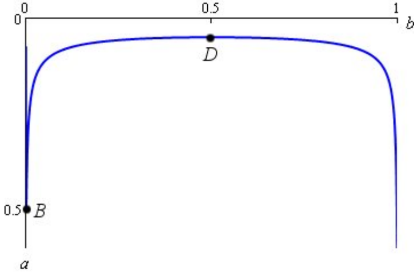

$$\delta = 1/40$$

Critical manifold S^δ

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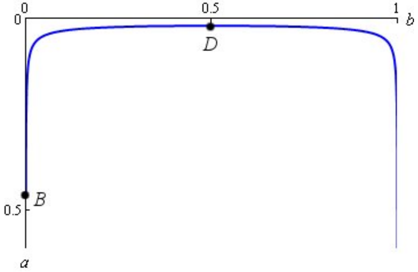
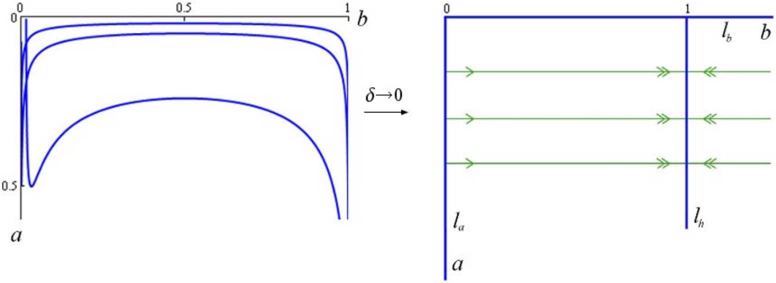


$$\delta = 1/100$$

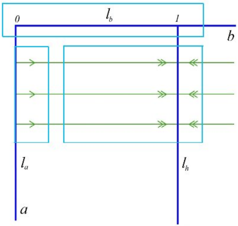
$(\varepsilon, \delta) = (0, 0)$ case



The folded critical manifold S^δ collapses to the more singular "manifold'

$$S^0 = l_a \cup l_b \cup l_h$$

Scaling regimes



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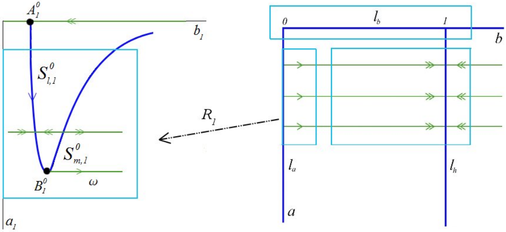

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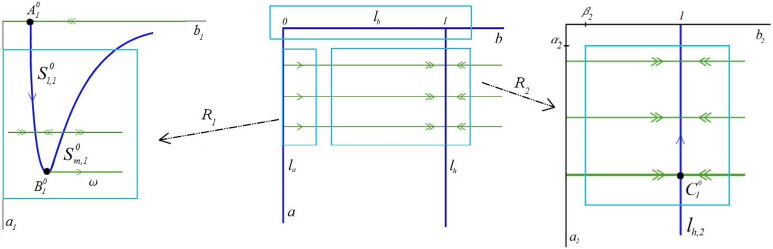

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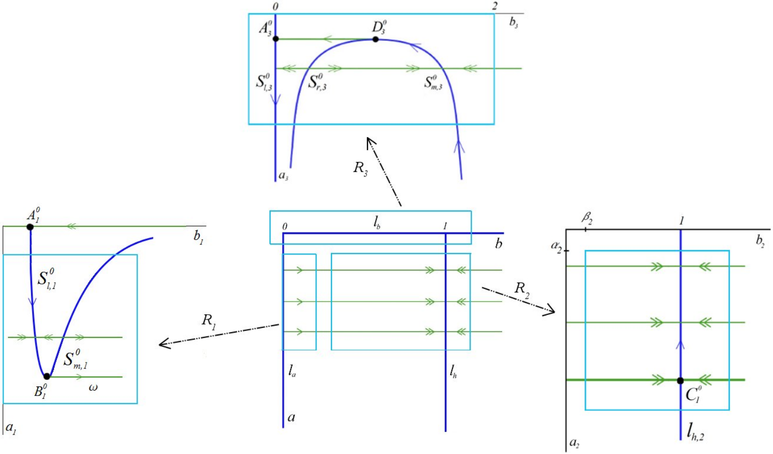
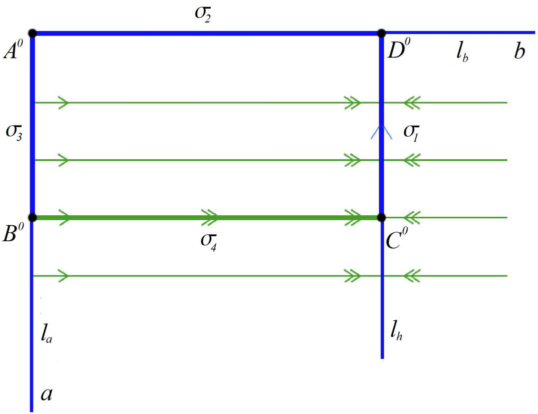


Singular cycle



$$\Gamma_0^0 := \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$$

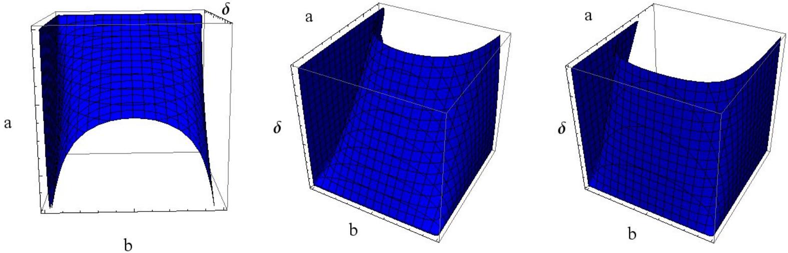
Extended system

$$\begin{aligned} a' &= \tilde{\varepsilon}\delta[a^2b^2(\mu-1)+\mu\delta^2] \\ b' &= a^2b^2(1-b)+\delta^2(a^2b^2-b+\delta^2) \\ \delta' &= 0 \end{aligned}$$

as a three-dimensional vector field $X_{\tilde{\varepsilon}}$ defined on $\mathbb{R}^3$

$\tilde{\varepsilon}$ is the singular perturbation parameter causing the slow-fast structure

Two-dimensional critical manifold S



For $\delta > 0$: $S = S_l \cup S_m \cup S_r$ with folds along the curves $F_B := \{(B^\delta, \delta) : \ \delta \in [0, \delta_0]\}$ and $F_D := \{(D^\delta, \delta) : \ \delta \in [0, \delta_0]\}$

For $\delta = 0$: S limits on $S^0 \times \{0\}$ with $S^0 = l_a \cup l_b \cup l_h$

Strategy

1. Main task
   - analyze the dynamics close to the degenerate lines $l_a \cup \{0\}$ and $l_b \cup \{0\}$
2. How ?
   - perform two blow-ups of the $\delta = 0$ degenerate critical manifold with respect to δ
3. Result
   - complete desingularization of the problem such that uniform results in ε become possible

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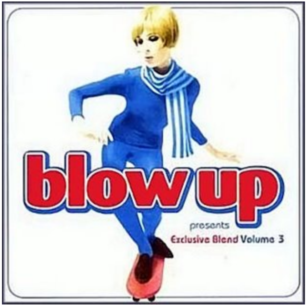
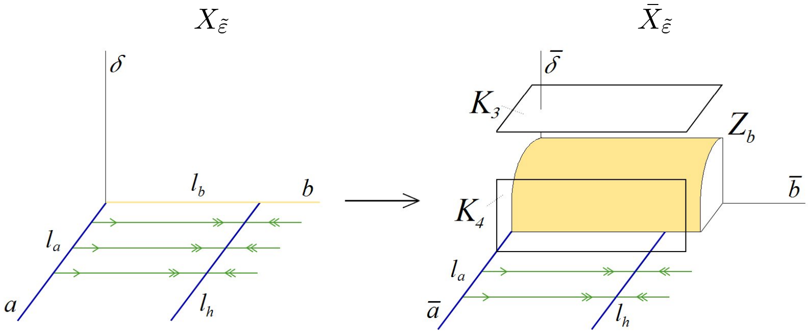
Blow-up presents



Blow-up of the non-hyperbolic line $l_b \times \{0\}$



$$a = r\bar{a}, \qquad b = \bar{b}, \qquad \delta = r\bar{\delta}, \qquad (\bar{a}, \bar{\delta}, r, \bar{b}) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$$

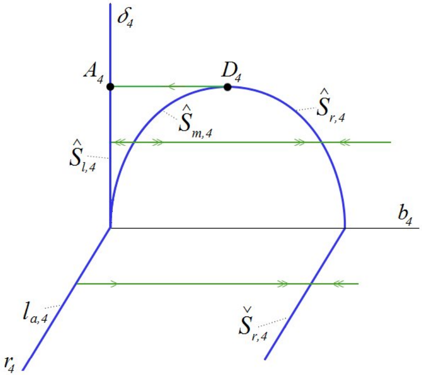
Dynamics in K_4

Slow-fast system

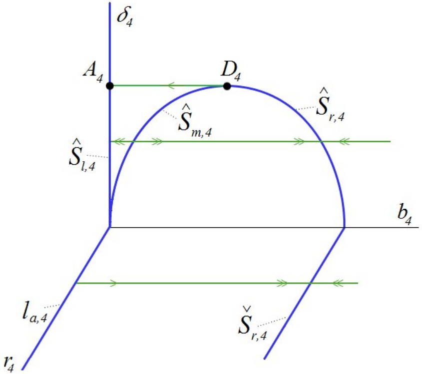
$$\begin{aligned} r_4' &= \tilde{\varepsilon} g_1(b_4, \delta_4, r_4) \\ \delta_4' &= \tilde{\varepsilon} g_2(b_4, \delta_4, r_4) \\ b_4' &= f(b_4, \delta_4, r_4) \end{aligned}$$

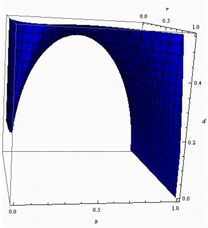
with slow variables r_4, δ_4
and fast variable b_4

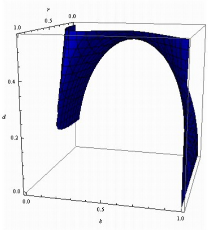
$$0 < \tilde{\varepsilon} \ll 1$$



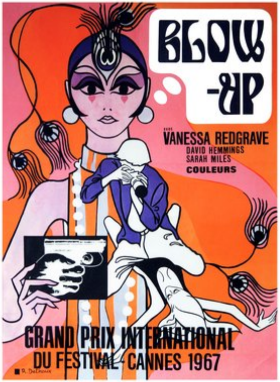
Critical manifold in K_4



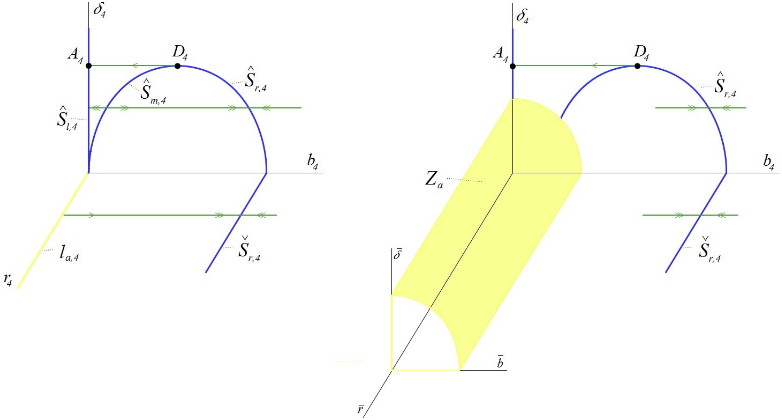




Blow-up II presents

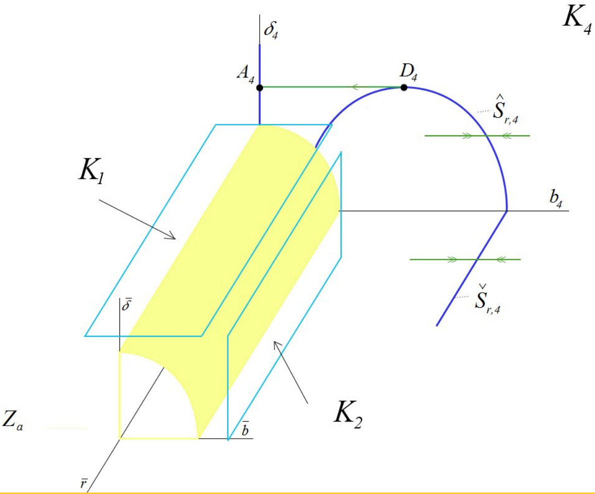


Blow-up of the non-hyperbolic line $l_{a,4}$

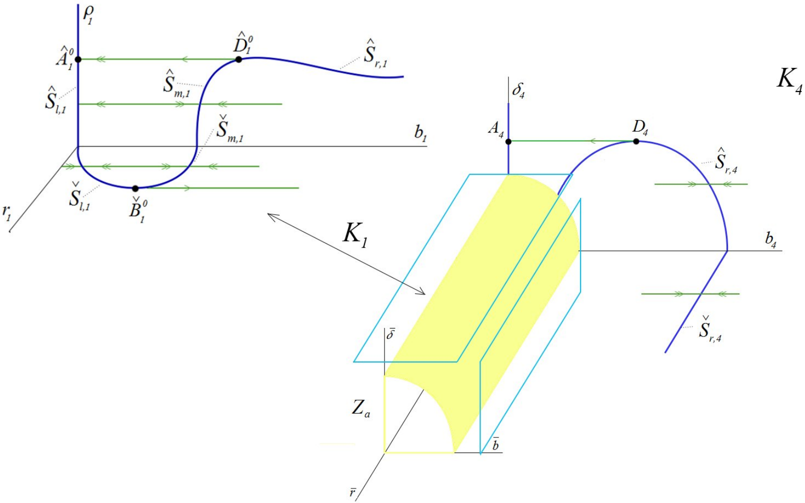


$$r_4 = \bar{r}, \qquad b_4 = \rho^2\bar{b}, \qquad \delta_4 = \rho\bar{\delta}, \qquad (\bar{b}, \bar{\delta}, \rho, \bar{r}) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$$

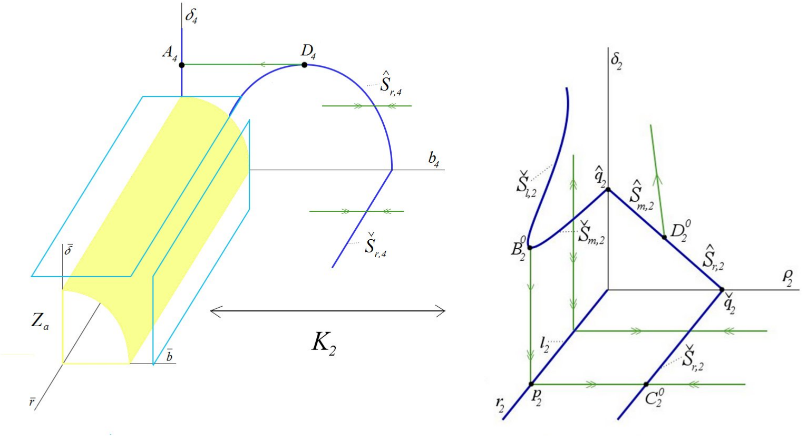
Dynamics in charts



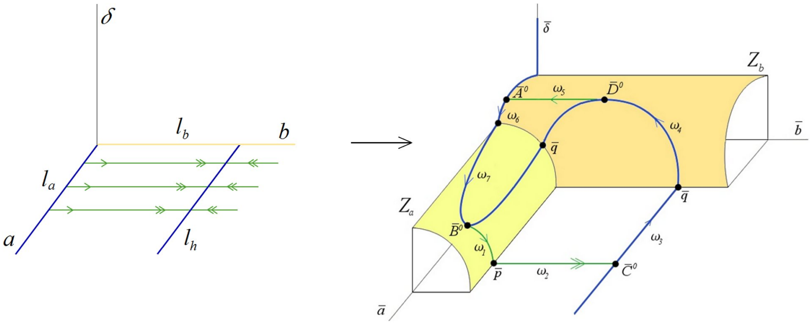
Dynamics in K_1



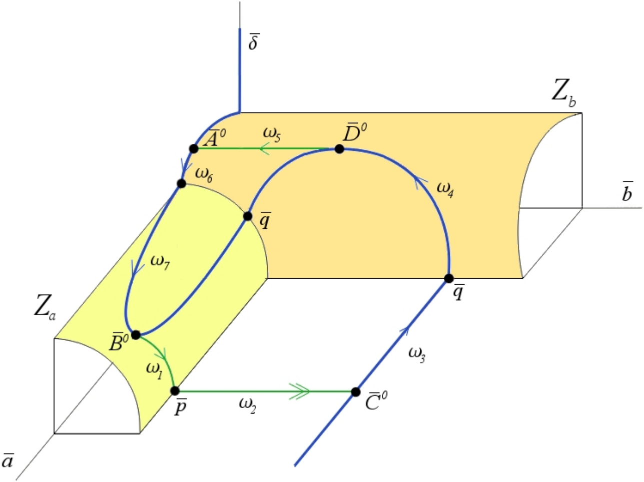
Dynamics in K_2



After two blow-ups...



Dynamics of the blown-up system



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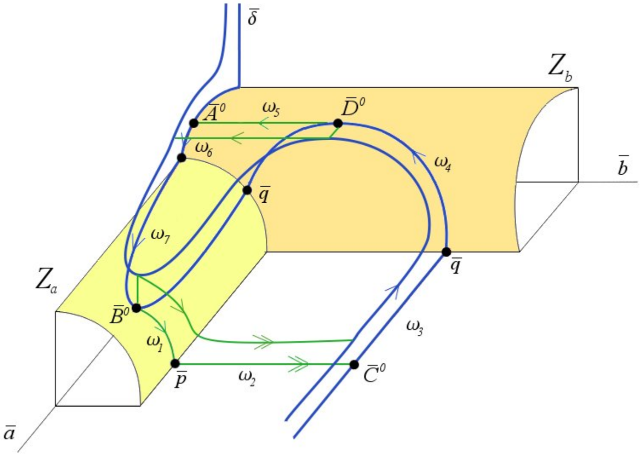

Summary

- singular behavior of the folded critical manifold S^δ for $\delta \to 0$ has been desingularized, its branched structure is visible
- blow-up construction prevents the non-uniform collapse of S^δ onto the degenerate S^0