Second Order Models of Traffic Flow: a Fluid Description of Traffic

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Collaborations with

- Aw, SIAP 2000: initial model, Riemann Pb
- Aw-Klar-Materne, SIAP 2002: Lagrangian view, rigorous derivation from microscopic models
- Greenberg-Klar, SIAP 2003: oscillations in model with ad-hoc relaxation
- Bagnerini, SIMA 2003: homogenized model, existence and uniqueness
- Berthelin-Degond-Delitala, To appear ARMA: formal asymptotic limit = sticky (incompressible) clusters

- Herty, SIMA 2006: Riemann Pb for junctions, imposed proportions between incoming roads
- Herty-Moutari, NHM 2006: Same pb, maximization of flux (priority to more aggressive drivers)
- Moutari, submitted: new hybrid scheme, with Lagrangian interfaces (regularly refreshed)
- Mauser-Moutari-Siebel, submitted: relaxed model with sometimes negative relaxation time
- Lebacque, Leclercq-Lesort (INRETS): comparisons with other Traffic Flow models, work in progress
- ....

Outline

- Introduction
- The fluid model
- Main properties
  - Lagrangian version
  - Godunov scheme
  - BV estimates
  - Scaling
  - Link with Microscopic Models (FLM)
- Continuous description of Junctions
  - Demand, supply
  - Homogenization pbs on outgoing roads ....

Introduction

3 classes of models:

- Fully discrete: ODE system (FLM), see further
- Kinetic
- Fluid:
  First Order: Lighthill-Whitham-Richards

$$\partial_t \rho + \partial_x(\rho v) = 0\ ,\ v = V(\rho)\ ,\ V'(\rho) < 0$$

  Second Order: Payne-Whitham (cf Gas Dynamics)

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0, \\ \partial_t v + v\,\partial_x v = -\rho^{-1} p'(\rho) + \ldots &:= -\tilde{p}'(\rho) + \ldots \end{aligned}$$

The Fluid Model

- Daganzo (Requiem, 95) PW is a terrible model!!
  Paradoxes: 1: $v < 0$ and 2: $\lambda_2 = v + c > v$!!
- Aw-Rascle (Resurrection, 2000), Zhang(2002).
  Fixing: $\partial_x p \longrightarrow \partial_t p + v\partial_x p$ with e.g. (new) $p := \tilde{p}$

$$\partial_t \rho + \partial_x(\rho v) = 0,$$
$$\partial_t w + v\,\partial_x w = 0,$$

with e.g. $w := v + p(\rho) \;=\;$ Lagrangian marker

- "Second order" (fluid) model : mass conservation equation, coupled with a Lagrangian marker (w), with or without influence on the propagation
- In contrast, possible additional Lagrangian markers may be passively advected
- Model by far too sophisticated for real applications ! But ...
- Very general : contains 1rst order models, affectation ... and it forces to clarify : Lagrangian vs Eulerian, discrete vs continuous, why homogenization ?... intellectually quite stimulating ! may be useful ...

The Eulerian System

- $$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0 \end{cases} \tag{1}$$
- in non conservative form, $\partial_t w + v\partial_x w = 0$: $w := v + p(\rho)$ is constant along trajectories
- here $\rho \mapsto p(\rho)$ is a known function (distance to equilibrium) s.t. $p'(\rho) > 0$ and λ_1 is GNL, i.e.

$$\forall \rho, \rho p''(\rho) + 2p'(\rho) \neq 0 \; (> 0 \text{ here})$$

Lagrangian System

- Strictly hyperbolic system, (except for $\rho = 0$ ...)
- Eigenvalues of 2x2 matrix :

$$\lambda_1(U) = v - \rho p'(\rho) \text{ and } \lambda_2(U) = v$$

- λ_1 : genuinely nonlinear:shock (braking) or rarefaction (acceleration),whose curves coincide here, since $[\rho w(v-\sigma)] = ((\rho(v-\sigma)_{\pm}) \cdot [w]$
- λ_2 is linearly degenerate : 2-contact discontinuity.
- Diagonalization: Riemann invariants (on road i) :

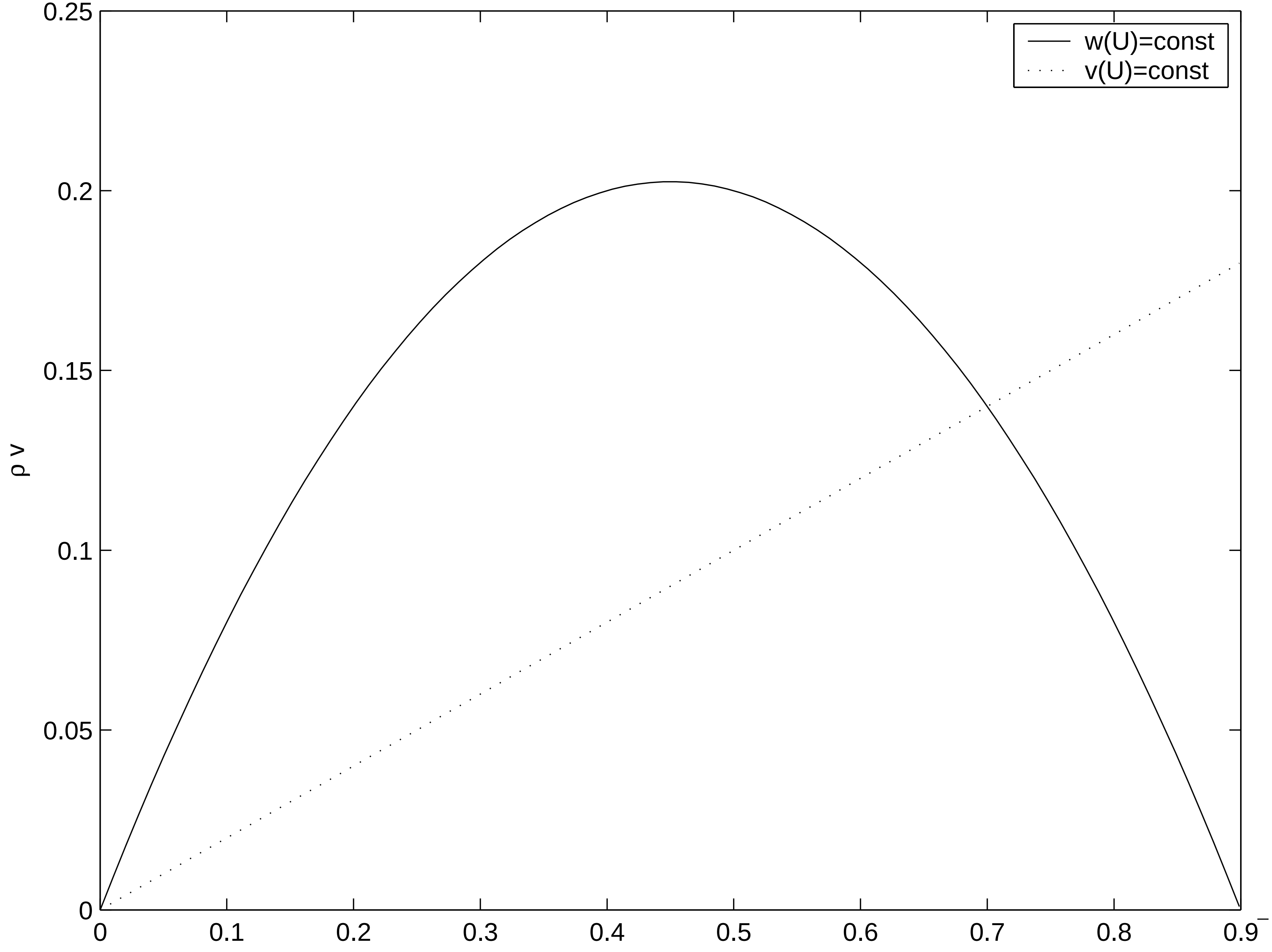
$$w(U) := w_i(U) = v + p_i(\rho) \text{ and } v(U) = v$$

$$\partial_t w + v\,\partial_x w = 0, \;\; \partial_t v + \lambda_{1,i}(U)\partial_x v \approx 0$$

Riemann Problem with initial data U^- and U^+

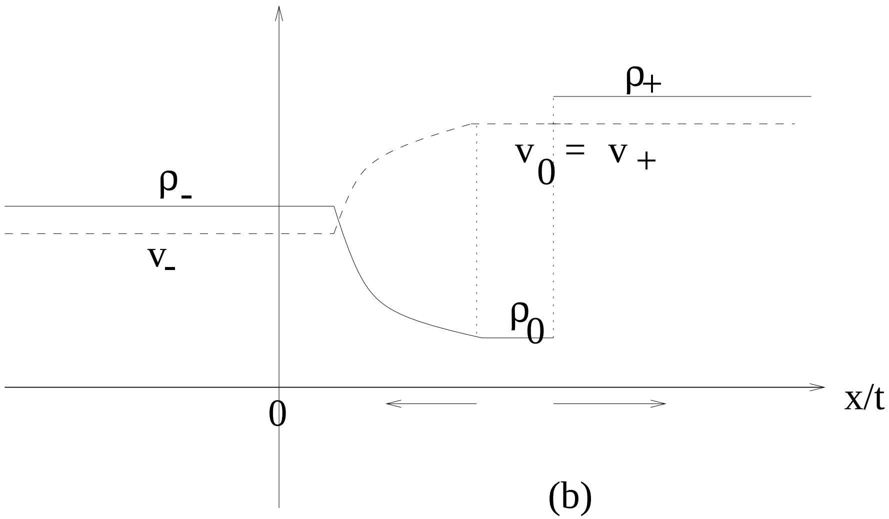
- $1-$ waves between U^- and U:
  - Rarefaction: $w(U) := v + p(\rho) = w(U^-)$, if $v > v^-$
  - or shock: $w(U) := v + p(\rho) = w(U^-)$, if $v > v^-$ (coinciding)
- $2-$ waves between U and U^+: contact discontinuity: $v = v^+$

Solution of Riemann Pb: connect U^- with $U := U^0 := (w^-, v^+)$ through a $1-$wave, then connect U^0 with U^+ through a $2-$wave, see Figure

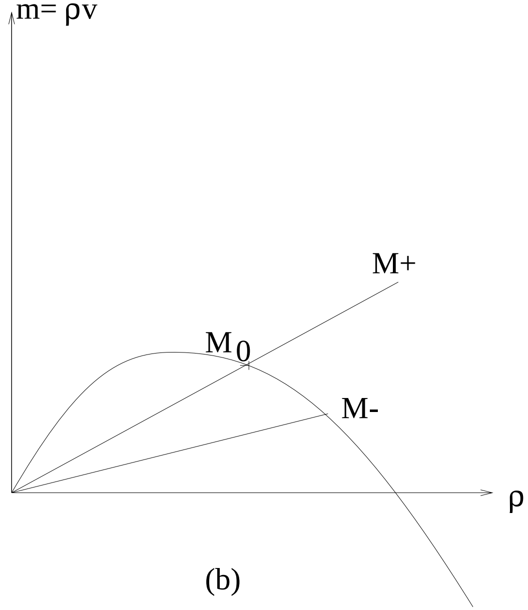


0.25
0.2
0.15
0.1
0.05
0
ρ v
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
w(U)=const
v(U)=const

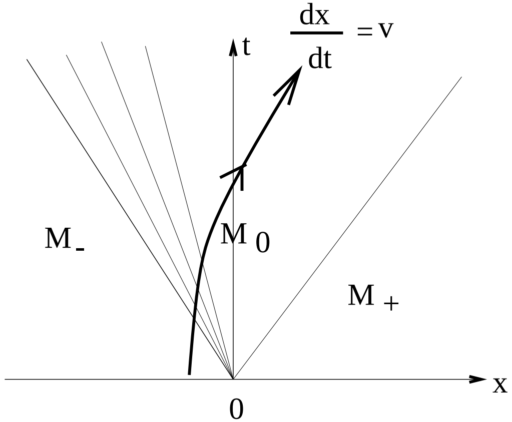
Riemann Pb on single road



(b)



(b)



(b)

Back to a junction

Estimates for the Riemann Pb

Corresponding estimates, see Figure in (w, v) plane:

- L^∞ estimates: bounded invariant (convex) regions

  $$\{0 \le v_{\text{min}} \le v \le v_{\text{max}}\} \cap \{w_{\text{min}} \le w \le w_{\text{max}}\}$$

  and $0 \le v, p(\rho) \le w \le w_{\text{max}}$: Paradox 1 is suppressed
- BV estimates: Set $d(U_1, U_2) := |v_2 - v_1| + |w_2 - w_1|$. Then,

  $$d(U^-, U^+) \le d(U^-, U^0) + d(U^0, U^+),$$

  (equality iff $U_- \to U_0 \to U_+$ in Riemann Pb), due to coinciding shock and rarefaction curves.

Motivation I: x or t dependence

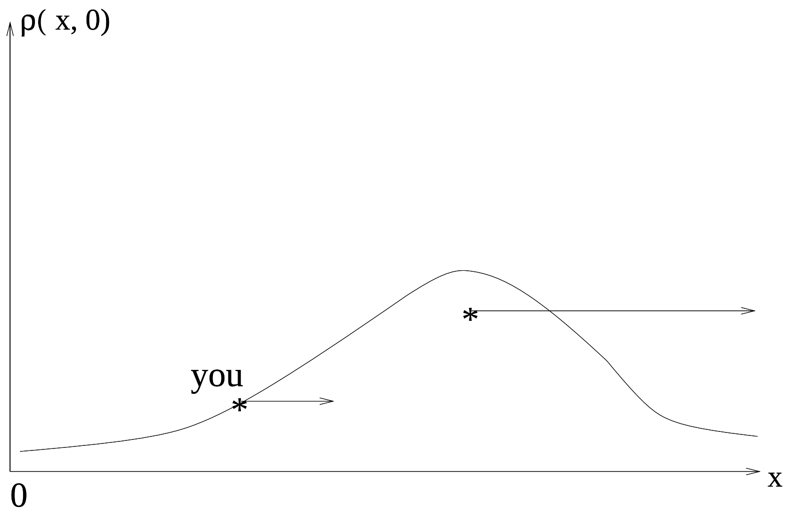
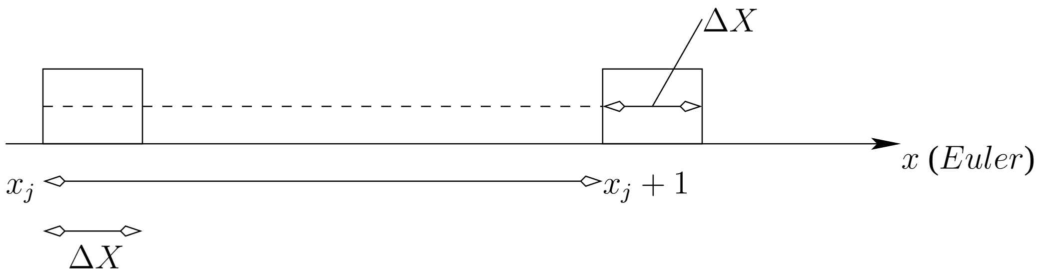


Figure 1: Do we react to flow variations in x or t : if the "wave" is faster than you, should you brake (cf gas dynamics), or accelerate (cf our model)?? Compare: $\partial_t v + v\,\partial_x v = -\rho^{-1}\,\partial_x\,p(\rho)$ or $= -(\partial_t + v\,\partial_x)(\tilde{p}(\rho))$

Motivation II: Lagrangian view

- Lagrangian mass coordinates



- $$X(x,t) = \int_{-\infty}^{x} \rho(y,t)dy;$$

  $$\rho_j = \frac{\Delta X}{x_{j+1} - x_j} \qquad \text{adimensional;}$$

- $$\tau_j := \frac{1}{\rho_j} = \frac{x_{j+1} - x_j}{\Delta X} \qquad \text{adimensional;}$$

Follow The Leader Model

$$\begin{cases} \dot{x}_j = v_j \Longrightarrow \dot{\tau}_j = \frac{v_j+1-v_j}{\Delta X} \\ \\ \dot{v}_j = -P'\left(\frac{x_j+1-x_j}{\Delta X}\right)\frac{v_j+1-v_j}{\Delta X} = -P'(\tau_j)\dot{\tau}_j \end{cases} \tag{2}$$

$$\Updownarrow$$

$$\begin{cases} \dot{\tau}_j = \frac{v_j+1-v_j}{\Delta X} \\ \\ \dot{w}_j == 0 \end{cases} \tag{3}$$

Compare with Godunov

Hyperbolic Scaling

$$(x,t) \to (\epsilon x, \epsilon t)$$

$$(X,t) \to (\epsilon X, \epsilon t)$$

$$\epsilon \to 0 \quad \text{Zoom} \Rightarrow \qquad \Delta X \to 0$$

$$\partial_{t|L} = \partial_{t|E} + v\partial_x$$

ρ, τ, v : scale invariant

The Lagrangian System

To Eulerian System

- (3) formally CV to: (4)

$$\begin{cases} \partial_t \tau - \partial_x v = 0\,, \; \tau := \rho^{-1}\,, \\ \partial_t w = 0\,, \; w = v + P(\tau) := v + p(\rho) \end{cases} \qquad (4)$$

- Rigorous derivation: Aw-Klar-Materne-Rascle (2002): Godunov for (4)$\equiv$ Euler for (2) ...
- Independent, formal M. Zhang (2002)
- First $\exists$ result (no scaling): J. Greenberg (2001), + Relax, (sub)"characteristic" case; Aw, PhD
- Lagrangian system (4) $\Leftrightarrow$ Eulerian system (1) (cf Wagner, 87)

Details on (4)

Details on Lagrangian system

- Example $P(\tau) = p(\rho) = \rho = \tau^{-1}$
- Eigenvalues become: $\lambda_1 = P'(\tau) < 0$ (GNL), and $\lambda_2 = 0$ (LD)
- **Same solution of Riemann Pb:** connect U^- with $U := U^0 := (w^-, v^+)$ through a $1-$wave, then connect U^0 with U^+ through a $2-$wave, see Figure
- For all entropy-flux pairs (η, q), $q \equiv q(v)$, and η convex in τ iff q is concave in v

Back

Lagrangian Godunov Scheme

- $\equiv$ Eulerian Godunov scheme with moving cells (away from vacuum)
- $\Rightarrow w \equiv w(X)$, constant in each cell, remains BV
- Only one wave (of first family) in each cell $(X_{j-1/2}, X_{j+1/2}) \times (t_n, t_{n+1})$ hence v is monotonous in the solution to each Riemann Pb $\Rightarrow$ BV estimates for v are preserved
- Interfaces between cells are trajectories, and $2-$contacts: in Riemann Pb at $(X_{j+1/2}, t_n)$, $v^n_{j+1/2} = v^n_{j+1}$

Godunov scheme

Consequently, $w_j^{n+1} = ... = w_j^0$ and

- in Eulerian moving coordinates, $x_{j+1/2}^{n+1} = x_{j+1/2}^n + \frac{\Delta t}{\Delta X}\, v_{j+1}^n$
- $\Rightarrow$ in Lagrangian coordinates $\tau_j^{n+1} = \frac{x_{j+1/2}^{n+1} - x_{j-1/2}^{n+1}}{\Delta X} = \tau_j^n + \Delta t\, \frac{v_{j+1}^n - v_j^n}{\Delta X}$,
- Godunov scheme is exactly the explicit Euler scheme for (FLM). See (3)

Moreover, (BV estimates), commutation of limits: Godunov CV to Lagrangian system (4) when .... Now Godunov first CV (when ...) to (3), which then CV to (4) when ... Same limit by uniqueness

On a network

- Conservative form on each road:

$$\partial_t \begin{pmatrix} \rho_i \\ y_i \end{pmatrix} + \partial_x \begin{pmatrix} y_i - \rho_i p_i(\rho_i) \\ (y_i - \rho_i p_i(\rho_i)) y_i / \rho_i \end{pmatrix} = 0,$$

  and $y_i = \rho_i w_i = \rho(v_i + p_i(\rho_i))$.
- Rankine-Hugoniot conditions through a junction

$$\begin{aligned} \sum_{i \in \delta^-} (\rho_i v_i)(b_i^-, t) &= \sum_{j \in \delta^+} (\rho_j v_j)(a_j^+, t) \\ \sum_{i \in \delta^-} (\rho_i v_i w_i)(b_i^-, t) &= \sum_{j \in \delta^+} (\rho_j v_j w_j)(a_j^+, t) \end{aligned}$$

Weak (entropy) solution on a network must :

- be a weak (entropy) solution on each road i
- conserve total mass and (pseudo)- "momentum", at junctions $b_i = a_j$ ($\Leftrightarrow$ Rankine-Hugoniot)
- $\forall i \in \delta^-$: incoming, $j \in \delta^-$: outgoing road, unknown limit values U_i^+ at $b_i - 0$ and U_j^- at $a_j + 0$ (Attention !!), to be determined below.
- For Riemann Pb, centered waves have charact speed < 0 on ingoing, > 0 on outgoing.
  RP, single road

Incoming 1/2-Riemann Problem

- We want to connect the left Riemann data $U_i^- = (\rho_i^-, v_i^-)$ through a 1-wave of negative speed to a state

$$U_i^+ = \{w_i(U) := v_i + p_i(\rho) = w_i(U_i^-)\} \cap \{\rho v = q\}$$

- The actual flux q and U_i^+ are still unknown
- Definition (Daganzo, Lebacque): Demand of U_i^- := maximal possible such flux q.
- Recall: Actual $U_i^+ = U_i(b_i - 0, t)$ (in) and $U_j^- = U_j(a_j + 0, t)$ (out) must satisfy (RH)

Demand

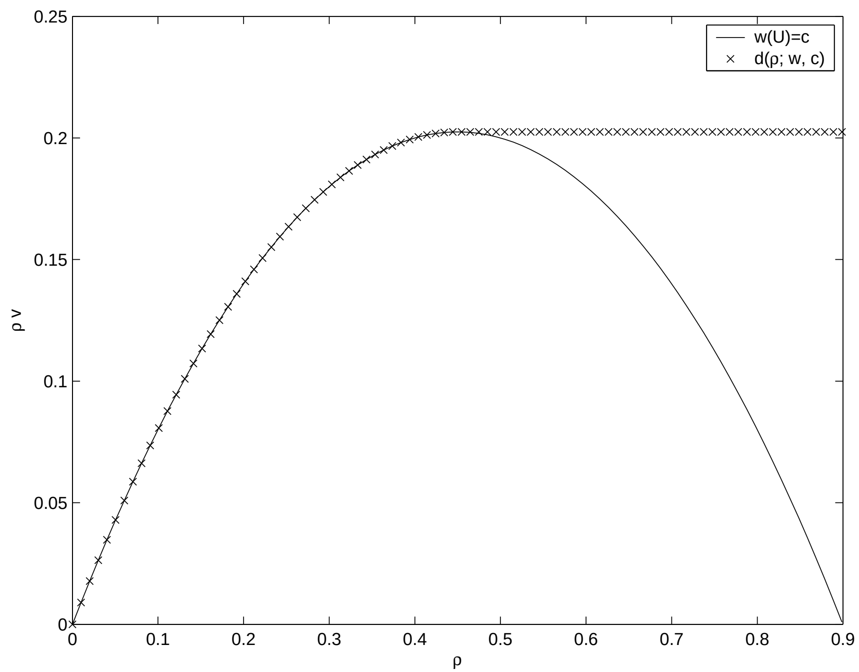


Figure 2: Demand function for $w(U) = v + p(\rho) = C$

Demand: e.g. in uncongested regime, i.e. if $\rho_i^- < \tilde{\rho}$ (the sonic point), then the maximal flux at a point U on curve $w(U) = w(U_i^-)$ which can be connected with U_i^- by a wave of speed ≤ 0 is reached for $U = U_i^-$ itself. Conversely, if $\rho_i^- > \tilde{\rho}$ ....

Outgoing Half Riemann Pb

- First connect right Riemann data $U_j^+ = (\rho_j^+, v_j^+)$ through a 2-contact discontinuity (of speed $v_j^+ > 0$) to a first intermediate state U_j^*, still unknown
- Here, U_j^* comes from road i, but is on road j. Therefore $w_j(U_j^*) := w_j^* := w_i(U_i^-)$!!. So

$$U_j^* = \{w_j(U) := v + p_j(\rho) = w_j^*\} \cap \{v = v_j^+\}$$

- Now, define supply associated with state U_j^* and the above curve, as for the demand:

- we want to connect U_j^* (on the right), through a 1-wave of positive speed to a state

$$U_j^- = \{w_j(U) = w_j^*\} \cap \{\rho_j v = q\},$$

- Definition: Supply of U_j^* $:=$ maximal possible such flux q
- **Full solution of Riemann Pb**: Set $q := \min(d_i(U_i^-), s_j(U_j^*))$
- That defines two (in fact, one (a.e.)) possible state on each road, e.g. $U_i^+ := U_i^-$ if $q = q_i^-$
- The corresponding solution is uniquely defined: $U_i^- ... U_i^+ || U_j^- ... U_j^* ... U_j^+$

Supply

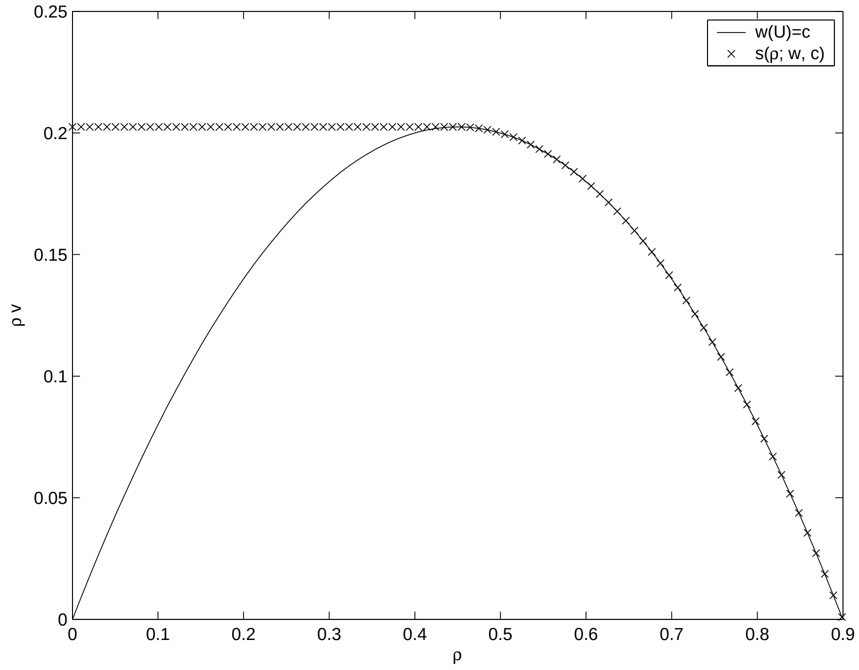
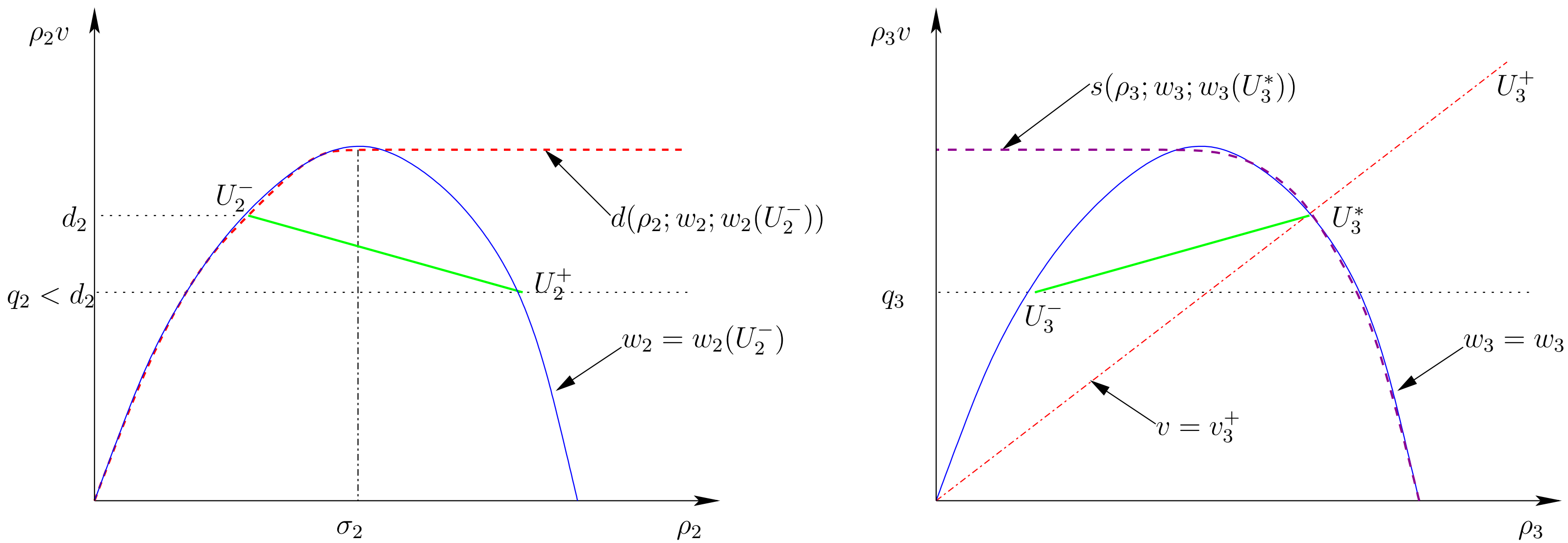


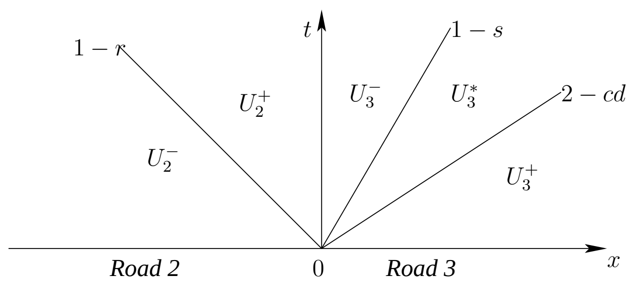
Figure 3: Supply function for $w(U) = v + p(\rho) = C$ e.g. in uncongested regime, i.e. if $\rho_j^* < \tilde{\rho}$ (the sonic point), then the maximal flux at a point U on curve $w(U) = w(U_j^*)$ which can be connected with U_j^* by a wave of nonnegative speed is reached for $U = \tilde{U}$.

Homogenized Supply

1-1 Junction: Example

Ex of solution on (incoming) road 2 : $p = p_2, w = w_2^-$
or on (outgoing) road 3 : $p = p_3, w = v + p_3(\rho) = w_2^-$!!





1/2 Riemann Pb

Toward Homogenization

- Example : two incoming roads 1 and 2, with resp. black and white cars, with equal priority, and one outgoing road 3
- Then cars mix up on road 3, with average grey color

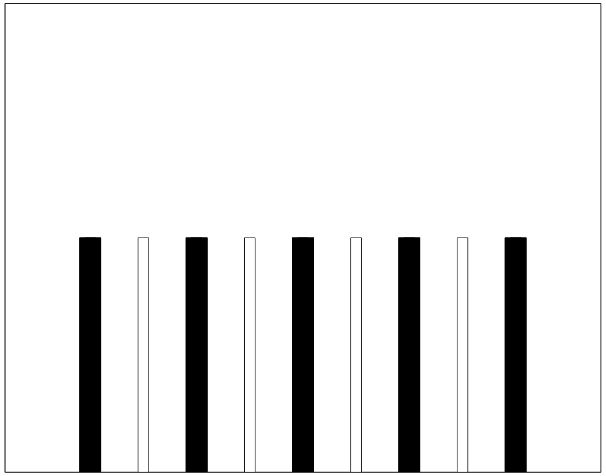


Figure 4: On an outgoing road ...

Figure

Back to Homogenization ...

(with P. Bagnerini)

$$\begin{aligned} \partial_t \tau^\varepsilon - \partial_X v^\varepsilon &= 0 \\ \partial_t w^\varepsilon &= 0 \\ w^\varepsilon = v^\varepsilon + P(\tau^\varepsilon)&, \end{aligned}$$

with initial data (or incoming boundary conditions, at a junction) oscillating in w

$$v_0^\varepsilon \longrightarrow v_0^* \, ; \, w_0^\varepsilon \rightharpoonup w_0^* := w^*$$

Corresponding solution : $v^* =$ strong limit of v^ε : no oscillation, v^* is a BV-function, master unknown function ...

- But oscillations are preserved for $w^\varepsilon \equiv w_0^\varepsilon$ and for any function $F(v^\varepsilon, w^\varepsilon)$
- For any F, weak limit (= "average") is described by Young measure :
  $< \nu_{X,t},\ F(v,w) >:= \int F(v,w) d\nu_{X,t}(v,w)$
- Since v^ε strongly converges, and since w is time-independent, the above integral equals
  $< \mu_X,\ F(v^*(X,t),w) >:=$
  $\int F(v^*(X,t),w) d\mu_X(w)$
- ν and μ : probability measures, resp. in v, w and in w

2-1Junction

- The homogenized w is: $w^* :=< \mu_X, w >$
- Since $P(\tau) := p(\tau^{-1})$ is strictly monotonous, $\tau = P^{-1}(w(X) - v)$. Therefore the homogenized τ, i.e.

$$\begin{cases} \tau^*(X,t) =< \mu_X,\ P^{-1}(w - v^*(X,t)) >, \\ := T(X, v^*(X,t)) \end{cases} \tag{5}$$

satisfies:

$$\partial_t T(X, v^*(X,t)) - \partial_X v^* = 0 \ldots \tag{6}$$

- ... In (6), τ^* is naturally a function of v^* and X. We could invert again the roles and write

$$\partial_t \tau^* - \partial_X F(X, \tau*) = 0, \tag{7}$$

  scalar conservation law whose flux is discontinuous in X

- Integration in X of (7) $\longrightarrow$ Hamilton-Jacobi equation. See Lions-Papanicolaou-Varadhan for homogenization in periodic case.

Details on Homog System

- Here, by monotonicity, we deal directly with (6), and use informations on Lagrangian system (4)
  - any (convex) entropy $\eta \leftrightarrow$ (concave) flux $q(v)$
  - Def: v^* is an entropy solution to (5) if $\forall k$,

$$\partial_t |T(X, v^*(X,t)) - T(X,k)| - \partial_X |v^*(X,t) - k| \leq 0,$$

which is equivalent **(!!)** to

$$\partial_t < \mu_X, |P^{-1}(w - v^*(X,t)) - P^{-1}(w-k)| > - \partial_X |v^*(X,t) - k| \leq 0!!$$

  - Theorem: $\exists$ a unique entropy solution to (5)

Example: 2-1 Junction

- Here, incoming roads 1 and 2 merge on outgoing road 3, with $\%\ \ \alpha$ and $1-\alpha$.
- Typically, if e.g. $\alpha = 1/2$, then $\forall F(v,w)$, $<\mu_X,\ F(v,w)>=\ \frac{1}{2}(F(v,w_1^-)+\ F(v,w_2^-))$
- Again, since $P_3(\tau) := p_3(\tau^{-1})$ is monotonous, the homogenized τ is (for $\alpha = 1/2$):

$$\tau^*(X,t) =<\mu_X,\ P_3^{-1}(w-v^*(X,t))>$$
$$:= \frac{1}{2}(P_3^{-1}(w_1 - v^*(X,t)) +\ P_3^{-1}(w_2 - v^*(X,t)))$$

- For any given v, the 1 and 2-drivers share the spacing, see Figure.

Homog.

Therefore, on outgoing road 3

- On outgoing road 3, (for $\alpha = 1/2$), set $w\dagger := w^* = (w_1 + w_2)/2$, rewrite homogenized τ as

$$\tau^* := \frac{1}{2}(P^{-1}(w_1 - v^*(X,t)) + P^{-1}(w_2 - v^*(X,t)))$$
$$= T(X, v^*(X,t)) \text{ , here, } P = P_3$$

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$$\tau^* := \frac{1}{2}(P^{-1}(w_1 - v^*(X,t)) + P^{-1}(w_2 - v^*(X,t))) = T(X, v^*(X,t)) \text{ , here, } P = P_3$$

- That defines a unique homogenized $\tau := \tau^*$ as a function of $v := v^*$, hence a curve in $(\rho, \rho v)$ plane

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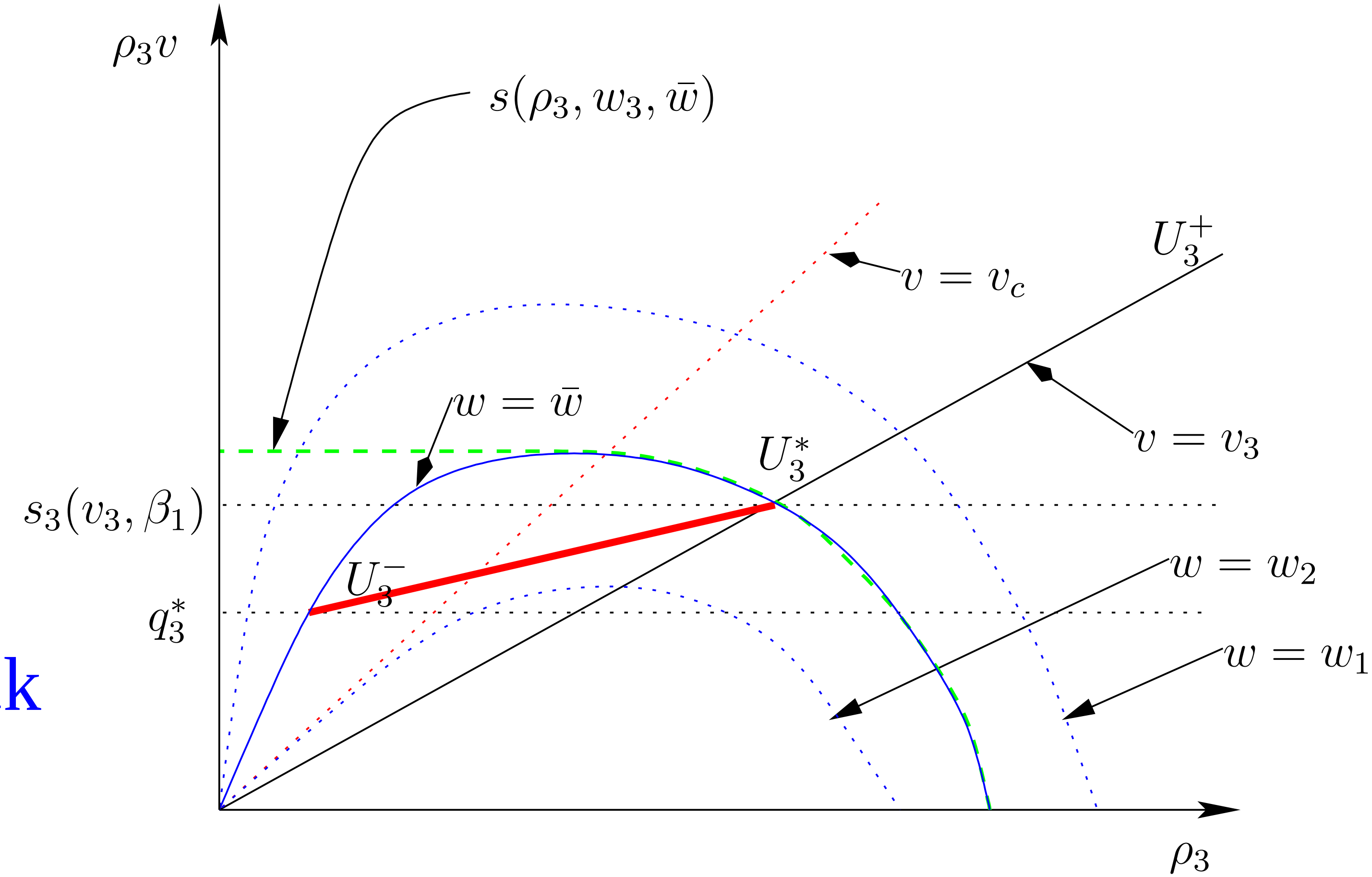
- That defines a unique homogenized $\tau := \tau^*$ as a function of $v := v^*$, hence a curve in $(\rho, \rho v)$ plane
- define the corresponding supply, see Figure below.

Supply

Homogenized supply

The supply on road 3 corresponds to this curve and to the unique point U_3^* on this curve with velocity v_3^+, with $p = p_3^*, w = w^*$



Back

Conclusion

- If several incoming roads and if influence of w on the propagation, then homogenization is needed
- Other ingredients are unchanged, e.g. here, in a $2-1$ junction with equal fluxes, compare d_1, d_2 (incoming) to (homogenized) outgoing $\frac{1}{2}\, s_3$
- Similar arguments in general case with imposed ratios between incoming fluxes,
- But maximization of flux must be restricted to states which also conserve each color !!
- Calculations not that complicated, like for two resistances in parallel, and can be strongly simplified (work in progress) ...