Rigorous asymptotic expansions for Lagerstrom's model equation—a geometric approach

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Abstract

The present work is a continuation of the geometric singular perturbation analysis of the Lagerstrom model problem which was commenced in [PS04]. We establish the same framework here, reinterpreting Lagerstrom's equation as a dynamical system which is subsequently analyzed by means of methods from dynamical systems theory as well as of the blow-up technique. We show how rigorous asymptotic expansions for the Lagerstrom problem can be obtained using geometric methods, thereby establishing a connection to the method of matched asymptotic expansions. We explain the structure of these expansions and demonstrate that the occurrence of the well-known logarithmic (switchback) terms therein is caused by a resonance phenomenon.

1 Introduction

Singular perturbation problems in general and singularly perturbed differential equations in particular are characterized by the presence of at least two fundamentally different scales. The existence of these independent scales gives one a small parameter and thus permits one to use perturbation methods. The aim of these methods is to obtain uniformly valid approximations. It is, however, the essence of a singular perturbation problem that a straightforward perturbation fails to be uniformly valid. Indeed, it is typical of singular perturbation techniques that one works with approximations which are valid in restricted domains only.

Traditionally, this type of problems has been treated using the method of *matched asymptotic expansions*: one proceeds by constructing two (or more) asymptotic expansions which together cover the entire domain, although neither is uniformly valid there. To obtain a uniformly valid approximation on the entire domain, these individual expansions have to be matched; the essence of *matching* lies in comparing two expansions on a suitable domain of overlap. An excellent account of the fundamental notions and concepts in perturbation theory is given in [LC72].

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More recently, an alternative approach to such problems, known as *geometric singular perturbation theory*, has emerged (see [Fen79] or [Jon95] for details and references). This approach, which in general requires certain hyperbolicity assumptions, is based on methods from the theory of dynamical systems, in particular on invariant manifold theory.

A classical singular perturbation problem from fluid dynamics occurs in the asymptotic treatment of viscous flow past a solid at low Reynolds number, see e.g. [vD75]. Though first attempts at clarification date back to [Sto51], it was not until a century later that the conceptual structure of the problem was at last resolved in [Kap57], [KL57], and [PP57]. To illustrate the mathematical ideas and techniques used by Kaplun in his asymptotic treatment of low Reynolds number flow, [Lag66] proposed an analytically rather simple model problem which was subsequently analyzed by Lagerstrom himself as well as by numerous other workers, see [Lag88] and the references therein.

Much of the interest in Lagerstrom's model problem has been directed at the development of matched asymptotic expansions to describe its solutions, see e.g. [KC81], [Lag88], or [HTB90]. Our goal is to show how such expansions can be obtained using geometric methods, thereby establishing a connection between the two approaches. As observed already by [Fen79], for layer-type problems finding outer solutions is equivalent to computing expansions of slow manifolds. However, the well-developed geometric theory is not applicable at points where hyperbolicity is lost. In several instances it has been possible to extend the geometric approach past such points using the blow-up technique. Blow-up is essentially a sophisticated rescaling which allows one to analyze the dynamics near a singularity; details can be found in [DR96]. In particular, blow-up has been employed by [KS01] and [vGKS] to give a detailed geometric analysis of the singularly perturbed planar fold. Apart from deriving asymptotic expansions of slow manifolds continued beyond the fold point, they also explained the structure of these expansions and gave an algorithm for the computation of its coefficients. Our line of attack is very similar to theirs.

A distinctive feature of asymptotic expansions for the singularly perturbed planar fold as well as for Lagerstrom's model is the occurrence of logarithmic terms. The nature of the expansions in these and similar problems is both complicated and unexpected, as the governing equations typically give no immediate hint of the presence of such terms; indeed, this is why they are often so tricky to obtain. Traditionally, logarithmic terms have been accounted for under the notions of *switchback* and *integrated effects*. We show that the occurrence of logarithms in the expansions for Lagerstrom's model equation is caused by a resonance phenomenon. At this point we conjecture that similar resonance phenomena are responsible for the occurrence of logarithmic terms in many other singular perturbation problems, at least after reinterpretation in a dynamical systems framework.

This article is organized as follows: Section 2 contains some background information on the Lagerstrom model problem as well as on the blow-up transformation used in our analysis; in Section 3 we derive asymptotic expansions for Lagerstrom's model, whereas Section 4 briefly indicates how these expansions are related to the classical ones known from the literature. This work should be regarded as a continuation of [PS04].

2 Background information

2.1 Lagerstrom's model equation

In its simplest form, Lagerstrom's model equation is given by the non-autonomous second-order boundary value problem

$$\ddot{u} + \frac{n-1}{x}\dot{u} + u\dot{u} = 0 \tag{2.1a}$$

$$u(\varepsilon) = 0, \qquad u(\infty) = 1 \tag{2.1b}$$

with $n \in \mathbb{N}$, $0 < \varepsilon \leq x \leq \infty$, and the overdot denoting differentiation with respect to x. Equivalently, by introducing the rescaling

$$\xi = \frac{x}{\varepsilon}, \tag{2.2}$$

one can write (2.1) as

$$u'' + \frac{n-1}{\xi}u' + \varepsilon u u' = 0 \tag{2.3a}$$

$$u(1) = 0, \qquad u(\infty) = 1 \tag{2.3b}$$

with $1 \leq \xi \leq \infty$ and the prime denoting differentiation with respect to ξ.

Originally, (2.1) and (2.3) are the versions of the model which was first introduced in [Lag66] and [KL57] to elucidate certain basic ideas used in the asymptotic treatment of viscous flow past a solid at low Reynolds number. In the following we will only consider the cases $n = 2$ and $n = 3$, which represent the physically relevant settings of flow in two and three dimensions, respectively. For more background information and further references on Lagerstrom's model equation we refer to [PS04].

As in [PS04], replacing $\xi \in [1, \infty)$ by $\eta := \xi^{-1} \in (0, 1]$, appending the (trivial) equation $\varepsilon' = 0$, and setting $u' = v$ yields

$$\begin{aligned} u' &= v \\ v' &= -(n-1)\eta v - \varepsilon u v \\ \eta' &= -\eta^2 \\ \varepsilon' &= 0 \end{aligned} \tag{2.4}$$

in extended phase space, with boundary conditions given by

$$u(1) = 0, \qquad \eta(1) = 1, \qquad u(\infty) = 1; \tag{2.5}$$

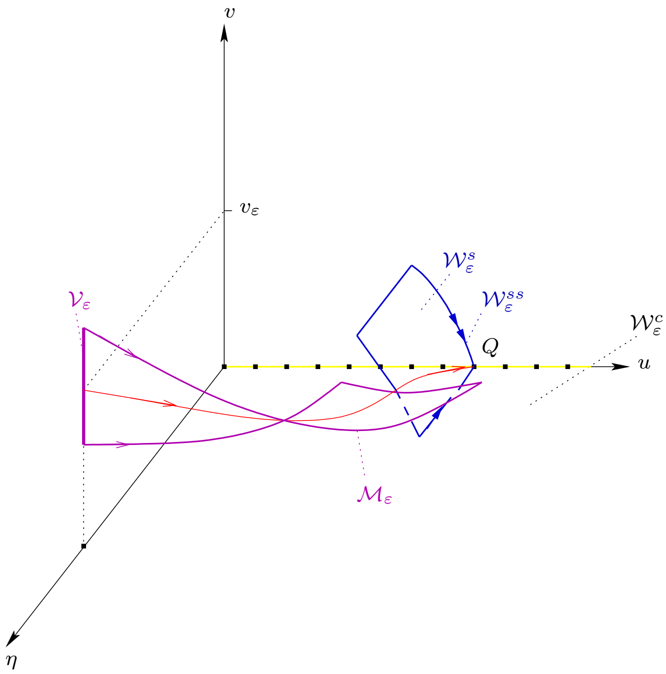

Figure 1: Geometry of system (2.4) for $\varepsilon > 0$ fixed.

obviously, (2.5) means that $v(\infty) = 0$ for the solution to (2.4), whereas $v(1)$ still is to be determined.

For $\varepsilon > 0$ fixed, let $\mathcal{V}_\varepsilon$ be defined by

$$\mathcal{V}_\varepsilon := \{(0, v, 1) |\ v \in [\underline{v}, \overline{v}]\}, \tag{2.6}$$

where $0 \leq \underline{v} < \overline{v} < \infty$, and let the point Q be given by $Q := (1, 0, 0)$ (see again [PS04]). Note that $\mathcal{V}_\varepsilon$ and Q correspond to the inner and outer boundary conditions in (2.3b), respectively. The saturation of $\mathcal{V}_\varepsilon$ under the flow induced by (2.4) we denote by $\mathcal{M}_\varepsilon$. Correspondingly, the manifolds $\mathcal{V}$ and $\mathcal{M}$ in extended phase space are defined by $\mathcal{V} := \bigcup_{\varepsilon \in [0,\varepsilon_0]} \mathcal{V}_\varepsilon \times \{\varepsilon\}$ and $\mathcal{M} := \bigcup_{\varepsilon \in [0,\varepsilon_0]} \mathcal{M}_\varepsilon \times \{\varepsilon\}$; here the parameter ε is not fixed, but is allowed to vary in some interval $[0, \varepsilon_0]$ with $\varepsilon_0 > 0$ small.

The equilibria of (2.4) are located on the line $l := \{(u, v, \eta, \varepsilon) |\ u \in \mathbb{R}^+, v = \eta = \varepsilon = 0\}$; obviously, $Q \in l$. For $\varepsilon > 0$, the one-dimensional strongly stable manifold of Q, which we call $\mathcal{W}_\varepsilon^{ss}$, can be computed explicitly due to the simple structure of (2.4) for $\eta = 0$, whence e.g.

$$v(u) = \frac{\varepsilon}{2}\left(1 - u^2\right); \tag{2.7}$$

here we have used $v(1) = 0$. The following result can be found in [PS04]:

Proposition 2.1 ([PS04]). *Let $k \in \mathbb{N}$ be arbitrary, and let $\varepsilon > 0$.*

1. *There exists an attracting two-dimensional center manifold $\mathcal{W}^c_\varepsilon$ of (2.4) which is given by $\{v = 0\}$.*

2. *For $|u-1|, v$, and η sufficiently small, there is a stable invariant $\mathcal{C}^k$-smooth foliation $\mathcal{F}^s_\varepsilon$ with base $\mathcal{W}^c_\varepsilon$ and one-dimensional $\mathcal{C}^k$-smooth fibers.*

Given Proposition 2.1, one can define the stable manifold $\mathcal{W}^s_\varepsilon$ of Q as

$$\mathcal{W}^s_\varepsilon := \bigcup_{P \in \Upsilon} F^s_\varepsilon(P), \tag{2.8}$$

where $\Upsilon := \{(1,0,\eta) |\ 0 \leq \eta \ll 1\}$, i.e., as a union of fibers $F^s_\varepsilon \in \mathcal{F}^s_\varepsilon$ with base points in the weakly stable orbit Υ. The situation is illustrated in Figure 1.

The main result in [PS04] is the following theorem on the existence and (local) uniqueness of solutions to (2.4) (and, consequently, to (2.1)):

Theorem 2.1 ([PS04]). *For $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small and $n = 2, 3$, there exists a locally unique solution to (2.4),(2.5).*

The proof is constructive, and is performed by tracking $\mathcal{M}_\varepsilon$ through phase space and showing that its intersection with $\mathcal{W}^s_\varepsilon$ is transverse under the resulting flow. As we are only interested in small values of ε, we took a perturbational approach: given transversality for $\varepsilon = 0$, we concluded that the intersection remained transverse for $\varepsilon > 0$ small. On a technical level, the tracking was done along singular orbits of (2.4) connecting $\mathcal{V}_0$ to Q. These orbits, which we denoted by Γ, were used as templates for orbits of the full problem obtained for $\varepsilon > 0$. However, due to the non-hyperbolic character of the problem for $\varepsilon = 0$, we could not deduce the existence of a stable manifold $\mathcal{W}^s_0$ from standard invariant manifold theory. It was shown in [PS04], however, that $\mathcal{W}^{ss}$ and $\mathcal{W}^s$ can still be smoothly defined down to $\varepsilon = 0$ by using blow-up techniques.

2.2 The blow-up transformation

The *(polar) blow-up transformation* Φ introduced in [PS04] to analyze the dynamics of (2.4) near l is given by

$$\Phi : \begin{cases} \mathbb{R} \times B \to \mathbb{R}^4 \\ (\bar{u}, \bar{v}, \bar{\eta}, \bar{\varepsilon}, \bar{r}) \mapsto (\bar{u}, \bar{r}\bar{v}, \bar{r}\bar{\eta}, \bar{r}\bar{\varepsilon}), \end{cases} \tag{2.9}$$

where $B := \mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{S}^2$ denotes the two-sphere in $\mathbb{R}^3$, i.e., $\mathbb{S}^2 = \{(\bar{v}, \bar{\eta}, \bar{\varepsilon}) |\ \bar{v}^2 + \bar{\eta}^2 + \bar{\varepsilon}^2 = 1\}$. Note that obviously $\Phi^{-1}(l) = \mathbb{R} \times \mathbb{S}^2 \times \{0\}$, which is the blown-up locus obtained by setting $\bar{r} = 0$. Moreover, for $\bar{r} \neq 0$, i.e., away from $\Phi^{-1}(l)$, Φ is a $\mathcal{C}^\infty$-diffeomorphism. We will only be interested in $\bar{r} \in [0, r_0]$ with $r_0 > 0$ small.

The reason for introducing (2.9) is that degenerate equilibria, such as those in l, are often amenable to analysis by means of blow-up techniques. The blow-up is a (singular) coordinate transformation whereby the degenerate equilibrium is blown up

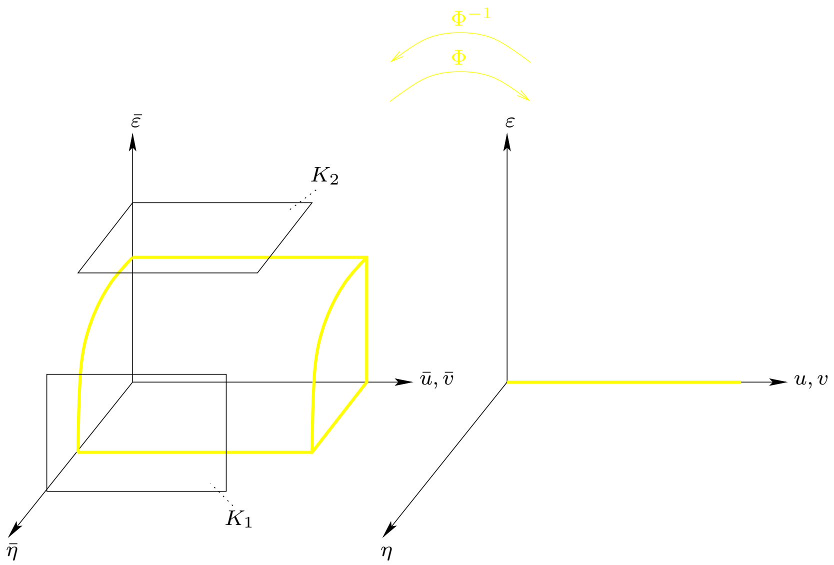

Figure 2: The blow-up transformation Φ.

to some n-sphere. Transverse to the sphere and even on the sphere itself one often gains enough hyperbolicity to allow a complete analysis by standard techniques. For a general discussion of blow-up we refer to [DR91] and to [Dum93], whereas applications to singular perturbation problems can be found in [DS95] and [DR96] as well as in [KS01] and [vGKS].

The vector field on $\mathbb{R} \times B$, which is induced by the vector field corresponding to (2.4), is most conveniently studied by introducing different charts for the manifold $\mathbb{R} \times B$. In the following, we will be concerned only with two charts K_1 and K_2 corresponding to $\bar{\eta} > 0$ and $\bar{\varepsilon} > 0$ in (2.9), respectively, see Figure 2. The reason is that these two charts correspond precisely to the inner and outer regions in the classical approach, see [PS04].

The *directional blow-up* in the direction of positive η (i.e., in K_1) is given by

$$\Phi_1 : \begin{cases} \mathbb{R}^4 \to \mathbb{R}^4 \\ (u_1, v_1, r_1, \varepsilon_1) \mapsto (u_1, r_1 v_1, r_1, r_1 \varepsilon_1), \end{cases} \tag{2.10}$$

whence

$$u = u_1, \quad v = r_1 v_1, \quad \eta = r_1, \quad \varepsilon = r_1 \varepsilon_1. \tag{2.11}$$

After transformation to K_1 and desingularization, the equations in (2.4) have the following form:

$$\begin{aligned} u_1' &= v_1 \\ v_1' &= (2-n)v_1 - \varepsilon_1 u_1 v_1 \\ r_1' &= -r_1 \\ \varepsilon_1' &= \varepsilon_1. \end{aligned} \tag{2.12}$$

The desingularization, which is necessary to obtain a non-trivial flow for $r_1 = 0$, is performed by dividing out the common factor r_1 on both sides of the equations; it corresponds to a rescaling of time, leaving the phase portrait unchanged. The equilibria of (2.12) are easily seen to lie in $l_1 := \{(u_1, v_1, r_1, \varepsilon_1)|\ u_1 \in \mathbb{R}^+, v_1 = r_1 = \varepsilon_1 = 0\}$. A simple computation shows that the corresponding eigenvalues are given by -1, 0, and 1 both for $n = 3$ and for $n = 2$; these eigenvalues obviously are in *resonance.* In fact, it is these resonances in K_1 which are responsible for the occurrence of logarithmic switchback terms in the Lagerstrom model, as will become clear later on.

Similarly, in chart K_2, (2.9) is given by

$$\Phi_2 : \begin{cases} \mathbb{R}^4 \to \mathbb{R}^4 \\ (u_2, v_2, \eta_2, r_2) \mapsto (u_2, r_2 v_2, r_2 \eta_2, r_2) \end{cases} \tag{2.13}$$

respectively

$$u = u_2, \quad v = r_2 v_2, \quad \eta = r_2 \eta_2, \quad \varepsilon = r_2, \tag{2.14}$$

which is simply an ε-dependent rescaling of the original variables, since $r_2 = \varepsilon$. Desingularizing once again, we obtain for the blown-up vector field in K_2

$$\begin{aligned} u_2' &= v_2 \\ v_2' &= (1-n)\eta_2 v_2 - u_2 v_2 \\ \eta_2' &= -\eta_2^2 \\ r_2' &= 0; \end{aligned} \tag{2.15}$$

these equations are simple insofar as r_2 occurs only as a parameter. The equilibria of (2.15) are given by $l_2 := \{(u_2, v_2, \eta_2, r_2)|\ u_2 \in \mathbb{R}^+, v_2 = \eta_2 = 0, r_2 \in [0, r_0]\}$.

The following observation, which is valid in both cases alike, is the starting point of our analysis:

Proposition 2.2 ([PS04]). *Let $k \in \mathbb{N}$ be arbitrary.*

1. *There exists an attracting three-dimensional center manifold $\mathcal{W}_2^c$ of (2.4) which is given by $\{v_2 = 0\}$.*
2. *For $|u_2 - 1|, v_2, \eta_2,$ and r_2 sufficiently small, there is a stable invariant $\mathcal{C}^k$-smooth foliation $\mathcal{F}_2^s$ with base $\mathcal{W}_2^c$ and one-dimensional $\mathcal{C}^k$-smooth fibers.*

Let $\mathcal{W}_2^{ss}$ denote the fiber $F_2^s(Q_2) \in \mathcal{F}_2^s$ with base point $Q_2 := (1, 0, 0, 0)$; note that for any $r_2 = \varepsilon \in [0, \varepsilon_0]$ fixed, Q_2 and $\mathcal{W}_2^{ss}$ correspond to the original Q and its stable fiber $\mathcal{W}_\varepsilon^{ss}$, respectively.

Remark 2.1. Just as was the case with $\mathcal{W}_\varepsilon^{ss}$, $\mathcal{W}_2^{ss}$ also is known explicitly: given $v_2(1) = 0$, one obtains from (2.15) with $\eta_2 = 0$ that

$$v_2(u_2) = \frac{1}{2}\left(1 - u_2^2\right); \tag{2.16}$$

hence $\mathcal{W}_2^{ss}$ is independent of both ε and n. □

As in [PS04], let the orbit γ_2 be defined by

$$\gamma_2(\xi_2) := \left\{ \left(1, 0, \xi_2^{-1}, 0\right) \middle| \, \xi_2 \in (0, \infty) \right\}, \tag{2.17}$$

and let $\Gamma_2 := \gamma_2 \cup \{Q_2\}$; note that $\gamma_2(\xi_2) \to Q_2$ as $\xi_2 \to \infty$. With Proposition 2.2 it then follows:

Proposition 2.3 ([PS04]). *The manifold $\mathcal{W}_2^s$ defined by*

$$\mathcal{W}_2^s := \bigcup_{P_2 \in \Gamma_2} F_2^s(P_2) \tag{2.18}$$

is an invariant, $\mathcal{C}^k$-smooth manifold, namely the stable manifold of Q_2.

Tracking the manifold $\mathcal{W}_2^s$ along the singular orbit $\overline{\Gamma}$ to the inner boundary in K_1 defines a global manifold $\overline{\mathcal{W}}^s$ which determines the solution to (2.4),(2.5) as given by Theorem 2.1, see Figure 3.

The relation between charts K_1 and K_2 on their overlap domain can be described as follows:

Lemma 2.1 ([PS04]). *Let κ_{12} denote the change of coordinates from K_1 to K_2, and let $\kappa_{21} = \kappa_{12}^{-1}$ be its inverse. Then κ_{12} is given by*

$$u_2 = u_1, \quad v_2 = v_1 \varepsilon_1^{-1}, \quad \eta_2 = \varepsilon_1^{-1}, \quad r_2 = r_1 \varepsilon_1 \tag{2.19}$$

and κ_{21} is given by

$$u_1 = u_2, \quad v_1 = v_2 \eta_2^{-1}, \quad r_1 = r_2 \eta_2, \quad \varepsilon_1 = \eta_2^{-1}. \tag{2.20}$$

Remark 2.2 *(Notation).* Let us introduce the following notation: for any object $\square$ in the original setting, let $\overline{\square}$ denote the corresponding object in the blow-up; in charts $K_i, \ i = 1, 2$, the same object will appear as $\square_i$ when necessary. $\square$

Moreover, as in [PS04] we define the *sections* Σ_1^{in}, Σ_1^{out}, and Σ_2^{in}, where

$$\Sigma_1^{in} := \{(u_1, v_1, r_1, \varepsilon_1) | \, u_1 \geq 0, v_1 \geq 0, \varepsilon_1 \geq 0, r_1 = \rho\} \tag{2.21a}$$

$$\Sigma_1^{out} := \{(u_1, v_1, r_1, \varepsilon_1) | \, u_1 \geq 0, v_1 \geq 0, r_1 \geq 0, \varepsilon_1 = \delta\} \tag{2.21b}$$

$$\Sigma_2^{in} := \left\{(u_2, v_2, \eta_2, r_2) \middle| \, u_2 \geq 0, v_2 \geq 0, r_2 \geq 0, \eta_2 = \delta^{-1}\right\} \tag{2.21c}$$

with $0 < \rho, \delta \ll 1$ arbitrary, but fixed; obviously $\kappa_{12}\left(\Sigma_1^{out}\right) = \Sigma_2^{in}$.

3 Rigorous asymptotic expansions

We now set out to derive asymptotic expansions for the function $v_{1_\varepsilon} := v_1\big|_{\xi=1}$ as defined by the unique solution to (2.4),(2.5) given in Theorem 2.1, see Figure 4. It is well-known that to leading order, the method of matched asymptotic expansions gives

$$v_{1_\varepsilon} = 1 - \varepsilon \ln \varepsilon - (\gamma + 1)\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \tag{3.1}$$

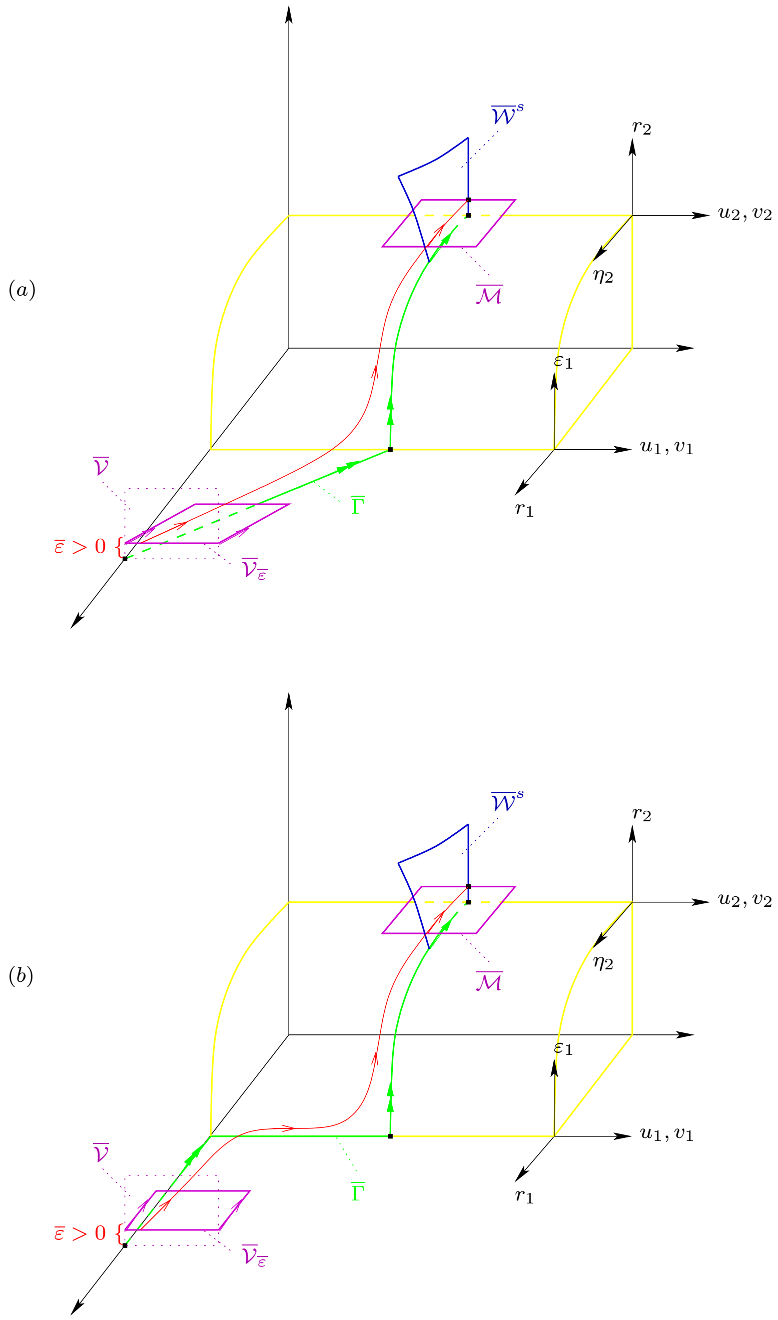

Figure 3: Geometry of the blown-up system for (a) $n = 3$ and (b) $n = 2$.

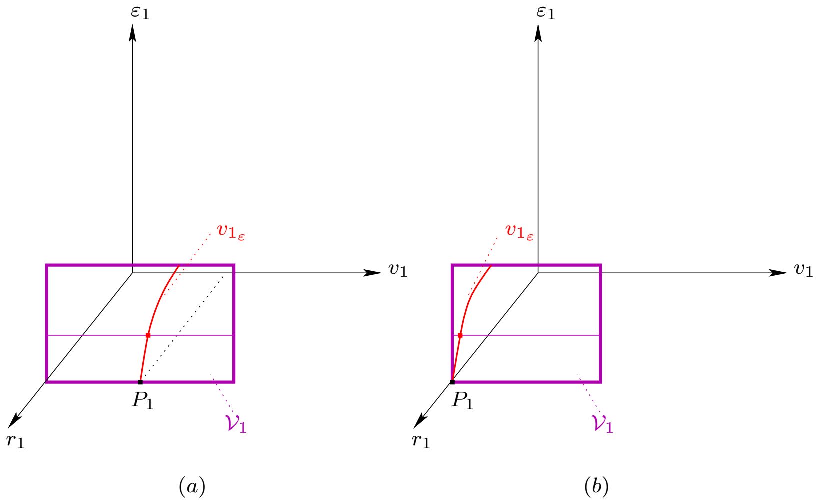

Figure 4: v_{1_ε} for (a) $n = 3$ and (b) $n = 2$.

for $n = 3$ and

$$v_{1_\varepsilon} = -\frac{1}{\ln \varepsilon} + \frac{\gamma}{(\ln \varepsilon)^2} + \mathcal{O}\left(\frac{1}{(\ln \varepsilon)^3}\right) \tag{3.2}$$

for $n = 2$, respectively, see e.g. [Lag88]. Classically, the somewhat surprising occurrence of logarithmic terms in these expansions has been accounted for under the notion of switchback; we will show that these terms are in fact due to a resonance phenomenon. Incidentally, note that v_{1_ε} equals $\frac{du}{d\xi}\big|_{\xi=1}$, the analogue of the drag on the solid, which is a quantity of considerable interest in the original fluid dynamical problem.

Our approach is rigorous in the sense that the expansions we compute are approximations to a well-defined geometric object, namely, to the invariant manifold $\overline{\mathcal{W}}^s$ introduced in the previous section. Roughly speaking, our strategy is the following: we begin by computing expansions for $\mathcal{W}_2^s$ in K_2; these expansions, when translated to K_1 and evaluated at the inner boundary there, will provide us with expansions for v_{1_ε}. We again distinguish between $n = 3$ and $n = 2$ here, the case $n = 3$ being considerably simpler.

Remark 3.1. This strategy, which is slightly different from the strategy applied in [PS04], is somewhat more efficient as far as computing expansions for v_{1_ε} is concerned. Later on we will indicate how asymptotic *solution* expansions for (2.4),(2.5) can be obtained. □

The complicated structure of the expansions in K_1 arises as $\overline{\mathcal{W}}^s$ passes near the line of equilibria l_1 in K_1. As indicated above, the logarithmic terms in (3.1) respectively (3.2) are due to the resonant eigenvalues -1, 0, and 1 which occur in K_1, see [PS04]. We now present a simple argument to substantiate our claim: for $n = 3$, consider the equations in K_1 given by (2.12). After introducing the new variable $\tilde{v}_1 = e^{\xi_1} v_1$, one obtains for the first two equations in (2.12)

$$\begin{aligned} u_1' &= e^{-\xi_1} \tilde{v}_1 \\ \tilde{v}_1' &= -\varepsilon_1 u_1 \tilde{v}_1. \end{aligned} \tag{3.3}$$

Integration of (3.3) yields[1]

$$\begin{aligned} u_1(\xi_1) &= u_{1_0} + \int_0^{\xi_1} e^{-\xi'} \tilde{v}_1(\xi')\, d\xi' \\ \tilde{v}_1(\xi_1) &= v_{1_0} - \varepsilon_{1_0} \int_0^{\xi_1} e^{\xi'} u_1(\xi') \tilde{v}_1(\xi')\, d\xi', \end{aligned} \tag{3.4}$$

where we have used $\varepsilon_1 = \varepsilon_{1_0} e^{\xi_1}$ and u_{1_0}, v_{1_0}, and ε_{1_0} are constants. Note that near l_1, v_1 and ε_1 are small, which implies $\tilde{v}_1 = \mathcal{O}(1)$ there. Hence a *Picard iteration scheme* can be applied to (3.4), with the starting point given by $\big(u_1^{(0)}, v_1^{(0)}\big) = (u_{1_0}, v_{1_0})$. In fact, one easily sees that (3.4) defines a contraction operator for u_1 and $\tilde{v}_1$ in $\mathcal{L}^\infty\big[0, \ln \frac{\varepsilon_1}{\varepsilon_{1_0}}\big]$, which ensures convergence of the scheme.[2] A straightforward computation gives

$$u_1^{(1)} = u_{1_0} + v_{1_0}\big(1 - e^{-\xi_1}\big) \tag{3.5a}$$

$$\tilde{v}_1^{(1)} = v_{1_0} + \varepsilon_{1_0} u_{1_0} v_{1_0}\big(1 - e^{\xi_1}\big) \tag{3.5b}$$

$$u_1^{(2)} = u_1^{(1)} + \varepsilon_{1_0} u_{1_0} v_{1_0}\big(1 - \xi_1 - e^{-\xi_1}\big) \tag{3.5c}$$

$$\begin{aligned} \tilde{v}_1^{(2)} = \tilde{v}_1^{(1)} &+ \varepsilon_{1_0} v_{1_0}^2\big(1 + \xi_1 - e^{\xi_1}\big) + \frac{1}{2}\varepsilon_{1_0} u_{1_0}^2 v_{1_0}\big(1 - 2e^{\xi_1} + e^{2\xi_1}\big) + \\ &+ \frac{1}{2}\varepsilon_{1_0} u_{1_0} v_{1_0}^2\big(3 + 2\xi_1 - 4e^{\xi_1} + e^{2\xi_1}\big) \end{aligned} \tag{3.5d}$$

for the first two iterates in (3.4). As $\xi_1 = \ln \frac{\varepsilon_1}{\varepsilon_{1_0}}$, this then generates a logarithmic term in ε_1 after rewriting (3.5c) as a function of ε_1. Similarly, the products of powers of ξ_1 and e^{ξ_1} which occur for higher iterates in (3.4) will give rise to products of powers of $\ln \varepsilon_1$ and ε_1 after those iterates have been rewritten in terms of ε_1.

In fact, it is thus possible to obtain successive approximations to the *transition map* Π from Σ_1^{in} to Σ_1^{out} for (2.12), see [KS01]; the above computation gives the leading order behaviour of Π. Note that it is precisely the resonant terms in (2.12) which cannot be eliminated by a normal form transformation and which preclude the existence of a linearizing transformation for (2.12) proper, see e.g. [CLW94].

[1] Without loss of generality, we set $\xi_{1_0} = 0$ here.

[2] Incidentally, the introduction of $\tilde{v}_1$ in (2.12) is required for the proof of contractivity of (3.4).

3.1 The case $n = 3$

3.1.1 Expansions in chart K_2

As the equations in K_2 are completely independent of r_2, we can simply omit the last equation in (2.15), which leaves us with the essentially three-dimensional system

$$\begin{aligned} u_2' &= v_2 \\ v_2' &= -2\eta_2 v_2 - u_2 v_2 \\ \eta_2' &= -\eta_2^2. \end{aligned} \tag{3.6}$$

Remark 3.2. In contrast to what is usually done in the literature, we do not intend to derive asymptotic expansions for the *solutions* to (2.4) here, but rather for the *manifold* $\mathcal{W}^s$ as defined in Section 2. As the solutions to (2.4),(2.5) clearly do depend on ε, however, any ansatz aimed at obtaining solution expansions would of course have to take into account this dependence on ε. Note that in our approach, ε enters only in chart K_1, see below. □

Given Proposition 2.2, we can make an ansatz for the expansion of $\mathcal{W}_2^s$ of the form

$$v_2(u_2, \eta_2) = \sum_{j=0}^{\infty} C_j(\eta_2)(u_2 - 1)^j, \tag{3.7}$$

where

$$C_j(\eta_2) := \frac{1}{j!} \frac{\partial^j}{\partial u_2^j} v_2(u_2, \eta_2)\Big|_{u_2=1}, \tag{3.8}$$

see [vGKS]. Hence

$$\left| v_2(u_2, \eta_2) - \sum_{j=0}^{N} C_j(\eta_2)(u_2 - 1)^j \right| = \mathcal{O}\left((u_2 - 1)^{N+1}\right) \tag{3.9}$$

for any $N \in \mathbb{N}$, and the above estimate is uniform for η_2 bounded.

Remark 3.3. An equally valid ansatz would be to set

$$u_2(v_2, \eta_2) = \sum_{j=0}^{\infty} D_j(\eta_2) v_2^j, \tag{3.10}$$

with

$$D_j(\eta_2) := \frac{1}{j!} \frac{\partial^j}{\partial v_2^j} u_2(v_2, \eta_2)\Big|_{v_2=0}. \tag{3.11}$$

However, the reason for considering (3.7) and not (3.10) is that ultimately we are interested in deriving an expansion for v_{1_ε}. Given Lemma 2.1, it is therefore the ansatz in (3.7) we have to use. □

Rewriting (3.6) with η_2 as the independent variable and omitting the subscript 2, we obtain

$$\frac{du}{d\eta} = -\frac{v}{\eta^2} \tag{3.12a}$$

$$\frac{dv}{d\eta} = \frac{2}{\eta}v + \frac{u-1}{\eta^2}v + \frac{v}{\eta^2}; \tag{3.12b}$$

inserting (3.7) into (3.12b) yields

$$\begin{aligned}\sum_{j=0}^{\infty}\left[\frac{dC_j}{d\eta}(u-1)^j - C_j j(u-1)^{j-1}\left(\frac{1}{\eta^2}\sum_{k=0}^{\infty}C_k(u-1)^k\right)\right] = \\ = \frac{2}{\eta}\sum_{j=0}^{\infty}C_j(u-1)^j + \frac{1}{\eta^2}\sum_{j=0}^{\infty}C_j(u-1)^{j+1} + \frac{1}{\eta^2}\sum_{j=0}^{\infty}C_j(u-1)^j,\end{aligned} \tag{3.13}$$

where we have used (3.12a). Collecting powers of $u-1$ in (3.13) gives a recursive sequence of differential equations for the coefficient functions in (3.7),

$$C_1' - \frac{C_1}{\eta}\left(\frac{C_1}{\eta} + \frac{1}{\eta} + 2\right) = 0 \tag{3.14a}$$

$$C_j' - \frac{C_j}{\eta}\left(2 + \frac{1}{\eta}\right) - \frac{j+1}{\eta^2}C_1C_j = \frac{1}{\eta^2}C_{j-1} + \frac{1}{\eta^2}\sum_{\substack{k+l=j+1\\ k,l\geq 2}} kC_kC_l, \quad j \geq 2, \tag{3.14b}$$

with initial conditions

$$C_1(0) = -1, \quad C_2(0) = -\frac{1}{2}, \quad C_j(0) = 0,\ j \geq 3; \tag{3.15}$$

note that $C_0 \equiv 0$ due to $v(1) = 0$. (3.15) is obtained from Remark 2.1, as $\mathcal{W}^{ss}$ is given by

$$v(u,0) = -(u-1) - \frac{1}{2}(u-1)^2. \tag{3.16}$$

We will first explicitly solve these equations for $j = 1$ and afterwards derive the general form of the solution for j arbitrary.

Remark 3.4. Most of the following computations have been performed with the help of the computer algebra package Maple, see e.g. [Cor02]. □

From (3.14a) it follows that

$$C_1(\eta) = -\frac{\eta^2}{\eta - e^{\eta^{-1}}\tilde{E}_1(\eta^{-1}) - \gamma_1 e^{\eta^{-1}}}, \tag{3.17}$$

where $\tilde{E}_k$ is in general defined by[3]

$$\tilde{E}_k(z) := \int_1^{\infty} e^{-z\tau}\tau^{-k}\,d\tau, \qquad z \in \mathbb{C},\ \Re(z) > 0,\ k \in \mathbb{N}, \tag{3.18}$$

[3]Here $\Re(z)$ denotes the real part of z.

see e.g. [AS64], and $\gamma_1 \in \mathbb{R}$ is some constant to be determined. With (3.15) and de l'Hôspital's rule we obtain $\lim_{\eta\to 0} C_1(\eta) = -1$ for $\gamma_1 = 0$; hence indeed $\gamma_1 = 0$. Due to

$$\tilde{E}_2\left(\eta^{-1}\right) = e^{-\eta^{-1}} - \eta^{-1}\tilde{E}_1\left(\eta^{-1}\right) \tag{3.19}$$

we can write

$$C_1(\eta) = -\frac{\eta e^{-\eta^{-1}}}{\tilde{E}_2\left(\eta^{-1}\right)}. \tag{3.20}$$

As we are interested in (3.7) for $\eta \to \infty$ (which corresponds to the overlap domain between the two charts K_1 and K_2), we expand $\tilde{E}_2\left(\eta^{-1}\right)^{-1}$ about $\eta = \infty$ to obtain an indication as to what the C_js might look like in general:

$$\begin{aligned}\tilde{E}_2\left(\eta^{-1}\right)^{-1} = 1 + (1-\gamma)\eta^{-1} + \eta^{-1}\ln\eta + \left(\gamma^2 - 2\gamma + \frac{3}{2}\right)\eta^{-2} + \\ + 2\gamma\eta^{-2}\ln\eta + \eta^{-2}(\ln\eta)^2 + \mathcal{O}\left(\eta^{-3}\right),\end{aligned} \tag{3.21}$$

which implies

$$C_1(\eta) = \eta e^{-\eta^{-1}} \sum_{k,l=0}^{\infty} \gamma_{kl}^1 \eta^{-k}(\ln\eta)^l. \tag{3.22}$$

We will show that C_j can in fact be expanded as in (3.22) for any $j \in \mathbb{N}$. To that end, note that e.g. for $j = 2$, equation (3.14b) becomes

$$C_2' - \frac{C_2}{\eta}\left(2 + \frac{1}{\eta}\right) + \frac{3e^{-\eta^{-1}}}{\eta\tilde{E}_2\left(\eta^{-1}\right)}C_2 = -\frac{e^{-\eta^{-1}}}{\eta\tilde{E}_2\left(\eta^{-1}\right)}, \tag{3.23}$$

which has the solution

$$\begin{aligned}C_2(\eta) = \left(-\int e^{3\int e^{-\eta'^{-1}}\eta'^{-1}\tilde{E}_2\left(\eta'^{-1}\right)^{-1}d\eta'}\eta^{-3}\tilde{E}_2\left(\eta^{-1}\right)^{-1}d\eta + \gamma_2\right)\times \\ \times\eta^2 e^{-\eta^{-1}} e^{-3\int e^{-\eta^{-1}}\eta^{-1}\tilde{E}_2\left(\eta^{-1}\right)^{-1}d\eta};\end{aligned} \tag{3.24}$$

here we have used (3.20). (3.24) obviously cannot be integrated in closed form. Still, one can derive the following result concerning the structure not only of C_2, but of any C_j with $j \geq 2$:

Proposition 3.1. *For $j \geq 1$, the solution $C_j(\eta)$ to (3.14),(3.15) can be written as*

$$C_j(\eta) = \eta e^{-\eta^{-1}} \sum_{k,l=0}^{\infty} \gamma_{kl}^j \eta^{-k}(\ln\eta)^l. \tag{3.25}$$

Here $\gamma_{kl}^j \in \mathbb{R}$ are constants to be determined from (3.15).

Proof. The proof is by an induction argument: for $i = 1$, the assertion is obviously valid, see (3.22); let us assume that it holds for $i = 1, \dots, j-1$. For the homogeneous solution to (3.14b) one finds

$$C_j^{hom}(\eta) = \gamma_j \eta^2 e^{-\eta^{-1}} e^{-(j+1)\int e^{-\eta^{-1}} \eta^{-1} \tilde{E}_2\left(\eta^{-1}\right)^{-1} d\eta}; \tag{3.26}$$

the integrand in (3.26) can be expanded as

$$-\frac{e^{-\eta^{-1}}}{\eta \tilde{E}_2\left(\eta^{-1}\right)} = -\eta^{-1} + \gamma\eta^{-2} - \eta^{-2}\ln\eta + \mathcal{O}\left(\eta^{-3}\right), \tag{3.27}$$

whence

$$e^{(j+1)\int\left[-\eta^{-1} + \gamma\eta^{-2} - \eta^{-2}\ln\eta + \mathcal{O}\left(\eta^{-3}\right)\right] d\eta} = \mathcal{O}\left(\eta^{-j-1}\right), \quad j \geq 2. \tag{3.28}$$

For (3.26) the claim now follows from (3.21), (3.28), and the following lemma:

Lemma 3.1. *For any $\alpha, \beta \in \mathbb{Z}$,*

$$\int z^\alpha (\ln z)^\beta \, dz = \begin{cases} \dfrac{z^{\alpha+1}(\ln z)^\beta}{\alpha+1} - \dfrac{\beta}{\alpha+1}\displaystyle\int z^\alpha (\ln z)^{\beta-1} \, dz, & \alpha \neq -1 \\ \dfrac{(\ln z)^{\beta+1}}{\beta+1}, & \alpha = -1. \end{cases} \tag{3.29}$$

For the particular solution, note first that by the induction hypothesis, the right-hand side of (3.14b) can be written as

$$\frac{1}{\eta^2} C_{j-1} + \frac{1}{\eta^2} \sum_{\substack{k+l=j+1 \\ k,l\geq 2}} k C_k C_l = e^{-\eta^{-1}} \sum_{m,n=0}^{\infty} \tilde{\gamma}_{mn}^j \eta^{-m} (\ln\eta)^n. \tag{3.30}$$

A particular solution to

$$C_j' - \frac{C_j}{\eta}\left(2 + \frac{1}{\eta}\right) + (j+1)\frac{e^{-\eta^{-1}}}{\eta\tilde{E}_2\left(\eta^{-1}\right)} C_j = e^{-\eta^{-1}} \eta^{-m} (\ln\eta)^n \tag{3.31}$$

is given by

$$\begin{aligned} C_j^{part}(\eta) = \int \eta^{-m-2} (\ln\eta)^n e^{(j+1)\int e^{-\eta'^{-1}} \eta'^{-1} \tilde{E}_2\left(\eta'^{-1}\right)^{-1} d\eta'} d\eta \times \\ \times \eta^2 e^{-\eta^{-1}} e^{-(j+1)\int e^{-\eta^{-1}} \eta^{-1} \tilde{E}_2\left(\eta^{-1}\right)^{-1} d\eta}; \end{aligned} \tag{3.32}$$

With (3.21), (3.28), and Lemma 3.1, this concludes the proof, as $m, n \geq 0$ and $j \geq 2$. □

We can even obtain a somewhat more precise result on the structure of C_j, $j \geq 1$. Let (k, l) denote the *index* of a term $\eta^{-k}(\ln\eta)^l$ in (3.25); given this notation, we have the following

Proposition 3.2. *A term with index (k, l) can occur in (3.25) only if $l \leq k$.*

Proof. The proof is again by induction: for $i = 1$, the assertion is obvious from (3.20) and (3.21). Given the assertion for $i = 1, \dots j-1$, it follows immediately from (3.28) and Lemma 3.1 that it holds for the homogeneous part (3.26) of C_j, as well. To prove the assertion for (3.32), we proceed as follows: as in [vGKS] we define a map, say, $\iota(m, n)$, which assigns to the index of a term in (3.30) the set of indices of the terms it generates in (3.32). By (3.28) and the proof of Proposition 3.1 one then easily sees that with Lemma 3.1,

$$\iota(m,n) = \begin{cases} \{(m+m', n+n'), (m+m', n+n'-1), \dots, (m+m', 0)\}, & m \neq j \\ \{(j+m'+1, n+n'+1)\}, & m = j; \end{cases} \tag{3.33}$$

here $m', n' \in \mathbb{N}$ with $n' \leq m'$. This completes the proof, as $n \leq m$ by assumption. □

3.1.2 Expansions in chart K_1

Given Proposition 3.1, we are able to derive asymptotic expansions for $\mathcal{W}_2^s$ in K_2 for $\eta_2 \to \infty$. To get the desired expansion for v_{1_ε}, however, we need to know what these expansions correspond to in K_1. First, note that with Lemma 2.1, (3.7) becomes

$$\varepsilon_1^{-1} v_1(u_1, \varepsilon_1) = \sum_{j=0}^{\infty} A_j(\varepsilon_1)(u_1 - 1)^j, \tag{3.34}$$

where

$$A_j(\varepsilon_1) = \frac{e^{-\varepsilon_1}}{\varepsilon_1} \sum_{\substack{k,l=0 \\ l \leq k}}^{\infty} \alpha_{kl}^j \varepsilon_1^k (\ln\varepsilon_1)^l; \tag{3.35}$$

here $\alpha_{kl}^j = (-1)^l \gamma_{kl}^j$. It remains to show that (3.35) does indeed make sense for $\varepsilon_1 \to 0$ (which is equivalent to $\eta_2 \to \infty$ in K_2). To that end, we assume that a curve of initial conditions in Σ_1^{out} of the form

$$(u_1, v_1, \varepsilon_1) = \left(u_1^{out}, v_1^{out}\left(u_1^{out}\right), \delta\right), \qquad v_1^{out}(1) = 0 \tag{3.36}$$

is given, and we investigate the corresponding invariant manifold consisting of segments of solutions of (2.12). By variation of constants integrating backwards from

Σ_1^{out}, this manifold can be represented as follows:

$$u_1\big(\xi_1, u_1^{out}\big) = u_1^{out} - \int_{\xi_1}^{\Xi} v_1\big(\xi', u_1^{out}\big)\, d\xi' \tag{3.37a}$$

$$v_1\big(\xi_1, u_1^{out}\big) = \frac{\delta}{\varepsilon} v_1^{out}\big(u_1^{out}\big)\, e^{-\xi_1} + e^{-\xi_1} \int_{\xi_1}^{\Xi} e^{\xi'} \varepsilon_1(\xi') u_1\big(\xi', u_1^{out}\big)\, v_1\big(\xi', u_1^{out}\big)\, d\xi' \tag{3.37b}$$

$$\varepsilon_1(\xi_1) = \varepsilon e^{\xi_1}, \tag{3.37c}$$

where $\Xi = \ln \frac{\delta}{\varepsilon}$. We have the following result:

Proposition 3.3. *Let $v_1^{out}\big(u_1^{out}\big)$ be $\mathcal{C}^k$-smooth for some $k \in \mathbb{N}$. Then, for $j = 0, \ldots, k$, $\frac{\partial^j}{\partial u_1^j} v_1(u_1, \varepsilon_1)$ exists and is continuous for $\varepsilon_1 \in [0, \delta]$ and $|u_1 - 1| \leq \beta$ with $\beta > 0$ sufficiently small.*

Proof. Changing the integration variable to ε_1 in (3.37), we obtain

$$u_1(\varepsilon_1) = u_1^{out} - \int_{\varepsilon_1}^{\delta} v_1(u_1(\varepsilon'), \varepsilon')\, \frac{d\varepsilon'}{\varepsilon'} \tag{3.38a}$$

$$v_1(u_1, \varepsilon_1) = \frac{\delta}{\varepsilon_1} v_1^{out} + \frac{1}{\varepsilon_1} \int_{\varepsilon_1}^{\delta} \varepsilon' u_1(\varepsilon') v_1(u_1(\varepsilon'), \varepsilon')\, d\varepsilon'; \tag{3.38b}$$

here $\varepsilon' = \varepsilon e^{\xi'}$. Then (3.38) together with

$$u_1(\varepsilon') \sim u_1^{out} + v_1^{out}\left(1 - \frac{\delta}{\varepsilon'}\right), \qquad v_1(u_1(\varepsilon'), \varepsilon') \sim \frac{\delta}{\varepsilon'} v_1^{out} \tag{3.39}$$

implies that v_1 remains continuous in (u_1, ε_1) for $\varepsilon_1 \to 0$. Differentiating (3.38) formally with respect to u_1 yields

$$1 = \frac{du_1^{out}}{du_1} - \int_{\varepsilon_1}^{\delta} \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1}\, d\varepsilon' \tag{3.40a}$$

$$\frac{\partial v_1(u_1, \varepsilon_1)}{\partial u_1} = \frac{\delta}{\varepsilon_1} \frac{dv_1^{out}}{du_1} + \frac{1}{\varepsilon_1} \int_{\varepsilon_1}^{\delta} \varepsilon' \left[\frac{du_1(\varepsilon')}{du_1} v_1(u_1(\varepsilon'), \varepsilon') + u_1(\varepsilon') \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1}\right] d\varepsilon'; \tag{3.40b}$$

as we have $\frac{d\varepsilon'}{d\varepsilon_1} = \frac{\varepsilon'}{\varepsilon_1}$, it follows that

$$\frac{du_1(\varepsilon')}{du_1} = \frac{v_1(u_1(\varepsilon'), \varepsilon')}{v_1(u_1(\varepsilon_1), \varepsilon_1)}, \tag{3.41}$$

whence

$$\frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1} = \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1(\varepsilon')} \frac{v_1(u_1(\varepsilon'), \varepsilon')}{v_1(u_1(\varepsilon_1), \varepsilon_1)}. \tag{3.42}$$

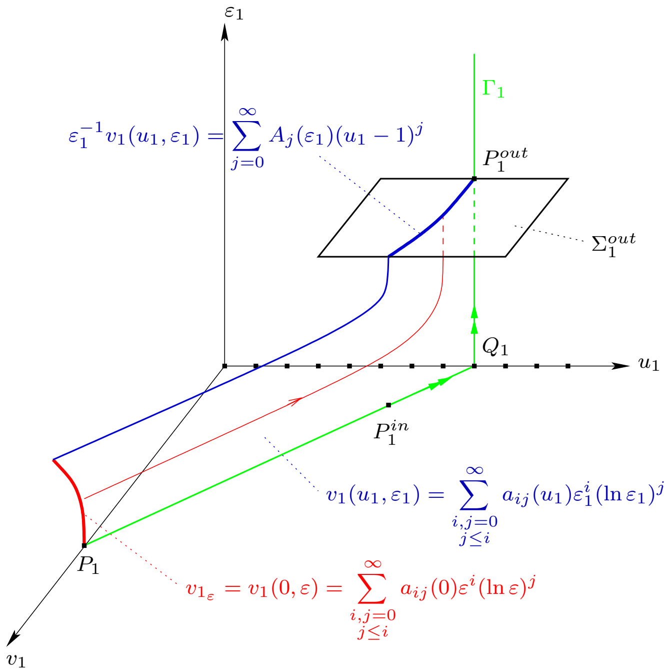

Figure 5: Strategy for deriving an expansion for v_{1_ε} in K_1 $(n = 3)$.

This, together with (3.40a), gives us a formula for $\frac{dv_1^{out}}{du_1} = \frac{dv_1^{out}}{du_1^{out}}\frac{du_1^{out}}{du_1}$,

$$\frac{dv_1^{out}}{du_1} = \frac{dv_1^{out}}{du_1^{out}}\left(1 + \frac{1}{v_1(u_1,\varepsilon_1)}\int_{\varepsilon_1}^{\delta}\frac{\partial v_1(u_1(\varepsilon'),\varepsilon')}{\partial u_1(\varepsilon')}\frac{v_1(u_1(\varepsilon'),\varepsilon')}{\varepsilon'}\,d\varepsilon'\right); \tag{3.43}$$

in sum, we obtain

$$\begin{aligned}\frac{\partial v_1(u_1,\varepsilon_1)}{\partial u_1} &= \frac{\delta}{\varepsilon_1}\frac{dv_1^{out}}{du_1^{out}}\left(1 + \frac{1}{v_1(u_1,\varepsilon_1)}\int_{\varepsilon_1}^{\delta}\frac{\partial v_1(u_1(\varepsilon'),\varepsilon')}{\partial u_1(\varepsilon')}\frac{v_1(u_1(\varepsilon'),\varepsilon')}{\varepsilon'}\,d\varepsilon'\right) + \\ &+ \frac{1}{\varepsilon_1 v_1(u_1,\varepsilon_1)}\int_{\varepsilon_1}^{\delta}\varepsilon'\left[v_1(u_1(\varepsilon'),\varepsilon') + u_1(\varepsilon')\frac{\partial v_1(u_1(\varepsilon'),\varepsilon')}{\partial u_1(\varepsilon')}\right]v_1(u_1(\varepsilon'),\varepsilon')\,d\varepsilon'.\end{aligned} \tag{3.44}$$

Suppose now that $v_1^{out}(u_1^{out})$ is $\mathcal{C}^1$-smooth; using a standard fixed point argument, one can show that (3.44) has a unique solution $\frac{\partial v_1(u_1,\varepsilon_1)}{\partial u_1}$ which is continuous in (u_1, ε_1). This concludes the proof for $k = 1$; the argument for $k \geq 2$ is similar. □

Inspired by (3.34) and (3.35), we feel induced to attempt an expansion of $v_1(u_1, \varepsilon_1)$ as

$$v_1(u_1,\varepsilon_1) = \sum_{\substack{i,j=0\\ j\le i}}^{\infty} a_{ij}(u_1)\varepsilon_1^i(\ln\varepsilon_1)^j, \tag{3.45}$$

see Figure 5; the requirement that $j \leq i$ in the summation in (3.45) is a consequence of Proposition 3.2. The coefficient functions a_{ij}, $0 \leq j \leq i$, will be determined uniquely by the demand for smoothness in u_1 and by the requirement that, seen as a double expansion, (3.45) should agree with (3.34). To that end, let us introduce u_1 as the independent variable in (2.12), whence[4]

$$\begin{aligned} \frac{dv}{du} &= -1 - \varepsilon u \\ \frac{d\varepsilon}{du} &= \frac{\varepsilon}{v}. \end{aligned} \tag{3.46}$$

Remark 3.5. With (3.10) instead of (3.7) in K_2, we would now have

$$u(v, \varepsilon) = \sum_{\substack{i,j=0 \\ j \leq i}}^{\infty} b_{ij}(v) \varepsilon^i (\ln \varepsilon)^j \tag{3.47}$$

instead of (3.45). One can easily check that the following considerations would then go through just the same, with only a few minor adjustments required. □

By proceeding just as in K_2 and multiplying the resulting equations with v, we obtain

$$\begin{aligned} \sum_{\substack{i,j=0 \\ j \leq i}}^{\infty} \left[a'_{ij} \varepsilon^i (\ln \varepsilon)^j \left(\sum_{\substack{k,l=0 \\ l \leq k}}^{\infty} a_{kl} \varepsilon^k (\ln \varepsilon)^l \right) + a_{ij} \varepsilon^i (\ln \varepsilon)^{j-1} (i \ln \varepsilon + j) \right] = \\ = (-1 - \varepsilon u) \sum_{\substack{i,j=0 \\ j \leq i}}^{\infty} a_{ij} \varepsilon^i (\ln \varepsilon)^j, \end{aligned} \tag{3.48}$$

where $' = \frac{d}{du}$ now. We do not bother to look for the general solution to (3.48) right now, but will for the moment only consider the first few terms in (3.45); hence e.g. for $0 \leq j \leq i \leq 2$,

$$a'_{00} a_{00} = -a_{00} \tag{3.49a}$$

$$a'_{11} a_{00} + a'_{00} a_{11} + a_{11} = -a_{11} \tag{3.49b}$$

$$a'_{10} a_{00} + a'_{00} a_{10} + a_{10} + a_{11} = -a_{10} - u a_{00} \tag{3.49c}$$

$$a'_{22} a_{00} + a'_{11} a_{11} + a'_{00} a_{22} + 2a_{22} = -a_{22} \tag{3.49d}$$

$$a'_{21} a_{00} + a'_{11} a_{10} + a'_{10} a_{11} + a'_{00} a_{21} + 2a_{21} + 2a_{22} = -a_{21} - u a_{11} \tag{3.49e}$$

$$a'_{20} a_{00} + a'_{10} a_{10} + a'_{00} a_{20} + 2a_{20} + a_{21} = -a_{20} - u a_{10}. \tag{3.49f}$$

Note that these equations can be solved recursively: (3.49a) yields either $a_{00} \equiv 0$ or $a'_{00} = -1$; however, for (3.34) and (3.45) to agree when seen as double expansions, we have to take the latter, see (3.20), whence

$$a_{00} = -u + \alpha^{00}. \tag{3.50}$$

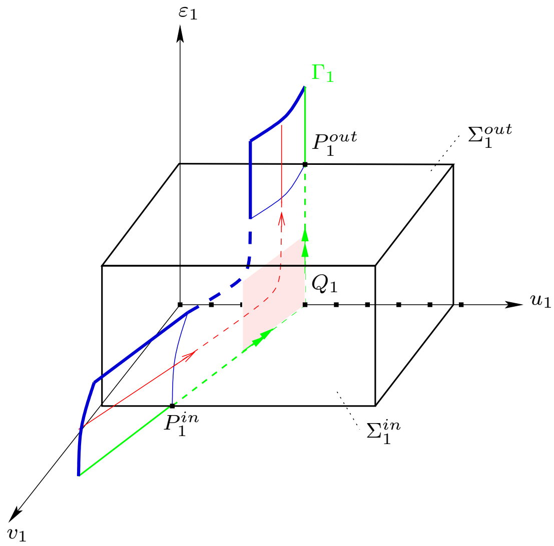

Figure 6: Overlap domain (shaded) of expansions (3.34) and (3.45).

Here α^{00} is a constant which is to be determined from (3.34); indeed, it follows from (3.20) and (3.21) that an expansion for v is given by

$$v(u,\varepsilon) = \left[-1 + \gamma\varepsilon + \varepsilon \ln \varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right](u-1) + \mathcal{O}\left((u-1)^2\right). \tag{3.51}$$

To obtain agreement to lowest order between (3.34) and (3.45), we hence have to take $\alpha^{00} = 1$.

By plugging (3.50) into (3.49b) and solving the resulting equation

$$-a'_{11}(u-1) + a_{11} = 0, \tag{3.52}$$

one then has

$$a_{11} = \alpha^{11}(u-1). \tag{3.53}$$

Similarly, (3.50) and (3.53) together with (3.49c) give

$$-a'_{10}(u-1) + a_{10} = (u - \alpha^{11})(u-1), \tag{3.54}$$

which has the solution

$$a_{10} = -(u-1)^2 - (1-\alpha^{10})(u-1) - (1-\alpha^{11})(u-1)\ln(u-1); \tag{3.55}$$

for (3.55) to be smooth, α^{11} has to be chosen such that the $\ln(u-1)$-terms in (3.56) vanish, which implies $\alpha^{11} = 1$. The requirement that (3.34) and (3.45) should agree then gives $\alpha^{10} = 1+\gamma$, see (3.51):

$$a_{10} = -(u-1)^2 + \gamma(u-1). \tag{3.56}$$

[4] Again the subscript 1 will be omitted.

For (3.49d), (3.49e), and (3.49f), one obtains by the same procedure

$$a_{22} = (u-1)^2 - (u-1) \tag{3.57a}$$

$$a_{21} = (2\gamma+1)(u-1)^2 - (2\gamma-1)(u-1) \tag{3.57b}$$

$$a_{20} = (u-1)^3 + (1+\alpha^{20})(u-1)^2 - (\gamma^2-\gamma+1)(u-1), \tag{3.57c}$$

where $\alpha^{22} = 1$ and $\alpha^{21} = 2\gamma+1$ have again been chosen such that the $\ln(u-1)$-terms in (3.57b) and (3.57c) cancel and α^{20} has to be determined by comparing the leading-order terms in (3.34) and (3.45).

Remark 3.6. The above procedure is in fact closely related to the approach one would classically take when matching (3.34) and (3.45). As these two expansions have to agree on the overlap domain between the two charts K_1 and K_2, it is there one would have to define an *intermediate variable*. Note that in [HTB90], say, logarithmic switchback is handled using a modified version of the block matching principle introduced by [vD75]: terms are matched in blocks according to the powers of ε they contain, with no distinction being made for any additional logarithmic factors. Our approach seems to justify this principle, as the logarithmic factors in (3.45) are determined simply by the requirement that a_{ij} be smooth. □

One expects, of course, that the above procedure can be carried out to any order in i and j in (3.48), where for i fixed one starts with $j = i$ and then proceeds recursively down to $j = 0$. That this is indeed possible is contained in the following result:

Proposition 3.4. *There exist unique smooth functions $a_{ij}(u_1)$ such that (3.34) and (3.45), seen as double expansions, are the same.*

Proof. We proceed just as in computing the first few coefficients of (3.45) above: for fixed i we successively solve (3.48), starting with $j = i$. By induction we will establish

$$a_{ij} = \sum_{k=1}^{i} \alpha_k^{ij}(u-1)^k, \qquad 1 \le j \le i, \tag{3.58a}$$

$$a_{i0} = \sum_{k=1}^{i+1} \alpha_k^{i0}(u-1)^k, \qquad j = 0 \tag{3.58b}$$

for any $i \ge 1$ and $0 \le j \le i$, with some constants $\alpha_k^{ij} \in \mathbb{R}$ to be chosen appropriately. Indeed, for $i = 1$ the claim follows from (3.53) and (3.56) by inspection. Let us assume that (3.58) is valid for a_{kl}, where $k = 1, \dots, i-1$ and $l \le k$. By plugging (3.50) into (3.48) and collecting powers of $\varepsilon \ln \varepsilon$, one obtains

$$-a'_{ij}(u-1) + i a_{ij} = -(j+1)a_{i,j+1} - u a_{i-1,j} - \sum_{\substack{k+m=i \\ l+n=j \\ l \le k \le i-1,\ n \le m \le i-1}} a'_{kl} a_{mn}; \tag{3.59}$$

here we have used $a'_{00} = -1$. The homogeneous solution to (3.59) is given by

$$\alpha^{ij}(u-1)^i, \qquad 0 \le j \le i \tag{3.60}$$

with some constant α^{ij}; to complete the proof, we have to consider the following cases:

- for $j = i$, equation (3.59) becomes

$$-a'_{ii}(u-1) + i a_{ii} = -\sum_{\substack{k+m=i \\ k,m\ge 1}} a'_{kk} a_{mm}, \tag{3.61}$$

  which has the solution

$$a_{ii} = \alpha^{ii}(u-1)^i + \sum_{k=1}^{i-1} \alpha_k^{ii}(u-1)^k, \tag{3.62}$$

  as by (3.58)

$$-\sum_{\substack{k+m=i \\ k,m\ge 1}} a'_{kk} a_{mm} = -\sum_{k=1}^{i-1} \tilde{\alpha}_k^{ii}(u-1)^k \tag{3.63}$$

  and a term $-\tilde{\alpha}_k^{ii}(u-1)^k$ gives a particular solution of the form

$$-\frac{\tilde{\alpha}_k^{ii}}{i-k}(u-1)^k, \qquad 1 \le k \le i-1; \tag{3.64}$$

  the constant α^{ii} remains to be determined in one of the next steps.

- for $j = i-1$ (which is indeed representative of all further cases), one obtains

$$\begin{aligned} -a'_{i,i-1}(u-1) + i a_{i,i-1} = &-i a_{ii} - u a_{i-1,i-1} - \\ &- \sum_{\substack{k+m=i \\ k,m\ge 1}} \left[a'_{kk} a_{m,m-1} + a_{kk} a'_{m,m-1} \right], \end{aligned} \tag{3.65}$$

  where the homogeneous solution is again given by (3.60). As for the inhomogeneity, note that terms of the form $-i\alpha_k^{ii}(u-1)^k$ in $-i a_{ii}$ generate particular solutions of the form

$$i\alpha^{ii}(u-1)^i \ln(u-1), \qquad k = i, \tag{3.66a}$$

$$-\frac{i\alpha_k^{ii}}{i-k}(u-1)^k, \qquad 1 \le k \le i-1. \tag{3.66b}$$

  Similarly, for the terms $-\alpha_k^{i-1,i-1} u(u-1)^k$ in $-u a_{i-1,i-1}$ one obtains

$$\alpha_{i-1}^{i-1,i-1}(u-1)^{i-1}((u-1)\ln(u-1) - 1), \qquad k = i-1, \tag{3.67a}$$

$$-\frac{\alpha_{i-2}^{i-1,i-1}}{(i-k)(i-k-1)}(u-1)^k(-1 + (i-k)u), \qquad 1 \le k \le i-2. \tag{3.67b}$$

By the induction hypothesis, for the remaining terms one has

$$-\sum_{\substack{k+m=i \\ k,m\geq 1}} \left[a'_{kk}a_{m,m-1} + a_{kk}a'_{m,m-1}\right] = -\sum_{k=0}^{i} \tilde{\alpha}_k^{i,i-1}(u-1)^k \tag{3.68}$$

for some constants $\tilde{\alpha}_k^{i,i-1} \in \mathbb{R}$. Just as above, the terms $-\tilde{\alpha}_k^{i,i-1}(u-1)^k$ give rise to terms of the form

$$\tilde{\alpha}_i^{i,i-1}(u-1)^i \ln(u-1), \qquad k = i, \tag{3.69a}$$

$$-\frac{\tilde{\alpha}_k^{i,i-1}}{i-k}(u-1)^k, \qquad 1 \leq k \leq i-1. \tag{3.69b}$$

In sum, one thus has

$$\begin{aligned} a_{i,i-1} = \alpha^{i,i-1}(u-1)^i + \sum_{k=1}^{i-1} \alpha_k^{i,i-1}(u-1)^k + \\ + \left(i\alpha^{ii} + \alpha_{i-1}^{i-1,i-1} + \tilde{\alpha}_i^{i,i-1}\right) \ln(u-1)(u-1)^i, \end{aligned} \tag{3.70}$$

where α^{ii} is now chosen such that (3.70) is smooth, i.e., such that the $\ln(u-1)$-terms cancel, and $\alpha^{i,i-1}$ still is at our disposal.

- for $1 \leq j \leq i-2$ in general, one repeats the same procedure, i.e., one solves (3.59) and subsequently fixes $\alpha^{i,j+1}$ appropriately so as to eliminate any $\ln(u-1)$-terms in a_{ij}, guaranteeing the smoothness of a_{ij}.
- in the final step, for $j = 0$, additional terms of $\mathcal{O}\left((u-1)^{i+1}\right)$ are generated as claimed due to the terms $-\alpha_i^{i-1,0}u(u-1)^i$ and $-\tilde{\alpha}_i^{i0}(u-1)^{i+1}$ in the right-hand side of (3.59) giving
$$\alpha_i^{i-1,0}\left(u + \ln(u-1)\right)(u-1)^i \tag{3.71}$$
and
$$\tilde{\alpha}_i^{i0}u(u-1)^i \tag{3.72}$$
in a_{i0}, respectively. One is then left with α^{i1} and α^{i0}, which one chooses such as to make sure that a_{i0} is smooth and that (3.34) and (3.45) agree if both are seen as double expansions, which concludes the proof.

□

We are now ready to formulate the main result of this section, namely, to give an expansion of v_{1_ε} for $n = 3$:

Proposition 3.5. *For $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small, $v_{1_\varepsilon} = v_1(0, \varepsilon)$ can be expanded as*

$$v_{1_\varepsilon} = 1 - \varepsilon \ln \varepsilon - (\gamma + 1)\varepsilon + 2\varepsilon^2(\ln \varepsilon)^2 + 4\gamma\varepsilon^2 \ln \varepsilon + \mathcal{O}\left(\varepsilon^3\right). \tag{3.73}$$

Proof. By plugging (3.50), (3.53), (3.56), (3.57a), and (3.57b) into (3.45), one obtains the following expansion for v_1:

$$\begin{aligned} v_1(u_1,\varepsilon_1) = &-(u_1-1) + (u_1-1)\varepsilon_1 \ln\varepsilon_1 + \left[-(u_1-1)^2+\gamma(u_1-1)\right]\varepsilon_1 + \\ &+ \left[(u_1-1)^2-(u_1-1)\right]\varepsilon_1^2(\ln\varepsilon_1)^2 + \\ &+ \left[(2\gamma+1)(u_1-1)^2-(2\gamma-1)(u_1-1)\right]\varepsilon_1^2\ln\varepsilon_1 + \mathcal{O}\left(\varepsilon_1^3\right). \end{aligned} \tag{3.74}$$

The assertion now follows with $u_1 = 0$ and $\varepsilon_1 = \varepsilon$ in (3.74). □

Remark 3.7. Note that the expansion in (3.73) is equally valid in the original setting of (2.4), i.e., $v_\varepsilon = v_{1_\varepsilon}$, as is easily seen by performing the appropriate blow-down transformation, which is trivial here. Analogous results have been obtained in the literature, see e.g. [Lag88]. □

3.2 The case $n = 2$

Although the case $n = 2$ is potentially more difficult, we can use the same strategy as before to compute expansions for v_{1_ε}. Not surprisingly, however, computationally the analysis is more involved now, due to the extensive switchback which arises in the matching process for $n = 2$.

3.2.1 Expansions in chart K_2

As the situation in K_2 is very similar to that for $n = 3$, we will not go into too many details: given the ansatz

$$v_2(u_2,\eta_2) = \sum_{j=0}^{\infty} C_j(\eta_2)(u_2-1)^j, \tag{3.75}$$

which we plug into (3.6) rewritten with η_2 as the independent variable,

$$\frac{du}{d\eta} = -\frac{v}{\eta^2} \tag{3.76a}$$

$$\frac{dv}{d\eta} = \frac{v}{\eta} + \frac{u-1}{\eta^2}v + \frac{v}{\eta^2}, \tag{3.76b}$$

we obtain a recursive sequence of equations for $C_j,\ j \geq 1$, by comparing powers of $u - 1$:

$$C_1' - \frac{C_1}{\eta}\left(\frac{C_1}{\eta} + \frac{1}{\eta} + 1\right) = 0 \tag{3.77a}$$

$$C_j' - \frac{C_j}{\eta}\left(1 + \frac{1}{\eta}\right) - \frac{j+1}{\eta^2}C_1C_j = \frac{1}{\eta^2}C_{j-1} + \frac{1}{\eta^2}\sum_{\substack{k+l=j+1\\ k,l\geq 2}} kC_kC_l, \quad j \geq 2. \tag{3.77b}$$

The initial conditions are again given by

$$C_1(0) = -1, \quad C_2(0) = -\frac{1}{2}, \quad C_j(0) = 0,\ j \geq 3; \tag{3.78}$$

moreover, $C_0 \equiv 0$ just as for $n = 3$. From (3.77a) we have

$$C_1(\eta) = -\frac{\eta e^{-\eta^{-1}}}{\tilde{E}_1\left(\eta^{-1}\right) + \gamma_1}; \tag{3.79}$$

as $\lim_{\eta \to 0} C_1(\eta) = -1$ for $\gamma_1 = 0$, we conclude that $\gamma_1 = 0$ again. The expansion of $\tilde{E}_1\left(\eta^{-1}\right)^{-1}$ about $\eta = \infty$ is given by

$$\begin{aligned}\tilde{E}_1\left(\eta^{-1}\right)^{-1} = (\ln\eta - \gamma)^{-1} - \eta^{-1}(\ln\eta - \gamma)^{-2} + \frac{1}{4}\eta^{-2}(\ln\eta - \gamma)^{-2} + \\ + \eta^{-2}(\ln\eta - \gamma)^{-3} + \mathcal{O}\left(\eta^{-3}\right),\end{aligned} \tag{3.80}$$

whence

$$C_1(\eta) = \eta e^{-\eta^{-1}} \sum_{k,l=0}^{\infty} \gamma_{kl}^1 \eta^{-k}(\ln\eta - \gamma)^{-l}; \tag{3.81}$$

here γ is Euler's constant, as before. For $j = 2$, (3.77b) gives

$$C_2' - \frac{C_2}{\eta}\left(1 + \frac{1}{\eta}\right) + \frac{3e^{-\eta^{-1}}}{\eta\tilde{E}_1\left(\eta^{-1}\right)} C_2 = -\frac{e^{-\eta^{-1}}}{\eta\tilde{E}_1\left(\eta^{-1}\right)}, \tag{3.82}$$

which has the solution

$$C_2(\eta) = \left(-\int \eta^{-2}\tilde{E}_1\left(\eta^{-1}\right)^2 d\eta + \gamma_2\right)\eta e^{-\eta^{-1}}\tilde{E}_1\left(\eta^{-1}\right)^{-3}. \tag{3.83}$$

Although (3.83) cannot be integrated in closed form, we still have the following result, which is similar to the one derived for $n = 3$:

Proposition 3.6. *For $j \geq 1$, the solution $C_j(\eta)$ to (3.77),(3.78) can be written as*

$$C_j(\eta) = \eta e^{-\eta^{-1}} \sum_{k,l=0}^{\infty} \gamma_{kl}^j \eta^{-k}(\ln\eta - \gamma)^{-l}. \tag{3.84}$$

Here $\gamma_{kl}^j \in \mathbb{R}$ are constants to be determined from (3.15).

Proof. The proof is very similar to that of Proposition 3.1: for $i = 1$, the assertion holds by (3.79); let it be valid for $i = 1, \dots, j-1$. The homogeneous solution to (3.77b) is given by

$$C_j^{hom}(\eta) = \gamma_j \frac{\eta e^{-\eta^{-1}}}{\tilde{E}_1\left(\eta^{-1}\right)^{j+1}}, \tag{3.85}$$

where $\tilde{E}_1\left(\eta^{-1}\right)^{-j-1}$ can be expanded as

$$\tilde{E}_1\left(\eta^{-1}\right)^{-j-1} = (\ln\eta-\gamma)^{-j-1} - (j+1)\eta^{-1}(\ln\eta-\gamma)^{-j-2} + \\ + \frac{1}{4}(j+1)\eta^{-2}(\ln\eta-\gamma)^{-j-2} + \frac{1}{2}(j+1)(j+2)\eta^{-2}(\ln\eta-\gamma)^{-j-3} + \mathcal{O}\left(\eta^{-3}\right). \tag{3.86}$$

Given the induction hypothesis, the right-hand side of (3.77b) has the following form:

$$\frac{1}{\eta^2}C_{j-1} + \frac{1}{\eta^2}\sum_{\substack{k+l=j+1\\k,l\geq 1}} kC_kC_l = e^{-\eta^{-1}}\sum_{m,n=0}^{\infty}\tilde{\gamma}^j_{mn}\eta^{-m}(\ln\eta-\gamma)^{-n}; \tag{3.87}$$

a particular solution of (3.77b) corresponding to a term $e^{-\eta^{-1}}\eta^{-m}(\ln\eta-\gamma)^{-n}$ is given by

$$C_j^{part}(\eta) = \int \eta^{-m-1}(\ln\eta-\gamma)^{-n}\tilde{E}_1\left(\eta^{-1}\right)^{j+1}d\eta\cdot\eta e^{-\eta^{-1}}\tilde{E}_1\left(\eta^{-1}\right)^{-j-1}. \tag{3.88}$$

With the substitution $\eta' = \frac{\eta}{\gamma}$ and (3.86), the integrand in (3.88) can be written as

$$\sum_{\substack{m'=m+1\\n'=n-j-1}}^{\infty}\tilde{\gamma}^{mn}_{m'n'}\eta'^{-m'}(\ln\eta')^{-n'}; \tag{3.89}$$

the claim now follows from Lemma 3.1 with η' again replaced by η. □

Note, however, that we can state no analogue to Proposition 3.2 here; in fact, it seems that any index pair (k,l) can occur in (3.45) now.

3.2.2 Expansions in chart K_1

As in the case $n=3$, in order to obtain an expansion for v_{1_ε}, it now remains to translate (3.7) to K_1. Lemma 2.1 again gives

$$\varepsilon_1^{-1}v_1(u_1,\varepsilon_1) = \sum_{j=0}^{\infty}A_j(\varepsilon_1)(u_1-1)^j, \tag{3.90}$$

see (3.34), where

$$A_j(\varepsilon_1) = \frac{e^{-\varepsilon_1}}{\varepsilon_1}\sum_{k,l=0}^{\infty}\alpha^j_{kl}\frac{\varepsilon_1^k}{(\ln\varepsilon_1+\gamma)^l} \tag{3.91}$$

with $\alpha^j_{kl} = (-1)^l\gamma^j_{kl}$.

Remark 3.8 *(Transcendentally small terms).* Terms in powers of ε_1, as well as terms in powers of ε_1 multiplied by powers of $(\ln \varepsilon_1 + \gamma)^{-1}$, are smaller than all positive powers of $(\ln \varepsilon_1 + \gamma)^{-1}$: they are said to be beyond all orders of $(\ln \varepsilon_1 + \gamma)^{-1}$, or to be transcendentally small terms. In the original setting of flow around a circular cylinder, [Ski75] showed how to calculate a few of these terms. He pointed out, however, that they are in fact negligible: it is the only very slight asymmetry in the flow field which indicates the relative insignificance of these inertial terms for low Reynolds numbers, see [Ski75] for a detailed analysis. □

As for $n = 3$, using variation of constants we can again write

$$u_1\big(\xi_1, u_1^{out}\big) = u_1^{out} - \int_{\xi_1}^{\Xi} v_1\big(\xi', u_1^{out}\big)\, d\xi' \tag{3.92a}$$

$$v_1\big(\xi_1, u_1^{out}\big) = v_1^{out}\big(u_1^{out}\big) + \int_{\xi_1}^{\Xi} \varepsilon_1(\xi') u_1\big(\xi', u_1^{out}\big)\, v_1\big(\xi', u_1^{out}\big)\, d\xi' \tag{3.92b}$$

$$\varepsilon_1(\xi_1) = \varepsilon e^{\xi_1} \tag{3.92c}$$

for the manifold consisting of segments of solutions to (2.12) given the initial curve (3.36). In analogy to Proposition 3.3 we now have

Proposition 3.7. *Let $v_1^{out}\big(u_1^{out}\big)$ be $\mathcal{C}^k$-smooth for some $k \in \mathbb{N}$. Then, for $j = 0, \ldots, k$, $\frac{\partial^j}{\partial u_1^j} v_1(u_1, \varepsilon_1)$ exists and is continuous for $\varepsilon_1 \in [0, \delta]$ and $|u_1 - 1| \le \beta$ with $\beta > 0$ sufficiently small.*

Proof. The proof is the same as for $n = 3$, with the relevant relations given by

$$u_1(\varepsilon_1) = u_1^{out} - \int_{\varepsilon_1}^{\delta} v_1(u_1(\varepsilon'), \varepsilon')\, \frac{d\varepsilon'}{\varepsilon'} \tag{3.93a}$$

$$v_1(u_1, \varepsilon_1) = \frac{\delta}{\varepsilon_1} v_1^{out} + \frac{1}{\varepsilon_1} \int_{\varepsilon_1}^{\delta} u_1(\varepsilon') v_1(u_1(\varepsilon'), \varepsilon')\, d\varepsilon' \tag{3.93b}$$

and

$$\begin{aligned}\frac{\partial v_1(u_1, \varepsilon_1)}{\partial u_1} &= \frac{dv_1^{out}}{du_1^{out}} \left(1 + \frac{1}{v_1(u_1, \varepsilon_1)} \int_{\varepsilon_1}^{\delta} \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1(\varepsilon')} \frac{v_1(u_1(\varepsilon'), \varepsilon')}{\varepsilon'}\, d\varepsilon' \right) + \\ &+ \frac{1}{v_1(u_1, \varepsilon_1)} \int_{\varepsilon_1}^{\delta} \left[v_1(u_1(\varepsilon'), \varepsilon') + u_1(\varepsilon') \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1(\varepsilon')} \right] v_1(u_1(\varepsilon'), \varepsilon')\, d\varepsilon'\end{aligned} \tag{3.94}$$

now, respectively. □

To derive an expansion for $v_1(u_1, \varepsilon_1)$ of the form

$$v_1(u_1, \varepsilon_1) = \sum_{i,j=0}^{\infty} a_{ij}(u_1) \frac{\varepsilon_1^i}{(\ln \varepsilon_1 + \gamma)^j}, \tag{3.95}$$

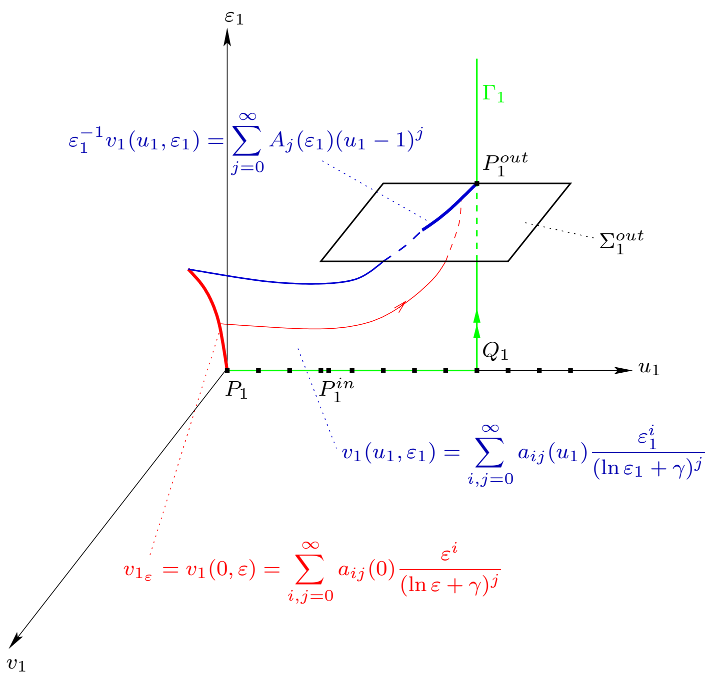

Figure 7: Strategy for deriving an expansion for v_{1_ε} in K_1 ($n = 2$).

we have to consider

$$\begin{aligned} \frac{dv}{du} &= -\varepsilon u \\ \frac{d\varepsilon}{du} &= \frac{\varepsilon}{v}, \end{aligned} \tag{3.96}$$

which yields

$$\sum_{i,j=0}^{\infty} \left[a'_{ij} \frac{\varepsilon^i}{(\ln \varepsilon + \gamma)^j} \left(\sum_{k,l=0}^{\infty} a_{kl} \frac{\varepsilon^k}{(\ln \varepsilon + \gamma)^l} \right) + a_{ij} \frac{\varepsilon^i}{(\ln \varepsilon + \gamma)^j} \left(i - \frac{j}{\ln \varepsilon + \gamma} \right) \right] = \\ = -\varepsilon u \sum_{i,j=0}^{\infty} a_{ij} \frac{\varepsilon^i}{(\ln \varepsilon + \gamma)^j}. \tag{3.97}$$

Collecting powers of $\varepsilon(\ln \varepsilon + \gamma)^{-1}$ in (3.97), we obtain the following sequence of equations for a_{ij} with $0 \leq i + j \leq 3$:

$$a'_{00}a_{00} = 0 \tag{3.98a}$$
$$a'_{01}a_{00} + a'_{00}a_{01} = 0 \tag{3.98b}$$
$$a'_{10}a_{00} + a'_{00}a_{10} + a_{10} = -ua_{00} \tag{3.98c}$$
$$a'_{02}a_{00} + a'_{00}a_{02} + a'_{01}a_{01} - a_{01} = 0 \tag{3.98d}$$
$$a'_{11}a_{00} + a'_{00}a_{11} + a'_{10}a_{01} + a'_{01}a_{10} + a_{11} = -ua_{01} \tag{3.98e}$$
$$a'_{20}a_{00} + a'_{00}a_{20} + a'_{10}a_{10} + 2a_{20} = -ua_{10} \tag{3.98f}$$
$$a'_{03}a_{00} + a'_{00}a_{03} + a'_{02}a_{01} + a'_{01}a_{02} - 2a_{02} = 0 \tag{3.98g}$$
$$a'_{12}a_{00} + a'_{00}a_{12} + a'_{11}a_{01} + a'_{01}a_{11} + a'_{10}a_{02} + a'_{02}a_{10} + a_{12} - a_{11} = -ua_{02} \tag{3.98h}$$
$$a'_{21}a_{00} + a'_{00}a_{21} + a'_{20}a_{01} + a'_{01}a_{20} + a'_{11}a_{10} + a'_{10}a_{11} + 2a_{21} = -ua_{11} \tag{3.98i}$$
$$a'_{30}a_{00} + a'_{00}a_{30} + a'_{20}a_{10} + a'_{10}a_{20} + 3a_{30} = -ua_{20}. \tag{3.98j}$$

Here we have proceeded by diagonalization: as we do not have such precise information on the structure of (3.84) as we had for $n = 3$, a simple recursion will not work. We thus have to take a different approach, comparing coefficients in (3.97) for $i + j = p$ constant.

Let us illustrate the procedure by explicitly solving the first few equations in (3.98): from (3.98a) we have $a_{00} \equiv 0$ or $a'_{00} = 0$; however, for (3.90) and (3.95) to agree, we have to take the former, which substantially simplifies all further arguments. Equation (3.98b) is vacuous, as indeed $0 = 0$, whereas (3.98c) then immediately yields $a_{10} \equiv 0$. Given $a_{00} \equiv 0$, (3.98d) implies either $a_{01} \equiv 0$ or $a'_{01} = 1$. Here the requirement that (3.90) and (3.95) agree to lowest order fixes $a'_{01} = 1$, whence

$$a_{01} = u + \alpha^{01} \tag{3.99}$$

for some $\alpha^{01} \in \mathbb{R}$. In fact, with (3.79) and (3.80) one finds

$$\begin{aligned} v(u,\varepsilon) = \Bigg[&\frac{1}{\ln\varepsilon+\gamma} - \frac{\varepsilon}{\ln\varepsilon+\gamma} + \frac{\varepsilon}{(\ln\varepsilon+\gamma)^2} + \frac{1}{2}\frac{\varepsilon^2}{\ln\varepsilon+\gamma} - \frac{5}{4}\frac{\varepsilon^2}{(\ln\varepsilon+\gamma)^2} + \\ &+ \frac{\varepsilon^2}{(\ln\varepsilon+\gamma)^3} + \mathcal{O}\left(\varepsilon^3\right)\Bigg](u-1) + \mathcal{O}\left((u-1)^2\right), \end{aligned} \tag{3.100}$$

which implies $\alpha^{01} = -1$. Plugging (3.99) into (3.98e) then gives

$$a_{11} = -(u-1)^2 - (u-1); \tag{3.101}$$

moreover, it follows from (3.98f) that $a_{20} \equiv 0$, as well. Again with (3.99), equation (3.98g) becomes

$$a'_{02}(u-1) - a_{02} = 0, \tag{3.102}$$

which has the solution

$$a_{02} = \alpha^{02}(u-1) \tag{3.103}$$

for some constant α^{02}. With reference to (3.100) one can fix α^{02}, whence $\alpha^{02} = 0$. As for a_{12} and a_{21}, one easily obtains from (3.98h) and (3.98i) that

$$a_{12} = 2(u-1)^2 + u - 1 \tag{3.104}$$

and

$$a_{21} = \frac{1}{2}(u-1)^3 + (u-1)^2 + \frac{1}{2}(u-1), \tag{3.105}$$

respectively, whereas $a_{30} \equiv 0$. As for $n = 3$, we can now prove the following general result:

Proposition 3.8. *There exist unique smooth functions $a_{ij}(u_1)$ such that (3.90) and (3.95), seen as double expansions, are the same.*

Proof. The proof differs significantly from the one we gave for $n = 3$, although it is again by induction, now on the sum $i + j$ instead of on i alone, however. We will show

$$a_{ij} = \sum_{k=1}^{i+j} \alpha_k^{ij} (u-1)^k, \qquad i, j \geq 1, \tag{3.106a}$$

$$a_{0j} = \sum_{k=1}^{j-1} \alpha_k^{0j} (u-1)^k, \qquad j \geq 2, \tag{3.106b}$$

$$a_{i0} \equiv 0, \qquad i \geq 2; \tag{3.106c}$$

indeed, for $i + j = 2$, the assertion is obvious from (3.101) and (3.103). Let (3.106) be valid for $i + j = 2, \ldots, p-1$; we have to show that it is valid for $i + j = p$, as well. Collecting powers of $\varepsilon(\ln \varepsilon + \gamma)^{-1}$ in (3.97), we obtain

$$\sum_{\substack{k+m=i \\ l+n=j}} \left[a'_{kl} a_{mn} + a'_{mn} a_{kl}\right] + i a_{ij} - (j-1) a_{i,j-1} = -u a_{i-1,j}. \tag{3.107}$$

Plugging $a_{00} \equiv 0$ and (3.99) into (3.107) gives the following equation for a_{0p}:

$$a'_{0p}(u-1) - (p-1) a_{0p} = - \sum_{\substack{k+l=p+1 \\ k,l \geq 2}} [a'_{0k} a_{0l} + a'_{0l} a_{0k}]; \tag{3.108}$$

by the induction hypothesis, the above sum can be written as

$$-\sum_{k=1}^{p-2} \tilde{\alpha}_k^{0p} (u-1)^k. \tag{3.109}$$

As terms of the form $-\tilde{\alpha}_k^{0p}(u-1)^k$ generate particular solutions of the form

$$-\frac{\tilde{\alpha}_k^{0p}}{k-p+1}(u-1)^k, \qquad 1 \leq k \leq p-2 \tag{3.110}$$

in (3.108) and the homogeneous solution is given by

$$\alpha^{0p}(u-1)^{p-1}, \tag{3.111}$$

for a_{0p} the claim follows. Note that the constant α^{0p} has to be chosen such that (3.90) and (3.95) agree when seen as double expansions; the smoothness of a_{0p} is granted irrespective of the choice of α^{0p}. For a_{ij} with $i+j=p$ and $i \geq 1$, (3.107) yields

$$ia_{ij} = ja_{i,j-1} - ua_{i-1,j} - \sum_{\substack{k+m=i \\ l+n=j}} [a'_{kl}a_{mn} + a'_{mn}a_{kl}]; \tag{3.112}$$

due to the fact that $a_{00} \equiv 0$, this completely determines a_{ij}. Moreover, it follows from the induction hypothesis that a_{ij} is of the desired form and that $a_{p0} \equiv 0$, respectively, which concludes the proof. □

Remark 3.9. The above proof shows that once the leading-order behaviour in (3.95) is determined, the transcendentally small terms in (3.95) are given as solutions not of differential, but of algebraic equations. Our approach thus immediately provides us with these terms, whereas in the classical approach, quite cumbersome computations are required for their determination, as matching is typically done only up to transcendentally small quantities there. □

We can now give an expansion of v_{1_ε} for $n=2$:

Proposition 3.9. *For $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small, $v_{1_\varepsilon} = v_1(0,\varepsilon)$ can be expanded as*

$$v_{1_\varepsilon} = -\frac{1}{\ln\varepsilon + \gamma} + \mathcal{O}\left(\frac{1}{(\ln\varepsilon+\gamma)^2}\right). \tag{3.113}$$

Proof. As for $n=3$, (3.95) gives

$$\begin{aligned} v_1(u_1,\varepsilon_1) = (u_1-1)\frac{1}{\ln\varepsilon_1+\gamma} + \left[-(u_1-1)^2 - (u_1-1)\right]\frac{\varepsilon_1}{\ln\varepsilon_1+\gamma} + \\ + \left[\frac{1}{2}(u_1-1)^3 + (u_1-1)^2 + \frac{1}{2}(u_1-1)\right]\frac{\varepsilon_1^2}{\ln\varepsilon_1+\gamma} + \mathcal{O}\left(\frac{1}{(\ln\varepsilon_1+\gamma)^2}\right), \end{aligned} \tag{3.114}$$

whence we obtain the assertion with $u_1 = 0$. □

Remark 3.10. As $-(\ln\varepsilon+\gamma)^{-1}$ can be expanded as

$$-\frac{1}{\ln\varepsilon+\gamma} = -\frac{1}{\ln\varepsilon}\sum_{j=0}^{\infty}\left(-\frac{\gamma}{\ln\varepsilon}\right)^j \tag{3.115}$$

for $0 < \varepsilon < \varepsilon_0$ sufficiently small, v_{1_ε} can be written as

$$v_{1_\varepsilon} = -\frac{1}{\ln\varepsilon} + \frac{\gamma}{(\ln\varepsilon)^2} + \mathcal{O}\left(\frac{1}{(\ln\varepsilon)^3}\right), \tag{3.116}$$

which agrees with the expansion found e.g. in [HTB90]. In fact, as was pointed out by [LC72], the expansion is more compact if it is *telescoped*, i.e., arranged in powers of $(\ln\varepsilon+\gamma)^{-1}$. However, this arrangement is not helpful numerically, as (3.113) becomes undefined for $\varepsilon = 0.5614\ldots$, whereas (3.116) allows for values of ε up to 1. □

4 Asymptotic solution expansions

4.1 Expansions in chart K_1.

Given the expansion for v_{1_ε} from Proposition 3.5, which we can write as

$$v_{1_\varepsilon} = \sum_{\substack{i,j=0\\ j\le i}}^{\infty} \beta_{ij}\varepsilon^i(\ln\varepsilon)^j \tag{4.1}$$

with constants $\beta_{ij}\in\mathbb{R}$, it makes sense to set up expansions

$$u_1(r_1,\varepsilon_1) = \sum_{\substack{i,j=0\\ j\le i}}^{\infty} a_{ij}(r_1)\varepsilon_1^i(\ln\varepsilon_1)^j,\quad v_1(r_1,\varepsilon_1) = \sum_{\substack{i,j=0\\ j\le i}}^{\infty} b_{ij}(r_1)\varepsilon_1^i(\ln\varepsilon_1)^j, \tag{4.2}$$

where we regard both u_1 and v_1 as functions of (r_1,ε_1) now. The ansatz in (4.2) is closer to the classical approach than what was done in the previous sections; however, as the techniques we apply are very similar to the ones used before, we only sketch the procedure here, leaving out most of the details. Rewriting (2.12) with r_1 as the independent variable now and omitting the subscript 1 again, we obtain

$$\begin{aligned}\frac{du}{dr} &= -\frac{v}{r}\\ \frac{dv}{dr} &= \frac{v}{r}+\frac{\varepsilon uv}{r}\\ \frac{d\varepsilon}{dr} &= -\frac{\varepsilon}{r}.\end{aligned} \tag{4.3}$$

With (4.3) we get

$$\sum_{\substack{i,j=0\\ j\le i}}^{\infty}\left[a'_{ij}\varepsilon^i(\ln\varepsilon)^j - \frac{a_{ij}}{r}\varepsilon^i(\ln\varepsilon)^{j-1}(i\ln\varepsilon+j)\right] = -\frac{1}{r}\sum_{\substack{i,j=0\\ j\le i}}^{\infty} b_{ij}\varepsilon^i(\ln\varepsilon)^j \tag{4.4a}$$

$$\begin{aligned}\sum_{\substack{i,j=0\\ j\le i}}^{\infty}\left[b'_{ij}\varepsilon^i(\ln\varepsilon)^j - \frac{b_{ij}}{r}\varepsilon^i(\ln\varepsilon)^{j-1}(i\ln\varepsilon+j)\right] &= \frac{1}{r}\sum_{\substack{i,j=0\\ j\le i}}^{\infty} b_{ij}\varepsilon^i(\ln\varepsilon)^j +\\ &+\frac{1}{r}\left(\sum_{\substack{i,j=0\\ j\le i}}^{\infty} a_{ij}\varepsilon^i(\ln\varepsilon)^j\right)\cdot\left(\sum_{\substack{k,l=0\\ l\le k}}^{\infty} b_{kl}\varepsilon^k(\ln\varepsilon)^l\right);\end{aligned} \tag{4.4b}$$

comparing powers of $\varepsilon \ln \varepsilon$ yields the following recursive sequence of differential equations,

$$ra'_{ij} - ia_{ij} - (j+1)a_{i,j+1} = -b_{ij} \tag{4.5a}$$

$$rb'_{ij} - ib_{ij} - (j+1)b_{i,j+1} = b_{ij} + \sum_{\substack{k+m=i-1 \\ l+n=j \\ l\leq k,\ n\leq m}} a_{kl}b_{mn}, \tag{4.5b}$$

with the initial conditions given by

$$a_{ij}(1) = 0, \quad b_{ij}(1) = \beta_{ij}, \qquad 0 \leq j \leq i. \tag{4.6}$$

Remark 4.1. For v_1, it follows directly from (4.1) that the ansatz in (4.2) is plausible, whereas for u_1, it can be justified a posteriori using (4.5a). □

For the first few coefficients in (4.2) we thus have

$$ra'_{00} = -b_{00} \tag{4.7a}$$

$$rb'_{00} = b_{00} \tag{4.7b}$$

$$ra'_{11} - a_{11} = -b_{11} \tag{4.7c}$$

$$rb'_{11} - b_{11} = b_{11} \tag{4.7d}$$

$$ra'_{10} - a_{10} - a_{11} = -b_{10} \tag{4.7e}$$

$$rb'_{10} - b_{10} - b_{11} = b_{10} + a_{00}b_{00}; \tag{4.7f}$$

with Proposition 3.5, the solutions to (4.7) are easily found to be given by

$$a_{00} = 1 - r, \qquad b_{00} = r \tag{4.8a}$$

$$a_{11} = -r(1-r), \qquad b_{11} = -r^2 \tag{4.8b}$$

$$a_{10} = (1-\gamma)r(1-r) + 2r^2 \ln r, \qquad b_{10} = -r(1+\gamma r) - 2r^2 \ln r, \tag{4.8c}$$

which allows us to state the following result:

Proposition 4.1. *For $n = 3$, the solution to (2.1) can be expanded as*

$$u(\xi, \varepsilon) = 1 - \frac{1}{\xi} + \varepsilon(1 - \gamma - \ln \varepsilon)\left(1 - \frac{1}{\xi}\right) - \varepsilon\left(1 + \frac{1}{\xi}\right)\ln \xi + \mathcal{O}\left(\varepsilon^2\right) \tag{4.9}$$

*(*inner expansion*); here ξ is as defined in (2.2).*

Proof. The result is immediate from (4.2) and (4.8) after one has applied the appropriate blow-down transformations $r_1 = \xi^{-1}$ and $\varepsilon_1 = \varepsilon\xi$, as

$$u_1(r_1, \varepsilon_1) = 1 - r_1 - r_1(1-r_1)\varepsilon_1 \ln \varepsilon_1 + (1-\gamma)r_1(1-r_1)\varepsilon_1 + \\ + 2r_1^2 \ln r_1 \varepsilon_1 + \mathcal{O}\left(\varepsilon_1^2\right). \tag{4.10}$$

□

Similarly, for $n = 2$ the equations

$$\begin{aligned} \frac{du}{dr} &= -\frac{v}{r} \\ \frac{dv}{dr} &= \frac{\varepsilon uv}{r} \\ \frac{d\varepsilon}{dr} &= -\frac{\varepsilon}{r} \end{aligned} \tag{4.11}$$

in combination with an ansatz of the form

$$u_1(r_1, \varepsilon_1) = \sum_{i,j=0}^{\infty} a_{ij}(r_1) \frac{\varepsilon_1^i}{(\ln \varepsilon_1 + \gamma)^j}, \quad v_1(r_1, \varepsilon_1) = \sum_{i,j=0}^{\infty} b_{ij}(r_1) \frac{\varepsilon_1^i}{(\ln \varepsilon_1 + \gamma)^j} \tag{4.12}$$

lead to the recursive sequence of equations

$$ra'_{ij} - ia_{ij} + (j-1)a_{i,j-1} = -b_{ij} \tag{4.13a}$$

$$rb'_{ij} - ib_{ij} + (j-1)b_{i,j-1} = \sum_{\substack{k+m=i-1 \\ l+n=j}} a_{kl} b_{mn}; \tag{4.13b}$$

the initial conditions are again given by

$$a_{ij}(1) = 0, \quad b_{ij}(1) = \beta_{ij}, \qquad i, j \geq 0 \tag{4.14}$$

for some $\beta_{ij} \in \mathbb{R}$. To leading order one finds

$$a_{01} = \ln r, \qquad b_{01} = -1, \tag{4.15}$$

whence one obtains the following

Proposition 4.2. *For $n = 2$, the inner expansion of the solution to (2.1) is given by*

$$u(\xi, \varepsilon) = -\frac{\ln \xi}{\ln \varepsilon\xi + \gamma} + \mathcal{O}\left(\frac{1}{(\ln \varepsilon\xi + \gamma)^2}\right). \tag{4.16}$$

Proof. The proof is analogous to that for $n = 3$. □

Remark 4.2. By expanding $(\ln \varepsilon\xi + \gamma)^{-1}$ for $0 < \varepsilon < \varepsilon_0$ small, one can write

$$u(\xi, \varepsilon) = -\frac{\ln \xi}{\ln \varepsilon + \gamma} + \mathcal{O}\left(\frac{1}{(\ln \varepsilon + \gamma)^2}\right) \tag{4.17}$$

or

$$u(\xi, \varepsilon) = -\frac{\ln \xi}{\ln \varepsilon} + \frac{\gamma \ln \xi}{(\ln \varepsilon)^2} + \frac{(\ln \xi)^2}{(\ln \varepsilon)^2} + \mathcal{O}\left(\frac{1}{(\ln \varepsilon)^3}\right), \tag{4.18}$$

which are the expansions usually found in the literature, see e.g. [LC72] or [HTB90]. □

4.2 Expansions in chart K_2

To derive solution expansions in K_2, we would have to proceed as above, i.e., we would set out by determining the structure of the coefficients in (4.2) respectively (4.12) in general, which we would then use to rearrange (4.2) and (4.12) with respect to a new basis. It is these expansions which would provide us with the proper ansatz for the corresponding expansions in K_2. In sum, we expect to obtain the following result, which we cite for reference only, see [LC72]:

Proposition 4.3. *The outer solution expansion for (2.1) is given by*

$$u(x,\varepsilon) = 1 - \varepsilon E_2(x) + \mathcal{O}(\varepsilon^2) \tag{4.19}$$

for $n = 3$ *and by*

$$u(x,\varepsilon) = 1 + \frac{1}{\ln\varepsilon + \gamma} E_1(x) + \mathcal{O}\left(\frac{1}{(\ln\varepsilon + \gamma)^2}\right) \tag{4.20}$$

for $n = 2$*, respectively; here* E_k *is defined by*

$$E_k(z) := \int_z^\infty e^{-t} t^{-k}\, dt, \qquad z \in \mathbb{C},\ \Re(z) > 0,\ k \in \mathbb{N}. \tag{4.21}$$

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