

# The nonlinear Schrödinger equation with a strongly anisotropic harmonic potential

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## Abstract

The nonlinear Schrödinger equation with general nonlinearity of polynomial growth and harmonic confining potential is considered. More precisely, the confining potential is strongly anisotropic, i.e., the trap frequencies in different directions are of different orders of magnitude. The limit as the ratio of trap frequencies tends to zero is carried out. A concentration of mass on the ground state of the dominating harmonic oscillator is shown to be propagated, and the lower dimensional modulation wave function again satisfies a nonlinear Schrödinger equation. The main tools of the analysis are energy and Strichartz estimates as well as two anisotropic Sobolev inequalities. As an application, the dimension reduction of the 3-dimensional Gross-Pitaevskii equation is discussed, which models the dynamics of Bose-Einstein condensates.

**Key words:** Energy estimates, Strichartz estimates, anisotropic Sobolev embedding, Gross-Pitaevskii equation, Bose-Einstein condensate.

## 1 Introduction and Main Result

The major goal of the present work is the analysis of the space dimension reduction of the  $n + d$  dimensional nonlinear Schrödinger equation with external confining potential. We are interested in the case where the external potential is anisotropic and strongly confining in  $d$  directions. This work follows the approach used in [BMP] for analyzing a dimension reduction (from dimension 3 to dimension 2) for the Schrödinger-Poisson system and where an asymptotics for strong partial confinement

was introduced. In other words, we deal with the asymptotic behavior of solutions of the  $n + d$ -dimensional Schrödinger equation

$$i\psi_t = -\frac{1}{2}\Delta\psi + V^\varepsilon(x, z)\psi + f(\delta|\psi|)\psi, \quad (1)$$

$$\psi_I(x, z) = \psi(0, x, z), \quad (2)$$

$$V^\varepsilon(x, z) = \frac{|x|^2}{2} + \frac{|z|^2}{2\varepsilon^4}, \quad x \in \mathbb{R}^n, \quad z \in \mathbb{R}^d,$$

as  $\varepsilon \rightarrow 0$ . Here  $V^\varepsilon$  is the trapping harmonic potential and  $\Delta = \Delta_x + \Delta_z$  is the Laplacian in  $\mathbb{R}^{n+d}$  and  $\psi$  satisfies the normalization condition

$$\int_{\mathbb{R}^{n+d}} |\psi|^2 dx dz = 1, \quad (3)$$

which is preserved by (1). Since the sign of the function  $f$  is not specified, we are dealing with both focusing and defocusing nonlinearities. Performing the limit  $\varepsilon \rightarrow 0$  in this system will enable us to write a reduced model involving a nonlinear Schrödinger equation in dimension  $n$ . In Section 4, an application to the dynamics of Bose-Einstein condensates is presented; we justify mathematically the effective models which can be found in the physics literature [LP]. In this context, as it was remarked in [BJM], the use of such approximate models significantly reduces the complexity of numerical simulations.

In order to balance the kinetic and potential energy terms in the  $z$ -direction, we introduce the rescaling  $z \rightarrow \varepsilon z$ . In order to keep the wavefunction normalized we have to rescale by  $\psi \rightarrow \varepsilon^{-d/2}\psi^\varepsilon(t, x, z)$ . As we want to balance the nonlinearity with the terms of order 1 we choose  $\delta = \varepsilon^{d/2}$ , thus we consider weak nonlinearities. The rescaled problem reads

$$\begin{aligned} i\psi_t^\varepsilon &= H^\perp \psi^\varepsilon + \frac{1}{\varepsilon^2} H \psi^\varepsilon + f(|\psi^\varepsilon|)\psi^\varepsilon, \\ \psi^\varepsilon(t=0, x, z) &= \psi_I(x, z), \end{aligned} \quad (4)$$

with  $H^\perp = -\frac{1}{2}\Delta_x + \frac{|x|^2}{2}$  and  $H = -\frac{1}{2}\Delta_z + \frac{|z|^2}{2}$ , harmonic oscillators in the  $x$ - and  $z$ -directions, respectively.

We introduce a new time scale  $\tau = t/\varepsilon^2$ , so that we have the fast oscillations in  $z$  corresponding to the fast time scale  $\tau$ . If we let  $\varepsilon \rightarrow 0$  we formally obtain the equation

$$i\Psi_\tau = H\Psi,$$

which we can solve explicitly in terms of the spectral decomposition of  $H$ :

$$\Psi = \sum_{k \geq 0} \phi_k e^{-i\mu_k \tau} \omega_k(z).$$

Here  $(\mu_k, \omega_k(z))_{k \geq 0}$  are the eigenvalues and normalized (with respect to  $L^2(\mathbb{R}^d)$ ) eigenfunctions of  $H$ , and  $(\phi_k)_{(k \geq 0)}$  are coefficients independent of  $\tau$  and  $z$ . The

eigenvalue problem can be solved explicitly with the eigenvalues  $\mu_k = k + \frac{d}{2}$  (see [Te] Theorem 7.3). The eigenfunctions are products of a Gaussian with Hermite polynomials, and, in particular, the ground state eigenfunction is given by

$$\omega_0(z) = \left(\frac{1}{\pi}\right)^{d/4} e^{-\frac{|z|^2}{2}}.$$

By modulation, thus introducing the slow variables  $x$  and  $t$  we would have  $\phi_k$  depending on  $(t, x)$ . This motivates us to expand  $\psi^\varepsilon$  with respect to the eigenstates of  $H$ :

$$\psi^\varepsilon(t, x, z) = \sum_{k \geq 0} e^{-i\mu_k t/\varepsilon^2} \phi_k^\varepsilon(t, x) \omega_k(z). \quad (5)$$

Actually, our aim is to determine and justify approximations of the form

$$\psi^\varepsilon(t, x, z) \approx \varphi(t, x) e^{-i\mu_0 t/\varepsilon^2} \omega_0(z), \quad (6)$$

i.e., modulations of the ground state, under an assumption of well-prepared initial data (see (11) below). A formal analysis indicates that the general case, where the transport occurs on several modes, is more complicated and might involve coupling terms between the limiting  $n$ -dimensional Schrödinger equations (this is not the case for the Schrödinger-Poisson system [BMP], where the nonlinearity is weaker).

The projection  $\Pi$  onto the eigenspace generated by the groundstate  $\omega_0(z)$  is given by

$$\Pi \psi^\varepsilon(t, x, z) = e^{-i\mu_0 t/\varepsilon^2} \phi^\varepsilon(t, x) \omega_0(z)$$

with

$$\phi^\varepsilon(x, t) := e^{i\mu_0 t/\varepsilon^2} \int_{\mathbb{R}^d} \psi^\varepsilon(t, x, z) \omega_0(z) dz. \quad (7)$$

It is obvious that the projection has the following properties:

$$\partial_t \Pi = \Pi \partial_t, \quad \Pi H^\perp = H^\perp \Pi, \quad \Pi H = \mu_0 \Pi.$$

By projecting the equation (4) we obtain

$$i\phi_t^\varepsilon = H^\perp \phi^\varepsilon + e^{i\mu_0 t/\varepsilon^2} \int_{\mathbb{R}^d} f(|\psi^\varepsilon|) \psi^\varepsilon \omega_0 dz. \quad (8)$$

The nonlinearity can be written as

$$\begin{aligned} e^{i\mu_0 t/\varepsilon^2} \int_{\mathbb{R}^d} f(|\psi^\varepsilon|) \psi^\varepsilon \omega_0 dz &= \bar{f}(|\phi^\varepsilon|) \phi^\varepsilon + h^\varepsilon \\ \text{with } \bar{f}(|\phi|) &= \int_{\mathbb{R}^d} f(|\phi| \omega_0) \omega_0^2 dz \\ \text{and } h^\varepsilon &= e^{i\mu_0 t/\varepsilon^2} \int_{\mathbb{R}^d} [f(|\psi^\varepsilon|) \psi^\varepsilon \omega_0 - f(|\phi^\varepsilon| \omega_0) e^{-i\mu_0 t/\varepsilon^2} \phi^\varepsilon \omega_0^2] dz. \end{aligned} \quad (9)$$

Then the formal limit of (8) as  $\varepsilon \rightarrow 0$  is the  $n$ -dimensional Schrödinger equation

$$i\varphi_t = H^\perp \varphi + \bar{f}(|\varphi|)\varphi. \quad (10)$$

When the initial data for the full problem (4) are chosen compatible with the ansatz (6), i.e.,

$$\psi_I(x, z) = \varphi_I(x)\omega_0(z), \quad (11)$$

then appropriate initial conditions for the solution of (10) are

$$\varphi(0, x) = \varphi_I(x). \quad (12)$$

The main result of this work is a justification of the limit problem (10), (12) under the following assumptions on the initial data and on the nonlinearity:

**Assumption 1.** *The function  $\varphi_I$  satisfies*

$$\int_{\mathbb{R}^n} (|\nabla_x \varphi_I(x)|^2 + |x\varphi_I(x)|^2) dx < \infty, \quad \int_{\mathbb{R}^n} |\varphi_I(x)|^2 dx = 1.$$

**Assumption 2.** *The nonlinearity  $f$  satisfies*

$$|f(|u|)u - f(|v|)v| \leq C(|u|^\alpha + |v|^\alpha)|u - v|,$$

where either  $f \geq 0$  (defocusing case) and  $0 \leq \alpha < \frac{4}{n+d-2}$ , or  $0 \leq \alpha < \min\{\frac{4}{n+d-2}, \frac{4}{n}\}$ . Additionally,  $\alpha \leq \frac{2}{n-2}$  if  $n > 2$ .

**Remark** The assumptions are sufficient for proving existence and uniqueness of local solutions of both the full problem (4), (11) and the limit problem (10), (12) (see [Caz, Oh, Car2]). Note that the property of  $f$  required in Assumption 2 carries over to  $\bar{f}$ . In the repulsive case, global existence is a straightforward consequence of energy conservation (see next section). Without sign assumptions on the nonlinearity, the additional requirement  $\alpha < 4/n$  leads to global solvability of the limit problem [Oh]. Here, however, it is used for proving  $\varepsilon$ -independent estimates for the full problem on finite time intervals.

**Theorem 3.** *Let Assumptions 1 and 2 be satisfied and let  $\psi^\varepsilon$  and  $\varphi$  be the unique solutions of (4), (11) and (10), (12), respectively. Then for every  $T < \infty$  there exists a constant  $c_T$  such that*

$$\sup_{t \in (0, T)} \|\psi^\varepsilon(t, \cdot, \cdot) - e^{-i\mu_0 t/\varepsilon^2} \varphi(t, \cdot)\omega_0\|_{L^2(\mathbb{R}^{n+d})} \leq c_T \varepsilon.$$

The rest of the paper is organized as follows. In the following section, conservation of energy is used to derive uniform estimates of  $H^1$ -norms of the solution of the  $(n+d)$ -dimensional problem and its ground state contribution. Whereas for repulsive nonlinearities these results follow directly from the energy conservation, in the general case the nonlinearity needs to be controlled by an anisotropic generalization of the Gagliardo-Nirenberg inequality. Also the difference between the full solution and its

projection to the ground state is shown to be small. In Section 3, the difference between the ground state contribution and its formal limit is estimated. The main tools are Strichartz estimates [Caz, GV, S] and an anisotropic Sobolev inequality.

Section 4 deals with an application, the Gross-Pitaevskii equation, which has a cubic nonlinearity and models the dynamics of Bose-Einstein condensates. In this case, dimension reduction means to obtain disk-shaped or cigar-shaped condensates. Finally, in the Appendix the anisotropic Sobolev embedding and the anisotropic Gagliardo-Nirenberg inequality are proved.

## 2 Uniform estimates

In this section we derive some  $\varepsilon$ -independent estimates from energy conservation. The energy is defined by

$$E^\varepsilon[\psi^\varepsilon(t)] := \langle H^\perp \psi^\varepsilon(t), \psi^\varepsilon(t) \rangle + \frac{1}{\varepsilon^2} \langle H \psi^\varepsilon(t), \psi^\varepsilon(t) \rangle + 2\mathcal{F}[\psi^\varepsilon(t)],$$

where  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $L^2(\mathbb{R}^{n+d})$  and

$$\mathcal{F}[\psi] = \int_{\mathbb{R}^{n+d}} F(|\psi|) dx dz, \quad \text{with } F(s) = \int_0^s f(\sigma) \sigma d\sigma.$$

Note that the first two terms in the energy are nonnegative quadratic forms controlling the  $H^1$ -norms in the  $x$ - and  $z$ -directions, respectively.

With Assumption 1, the initial data (11) satisfy  $E^\varepsilon[\psi_I] < \infty$  for fixed  $\varepsilon$ . From [Caz] (Theorem 9.2.5 and Remark 9.2.7) and Assumption 2 we obtain local in time existence for the  $n + d$ -dimensional problem (4) as well as energy and mass conservation:

$$E^\varepsilon[\psi^\varepsilon(t)] = E^\varepsilon[\psi_I], \quad \|\psi^\varepsilon(t)\|_{2,2} = \|\psi_I\|_{2,2} = \|\varphi_I\|_2. \quad (13)$$

Considering the limit of  $E^\varepsilon$  when  $\varepsilon \rightarrow 0$ , we immediately obtain uniform bounds for the dominant term. The main difficulty consists in finding uniform bounds on  $\langle H^\perp \psi^\varepsilon(t), \psi^\varepsilon(t) \rangle$ . Once we have this, we can derive uniform bounds on the  $H^1$ -norm of  $\psi(t)$ .

For the notation of norms we use the following conventions:

**Definition 4.** *Let  $0 < T \leq \infty$ ,  $1 \leq p, q, r \leq \infty$ , and  $u(t, x)$ ,  $v(t, x, z)$  functions of  $t \in (0, T)$ ,  $x \in \mathbb{R}^n$ , and  $z \in \mathbb{R}^d$ . Then we define the norms*

$$\begin{aligned} \|u(t)\|_p &:= \|u(t, \cdot)\|_{L^p(\mathbb{R}^n)}, \\ \|u\|_{r(p)} &:= \left\| \|u(\cdot)\|_p \right\|_{L^r((0, T))}, \\ \|v(t)\|_{q,p} &:= \left\| \|v(t, \cdot)\|_p \right\|_{L^q(\mathbb{R}^d)}, \\ \|v\|_{r(q,p)} &:= \left\| \|v(\cdot)\|_{q,p} \right\|_{L^r((0, T))}, \end{aligned}$$

and the corresponding Banach spaces are denoted by  $L_x^p$ ,  $L_t^r L_x^p$ ,  $L_z^q L_x^p$ , and  $L_t^r L_z^q L_x^p$ .

Taking into account the expansion (5) of the  $(n + d)$ -dimensional wavefunction  $\psi^\varepsilon$  with respect to the orthonormal basis  $(\omega_k)_{k \geq 0}$  of eigenfunctions gives

$$\|\psi^\varepsilon(t)\|_{2,2}^2 = \sum_{k=0}^{\infty} \|\phi_k^\varepsilon(t)\|_2^2 \quad (14)$$

$$\|\nabla_x \psi^\varepsilon(t)\|_{2,2}^2 = \sum_{k=0}^{\infty} \|\nabla_x \phi_k^\varepsilon(t)\|_2^2 \quad (15)$$

At first sight, the energy equation seems to be of limited use, since it is dominated by the contributions in the  $z$ -direction. However, with the mass conservation this part can be written as

$$\begin{aligned} \langle H\psi^\varepsilon(t), \psi^\varepsilon(t) \rangle &= \sum_{k=0}^{\infty} \mu_k \|\phi_k^\varepsilon(t)\|_2^2 \\ &= \sum_{k=1}^{\infty} (\mu_k - \mu_0) \|\phi_k^\varepsilon(t)\|_2^2 + \mu_0 \|\varphi_I\|_2^2, \end{aligned} \quad (16)$$

and, on the other hand

$$\langle H\psi_I, \psi_I \rangle = \mu_0 \|\varphi_I\|_2^2. \quad (17)$$

By using (16) and (17) we can rewrite the energy conservation as

$$\begin{aligned} \langle H^\perp \psi^\varepsilon(t), \psi^\varepsilon(t) \rangle + \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} (\mu_k - \mu_0) \|\phi_k^\varepsilon(t)\|_2^2 + 2\mathcal{F}[\psi^\varepsilon(t)] \\ = \langle H^\perp \psi_I, \psi_I \rangle + 2\mathcal{F}[\psi_I]. \end{aligned} \quad (18)$$

In the case of defocusing nonlinearities all terms in this equation are nonnegative and we obtain straight away uniform boundedness of  $\psi^\varepsilon(t)$  in  $H^1(\mathbb{R}^{n+d})$  as well as the statement that the mass remains concentrated to the ground state as  $\varepsilon \rightarrow 0$ . The rest of this section is devoted to proving the same results (Lemma 5 and 6) without sign assumption on the nonlinearity.

By applying Lemma 7 from the Appendix with  $r = \alpha + 2$ , we can control the term coming from the nonlinearity:

$$|\mathcal{F}[\psi^\varepsilon(t)]| \leq \|\psi^\varepsilon(t)\|_{2+\alpha, 2+\alpha}^{2+\alpha} \leq c \|\nabla_x \psi^\varepsilon(t)\|_{2,2}^{n\alpha/2} \|\nabla_z \psi^\varepsilon(t)\|_{2,2}^{d\alpha/2}, \quad (19)$$

where here and in the following  $c$  denotes possibly different  $\varepsilon$ -independent, positive constants. Consequently, the energy conservation multiplied by  $\varepsilon^2$  yields

$$\varepsilon^2 \|\nabla_x \psi^\varepsilon(t)\|_{2,2}^2 + \|\nabla_z \psi^\varepsilon(t)\|_{2,2}^2 \leq c + c\varepsilon^2 \|\nabla_x \psi^\varepsilon(t)\|_{2,2}^{n\alpha/2} \|\nabla_z \psi^\varepsilon(t)\|_{2,2}^{d\alpha/2},$$

and, from the Young inequality,

$$\varepsilon^2 \|\nabla_x \psi^\varepsilon(t)\|_{2,2}^2 + \|\nabla_z \psi^\varepsilon(t)\|_{2,2}^2 \leq c + \varepsilon^2 \eta \|\nabla_x \psi^\varepsilon(t)\|_{2,2}^2 + \varepsilon^2 C(\eta) \|\nabla_z \psi^\varepsilon(t)\|_{2,2}^{\frac{2d\alpha}{4-n\alpha}}.$$

**Remark.** The constraint  $\alpha < \frac{4}{n}$  in Assumption 2 guarantees that the exponent remains positive.

With the choice  $\eta = \frac{1}{2}$  we deduce

$$\|\nabla_z \psi^\varepsilon(t)\|_{2,2}^2 \leq c + \varepsilon^2 c \|\nabla_z \psi^\varepsilon(t)\|_{2,2}^{\frac{2d\alpha}{4-n\alpha}}$$

In the case  $\theta = \frac{2d\alpha}{4-n\alpha} < 2$  we conclude that

$$\|\nabla_z \psi^\varepsilon\|_{\infty(2,2)} \leq c \quad (20)$$

For  $\theta > 2$  we obtain the result by the standard bootstrap argument, since  $\|\nabla_z \psi_I\|_{2,2}$  is independent of  $\varepsilon$  (see [Car1], Lemma 2.9). Using (19) with (20) in (18), we get

$$\|\nabla_x \psi^\varepsilon(t)\|_{2,2}^2 + \|x \psi^\varepsilon(t)\|_{2,2}^2 + \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} (\mu_k - \mu_0) \|\phi_k^\varepsilon(t)\|_2^2 \leq c + c \|\nabla_x \psi^\varepsilon(t)\|_{2,2}^{n\alpha/2}. \quad (21)$$

Since, by  $\alpha < 4/n$ , the exponent in the last term is smaller than 2, uniform boundedness of  $\|\nabla_x \psi^\varepsilon(t)\|_{2,2}$  follows.

It is now easy to prove the following two results on uniform boundedness and on the uniform smallness of the contributions from excited states.

**Lemma 5.** *Let the assumptions of Theorem 3 be satisfied, let  $\psi^\varepsilon$  be the solution of (4), (11), let  $\phi^\varepsilon$  be defined by (7) and let  $\varphi$  be the solution of (10), (12). Then*

$$\psi^\varepsilon \in L^\infty((0, \infty); H^1(\mathbb{R}^{n+d})), \quad \phi^\varepsilon, \varphi \in L^\infty((0, \infty); H^1(\mathbb{R}^n)),$$

*uniformly in  $\varepsilon$ .*

*Proof.* >From (16) and (21) it is immediately clear that  $\langle H^\perp \psi^\varepsilon(t), \psi^\varepsilon(t) \rangle + \langle H \psi^\varepsilon(t), \psi^\varepsilon(t) \rangle$  is uniformly bounded with respect to  $\varepsilon$  and  $t$ . The observation that this term dominates the  $H^1(\mathbb{R}^{n+d})$ -norm, completes the proof of the first statement of the lemma.

The representation of  $\psi^\varepsilon$  in terms of the eigenstates shows

$$\langle H^\perp \psi^\varepsilon(t), \psi^\varepsilon(t) \rangle \geq \frac{1}{2} (\|\nabla_x \phi^\varepsilon\|_2^2 + \|x \phi^\varepsilon\|_2^2),$$

which proves the statement for  $\phi^\varepsilon$ . Finally, the statement for the  $\varepsilon$ -independent  $\varphi$  is a consequence of the existence theory.  $\square$

**Lemma 6.** *With the assumptions of the previous lemma,*

$$\|(I - \Pi) \psi^\varepsilon\|_{\infty(p,2)} \leq c \varepsilon$$

*holds with an  $\varepsilon$ -independent constant  $c$  and with  $p \in [2, \frac{2d}{d-2}]$  if  $d \geq 3$ ,  $p \in [2, \infty)$  if  $d = 2$ , and  $p \in [2, \infty]$  if  $d = 1$ .*

*Proof.* Using (21) we obtain

$$\begin{aligned} \|(I - \Pi)\psi^\varepsilon(t)\|_{2,2}^2 &= \sum_{k=1}^{\infty} \|\phi_k^\varepsilon(t)\|_2^2 \\ &\leq \frac{1}{\mu_1 - \mu_0} \sum_{k=1}^{\infty} (\mu_k - \mu_0) \|\phi_k^\varepsilon(t)\|_2^2 \leq c\varepsilon^2, \end{aligned}$$

i.e., the statement of the lemma with  $p = 2$ . On the other hand we estimate

$$\begin{aligned} \|\nabla_z(I - \Pi)\psi^\varepsilon(t)\|_{2,2}^2 &\leq \langle H(I - \Pi)\psi^\varepsilon(t), (I - \Pi)\psi^\varepsilon(t) \rangle = \sum_{k=1}^{\infty} \mu_k \|\phi_k^\varepsilon(t)\|_2^2 \\ &= \sum_{k=1}^{\infty} (\mu_k - \mu_0) \|\phi_k^\varepsilon(t)\|_2^2 + \mu_0 \|(I - \Pi)\psi^\varepsilon(t)\|_{2,2}^2 \leq c\varepsilon^2. \end{aligned}$$

The result is now a consequence of the Sobolev imbedding  $H^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$  in  $z$ -space.  $\square$

### 3 Proof of the Main Result

The approximation error in Theorem 3 can be split into two parts:

$$\begin{aligned} \|\psi^\varepsilon - \varphi \omega_0 e^{-i\mu_0 t/\varepsilon^2}\|_{\infty(2,2)} &\leq \|(I - \Pi)\psi^\varepsilon\|_{\infty(2,2)} + \|\omega_0 e^{-i\mu_0 t/\varepsilon^2}(\phi^\varepsilon - \varphi)\|_{\infty(2,2)} \\ &= \|(I - \Pi)\psi^\varepsilon\|_{\infty(2,2)} + \|\phi^\varepsilon - \varphi\|_{\infty(2)}. \end{aligned}$$

The first term is taken care of by Lemma 6. The difference  $\chi^\varepsilon := \phi^\varepsilon - \varphi$  solves the problem

$$\begin{aligned} i\chi_t^\varepsilon &= H^\perp \chi^\varepsilon + g^\varepsilon + h^\varepsilon \\ \chi^\varepsilon(t=0) &= 0, \end{aligned} \tag{22}$$

where

$$g^\varepsilon = \bar{f}(|\phi^\varepsilon|)\phi^\varepsilon - \bar{f}(|\varphi|)\varphi$$

and  $h^\varepsilon$  given by (9).

For the nonlinear Schrödinger equation (22) with harmonic potential, a local dispersion result can be established ([Fuj1, Fuj2], [Caz] Lemma 9.2.4). This property allows us to use Strichartz estimates (see [Caz] Theorem 3.4.1, [Car2, KT]) and we obtain for any admissible pair  $(q^*, q)$  and a *bounded* time interval  $T < \infty$ :

$$\|\chi^\varepsilon\|_{\infty(2)} \leq c_T (\|g^\varepsilon\|_{q^*(q)} + \|h^\varepsilon\|_{q^*(q)}). \tag{23}$$

A pair  $(q, q^*)$  is admissible iff

$$\frac{2n}{n+2} \leq q \leq 2 \text{ for } n \geq 3, \quad 1 < q \leq 2 \text{ for } n = 2, \quad 1 \leq q \leq 2 \text{ for } n = 1, \tag{24}$$

$$q^* = \frac{4}{4 - n(2/q - 1)}. \tag{25}$$

Note that the definition of admissible pair is not the usual one.

**Remark.** We need a bounded time interval because the constant depends on the length of the time interval. A global statement can be done in the case that there is no confinement in  $x$  direction. For more details, see [Caz].

Assumption 2 implies the pointwise estimate

$$|g^\varepsilon| \leq c(|\phi^\varepsilon|^\alpha + |\varphi|^\alpha)|\chi^\varepsilon|.$$

Applying the Hölder inequality we obtain

$$\|g^\varepsilon(t)\|_q \leq c(\|\phi^\varepsilon\|_{2\alpha q/(2-q)}^\alpha + \|\varphi\|_{2\alpha q/(2-q)}^\alpha)\|\chi^\varepsilon\|_2.$$

The assumption  $\alpha \leq 2/(n-2)$  for  $n \geq 3$  allows to choose  $q$  such that both (24) is satisfied and  $H^1(\mathbb{R}^n) \hookrightarrow L_x^{2\alpha q/(2-q)}$ . Therefore we can use Lemma 5 to obtain

$$\|g^\varepsilon(t)\|_{q^*(q)} \leq c\|\chi^\varepsilon\|_{q^*(2)}. \quad (26)$$

For  $h^\varepsilon$  we also employ Assumption 2 to obtain a pointwise estimate:

$$|h^\varepsilon| \leq c \int_{\mathbb{R}^d} (|\psi^\varepsilon|^\alpha + |\Pi\psi^\varepsilon|^\alpha)|(I - \Pi)\psi^\varepsilon|\omega_0 dz.$$

Computing the  $L^q(\mathbb{R}^n)$ -norm and applying the Hölder inequality twice (to the  $x$ - and  $z$ -integrals, respectively), lead to:

$$\|h^\varepsilon\|_q \leq c(\|\psi^\varepsilon\|_{\alpha p', 2\alpha q/(2-q)}^\alpha + \|\phi^\varepsilon\|_{2\alpha q/(2-q)}^\alpha)\|(I - \Pi)\psi^\varepsilon\|_{p,2},$$

whereby  $p' = \frac{p}{p-1}$ .

Let us recall all the conditions on  $p$  and  $q$ :

- the assumptions of Lemma 6 for  $p$  and the condition (24) for  $q$  are satisfied
- the imbeddings  $H^1(\mathbb{R}^n) \hookrightarrow L_x^{2\alpha q/(2-q)}$  and  $H^1(\mathbb{R}^{n+d}) \hookrightarrow L_z^{\alpha p'} L_x^{2\alpha q/(2-q)}$  (see Lemma 7 in the Appendix) hold.

All this is possible since  $\alpha \leq 4/(n+d-2)$  and  $\alpha \leq 2/(n-2)$  for  $n \geq 3$ . As a consequence of Lemmata 5 and 6 we obtain

$$\|h^\varepsilon\|_{\infty(q)} \leq c\varepsilon. \quad (27)$$

With (26) and (27), the Strichartz estimate (23) becomes

$$\|\chi^\varepsilon\|_{\infty(2)} \leq c_T(\|\chi^\varepsilon\|_{q^*(2)} + \varepsilon).$$

Using this estimate on the time interval  $(0, t)$  with  $t \leq T$  gives

$$\|\chi^\varepsilon(t)\|_2^{q^*} \leq \tilde{c}_T \left( \int_0^t \|\chi^\varepsilon(s)\|_2^{q^*} ds + \varepsilon^{q^*} \right).$$

Now, an application of the Gronwall lemma concludes the proof of Theorem 3.

## 4 Application: the Gross-Pitaevskii equation

The three dimensional nonlinear Schrödinger equation with cubic nonlinearity and an external potential is called Gross-Pitaevskii equation (GPE). It models the temporal evolution of Bose-Einstein condensates (BEC) at temperatures much smaller than the critical condensation temperature [DGPS, LSY, PS]. In dimensional form, the GPE reads

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + \frac{m}{2}(\omega_x^2|x|^2 + \omega_z^2|z|^2)\psi + Ng|\psi|^2\psi, \quad (28)$$

where  $m$  is the atomic mass,  $\hbar$  is the Planck constant,  $N$  is the number of atoms in the condensate, and  $\omega_x, \omega_z$  are the trap frequencies in  $x$ - and  $z$ - directions, respectively. The parameter  $g$  describes the interaction between the atoms in the condensate and has the form  $g = \hbar^2 a/m$ , where  $a$  is the scattering length, positive for repulsive interactions and negative for attractive interactions. We consider the cases  $n = 1$  and  $n = 2$  with  $d = 3 - n$ . Characteristic lengths of the condensate in the  $x$ - and  $z$ -directions are  $a_x = \sqrt{\hbar/(m\omega_x)}$  and  $a_z = \sqrt{\hbar/(m\omega_z)}$ , respectively.

Let us write the equation (28) in dimensionless form. With the scaling

$$x = a_x\tilde{x}, \quad z = a_z\tilde{z}, \quad \psi = \frac{\tilde{\psi}}{\sqrt{a_x^n a_z^{3-n}}}, \quad t = \frac{\tilde{t}}{\omega_x},$$

and skipping the tildes we obtain

$$i\psi_t = -\frac{1}{2}\Delta_x\psi + \frac{|x|^2}{2}\psi + \frac{\omega_z}{\omega_x}\left(-\frac{1}{2}\Delta_z\psi + \frac{|z|^2}{2}\psi\right) + N\frac{a}{a_z^{3-n}a_x^{n-2}}|\psi|^2\psi.$$

In experiments it is observed that in a strongly anisotropic confinement the motion of particles is quenched in one or two directions. This means that by changing the shape of the confining potential lower dimensional BEC are obtained. They are called disk-shaped or cigar-shaped condensates, respectively. This is the motivation to consider the Gross-Pitaevskii equation with strongly anisotropic confining harmonic potential, thus

$$\varepsilon^2 := \frac{\omega_x}{\omega_z} \ll 1.$$

Furthermore we assume the case of weak coupling, namely

$$\gamma := \frac{Na}{a_z^{3-n}a_x^{n-2}} = \mathcal{O}(1).$$

We then have the equation

$$i\psi_t = -\frac{1}{2}\Delta_x\psi + \frac{|x|^2}{2}\psi + \frac{1}{\varepsilon^2}\left(-\frac{1}{2}\Delta_z\psi + \frac{|z|^2}{2}\psi\right) + \gamma|\psi|^2\psi,$$

where  $\gamma|\psi|^2 = f(|\psi|)$  with  $\gamma$  positive, if we consider repulsive interactions, e.g. for  $^{23}\text{Na}$  and  $^{87}\text{Rb}$ , or negative for attractive interactions, e.g. for  $^7\text{Li}$ . Obviously, Assumption 2 on  $f$  holds with  $\alpha = 2$ .

For *repulsive interactions* ( $\gamma > 0$ ) we have global existence of the solution of the  $(n+d)$ -dimensional Schrödinger equation if  $\alpha < 4/(n+d-2)$ . Since  $\alpha = 2$  we obtain the condition  $n+d < 4$ , which includes the physically interesting case  $n+d = 3$ . The limiting lower dimensional GPE is

$$i\varphi_t = H^\perp \varphi + \gamma_0 |\varphi|^2 \varphi \quad \text{with } \gamma_0 = \gamma \int_{\mathbb{R}^d} \omega_0^4(z) dz.$$

On the one hand, if we consider the strong confinement in one direction ( $d = 1$ ), we obtain a two dimensional approximate equation ( $n = 2$ ). In this case we speak about a **disk-shaped condensate**. On the other hand we consider a strong confinement in 2 dimensions ( $d = 2$ ). Accordingly, the approximate equation is 1 dimensional ( $n = 1$ ) and we call the condensate a **cigar-shaped condensate**. Theorem 3 can be applied in both cases.

In the case of *attractive interactions*, thus for  $\gamma < 0$ , we get stronger constraints on the dimensions, namely  $n = 1$  and  $d < 3$ . Thus, Theorem 3 can only be applied for the reduction from 3D to 1D (cigar-shape condensate).

## A Anisotropic Sobolev inequalities

In this section, we state anisotropic Sobolev embeddings and a generalized Gagliardo-Nirenberg inequality. The proof of this lemma, rather straightforward, is skipped. It uses standard Sobolev embeddings and Gagliardo-Nirenberg inequalities, combined with interpolation estimates. We generalize here a result of [BM] (see also [IR] where a similar Sobolev embedding is obtained). Recall that in this paper the whole dimension is  $n+d$  and the space variable is written  $(x, z)$ , where  $x \in \mathbb{R}^n$  and  $z \in \mathbb{R}^d$ .

**Lemma 7.** *Let  $2 \leq p, q \leq \infty$  be such that*

$$\frac{n}{p(n+d)} + \frac{d}{q(n+d)} \geq \frac{1}{2} - \frac{1}{n+d} \quad (29)$$

(with  $q < \infty$  if  $d = 2$ ;  $p < \infty$  if  $n = 2$  and strict inequality if  $n = d = 1$ ). Then

$$H^1(\mathbb{R}^{n+d}) \hookrightarrow L_z^q(\mathbb{R}^d; L_x^p(\mathbb{R}^n)).$$

Furthermore, for any  $r \in [2, \frac{2(n+d)}{n+d-2}]$  we have

$$\|u\|_{L_{x,z}^r} \leq C \|u\|_{L_{x,z}^2}^{1-(n+d)(\frac{1}{2}-\frac{1}{r})} \|\nabla_x u\|_{L_{x,z}^2}^{n(\frac{1}{2}-\frac{1}{r})} \|\nabla_z u\|_{L_{x,z}^2}^{d(\frac{1}{2}-\frac{1}{r})}.$$

**Acknowledgement.** Support by the European IHP network no. RNT2 2001 349 entitled "Hyperbolic and kinetic equations: asymptotics, numerics, applications" is acknowledged. The first and the second authors also acknowledge support by the ACI Nouvelles Interfaces des Mathématiques no. ACINIM 176-2004 entitled "MOQUA" and funded by the French ministry of research as well as the ACI Jeunes chercheurs

no. JC1035 “Modèles dispersifs vectoriels pour le transport à l’échelle nanométrique”. The work of the third and fourth author has been supported by the PhD program “Differential Equations” funded by the Austrian Science Fund, project no. W8. The third author also acknowledges support through project no. P16174-N05, Austrian Science Fund. The work of the fourth author has been partly supported by the “Wittgenstein 2000” Award of P. Markowich.

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