Long time properties of solutions to collisional kinetic equations

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit dem Langzeitverhalten von Lösungen kinetischer Transportgleichungen. Insbesondere wird die Konvergenz zu einem Gleichgewichtszustand studiert. Für die Untersuchung werden zwei verschiedene Methoden benutzt. Die Arbeit gliedert sich in drei Abschnitte.

I. Der erste Teil enthält eine allgemeine Einleitung und einen kurzen Überblick über die zwei Methoden. Des weiteren wird je ein Resultat vorgestellt anhand dessen die Methoden verglichen werden.

II. Im zweiten Teil wird die sogenannte Entropie Entropie-Dissipationsmethode angewandt, um den Trend der Lösung zum Gleichgewichtszustand zu untersuchen. Hierbei werden wiederum zwei Modelle betrachtet - ein lineares und ein nichtlineares Relaxationsmodell. Es zeigt sich der grosse Vorteil der Methode - die Robustheit gegenüber Nichtlinearität. In einem vereinfachten linearen Fall vergleichen wir das Ergebnis auch mit dem Optimalen, das sich durch Spektraltheorie ergibt. Die Spektraltheorie hat allerdings den Nachteil nur bedingt (i.e., nahe am Gleichgewicht) für nichtlineare Probleme geeignet zu sein. Demgegenüber zeigen sich aber auch die Nachteile des Entropie-Zugangs. Neben dem nicht optimalen Resultat (polynomiale Kovergenz anstatt exponentieller) beruht die Entropie-Methode auf *globalen* Schranken an die Lösung und ihre Regularität.

III. Da diese Schranken im allgemeinen schwierig zu etablieren sind, entwickeln wir im dritten Teil eine Methode, die sowohl Regularität der Lösung gleichmäßig in der Zeit als auch exponentielle Konvergenz liefern kann. Dieser Ansatz ist weniger stabil gegenüber Nichtlinearitäten. Wir wenden ihn (unter anderem) auf die Boltzmanngleichung nahe am Gleichgewicht an. Des weiteren zeigen wir, wie ein selbstkonsistentes Wechselwirkungs-Potential oder ein schwaches äußeres Feld berücksichtigt werden können.

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Contents

Part I

Introduction

1.1 General Introduction

In this work we study the long time properties of colissional or scattering non-homogenous kinetic equations. First we want to explain the general structure of these equations. $f(t,x,v)$ denotes the distribution of particles in space x and velocity v at time t. In the non relativistic case the evolution of the function f is guided by the equation

$$\partial_t f + v \cdot \nabla_x f = 0 \ ,$$

provided the particles do not interact with one another. The characteristic equations corresponding to this partial differential equation are the Newton-equations of classical mechanics. The operator $v \cdot \nabla_x$ is called the *transport operator* because it corresponds to the free transport of the particles. Now taking into account interaction between the particles or particles and a background medium we can write

$$\partial_t f + v \cdot \nabla_x f = Q(f) \ . \tag{1.1}$$

Q is refered to as the *collision operator*. We will only consider local collisions. In this case only particles in the same position interact (as opposed for example to many models of quantum mechanics) and the interaction does not change the position of the particle. Moreover the underlying collision mechanism is independent of x in the models we study. Thus the collision operator depends on the position only via f. In many cases this operator splits naturally into a gain and a loss therm. The loss part describes the quantity of particles that, due to interaction, change their velocity from the intervall $v + \Delta v$ to any other velocity and the gain part describes the opposite process. We do not consider effects like creation or fragmentation and the collision operators we study are mass preserving. Throughout this work we will study equation (1.1) spacially confined in the torus. This is not only a mathematical convenient setting but, under certain conditions, also equivalent to the interresting case of particles in a box with specular reflection boundary conditions. Moreover we need to precise the initial distribution of the particles $f_0(x,v) = f(0,x,v)$. Since f_0 represents a distribution of particles it is considered to be nonnegative and this property is perserved in time for the equations that we study.

It is natural to ask if there is a particle distribution which does not change due to collision, i.e. the amount of particles assuming velocity v during collision is the same as the amount of particles that change their velocity from v to a different value. In the cases we study such a distribution (in the kernel of Q) exists. This rises the question if and in what sence the distribution function $f(t,x,v)$ assumes such a form in the long time. In the cases that we study there is (due to conserved quantities) only one solution to the equation that is also in the kernel of Q for every initial data. We denote this equilibrium solution by f_∞. The main goal of this work is to derive strong convergence of f towards f_∞ with explicit rates. For this we utilize two different methods that will be introduced in the following two sections. We will also give examples of the type of equations that we study and the results that can be expected in this two sections.

1.2 Entropy Entropy-Dissipation Method

The Entropy Entropy-Dissipation method has proven to be a very efficient tool to study the long time behaviour of non homogeneous kinetic equations that feature a dissipative collision operator. Moreover it is possible to give rates of convergence toward the equilibrium solution. We shall briefly describe the concept of the method and give an overview of results. The collision operators we shall consider act only in the velocity variable (local, instantaneous collisions) and feature a non vanishing kernel. We study only equations that have a unique steady state f_∞. To a solution f of the equation we associate a local equilibrium f_l in the following way. The x-dependence is fixed by the fact that we impose the same local mass density $\rho(x) = \int f \mathrm{d}v = \int f_l \,\mathrm{d}v$, while the velocity dependence is such that f_l lies in the kernel of the collision operator Q. f_l can be seen as the projection of f onto the kernel of Q. This local equilibrium is in general no solution to the equation since it does not make the transport part of the equation vanish. The next step is to define a “distance like” positive functional $H(f|g)$ - the relative entropy with the following properties.
1. Dissipation property: A quantitative H-theorem holds.

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|f_\infty) \leq -C_1 H(f|f_l) \tag{2.2}$$

2. Additivity of the relative entropy with respect to the local equilibrium

$$H(f|f_l) + H(f_l|f_\infty) = H(f|f_\infty) \tag{2.3}$$

Geometrically speaking the projection onto f_l is orthogonal with respect to H. Note that the physical entropy is usually defined with the opposite sign, i.e. negative. This is why in physical literature the entropy is usually expected to grow in time which also leads to the term Entropy Entropy-Production method. However in the mathematical literature the positive sign (which we use here) is more common, probably because with this convention the entropy resembles a norm more closely.

Functionals fulfilling the first conditions can usually be taken from the space homogeneous case - it means no more than that entropy is dissipated as long as the solution is not in the kernel of Q. This first inequality is typically fulfilled by choosing H to be generated by a convex functional and applying Jensen's inequality.
Now this is not enough to show strong convergence (for a classic argument of deducing convergence in a weak sense from this inequality see [**14**]) since the solution can become close to a local equilibrium and thus arbitrary little entropy is dissipated. However we will see that the transport term in the equation and confinement - by a uniformly convex at infinity potential or in a torus - will provide enough convection to drive the solution away from the local equilibrium in little time. As the reader might imagine the sheer existence of a transport term is not always enough to provide convection - in addition one often needs the local temperature to be bounded away from zero. Even though this is not always the case for linear models in the torus (confine Chapter 2) an assumption of this type is already needed for linear models subject to a confining potential in whole space [**6, 16**] In a situation where a maximum principle holds this is usually ensured by the (not optimal) assumption

$$c f_\infty \le f_0 \ .$$

Now we study the time evolution of $H(f|f_l)$. If $H(f|f_l)$ vanishes so does its first time-derivative since the relative entropy is positive. Thus to study the evolution of the relative entropy with respect to the local equilibrium we calculate the second time derivative. Assuming that the potential is uniformly convex at infinity a logarithmic Sobolev type inequality holds and enables us - together with (2.3) - to write

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|f_l) \ge C_2 H(f|f_\infty) - R \ . \tag{2.4}$$

The exact form of Sobolev inequality of course depends on the situation. For example in the simplest case of the quadratic relative entropy (i.e., weighted L^2-norm) and confinement in a bounded domain it is just a Poincaré inequality.

To close the system of differential inequalities (2.2)(2.4) we have to estimate the remainder term R by $H(f|f_l)$ while keeping the positive constant C_2. Typically in this step it is necessary to bound moments of derivatives. This is done by interpolation of the type

$$\left\| \frac{1}{\sqrt{f_\infty}} (f - f_l) \right\|_{H^1} \le \left\| \frac{1}{\sqrt{f_\infty}} (f - f_l) \right\|_{L^2}^{1-\frac{1}{n}} \left\| \frac{1}{\sqrt{f_\infty}} (f - f_l) \right\|_{H^n}^{\frac{1}{n}} \tag{2.5}$$

and (again in the case where a maximum principle holds) the crude assumption

$$f_0 \leq C f_\infty \ ,$$

to treat the moments. Note that in the case of no potential and confinement by a bounded domain this interpolation is only done with respect to the x-variable and one can get away without the upper bound by carefully using the decay of the equilibrium solution (again confine Part II). In cases where we have a potential also the v-derivatives have to be controlled. Now assuming uniform in time bounds on the weighted H^n-norm and estimating $\|\frac{1}{M}(f - f_l)\|^2$ by $H(f|f_l)$ we arrive at the following differential inequality

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|f_l) \geq C_2 H(f|f_\infty) - C_3 H(f|f_l)^{1-\frac{1}{n}} \ . \tag{2.6}$$

This shows that the solution will be pushed out of local equilibrium as long as it is not in the global equilibrium. Quantitatively speaking Desvillettes and Villani proved in [**16**] that the system of differential inequalities (2.2),(2.6) yields the following decay of the relative entropy

$$H(f|f_\infty) \leq C t^{-(n-1)} H(f_0|f_\infty) \ . \tag{2.7}$$

Then by a Czisar-Cullback inequality one can conclude convergence to the global equilibrium in the L^1-norm, typically at the rate

$$\|f - f_\infty\|_1 \leq C t^{-\frac{n-1}{2}} \|f_0 - f_\infty\|_1 \ ,$$

for all initial data with bounded initial entropy.
Thus for all initial data fulfilling the bounds $c f_\infty \leq f_0 \leq C f_\infty$, the uniform regularity assumption $\forall t : \ \|f f_\infty^{-1/2}\|_{H^n} \leq C_n$ and having bounded initial entropy $H(f_0|f_\infty)$ with respect to a suitable entropy we derive convergence to equilibrium in L^1 at the polynomial rate $t^{-\frac{n-1}{2}}$. An important feature of this method is that all the constants can be given explicitly.

Now we shall give a short review of situations in which this method has been applied successfully. While a simpler version of this method - not having to deal with the problems of local equilibria - has been applied successfully to homogeneous diffusive and collisional models (see [**2**] for a review of results) we shall focus on non homogeneous situations.
Historically some ideas of the approach have been given by Grad in [**26**] for the Boltzmann equation. The first rigorous application to inhomogeneous equations was done by Desvillettes and Villani in [**16**]. The authors study the inhomogeneous Fokker-Plank equation (sometimes also called Kramers equation). Regularity in whole space with confining potential is derived using

the hypoellipticity of the collision operator. They also proved that the system of differential inequalities (2.2), (2.6) ensures convergence to equilibrium with explicit rate. We shall now focus more on applications to collisional models and also track the regularity issues in these results. First we will consider linear models. In [**6**] the method was used to show convergence for the Vlasov-Relaxation equation featuring the relaxation kernel

$$Q(f) = \rho M - f \ ,$$

with M being the Maxwellian with fixed temperature. The authors study the equation in whole space with a confining potential. The necessary uniform in time regularity result however is provided only in the case of quadratic potential using a characteristics approach that seems hard to generalize to different potentials. In the article [**21**] a quite general (not necessarily micro-reversible) kernel of the type

$$Q(f) = \int_B S(k', k) f'(k) - S(k, k') f(k) \mathrm{d}k'$$

is studied. The result applies to collision kernels that admit an equilibrium distribution that can be factorized into a x-dependent and a v-dependent part. This article is the basis of Chapter 2 and precise requirements can be found there. Regularity is not an issue since we study this linear equation in a bounded domain with periodic boundary conditions. In such a situation the H-theorem can be differentiated and shows that all derivatives that are bounded initially will not increase. The result of this article however can be generalized to the whole space with a confining potential straightforward, the only major difference being the estimate to extract the term $H(f|f_\infty)$ in the differential inequality (2.4) - here we need the uniform convexity at infinity of the potential V to allow for the use of a logarithmic Sobolev inequality, and changes in the conservation laws. The estimates for the remainder term - using bounds from above and below on the solution - are the same as in [**6**] or [**16**]. For this more general situation the propagation of regularity is of course not trivial anymore.

In both these articles the "entropy of choice" is the quadratic one. Despite the fact that the condition of bounded initial entropy is more restricting than with the logarithmic entropy (L^2 instead of $L^1 \log L^1$ for the logarithmic one) it is more natural in this linear framework to use the linearized version of the classic Boltzmann entropy.

Going to nonlinear equations a simple situation is the sublinear relaxation model for fermions including the Pauli principle in a heuristic way. We studied the following collision kernel

$$Q(f) = \int_{\mathbb{R}^d} [M(1-f)f' - M'(1-f')f] \mathrm{d}v'$$

in the article [**45**] that is the basis of the chapter on fermionic relaxation. In this article the regularity issue was completely left aside by assuming a priory boundedness of the x - derivatives of the solution. Again the study is in a bounded domain but here due to the nonlinear nature of the equation regularity is a delicate problem (see the comments in Chapter 4). The result can again be generalized to whole space with a confining potential - making boundedness of x - and v - derivatives necessary. In the end of Part II we prove some estimates on the collision kernel that ensure uniform in time regularity in the homogeneous case - however we failed to apply them directly in the space dependent setting. Part III of this work gives a regularity result for the complete situation - albeit only in the perturbative setting.
Finally the method has also been applied to the inhomogeneous Boltzmann equation in [**17**]. In this case the entropy method is much more involved. The local Maxwellian depends on x not only via the density but also via the local mean velocity and the local temperature. For this reason the system of differential inequalities (2.2),(2.6) can not be established and the authors overcome this problem by replacing (2.6) by three differential inequalities of order two. To close the system they use additivity properties as well as some differential inequalities of order one in time. Nevertheless the basic assumptions that are necessary for the convergence result to hold are the same - i.e., boundedness of the initial data from above and below and uniform in time regularity of the solution. Note that in this nonlinear setting the classic Boltzmann entropy is used since it fulfills the additivity property (2.3).
The problem of establishing convergence to equilibrium for the Boltzmann equation with *explicit rate* is especially interesting for physical as well as historical reasons. When Boltzmann established the H theorem he found time irreversibility in an equation describing (micro-reversible) particle interactions. The irreversibility of the Boltzmann equation - introduced by the molecular chaos assumption - also gave a connection between statistical physics and axiomatic thermodynamics. However it seemed to contradict Poincarés recurrence theorem proved later. This theorem roughly states that the N-particle system will become arbitrary close to its initial state during the time evolution again with probability one. In that case the entropy will of course be arbitrary close to the one of the initial state. This paradox was the basis of a discurs between Boltzmann and (mainly) Zermelo. The recurrence time is much larger (and in every-day situations way to large for the phenomena to be observed) than the timespan on which the assumptions used to deduce the Boltzmann equation are valid and thus the paradox is resolved because the Boltzmann equation is no longer an accurate description of the dynamics. This shows that it is important to give rates of convergence to be sure that the convergence takes place on a timescale on which the underlying equation is valid. A more qualified discussion of this topic can be found

in the lecture notes [**54**] that are based on the article [**17**] and surprisingly reader-friendly or in [**53**].

Now let us give a result from Chapter 3 to show precisely what results can be expected. The result concerns the fermionic relaxation model. A detailed introduction can be found in Chapter 3. For now we only want to write the equation

(2.8)

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^d} [M(1-f)f' - M'(1-f')f] \mathrm{d}v' \ , \quad f(0,x,v) = f_0(x,v) \ ,$$

where M is the normalized Maxwellian, and state the result.

Theorem. *Let* $\kappa_-, \kappa_+ > 0$ *be constants such that*

$$f_-(v) \le f_0(x,v) \le f_+(v) \,, \qquad \textit{with } f_\pm(v) = \frac{\kappa_\pm M(v)}{1+\kappa_\pm M(v)} \,,$$

for all $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$. *Then there is a unique solution* $f(t,x,v)$ *of* (2.8) *satisfying the same bounds. If this solution moreover satisfies*

$$\left\| M^{-1/2} \frac{\partial^{k_1+\ldots+k_d} f}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}}(t,\cdot,\cdot) \right\| \le c \,, \qquad \forall\, k_1 + \ldots + k_d \le n \quad \textit{and} \quad \forall\, t > 0 \,,$$

for a constant c *and a positive integer* n, *then there exists a constant* $\tilde{c} > 0$ *such that*

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)} \le \tilde{c} t^{(1-n)/2} \,.$$

1.3 Coercivity in Sobolev spaces

In Part 4.2 of this work we develop a method to derive exponential convergence for a large class of linear collision operators and also apply it to nonlinear problems in the perturbative situation. This method is mainly inspired from the energy-methods developed in [**28, 27**]. A more precise review of the literature can be found in the introduction of Part III.

The main requirements on the collision operator can be (not aiming for full generality) written as follows. We consider a linear collision operator L (defined and bounded on L^2) that is non positive and has a (non empty) finite dimensional kernel. The orthogonal projection onto this kernel is denoted by Π_l. We assume L to be coercive in the sense that $\exists \lambda > 0$:

$$\langle h, L(h) \rangle_{L^2_v} \le -\lambda \|h - \Pi_l(h)\|^2_{L^2_v} \ ,$$

which can be considered as a minimal requirement for exponential relaxation towards an equilibrium distribution. Moreover we restrict to collision operators which can be split $L = K - \Lambda$ into a "gain" part K that is regularizing in velocity

$$\forall \delta > 0 \; \exists\, C(\delta): \quad \langle \nabla_v K(h), \nabla_v h \rangle_{L^2_v} \leq \delta \|\nabla_v h\|^2_{L^2_v} + C(\delta)\|h\|^2_{L^2_v}$$

and a "loss" part that is again coercive

$$\exists\, \nu > 0: \quad \nu \|h\|^2_{L^2_v} \leq \langle \Lambda(h), h \rangle_{L^2_v} \ .$$

We assume further that a similar estimate to the one above holds true for velocity-derivatives of Λ,

$$\nu \|\nabla_v h\|^2_{L^2_v} - C\|h\|^2_{L^2_v} \leq \langle \nabla_v \Lambda(h), \nabla_v h \rangle_{L^2_v} \ .$$

Then we proof (roughly) that the linear operator $T = L - v \cdot \nabla_x$ generates a strongly continuous evolution semi-group obeying the estimate

$$\|e^{tT}(\mathrm{Id} - \Pi_g)\|_{H^1_{x,v}} \leq C_T \exp(-\tau_T t) \ ,$$

with explicit positive constants τ_T and C_T. Here Π_g denotes the projection on the global equilibrium f_∞. Remember that the global equilibrium is the unique function (that can be addressed by a specific initial data) in the kernel of L that solves also $T(f_\infty)$=0.
Moreover we show that there exists a Hilbert-norm $\mathcal{H}^1$, equivalent to the H^1 norm, for which the estimate

$$\langle Th, h \rangle_{\mathcal{H}^1} \leq -C' \|h - \Pi_g(h)\|^2_{H^1_{x,v}}$$

holds.
While these have to be understood as "a priori" estimates the global in time existence of a solution in H^1 follows by classical arguments. Now we will explain very briefly the idea of the proof and the importance of the various assumptions on the collisional operator.
The first observation is that due to the non-positivity of L the L^2 norm has to decay. Now for derivatives of h that include the velocity direction the regularizing effect of K together with the coercivity of Λ results in a damping effect of the collision operator. The main problem is thus to transfer some of the regularizing effect of L to the x-direction by utilizing the convection provided by the transport term and the confinement. This is obtained by including the mixed term $\langle \nabla_v \cdot, \nabla_x \cdot \rangle$ in the norm $\mathcal{H}^1$. It is easy to check the following commutator relation

$$[\nabla_v, -v \cdot \nabla_x] = -\nabla_x \ ,$$

which shows that, including the mixed derivatives in the norm, the transport term will transfer some of the damping effect of L on v-derivatives also to the x-variable. For this phenomenon the term "hypocoercivity" was introduced by C. Villani [**54**]. The proof is completed, roughly speaking, by adjusting the coefficients in the combination

$$\|\cdot\|^2_{\mathcal{H}^1} = A\|\cdot\|^2_{L^2} + \alpha\|\nabla_x\cdot\|^2_{L^2} + \beta\,\langle\nabla_v\cdot,\nabla_x\cdot\rangle_{L^2} + \gamma\|\nabla_v\cdot\|_{L^2}\ ,$$

decomposing h into a fluid and a kinetic part $h = \Pi_l(h) + (h - \Pi_l(h))$ and using Poincaré inequalities.

Part of the work in Part III consists of showing that (the linearization of) various physical collision operators have the properties assumed above. These are relaxation, semiclassical relaxation, the linearized Boltzmann operator for hard spheres and the linearized Landau operator for hard to moderately soft potentials. We also give the consequences of the abstract linear theorem for the full nonlinear models near equilibrium. Of these we shall quote here the one for semiclassical relaxation as an example of what can be expected. This is maybe not the most spectacular example but we can compare the result directly to the one derived by the entropy method in Chapter 3 of Part II.

Theorem. *Consider the semiclassical relaxation equation* (2.8) *in the torus with initial data* $0 \le f_0 \in L^1$. *Let* k *be such that* $E(k/2) > d/2$ *(where* E *denotes the integer value) and let the initial data satisfy*

$$\left\| M^{-1/2}\,(f_0 - f_\infty)\right\|_{H^k} \le \varepsilon$$

for some $0 < \varepsilon \le \varepsilon_0$*, where* ε_0 *can be given explicitly.*
Then there exists a unique global solution $0 \le f = f(t,x,v) \in \mathcal{C}([0,\infty[, H^k)$ *of the initial value problem* (2.8)*, such that*

$$\forall\, t \ge 0, \qquad \left\|M^{-1/2}(f(t,\cdot,\cdot) - f_\infty)\right\|_{H^k} \le C\,\exp\left[-\tau t\right]$$

for some explicit constants $C, \tau > 0$.

Note that we adopted the notation of Part II, i.e. M stands for the Maxwellian and d for the dimension of the space. In Part III we prove also that the same result holds true for the bosonic relaxation model as long as the mass is too small for condensation.

1.4 Comparison of the methods

To conclude with the introduction we want to compare the results we obtained by the different methods of Part II and Part III. While this is pretty obvious it certainly deserves a few words. When compared to the method of Part III the Entropy method has 2 major shortcomings. First it relays on uniform in time regularity estimates and second the convergence result is only "almost exponential", i.e. as $O(t^{-\infty})$. The convergence rate is from my point of view not a tremendous problem - and allows the method to be more robust in cases were exponential convergence can not always be expected (for example some cases included in [**17**]). However the assumption of uniform bounds on derivatives is a severe shortcoming of the results obtained by this method. On the contrary it is very robust against nonlinearity - as the reader can see from the similarity of the calculations in Chapter 2 and Chapter 3 - and at the moment the only method that can give results in situations far from equilibrium. The method that we describe in Part III on the other hand is superior in the result but - for nonlinear situations - restricted to the perturbative regime. From the recent work [**32**], concerning the linear relaxation equation, it seems that the limitation to "weak" external potentials can probably be overcome and in this respect the method could compete with the Entropy method.

This suggests that, while the Entropy method is still the method of choice for nonlinear models far from equilibrium, in the linear setting the results obtained in Part III are superior.

1.5 Outline

Part II of this work deals with the application of the Entropy Entropy-Production method. It consists roughly speaking of two articles. In the first one (Chapter 2), together with Christian Schmeiser and Klemens Fellner, we applied the method to a quite general class of linear relaxation models. It appeared in "Monatshefte für Mathematik" ([**21**]). The second article is a joint work with Christian Schmeiser and deals with the fermionic relaxation model. This constitutes Chapter 3 and appeared as [**45**]. In both these articles spatial confinement in the torus is assumed. To finish the part on the Entropy method a short discussion of regularity issues for the fermionic relaxation equation is given in Chapter 4. These are not very satisfactory results from the viewpoint of the requirements of the Entropy-method but they can give an idea of the problems in establishing uniform bounds on derivatives in general.
In Part III we developed a method to overcome some of this problems, moreover it can give exponential decay albeit only for linear models or in the perturbative setting. This Part of the thesis is a reproduction of a joint work with Clément Mouhot ([**43**]).

I preferred to leave the articles as they were published apart from the joint bibliography and maybe some changes in line-breaks. This facilitates the work for me as well as for the referees, some of whom might already have read one of the articles. Moreover it makes the collaborations in which the results were obtained more clear. For this reason I also let the cover-pages in place. However this comes at the prize of a not completely consistent notation. Namely I want to point out the following: In Part II the dimension of the space is denoted by d and in Part III by k. More severely the Maxwellian is written as M throughout most of Part II while M denotes the square root of the Maxwellian $\mathcal{M}$ in Part III. However since Part II and III are a priori independent the conscious reader should not find this a problem.

Part II

Entropy Entropy-Dissipation method

Chapter 2

Linear Relaxation

Convergence to global equilibrium for spatially inhomogeneous kinetic models of non-micro-reversible processes

Klemens Fellner[1], Lukas Neumann, and Christian Schmeiser[1]

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Abstract: We study the long-time asymptotics of linear kinetic models with periodic boundary conditions or in a rectangular box with specular reflection boundary conditions. An entropy dissipation approach is used to prove decay to the global equilibrium under some additional assumptions on the equilibrium distribution of the mass preserving scattering operator. We prove convergence at an algebraic rate depending on the smoothness of the solution. This result is compared to the optimal result derived by spectral methods in a simple one dimensional example.

Key words: kinetic transport equations, relative entropy, entropy dissipation, long-time asymptotics

AMS subject classification: 35B40, 35B35, 82C70

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2.1 Introduction

We investigate initial value problems of the form

$$\partial_t f + v(k)\cdot\nabla_x f = Q(f) \tag{1.1}$$

where $f = f(t,x,k) \geq 0$ denotes the particle distribution function, depending on time $t \geq 0$, position $x \in \mathbb{T}^d$ (a d-dimensional torus, $d \geq 1$) and momentum $k \in B \subset \mathbb{R}^d$. For the velocity-momentum relation $v = v(k) \in \mathbb{R}^d$ we assume continuity, oddness ($v(-k) = -v(k)$) and that flow in any direction is possible:

$$\forall z \in S^{d-1}\ \exists k \in B: \quad v(k)\cdot z \neq 0\,. \tag{1.2}$$

The set of admissible momentum vectors B is symmetric about the origin. The simplest example are free classical particles with $v(k) = k$ and B rotationally invariant, e.g. a ball, a sphere, or $B = \mathbb{R}^3$. Other examples are relativistic particles with $v(k) = k(1 + k^2/c^2)^{-1/2}$ and $B = \mathbb{R}^3$, or semi-classical particles moving in a periodic potential where $v(k) = \nabla_k \epsilon(k)$, $\epsilon(k)$ is the band diagram, and B the first Brillouin zone. In this case all functions of k satisfy periodic boundary conditions on ∂B. In all cases we assume B to be equipped with a measure, which we denote by $\mathrm{d}k$ for simplicity.

The scattering operator Q acts only on the variable k and is assumed linear and of the form

$$Q(f)(k) = \int_B [S(k',k)f' - S(k,k')f]\mathrm{d}k',$$

where $f' := f(k')$, and $S(k,k') \geq 0$ is the scattering rate.

We assume the existence of a normalized equilibrium distribution $M(k)$ with vanishing mean velocity and bounded velocity moments up to fourth order:

(a) $\quad M \in L^1(B,\mathrm{d}k),\ M \geq 0\,,$ $\qquad (b)$ $\quad Q(M) = 0\,,$

(c) $\quad \displaystyle\int_B M(k)\,\mathrm{d}k = 1\,,$ $\qquad (d)$ $\quad \displaystyle\int_B v(k)M(k)\,\mathrm{d}k = 0\,,$

(e) $\quad \displaystyle m_j := \int_B |v(k)|^j M(k)\,\mathrm{d}k < \infty\,,\ j \leq 4\,.$

A sufficient condition for $Q(M) = 0$ is the detailed balance or micro-reversibility condition $S(k',k)M' = S(k,k')M$, $\forall k,k' \in B$, which we shall *not* assume here.

It has been shown by Degond, Goudon, and Poupaud [**13**] (see also [**40**]) that even without micro-reversibility there is an entropy equation:

$$\int_B \frac{Q(f)f}{M}\mathrm{d}k = -\frac{1}{4}\int_B\int_B [S(k',k)M' + S(k,k')M]\left(\frac{f}{M} - \frac{f'}{M'}\right)^2 \mathrm{d}k'\mathrm{d}k\,, \tag{1.3}$$

showing that the bilinear form $\int_B \frac{Q(f)g}{M} \mathrm{d}k$ is non-positive (however it is non symmetric without detailed balance).

Motivated by the entropy equation we shall consider the weighted L^2-scalar product

$$\langle f, g \rangle_M = \int_B \frac{fg}{M} \, \mathrm{d}k \, .$$

For the scattering rate we assume the existence of positive constants γ and Γ such that

$$\gamma \leq \frac{S(k', k)}{M(k)} \leq \Gamma \qquad \forall k, k' \in B \, .$$

This implies the boundedness of $Q : L^2(B, \frac{\mathrm{d}k}{M}) \to L^2(B, \frac{\mathrm{d}k}{M})$ (actually $\|Q(f)\|_M \leq 2\Gamma \|f\|_M$) and

$$\text{(1.4)} \qquad \int_B f \, \mathrm{d}k = \langle f, M \rangle_M = 0 \quad \Longrightarrow \quad \gamma \|f\|_M^2 \leq - \langle Q(f), f \rangle_M \, ,$$

i.e. coercivity on the orthogonal complement of the kernel of Q, spanned by M.

As a consequence, for $g \in L^2(B, \frac{\mathrm{d}k}{M})$ with $\int_B g \mathrm{d}k = 0$ the problem

$$Q(f) = g \, , \qquad \int_B f \, \mathrm{d}k = 0 \, ,$$

has a unique solution f in $L^2(B, \frac{\mathrm{d}k}{M})$ (see [**13**]). The necessity of the solvability criterion $\int_B g \, \mathrm{d}k = 0$ follows from the mass conservation property of Q, also implying (by integration of (1.1) with respect to k) the macroscopic conservation law

$$\partial_t \rho + \nabla_x \cdot J = 0$$

for the macroscopic mass density $\rho(t, x) = \int_B f(t, x, k) \mathrm{d}k$, where J is the flux density $J(t, x) = \int_B v(k) f(t, x, k) \mathrm{d}k$.

Existence of a unique solution of (1.1) subject to an initial condition

$$\text{(1.5)} \qquad f(t = 0) = f_0 \in L^2 \left(\mathbb{T}^d ; L^1(B) \cap L^2 \left(B, \frac{\mathrm{d}k}{M} \right) \right)$$

follows by standard arguments.

We shall be concerned with the convergence as $t \to \infty$ of the solution of (1.1) to the global equilibrium $\rho_\infty M(k)$, where ρ_∞ is determined by conservation of total mass,

$$\rho_\infty = \frac{1}{\mu(\mathbb{T}^d)} \int_{\mathbb{T}^d} \rho(t, x) \, \mathrm{d}x = \frac{1}{\mu(\mathbb{T}^d)} \int_{\mathbb{T}^d} \int_B f_0(x, k) \, \mathrm{d}k \, \mathrm{d}x \, .$$

With the relative entropy

$$H(f|g) := \int_{\mathbb{T}^d} \|f-g\|_M^2 \mathrm{d}x,$$

the entropy equation (1.3) and the coercivity (1.4) imply

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|\rho_\infty M) \le -\gamma H(f|\rho M)\,. \tag{1.6}$$

Arguments, which are standard by now (see [**14**]) use this inequality for proving weak convergence of f to the global equilibrium $\rho_\infty M$.

In this work we shall prove *strong convergence* of smooth solutions *at an algebraic rate* depending on the smoothness. We shall use the entropy-entropy dissipation approach developed by Desvillettes and Villani [**16**]. In [**16**] it has been applied to a linear Fokker-Planck equation with a confining potential. There the necessary smoothness is produced by a hypo-ellipticity property. In the case considered here, regularization effects cannot be expected since the scattering operator is bounded. We will use smoothness assumptions on the initial data and propagation of regularity instead. A recent comparable study is [**6**], where the special (micro-reversible) case

$$v = k, \quad B = \mathbb{R}^3, \quad S(k', k) = M(k)$$

with a Maxwellian M of vanishing mean velocity and given constant temperature is considered. Similarly to [**16**], the whole space problem with confining potential is treated there.

It is important to point out that the approach of Desvillettes and Villani has the potential to deal with nonlinear problems. Recently it has been applied to the (gas dynamics) Boltzmann equation by the same authors.

Shortcomings of the approach are the required smoothness and the fact that exponential rates of convergence cannot be provided in general. This will be illustrated by a simple one-dimensional example at the end of this work, which can be solved by spectral analysis. As a result, exponential convergence to global equilibrium, even for unsmooth solutions, is shown.

A further comment is concerned with boundary conditions. Let $\mathbb{T}^d$ be represented by an interval in $\mathbb{R}^d$, centered around the origin, with periodic boundary conditions. Assume further that the initial datum f_0 is invariant under the transformations $(x_i, k_i) \mapsto (-x_i, -k_i), i = 1, \dots, d$. Then this symmetry will be propagated by (1.1), if $v(k)$ and $S(k, k')$ have the according symmetries. The crucial observation is that on the subinterval Ω of $\mathbb{T}^d$ defined by $x_i \ge 0, i = 1, \dots d$, distribution functions with the symmetry of f_0 satisfy specular reflection boundary conditions on $\partial\Omega$ [**26**]. This shows that our analysis applies to the case where the spatial domain is an *interval* with

specular reflection boundary conditions.
Smoothness with respect to x of f_0 as a function defined on $\mathbb{T}^d$ requires compatibility conditions along $\partial\Omega$. An example with $d = 1$ will be discussed in the last section. For general domains Ω with specular reflection boundary conditions, smoothness of the solution is a delicate question (see [**34**]).

2.2 The convergence result

In this section the following result will be proved:

2.1 Theorem. *Let* $n \geq 2$ *and* $f_0 \in L^1(\mathbb{T}^d \times B) \cap L^2\left(B, \frac{\mathrm{d}k}{M}; H^n(\mathbb{T}^d)\right)$. *Then there exists* $C > 0$, *such that the solution* f *of the initial value problem* (1.1), (1.5) *satisfies*

$$\|f - \rho_\infty M\|_{L^2(\mathbb{T}^d \times B, \mathrm{d}x\mathrm{d}k/M)} \leq C t^{(1-n)/2} .$$

As a preliminary step, we prove a smoothness result

2.2 Lemma. *Under the assumptions of the above theorem,* f *satisfies*

$$\|f(t,\cdot,\cdot)\|_{L^2\left(B,\mathrm{d}k/M;\ H^n(\mathbb{T}^d)\right)} \leq \|f_0\|_{L^2\left(B,\mathrm{d}k/M;\ H^n(\mathbb{T}^d)\right)} . \tag{2.7}$$

Proof. For $n = 0$ the result is a consequence of the entropy equation (1.3) which implies that the above norm of f is non increasing with time. Since the coefficients in the transport equation (1.1) are independent of x, the same equation holds for partial derivatives of f with respect to x. Therefore the left hand side of (2.7) also decays for positive n. □

The simplicity of the proof of this result strongly relies on the periodic boundary conditions, the linearity of the transport equation, and on the fact that the transport equation does not contain position dependent coefficients. In [**6**] and [**16**], where the whole space problem with a confining potential is treated, the proofs of results comparable to the Lemma are a major part of the analysis.
Now we proceed with the proof of the Theorem:

Proof. The main argument starts with the inequality (1.6). The problem is that it allows the decay to global equilibrium to stop as soon as a local equilibrium $f(t,x,k) = \rho(t,x)M(k)$ is reached. Therefore we have to prove that in such a situation f moves out of the local equilibrium as long as it is still away from the global equilibrium.
For this purpose, we compute derivatives with respect to time of the relative entropy of f with respect to the local equilibrium:

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|\rho M) = 2\int_{\mathbb{T}^d} \langle Q(f), f\rangle_M \,\mathrm{d}x + 2\int_{\mathbb{T}^d} \rho \nabla_x \cdot J \,\mathrm{d}x$$

Since $H(f|\rho M)$ is nonnegative, it is no surprise that the right hand side vanishes for $f = \rho M$. In the computation of the second order time derivative of $H(f|\rho M)$, we shall use the momentum balance equation

$$\partial_t J + \nabla_x \cdot P = \int_B v Q(f) \mathrm{d}k$$

with the pressure tensor $P := \int_B v \otimes v f \, \mathrm{d}k$. The right hand side can be interpreted as momentum relaxation term. Now we rewrite the pressure tensor as

$$P = \int_B v \otimes v (f - \rho M) \mathrm{d}k + \rho T \quad ,$$

where the temperature tensor $T := \int_B v \otimes v M \, \mathrm{d}k$ is positive definite as a consequence of (1.2) and of the continuity of $v(k)$.

A symmetrized version of the scattering operator is given by

$$Q^s(f) = \int_B \phi(k, k') \left(\frac{f'}{M'} - \frac{f}{M} \right) \mathrm{d}k' , \quad \phi(k, k') = \frac{S(k', k)M' + S(k, k')M}{2} .$$

The maps Q^s and Q produce the same quadratic form $\langle Q(f), f \rangle_M = \langle Q^s(f), f \rangle_M$, where the latter is derived from a symmetric bilinear form. Now it is straightforward to compute the second order time derivative of the relative entropy with respect to the local equilibrium:

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|\rho M) &= 2 \int_{\mathbb{T}^d} (\nabla_x \rho)^T T (\nabla_x \rho) \mathrm{d}x - 2 \int_{\mathbb{T}^d} (\nabla_x \cdot J)^2 \, \mathrm{d}x \\ &\quad +2 \int_{\mathbb{T}^d} \nabla_x \rho \cdot \int_B v \otimes v \nabla_x (f - \rho M) \mathrm{d}k \mathrm{d}x \\ &\quad -2 \int_{\mathbb{T}^d} \nabla_x \rho \cdot \int_B v Q(f) \mathrm{d}k \mathrm{d}x + 4 \int_{\mathbb{T}^d} \langle Q^s(f), Q(f) \rangle_M \, \mathrm{d}x \\ &\quad -4 \int_{\mathbb{T}^d} \langle Q^s(f), v \cdot \nabla_x f \rangle_M \, \mathrm{d}x . \end{aligned} \tag{2.8}$$

Note that when $f = \rho M$ all terms except the first one on the right hand side vanish. Exactly this term leads to the desired result that $\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|\rho M)$ is positive whenever $f = \rho M$, but f has not reached the global equilibrium $\rho_\infty M$. We estimate it from below by

$$\int_{\mathbb{T}^d} (\nabla_x \rho)^T T (\nabla_x \rho) \mathrm{d}x \geq K I(\rho M | \rho_\infty M),$$

where K is a positive constant coming from the positive definiteness of T, and

$$I(f|g) := H(\nabla_x f | \nabla_x g)$$

is the Fisher information. In estimating the remaining terms in (2.8), it is important that the bounds vanish for $f = \rho M$. For the first term we derive

$$\int_{\mathbb{T}^d} (\nabla_x \cdot J)^2 \mathrm{d}x = \int_{\mathbb{T}^d} \left(\nabla_x \cdot \int_B \sqrt{M(k)} v(k) \frac{(f - \rho M)}{\sqrt{M(k)}} \, \mathrm{d}k \right)^2 \mathrm{d}x \le m_2 I(f|\rho M),$$

where we have used that M has zero mean velocity. The other terms are estimated similarly by further applications of the Cauchy-Schwarz inequality:

$$\left| \int_{\mathbb{T}^d} \nabla_x \rho \cdot \int_B v \otimes v \nabla_x (f - \rho M) \mathrm{d}k \, \mathrm{d}x \right| \le \sqrt{m_4 I(\rho M|\rho_\infty M) I(f|\rho M)}\,,$$

$$\left| \int_{\mathbb{T}^d} \nabla_x \rho \cdot \int_B v Q(f) \mathrm{d}k \, \mathrm{d}x \right| \le 2\Gamma \sqrt{m_2 I(\rho M|\rho_\infty M) H(f|\rho M)}\,, \tag{2.9}$$

$$\left| \int_{\mathbb{T}^d} \langle Q^s(f), Q(f) \rangle_M \, \mathrm{d}x \right| = \left| \int_{\mathbb{T}^d} \langle Q^s(f - \rho M), Q(f - \rho M) \rangle_M \, \mathrm{d}x \right| \le 4\Gamma^2 H(f|\rho M)\,.$$

And finally, the most complicated term

$$\int_{\mathbb{T}^d} \langle Q^s(f), v \cdot \nabla_x f \rangle_M \, \mathrm{d}x = \int_{\mathbb{T}^d} \int_B \frac{1}{M} Q^s(f - \rho M) v \cdot \nabla_x (f - \rho M) \mathrm{d}k \, \mathrm{d}x + \int_{\mathbb{T}^d} \int_B Q^s(f) v \cdot \nabla_x \rho \, \mathrm{d}k \, \mathrm{d}x. \tag{2.10}$$

The first term on the right hand side is again split into two parts according to the gain and loss terms in Q^s. The second part originating from the loss term can be written as the integral of a divergence and, thus, vanishes by the divergence theorem due to the periodic boundary conditions. Therefore we have

$$\begin{aligned} &\int_{\mathbb{T}^d} \int_B \frac{1}{M} Q^s(f - \rho M) v \cdot \nabla_x (f - \rho M) \mathrm{d}k \, \mathrm{d}x \\ &= \int_{\mathbb{T}^d} \int_B \int_B \frac{\phi(k', k)}{M M'} (f' - \rho M') v \cdot \nabla_x (f - \rho M) \mathrm{d}k' \, \mathrm{d}k \, \mathrm{d}x\,. \end{aligned}$$

With the boundedness assumption on the scattering kernel and the Cauchy-Schwarz inequality the modulus of this term can be estimated by

$$\Gamma \sqrt{m_2 I(f|\rho M) H(f|\rho M)} \le \frac{\Gamma \sqrt{m_2}}{2} \big(I(f|\rho M) + H(f|\rho M) \big)\,.$$

The last term in (2.10) is estimated analogously to (2.9):

$$\left| \int_{\mathbb{T}^d} \nabla_x \rho \cdot \int_B v Q^s(f)\, \mathrm{d}k\, \mathrm{d}x \right| \le 2\Gamma \sqrt{m_2 I(\rho M|\rho_\infty M) H(f|\rho M)}\,.$$

Combining all these estimates, we have

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|\rho M) \ge{}& K I(\rho M|\rho_\infty M) - C_1 \sqrt{I(\rho M|\rho_\infty M)\big(H(f|\rho M) + I(f|\rho M)\big)} \\ &- C_2 H(f|\rho M) - C_3 I(f|\rho M)\,, \end{aligned} \tag{2.11}$$

with positive constants K, C_1, C_2, C_3. We simplify this inequality using the fact that $\forall\, \delta > 0\ \exists\, C_\delta > 0$, such that

$$\begin{aligned} &\sqrt{I(\rho M|\rho_\infty M)\big(H(f|\rho M) + I(f|\rho M)\big)} \le \\ &\qquad \delta I(\rho M|\rho_\infty M) + C_\delta H(f|\rho M) + C_\delta I(f|\rho M)\,, \end{aligned}$$

and the Poincaré inequality [**20**] yields

$$I(\rho M|\rho_\infty M) = \int_{\mathbb{T}^d} |\nabla_x \rho|^2 \mathrm{d}x \ge C \int_{\mathbb{T}^d} (\rho - \rho_\infty)^2\, \mathrm{d}x = C H(\rho M|\rho_\infty M)\,.$$

Choosing δ small enough, (2.11) takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|\rho M) \ge K H(\rho M|\rho_\infty M) - C_1 H(f|\rho M) - C_2 I(f|\rho M)\,, \tag{2.12}$$

with different, but still positive constants.

In the next step we use the additivity of the relative entropy, i.e.,

$$H(f|\rho M) + H(\rho M|\rho_\infty M) = H(f|\rho_\infty M)\,,$$

to derive

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|\rho M) \ge K H(f|\rho_\infty M) - \tilde{C}_1 H(f|\rho M) - C_2 I(f|\rho M)\,. \tag{2.13}$$

Now we need an estimate for the term $I(f|\rho M)$. This is done by standard interpolation [**48**] in the position variable:

$$\|\nabla_x u\|_{L^2(\mathbb{T}^d)} \le C \|u\|_{L^2(\mathbb{T}^d)}^{1-1/n} \|u\|_{H^n(\mathbb{T}^d)}^{1/n}\,, \qquad \forall\, u \in H^n(\mathbb{T}^d)\,,\ n \ge 1\,.$$

We derive

$$\begin{aligned} I(f|\rho M) &= \int_B \frac{1}{M} \|\nabla_x (f - \rho M)\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}k \\ &\le C \int_B \left(\frac{1}{M} \|f - \rho M\|_{L^2(\mathbb{T}^d)}^2 \right)^{1-1/n} \left(\frac{1}{M} \|f - \rho M\|_{H^n(\mathbb{T}^d)}^2 \right)^{1/n} \mathrm{d}k \\ &\le C H(f|\rho M)^{1-1/n} \left(\int_B \frac{1}{M} \|f - \rho M\|_{H^n(\mathbb{T}^d)}^2 \mathrm{d}k \right)^{1/n}\,, \end{aligned} \tag{2.14}$$

where the last estimate is due to the Hölder inequality. The boundedness of the last factor on the right hand side is ensured by the Lemma:

$$\int_B \frac{1}{M}\|f-\rho M\|^2_{H^n(\mathbb{T}^d)}\mathrm{d}k \le \|f(t,\cdot,\cdot)\|^2_{L^2\left(B,\frac{\mathrm{d}k}{M};H^n(\mathbb{T}^d)\right)} \le \|f_0\|^2_{L^2\left(B,\frac{\mathrm{d}k}{M};H^n(\mathbb{T}^d)\right)}\,.$$

Recalling (1.6), we have derived the following system of differential inequalities for the relative entropies with respect to local and global equilibrium:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(f|\rho_\infty M) &\le -\gamma H(f|\rho M)\,,\\ \frac{d^2}{\mathrm{d}t^2}H(f|\rho M) &\ge KH(f|\rho_\infty M) - C_1 H(f|\rho M) - C_2 H(f|\rho M)^{1-1/n}\,.\end{aligned} \tag{2.15}$$

Since $H(f|\rho M)$ is bounded, the term $C_1H(f|\rho M)$ can be dropped by increasing C_2.

Theorem 6.2 from [**16**] now ensures

$$H(f|\rho_\infty M) \le Ct^{-n+1}\,,$$

completing the proof. Note that for $n=\infty$ in (2.15), we would have exponential convergence by Theorem 6.2 of [**16**]. □

The Csiscár-Kullback inequality (compare [**3**]),

$$\|f\|_{L^1(\mathbb{T}^d\times B)} \le C\|f\|_{L^2(\mathbb{T}^d\times B,\mathrm{d}x\mathrm{d}k/M)}\,,$$

a simple consequence of the normalization of M and of the boundedness of $\mathbb{T}^d$, can be used to estimate decay of the L^1-norm rather than of the weighted L^2-norm:

2.3 Corollary. *Under the assumptions of the Theorem, the following estimate holds:*

$$\|f-\rho_\infty M\|_{L^1(\mathbb{T}^d\times B)} \le Ct^{(1-n)/2}\,.$$

2.3 A one-dimensional example

In this section we shall present a simple one dimensional problem that can be solved explicitly by spectral methods. We will show the relation between specular reflection and periodic boundary conditions and compare the decay estimate of the preceding section to the optimal result.

One advantage of the spectral theory approach is that it is not relying on smoothness of the solutions. While the assumption $f_0 \in L^2(B,H^1(\mathbb{T}^d))$ in Chapter 2 is necessary to give $\frac{d^2}{d^2t}H(f|\rho M)$ a meaning and even more regularity is needed to guarantee fast convergence, spectral theory shows *exponential* convergence for *every* L^2 solution.

We treat the following problem:

$$\partial_t f + k\partial_x f = \langle f \rangle - f \tag{3.16}$$

with $k \in \{+1, -1\}$, initial condition $f(t = 0) = f_0$, and periodic boundary conditions in x with period $2L$, i.e., $T^1 = (-L, L)$ is a possible choice. The expression $\langle f \rangle$ is the mean value

$$\langle f \rangle (t, x) := \frac{f(t, x, 1) + f(t, x, -1)}{2}.$$

This can be considered as a one dimensional neutron transport or radiative transfer equation and falls into the class of equations treated in this paper. We chose a simple one dimensional example because eigenfunctions and eigenvalues can be calculated explicitly, however our convergence result from Section 2 also applies to similar problems, i.e. for example mass preserving neutron transport or radiative transfer in higher dimensions. The entropy method was used to show exponential convergence for the homogeneous radiative transfer equation in [**22**]. For a review of neutron transport and spectral considerations leading to convergence results, see [**12**].

Equation (3.16) will be solved by spectral methods. The spectral problem

$$\lambda f + k\partial_x f = \langle f \rangle - f$$

can be written as the system of ordinary differential equations

$$\begin{aligned} \lambda \rho_\lambda + \partial_x j_\lambda &= 0\,, \\ \lambda j_\lambda + \partial_x \rho_\lambda &= -j_\lambda\,, \end{aligned} \tag{3.17}$$

by using the macroscopic density and flux

$$\rho(x) = f(x, +1) + f(x, -1)\,, \qquad J(x) := f(x, +1) - f(x, -1)\,.$$

The eigenvalues are solutions of

$$\lambda_l\,(\lambda_l + 1) = -\left(\frac{l\pi}{L}\right)^2, \qquad \text{i.e., } \lambda_{l,\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \left(\frac{l\pi}{L}\right)^2}, \qquad l \geq 0\,.$$

The eigenvalues $\lambda_{0,+} = 0$ and $\lambda_{0,-} = -1$ are simple with the eigenspaces $E_{0,+}$ =span$\{(1, 0)\}$ and $E_{0,-}$ =span$\{(0, 1)\}$. All the other eigenvalues have multiplicity two with the eigenspaces

$$E_{l,\pm} = \text{span}\left\{ \left(-\frac{l\pi}{\lambda_{l,\pm} L} \cos\frac{l\pi x}{L},\ \sin\frac{l\pi x}{L} \right),\ \left(\frac{l\pi}{\lambda_{l,\pm} L} \sin\frac{l\pi x}{L},\ \cos\frac{l\pi x}{L} \right) \right\}.$$

It is easily seen that the eigenfunctions form a basis of $L^2\big((-L,L)\big)^2$. Therefore, the initial value problem is solved completely for initial data $f_0(\cdot,\pm 1)\in L^2\big((-L,L)\big)$.
Observe that all eigenvalues have their real part in the interval $[-1,0]$. There is a spectral gap g of positive length between $\lambda_{0,+}=0$ and the nearest eigenvalue $\lambda_{1,+}$:

$$g=\begin{cases}\frac{1}{2} & \text{if } L<2\pi,\\ \frac{1}{2}-\sqrt{\frac{1}{4}-\left(\frac{\pi}{L}\right)^2} & \text{else.}\end{cases}$$

So we derived convergence of f as $t\to\infty$ (with respect to the L^2-norm) to

$$f_\infty=\frac{1}{4L}\int_{-L}^{L}(f_0(x,1)+f_0(x,-1))dx$$

at the exponential rate e^{-tg}.
As noted before, exponential convergence could be recovered in the entropy approach by showing that

$$I(f|\rho M)\le CH(f|\rho M) \tag{3.18}$$

holds and, thus, (2.15) is valid with $n=\infty$. However straightforward calculations show that this is not even possible in this simple case. It is well known [**3**] that validity of a Sobolev inequality of type (3.18) would imply the existence of a spectral gap of positive length. Our example, as many others, demonstrates that the converse is not true.

We conclude our work by commenting on specular reflection boundary conditions. Let us consider equation (3.16) on $\Omega=(0,L)$ with specular reflection boundary conditions

$$f(t,0,+1)=f(t,0,-1)\,,\qquad f(t,L,+1)=f(t,L,-1)\,.$$

Proceeding as proposed in the introduction, the initial data have to be continued to $(-L,L)$ by

$$f_0(x,k)=f_0(-x,-k)\,, \tag{3.19}$$

and the problem with periodic boundary conditions is solved. However, this periodic continuation has to satisfy regularity assumptions when the entropy approach is applied. It is of course *not* sufficient that $f_0(\cdot,\pm 1)\in H^n\big((0,L)\big)$. Additionally the initial data have to satisfy the following compatibility conditions:

$$\partial_x^m f_0(0,k)=(-1)^m\partial_x^m f_0(0,-k)\,,\qquad 0\le m\le n-1\,. \tag{3.20}$$

Conversely, for smooth periodic initial data satisfying (3.19), this symmetry (and, thus, also (3.20)) is propagated by the transport equation (being invariant under the map $(x, k) \to (-x, -k)$). Thus, if the solution is reduced to the interval $(0, L)$, it satisfies specular reflection boundary conditions.

Similar compatibility conditions arise in higher dimensional rectangular domains. For domains with curved boundaries and specular reflection boundary conditions, appropriate compatibility conditions are hard to formulate and are only expected to exist under convexity assumptions on the domain. Related results can be found in the recent work [**34**] on classical solutions of Vlasov-Poisson with specular reflection. There, only first order derivatives have to be controlled, which turns out to be difficult enough.

Chapter 3

Fermionic Relaxation

Convergence to global equilibrium for a kinetic fermion model

Lukas Neumann and Christian Schmeiser[1]

April 2004

Abstract: We study the long-time asymptotics of a kinetic models for fermions in a box with periodic boundary conditions. An entropy dissipation approach is used to prove decay to the global equilibrium for this nonlinear equation, that lacks dissipation in the position variable. We prove convergence at an algebraic rate depending on the smoothness of the solution. The result relies on some initial bounds and a uniform boundedness assumption for spatial derivatives of the solution.

Key words: Kinetic equations, Fermions, Fermi-Dirac-Distribution, semi-conductors, H-theorem, relative entropy, entropy dissipation, long time asymptotics

AMS subject classification: 82C10, 35B40, 82D37, 82B10, 82C21, 35Q40

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3.1 Introduction

We investigate the initial value problem

$$\partial_t f + v \cdot \nabla_x f = Q(f) \ , \quad f(0,x,v) = f_0(x,v) \ , \tag{1.1}$$

where $f = f(t,x,v) \geq 0$ denotes a particle distribution function, depending on time $t \geq 0$, position $x \in \mathbb{T}^d$ (where $\mathbb{T}^d$ is a d-dimensional torus, i.e. a rectangular box with periodic boundary conditions), and velocity $v \in \mathbb{R}^d$.

The particles are fermions and the scattering operator Q (acting only in the v-direction) is a simple model for the interaction of the particles with a nonmoving background medium with constant temperature

$$Q(f) = \int_{\mathbb{R}^d} [M(1-f)f' - M'(1-f')f] \mathrm{d}v' \, . \tag{1.2}$$

Here $f' = f(t,x,v')$ and $M(v) = (2\pi)^{-d/2} e^{-|v|^2/2}$ is the normalized Maxwellian. The factors $(1-f)$ and $(1-f')$ take into account the Pauli exclusion principle. The values of the distribution function have to respect the bounds $0 \leq f \leq 1$.

A kinetic equation with the same scattering operator, but also including acceleration of the particles by a given electric field, has been considered in [**46**]. Existence and uniqueness for initial value problems with $x \in \mathbb{R}^d$ (nonperiodic) have been proven and the macroscopic (diffusion) limit has been carried out.

A fermion Boltzmann equation modelling elastic particle-particle collisions has been studied by Dolbeault [**18**]. More elaborate models than (1.2) for the scattering of fermions due to a background medium or to a different species of particles have been considered in the modelling of charge transport in semiconductors ([**23**], [**38**], [**11**]). Existence and uniqueness and/or macroscopic limits are the subject of these studies.

Here we are interested in the long time behaviour of solutions of (1.1). It is characterized by two properties: conservation of total mass and entropy dissipation. The first is a consequence of the conservation property $\int_{\mathbb{R}^d} Q(f) \mathrm{d}v = 0$ of the scattering operator and of the periodic boundary conditions in position space:

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f(t,x,v) \mathrm{d}v \mathrm{d}x = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_0(x,v) \mathrm{d}v \mathrm{d}x \, .$$

Before stating the entropy dissipation property, we write the collision operator in the form

$$Q(f) = \int_{\mathbb{R}^d} MM'(1-f)(1-f')(F'-F) \mathrm{d}v' \, ,$$

with

$$F = \frac{f}{M(1-f)}\,.$$

Then, by the antisymmetry of the integrand with respect to v and v', it is easily shown that

$$\begin{aligned}(1.3)\quad &\int_{\mathbb{R}^d} Q(f)\chi(F)\mathrm{d}v = \\ &-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(F-F')(\chi(F)-\chi(F'))\mathrm{d}v'\mathrm{d}v \le 0\,,\end{aligned}$$

for arbitrary increasing functions χ. As a consequence, if an entropy is defined by $H_\infty = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} S_\infty(f,v)\mathrm{d}v\mathrm{d}x$ with

$$\frac{\partial S_\infty}{\partial f} = \chi\left(\frac{f}{M(v)(1-f)}\right)\,,$$

then the entropy dissipation equality

$$(1.4)\qquad \frac{\mathrm{d}H_\infty}{\mathrm{d}t} = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(f)\chi(F)\mathrm{d}v\mathrm{d}x$$

follows. If χ is chosen strictly increasing, then (by (1.3)) the entropy dissipation rate on the right hand side of (1.4) only vanishes if the (local) equilibrium condition $F = \kappa(t,x)$ is satisfied and the distribution function is the Fermi-Dirac distribution

$$(1.5)\qquad f(t,x,v) = f_l(t,x,v) = \frac{\kappa(t,x)M(v)}{1+\kappa(t,x)M(v)}\,.$$

Note that in this statement, $\kappa(t,x)$ can be chosen arbitrarily. In the following, however, we shall denote by f_l defined by (1.5) the local equilibrium distribution associated to an arbitrary (nonequilibrium) distribution f, where κ is chosen by fixing the position density:

$$(1.6)\qquad \int_{\mathbb{R}^d} f_l(t,x,v)\mathrm{d}v = \rho(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\mathrm{d}v\,.$$

Continuing our argument, we expect (because of the entropy dissipation equation) that for large times f approaches an equilibrium distribution, thus making the right hand side of the transport equation (1.1) vanish. The left hand side then only vanishes for constant κ. Thus, we expect f to converge to the global equilibrium

$$f_\infty(v) = \frac{\kappa_\infty M(v)}{1+\kappa_\infty M(v)}\,,$$

where κ_∞ is determined by mass conservation:

$$|\mathbb{T}^d| \int_{\mathbb{R}^d} f_\infty(v) \mathrm{d}v = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_0(x,v) \mathrm{d}v \mathrm{d}x \,.$$

From (1.4) a weak version of this statement can be proven (see, e.g., [**14**], [**1**]).

Recently, Desvillettes and Villani have developed a strategy for proving strong convergence to equilibrium for nonhomogeneous (in position) kinetic equations. It includes quantitative estimates on convergence rates. They have applied their approach to linear equations with a Fokker-Planck scattering operator and a confining potential [**16**] as well as, in a monumental work [**17**], to the Boltzmann equation of gas dynamics. Linear models have also been considered in [**21**] and [**6**].

In this work, the method of Desvillettes and Villani is applied to (1.1), (1.2). Its main point is to overcome the following difficulty: The right hand side of the entropy dissipation equation vanishes when the distribution is in local equilibrium. Thus, the entropy might stop decaying without f having reached the global equilibrium f_∞. As an input, the method requires certain bounds (uniform in time) on the distribution function and on its derivatives with respect to position. Whereas these could be proved for the linear problems in [**16**], [**21**], and [**6**], they have to be assumed for the Boltzmann equation. For the nonlinear problem (1.1), methods from [**46**] can be used to prove the propagation of bounds for the initial conditions in terms of Fermi-Dirac distributions. The boundedness of x-derivatives will be established in Part III, in the perturbative setting. At the end of this part we derive regulatity in the spatially homogenous setting.

In the following section, the boundedness result is proved, the method is outlined, and the main result is stated. The detailed (rather involved) computations and estimates are collected in section 3.

3.2 Preliminaries and main result

2.1 Theorem. *Assume there exist constants $\kappa_-, \kappa_+ > 0$ such that*

$$f_-(v) \le f_0(x,v) \le f_+(v) \,, \qquad \text{with } f_\pm(v) = \frac{\kappa_\pm M(v)}{1+\kappa_\pm M(v)} \,,$$

for all $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$. Then there is a unique solution $f(t,x,v)$ of (1.1), (1.2) satisfying the same bounds:

$$f_-(v) \le f(t,x,v) \le f_+(v) \,, \tag{2.7}$$

for all $t > 0$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$.

Proof. For details we refer to [**46**]. We only outline the proof of the bounds (2.7). The existence proof is based on a fixed point iteration on the set $\mathcal{V}$ of distribution functions satisfying (2.7). For $f \in \mathcal{V}$, we define the next iterate g by solving

$$\partial_t g + v \cdot \nabla_x g = M \int_{\mathbb{R}^d} f' \mathrm{d}v' - g \int_{\mathbb{R}^d} (Mf' + M'(1-f'))\mathrm{d}v' ,$$

$$g(0,x,v) = f_0(x,v) .$$

The difference $r = g - f_-$ satisfies

$$r(0,x,v) = f_0(x,v) - f_-(v) \geq 0$$

and

$$\begin{aligned} &\partial_t r + v \cdot \nabla_x r + r \int_{\mathbb{R}^d} (Mf' + M'(1-f'))\mathrm{d}v' \\ &= (1-f_-)M \int_{\mathbb{R}^d} f' \mathrm{d}v' - f_- \int_{\mathbb{R}^d} M'(1-f')\mathrm{d}v' \geq Q(f_-) = 0 . \end{aligned}$$

Nonnegativity of r and, thus, the lower bound $g \geq f_-$ follows. Analogously, $g \leq f_+$ and, therefore, $g \in \mathcal{V}$ is shown. □

Note that this ensures that $(1-f)$ is bounded away from zero, which we will make use of frequently.

In the following, relative entropies will be used for measuring the distance between distributions. Some arbitrariness comes from the freedom to choose the function χ in (1.4). We define the relative entropy of f with respect to g by

$$H(f|g) := \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} S(f,g)\mathrm{d}v\mathrm{d}x ,$$

with

$$S(f,g) = \int_g^f \ln \frac{z(1-g)}{g(1-z)} \mathrm{d}z = f \ln \frac{f(1-g)}{g(1-f)} + \ln \frac{1-f}{1-g} . \tag{2.8}$$

With this choice the relative entropy $H(f|f_\infty)$ coincides with the total entropy H_∞ defined in the introduction for $\chi(z) = \ln(z/\kappa_\infty)$. Until now this choice seems somewhat artificial, but we will further comment on it after explaining the strategy to derive the convergence result. We shall need the derivatives

$$\frac{\partial S}{\partial f} = \ln \frac{f(1-g)}{g(1-f)} , \qquad \frac{\partial^2 S}{\partial f^2} = \frac{1}{f(1-f)} . \tag{2.9}$$

By $S(f,f) = \frac{\partial S}{\partial f}(f,f) = 0$ and $\frac{\partial^2 S}{\partial f^2} > 0$, the relative entropy has the desired property to measure the distance between f and g. Actually, a stronger statement is true:

2.2 Lemma. *Let f and g satisfy (2.7). Then there exist constants $c_1, c_2 > 0$, such that*

$$c_1 \|f - g\|_M^2 \le H(f|g) \le c_2 \|f - g\|_M^2$$

with the weighted L^2-norm

$$\|f\|_M^2 := \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{f^2}{M} \,\mathrm{d}v\mathrm{d}x\,.$$

Proof. By (2.9) and the mean value theorem,

$$S(f,g) = \frac{(f-g)^2}{2\phi(1-\phi)}\,,$$

with ϕ lying between f and g. In particular, ϕ also satisfies (2.7). As a consequence

$$\frac{M}{c_2} \le 2\phi(1-\phi) \le \frac{M}{c_1}$$

holds with appropriate constants c_1, c_2, completing the proof. □

A second important property is what we would call a nonlinear version of the Pythagorean theorem:

2.3 Lemma. *The relative entropy is additive with respect to the local equilibrium,*

$$H(f|f_l) + H(f_l|f_\infty) = H(f|f_\infty)\,.$$

Proof. A straightforward computation gives

$$H(f|f_l) + H(f_l|f_\infty) = H(f|f_\infty) + \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f - f_l)\mathrm{d}v \,\ln\frac{\kappa_\infty}{\kappa}\,\mathrm{d}x\,.$$

The integral with respect to velocity vanishes because of (1.6). □

Also since

$$\frac{\partial S}{\partial f}(f, f_\infty) = \ln\frac{f}{\kappa_\infty M(1-f)}\,,$$

we can use (1.4), (1.3) with $\chi(z) = \ln(z/\kappa_\infty)$ to obtain

(2.10)

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|f_\infty) = -\frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(F-F')\ln\frac{F}{F'}\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x\,.$$

As mentioned above, the right hand side vanishes when f is a Fermi-Dirac distribution (such that F is independent of v). The basic idea of the Desvillettes-Villani approach is to show that in such a situation the entropy dissipation cannot remain zero as long as $f \neq f_\infty$. This is done by estimating the entropy dissipation in terms of the relative entropy of f with respect to the local equilibrium f_l and by deriving a second order differential inequality for $H(f|f_l)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(f|f_\infty) &\leq -c_3 H(f|f_l)\,, \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} H(f|f_l) &\geq c_4 H(f|f_\infty) - c_5 H(f|f_l)^{1-1/n}\,. \end{aligned} \tag{2.11}$$

It is the main contribution of this work to prove that these inequalities hold for appropriate $c_3, c_4, c_5 > 0$ and a positive integer n. This will be done in the following section. The proof requires the estimates from Theorem 2.1 and additional smoothness assumptions on the solution.

A result from [**16**] for systems of differential inequalities of the form (2.11) can then be used to get the desired convergence theorem:

2.4 Theorem. *Let the assumptions of Theorem 2.1 hold and let the solution f of (1.1) satisfy*

$$\left\| \frac{\partial^{k_1+\ldots+k_d} f}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}}(t,\cdot,\cdot) \right\|_M \leq c_6\,, \qquad \forall\, k_1+\ldots+k_d \leq n \quad \textit{and} \quad \forall\, t > 0\,, \tag{2.12}$$

for a constant c_6 and a positive integer n. Then there exists a constant $c_7 > 0$ such that

$$H(f|f_\infty) \leq c_7 t^{1-n}\,.$$

Remark 1: For the Fokker-Planck collision operator, a smoothness result like (2.12) was proven in [**16**] even for nonsmooth initial conditions by exploiting a hypoellipticity property. Here, one can only hope for propagation of regularity as in [**21**], assuming smoothness of the initial data. However we postpone this problem to a discussion i a seperate Chapter as well as Part III.

Remark 2: In principle we have the freedom to choose in (2.8) any entropy of the type

$$S_\chi(f,g) = \int_g^f \chi\left(\frac{z(1-g)}{g(1-z)}\right) \mathrm{d}z\,,$$

with an arbitrary monotone increasing function χ. Since the biggest difficulty is to deduce the second inequality in the system (2.11) we choose the relative entropy such that the expression for the derivative $\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_l)$ becomes as simple as possible (see (3.14) below), leading to the choice $\chi = \ln(\cdot/\kappa_\infty)$, which corresponds to (2.8).

2.5 Corollary. *With the assumptions of the previous theorem, there exists* $c_8 > 0$*, such that*

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} \le c_8 t^{(1-n)/2}\,.$$

Proof. The Cauchy-Schwarz inequality implies

$$\|g\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{|g|}{\sqrt{M}}\sqrt{M}\,\mathrm{d}v\mathrm{d}x \le \sqrt{|\mathbb{T}^d|}\,\|g\|_M\,.$$

The result now follows from Lemma 2.2 and from Theorem 2.4. □

3.3 Derivation of Differential Inequalities

Throughout this section c will be a positive real constant that may change from line to line.

3.1 Lemma. *Let the assumptions of Theorem 2.4 hold. Then there is a constant* $c_3 > 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_\infty) \le -c_3 H(f|f_l)\,.$$

Proof. We have to estimate the entropy production (2.10). Note that by the mean value theorem

$$\ln\frac{F}{F'} = \frac{F-F'}{\Phi}$$

holds, with Φ between F and F'. Also, by (2.7), we have $\kappa_- \le \Phi \le \kappa_+$. This gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_\infty) &\le -\frac{1}{2\kappa_+}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(F-F')^2\mathrm{d}v'\mathrm{d}v\mathrm{d}x\\ &\le -\frac{1}{2\kappa_+}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(1-f_l)(1-f_l')\\ &\qquad\qquad \times(F-F_l-(F'-F_l'))^2\mathrm{d}v'\mathrm{d}v\mathrm{d}x\,,\end{aligned}$$

where we have used $F_l = \frac{f_l}{M(1-f_l)} = \kappa = F_l'$. Expanding the square and using

$$F - F_l = \frac{f-f_l}{M(1-f)(1-f_l)}\,, \tag{3.13}$$

gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(f|f_\infty) \le &-\frac{1}{\kappa_+}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} M(1-f)(1-f_l)\mathrm{d}v\int_{\mathbb{R}^d}\frac{(f-f_l)^2}{M(1-f)(1-f_l)}\mathrm{d}v\mathrm{d}x\\ &+\frac{1}{\kappa_+}\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}^d}(f-f_l)\mathrm{d}v\right)^2\mathrm{d}x\,.\end{aligned}$$

The last term vanishes by the requirement (1.6) on the local equilibrium. From (2.7),

$$0 < 1 - f_+(0) \le 1 - f, 1 - f_l \le 1$$

follows and, therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|f_\infty) \le -c\|f - f_l\|_M^2 \,.$$

An application of Lemma 2.2 completes the proof. □

Now we shall prove the second inequality in (2.11). A straightforward computation gives

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|f_l) = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \left(\partial_t f \ln \frac{F}{F_l} - \partial_t f_l \frac{f - f_l}{f_l(1-f_l)}\right) \mathrm{d}v\mathrm{d}x$$
$$= \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \left(-v \cdot \nabla_x f \ln F + v \cdot \nabla_x f \ln \kappa + Q(f) \ln F - \frac{\partial_t \kappa}{\kappa}(f - f_l)\right) \mathrm{d}v\mathrm{d}x \,.$$

The first term on the right hand side vanishes by the divergence theorem (with respect to the x-variable) and the last one by (1.6), leaving

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|f_l) = \int_{\mathbb{T}^d} \nabla_x \cdot J \, \ln \kappa \, \mathrm{d}x + \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(f) \ln F \, \mathrm{d}v\mathrm{d}x = A + B \,, \tag{3.14}$$

with the flux density $J = \int_{\mathbb{R}^d} v f \, \mathrm{d}v$ (which vanishes for $f = f_l$).
For the computation of the time derivative of A we need the momentum balance equation

$$\partial_t J + \nabla_x \cdot P = \int_{\mathbb{R}^d} v Q(f) \mathrm{d}v \,,$$

where we shall split the pressure tensor into a local equilibrium part and a remainder:

$$P = \int_{\mathbb{R}^d} v \otimes v f \, \mathrm{d}v = \int_{\mathbb{R}^d} v \otimes v f_l \, \mathrm{d}v + \int_{\mathbb{R}^d} v \otimes v (f - f_l) \, \mathrm{d}v = P_l + \tilde{P} \,.$$

Differentiating (1.6) with respect to time and the continuity equation $\partial_t \rho + \nabla_x \cdot J = 0$ lead to

$$\frac{\partial_t \kappa}{\kappa} = \frac{-\nabla_x \cdot J}{\int_{\mathbb{R}^d} f_l(1 - f_l) \mathrm{d}v} \,.$$

With these preparations we obtain

$$
\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot (\nabla_x \cdot P_l)\mathrm{d}x + \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot (\nabla_x \cdot \tilde{P})\mathrm{d}x \\
&\quad - \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot \int_{\mathbb{R}^d} vQ(f)\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{T}^d} \frac{(\nabla_x \cdot J)^2}{\int_{\mathbb{R}^d} f_l(1-f_l)\mathrm{d}v}\,\mathrm{d}x\,. \qquad (3.15)
\end{aligned}
$$

Note that for $f = f_l$ all terms on the right hand side except the first vanish. This term is responsible for moving f out of local equilibrium as long as it is not in global equilibrium. For estimating it we need

$$
\nabla_x \cdot P_l = \int_{\mathbb{R}^d} v \otimes v\, f_l(1-f_l)\mathrm{d}v\, \frac{\nabla_x \kappa}{\kappa}\,. \qquad (3.16)
$$

The integral is an isotropic tensor which is positive definite since f_l satisfies (2.7):

$$
\int_{\mathbb{R}^d} v \otimes v\, f_l(1-f_l)\mathrm{d}v \geq \mathrm{Id} \int_{\mathbb{R}^d} v_i^2 f_-(1-f_+)\mathrm{d}v\,. \qquad (3.17)
$$

The first term on the right hand side of (3.15) can, thus, be estimated by

$$
\int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot (\nabla_x \cdot P_l)\mathrm{d}x \geq c\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)}\,.
$$

Now we estimate the remaining three terms in (3.15) one by one. First:

$$
\begin{aligned}
\left| \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot (\nabla_x \cdot \tilde{P})\mathrm{d}x \right| &\leq c \int_{\mathbb{T}^d} |\nabla_x \kappa| \int_{\mathbb{R}^d} |v|^2 |\nabla_x (f - f_l)|\mathrm{d}v\,\mathrm{d}x \\
&\leq c \int_{\mathbb{T}^d} |\nabla_x \kappa| \sqrt{\int_{\mathbb{R}^d} |v|^4 M\,\mathrm{d}v \int_{\mathbb{R}^d} \frac{|\nabla_x (f-f_l)|^2}{M}\mathrm{d}v}\;\mathrm{d}x \\
&\leq c\|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)} \|\nabla_x (f - f_l)\|_M\,.
\end{aligned}
$$

Second (similarly):

$$
\left| \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot \int_{\mathbb{R}^d} vQ(f)\mathrm{d}v\,\mathrm{d}x \right| \leq c\|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)} \|Q(f)\|_M\,.
$$

Now we need to quantify the behaviour of $Q(f)$ near f_l.

3.2 Lemma. *Q is a bounded operator and moreover*

$$
\|Q(f)\|_M \leq c\|f - f_l\|_M\,.
$$

Proof.

$$\|Q(f)\|_M^2 \le c \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} M \left(\int_{\mathbb{R}^d} M' |F - F'| \mathrm{d}v' \right)^2 \mathrm{d}v \, \mathrm{d}x \, .$$

The integral in parentheses can be estimated by

$$\begin{aligned} \int_{\mathbb{R}^d} M' |F - F'| \mathrm{d}v' &\le |F - F_l| + \int_{\mathbb{R}^d} M' |F' - F_l'| \mathrm{d}v' \\ &\le |F - F_l| + \sqrt{\int_{\mathbb{R}^d} M (F - F_l)^2 \mathrm{d}v} \, . \end{aligned}$$

Estimating the square of the sum by the sum of the squares we get

$$\|Q(f)\|_M^2 \le c \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} M (F - F_l)^2 \mathrm{d}v \, \mathrm{d}x \, .$$

Now the result of the lemma follows from (3.13) and the boundedness of $(1 - f)$ away from zero. □

Second, continued:

$$\left| \int_{\mathbb{T}^d} \frac{\nabla_x \kappa}{\kappa} \cdot \int_{\mathbb{R}^d} v Q(f) \mathrm{d}v \, \mathrm{d}x \right| \le c \|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)} \|f - f_l\|_M \, .$$

Third (last term in (3.15)):

$$\begin{aligned} &\int_{\mathbb{T}^d} \frac{(\nabla_x \cdot J)^2}{\int_{\mathbb{R}^d} f_l (1 - f_l) \mathrm{d}v} \mathrm{d}x \le c \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}^d} v \cdot \nabla_x (f - f_l) \mathrm{d}v \right)^2 \mathrm{d}x \\ &\le c \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |v|^2 M \mathrm{d}v \int_{\mathbb{R}^d} \frac{|\nabla_x (f - f_l)|^2}{M} \mathrm{d}v \, \mathrm{d}x = c \|\nabla_x (f - f_l)\|^2 \, . \end{aligned}$$

Collecting our results so far, we have proved

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} \ge c \|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)}^2 \; - \; &\tilde{c} \big(\|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)} \|\nabla_x (f - f_l)\|_M \\ &+ \|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)} \|f - f_l\|_M + \|\nabla_x (f - f_l)\|_M^2 \big) \, , \end{aligned}$$

implying

$$\frac{\mathrm{d}A}{\mathrm{d}t} \ge c \|\nabla_x \kappa\|_{L^2(\mathbb{T}^d)}^2 - \tilde{c} \big(\|f - f_l\|_M^2 + \|\nabla_x (f - f_l)\|_M^2 \big) \, . \tag{3.18}$$

The $\nabla_x \kappa$ term drives the solution out of local equilibria because it remains non zero as long as κ is different from the constant κ_∞. A Poincaré type estimate will help us to describe this by means of relative entropy.

3.3 Lemma. $\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)} \geq cH(f_l|f_\infty)$ *with* $c > 0$.

Proof. Since $\frac{d\rho}{d\kappa} = \frac{1}{\kappa}\int_{\mathbb{R}^d} f_l(1-f_l)dv$ is bounded from above and away from zero (by (2.7)),

$$\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)} \geq c\|\nabla_x \rho\|^2_{L^2(\mathbb{T}^d)}$$

with $c > 0$. Introducing $\rho_\infty = \int_{\mathbb{R}^d} f_\infty dv$ and noting that $\int_{\mathbb{T}^d}(\rho - \rho_\infty)dx = 0$, a Poincaré estimate gives

$$\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)} \geq c\|\rho - \rho_\infty\|^2_{L^2(\mathbb{T}^d)}dx\,,$$

with a possibly different, but still positive constant c. On the other hand,

$$|\rho - \rho_\infty| \geq c|\kappa - \kappa_\infty| = c\frac{|f_l - f_\infty|}{M(1-f_l)(1-f_\infty)} \geq c\frac{|f_l - f_\infty|}{M}\,,$$

where the first inequality follows again from the boundedness of $\frac{d\rho}{d\kappa}$ and the last from (2.7). This implies

$$\|\nabla_x \kappa\|^2_{L^2(\mathbb{T}^d)} \geq c\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} M(\kappa - \kappa_\infty)^2 dv dx \geq c\|f_l - f_\infty\|^2_M\,.$$

An application of Lemma 2.2 completes the proof. □

It remains to estimate the time derivative of the second term in (3.14). Using (1.3), this term can be written as

$$B = -\frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} MM'(1-f)(1-f')(F-F')\ln\frac{F}{F'}\,dv'dvdx\,.$$

The computation of the time derivative is facilitated by the fact that the integrand is symmetric with respect to f and f':

$$\begin{aligned}\frac{dB}{dt} \;&=\; \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M'\frac{1-f'}{1-f}\Bigg[M(1-f)(F-F')\ln\frac{F}{F'} \\ &\qquad\qquad - \ln\frac{F}{F'} - (F-F')\frac{1}{F}\Bigg]\partial_t f\,dv'dvdx\,.\end{aligned}$$

The term multiplying $\partial_t f$ in the integrand can by estimated (using (2.7)) by $cM'|F - F'|$. As a consequence,

$$\left|\frac{dB}{dt}\right| \leq c\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M'|F-F'|(|v|\,|\nabla_x f| + |Q(f)|)dv'dvdx$$

holds. With $|F - F'| \le |f - f_l|/M + |f' - f_l'|/M'$, the right hand side is bounded by the sum of four terms, which we estimate one by one. First:

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M'\frac{|f-f_l|\,|Q(f)|}{M}\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x \le \|f-f_l\|_M\|Q(f)\|_M\,.$$

Second:

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f'-f_l'|\,|Q(f)|\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x \le \|f-f_l\|_M\|Q(f)\|_M\,.$$

Third:

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f'-f_l'|\,|v|\,|\nabla_x f|\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x \le \int_{\mathbb{R}^d}|v|^2 M\,\mathrm{d}v\,\|f-f_l\|_M\|\nabla_x(f-f_\infty)\|_M\,.$$

The fourth term is the most difficult to estimate. Here we have to make a small concession on the exponent. We use $|f - f_l| \le cM$:

$$\begin{aligned}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M'|v|\frac{|f-f_l|\,|\nabla_x f|}{M}\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x \le \\ c\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\left|\frac{f-f_l}{\sqrt{M}}\right|^{1-\epsilon}\frac{|\nabla_x f|}{\sqrt{M}}|v|M^{\epsilon/2}\mathrm{d}v\mathrm{d}x \\ \le c\left(\int_{\mathbb{R}^d}|v|^{2/\epsilon}M\,\mathrm{d}v\right)^{\epsilon/2}\|f-f_l\|_M^{1-\epsilon}\|\nabla_x(f-f_\infty)\|_M\,.\end{aligned}$$

Since the Maxwellian has finite moments of arbitrary order, ϵ can be made arbitrarily small. Collecting the four estimates, using Lemma 3.2 and the fact that $\|f - f_l\|_M$ is bounded, we have

$$\left|\frac{\mathrm{d}B}{\mathrm{d}t}\right| \le c\left(\|f-f_l\|_M^2 + \|\nabla_x(f-f_\infty)\|_M\|f-f_l\|_M^{1-\epsilon}\right)\,.$$

This implies together with (3.14), (3.18), and Lemma 3.3

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(f|f_l) \ge cH(f_l|f_\infty) \quad - \quad & \tilde{c}\big(\|\nabla_x(f-f_l)\|_M^2 + \|f-f_l\|_M^2 \\ & +\|\nabla_x(f-f_\infty)\|_M\|f-f_l\|_M^{1-\epsilon}\big)\,.\end{aligned} \tag{3.19}$$

The next step is the derivation of bounds for the norms of the gradients. The interpolation inequality

$$\|\nabla_x u\|_{L^2(\mathbb{T}^d)} \le c\|u\|_{L^2(\mathbb{T}^d)}^{1-1/n}\|u\|_{H^n(\mathbb{T}^d)}^{1/n}$$

and the Hölder inequality imply

$$\begin{aligned}\|\nabla_x g\|_M^2 &= \int_{\mathbb{R}^d} \frac{1}{M}\|\nabla_x g\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}v \\ &\leq c \int_{\mathbb{R}^d} \left(\frac{1}{M}\|g\|_{L^2(\mathbb{T}^d)}^2\right)^{1-1/n} \left(\frac{1}{M}\|g\|_{H^n(\mathbb{T}^d)}^2\right)^{1/n} \mathrm{d}v \\ &\leq c\|g\|_M^{2(1-1/n)} \left(\int_{\mathbb{R}^d} \frac{1}{M}\|g\|_{H^n(\mathbb{T}^d)}^2 \mathrm{d}v\right)^{1/n} .\end{aligned}$$

By assumption (2.12) of Theorem 2.4 the last factor is bounded uniformly in time for $g = f - f_l$ and $g = f - f_\infty$. This gives

$$\|\nabla_x(f - f_l)\|_M^2 \leq c\|f - f_l\|_M^{2(1-1/n)}$$

and, with the Young inequality,

$$\begin{aligned}\|\nabla_x(f - f_\infty)\|_M \|f - f_l\|_M^{1-\epsilon} &\leq c\|f - f_\infty\|_M^{1-1/n}\|f - f_l\|_M^{1-\epsilon} \\ &\leq \delta\|f - f_\infty\|_M^2 + c_\delta\|f - f_l\|_M^{2(1-\epsilon)n/(n+1)}\end{aligned}$$

Now we choose $\epsilon = n^{-2}$ (such that $(1-\epsilon)n/(n+1) = 1 - 1/n$) and use the above inequalities and the Lemmata 2.2 and 2.3 in (3.19), to obtain the desired result

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(f|f_l) \geq c_4 H(f|f_\infty) - c_5 H(f|f_l)^{1-1/n} .$$

This completes the derivation of the differential inequalities (2.11) and, thus, the proof of Theorem 2.4.

Chapter 4

Comments on regularity for fermionic relaxation

In this chapter we shall prove some a priori bounds on the fermion collision operator. These are enough to conclude the uniform in time regularity in a spatially homogeneous setting, which indicates that the assumption of the theorem in the last chapter is at least not completely unrealistic.

We will only consider the physical case of dimension three in space and velocity. The results can be generalized to arbitrary dimensions but since we will use embeddings it is more convenient to restrict to a specific dimension. For convenience we use the following rescaling of the equation

$$f = \sqrt{M} F \ .$$

Then the equation can be written as

$$\partial_t F + v \cdot \nabla_x F = \tilde{Q}_f(F) \ . \tag{0.1}$$

The rescaled collision operator $\tilde{Q}_f$ (where the subscript f stands for fermion) can be split according to

$$\tilde{Q}_f(F) = Q_r(F) + B(F, F)$$

with the linear relaxation operator

$$Q_r(F) = \sqrt{M} \int \sqrt{M'} F' \mathrm{d}v' - F$$

and the bilinear part

$$B(F, F) = \int \sqrt{M'} F F' (M' - M) \mathrm{d}v' \ .$$

We recall from Chapter 3 that due to the maximum principle the following bounds on F propagate

$$\frac{\kappa_- M}{1+\kappa_- M} \leq F\sqrt{M} \leq \frac{\kappa_+ M}{1+\kappa_+ M} \ . \tag{0.2}$$

We denote by $k \in \mathbb{N}$ the total number of derivatives. Further we use the short notation $\partial^s_{s'} = \partial^s_x \partial^{s'}_v$ - where s and s' stand for two multi-index with $|s|+|s'| = k$. By H^k_p we denote the Sobolev space of functions with derivatives up to order k in L^p and write, as usual, H^k for H^k_2 . This implicitly refers to Sobolev-spaces that are periodic in x - however we will not make use of this.

4.1 Fermion Collision Estimates

First we will give a general estimate on the collision operator.

1.1 Theorem. *Let F be a function obeying the bound* (0.2). *For all multi-index s, s' with $k = |s| + |s'| \geq 3$ the rescaled collision operator $\tilde{Q}_f$ fulfills the following bound in the physical dimension* 3*:*

$$\left|\int \partial^s_{s'} F \partial^s_{s'} \tilde{Q}_f(F) \mathrm{d}(v,x)\right| \leq C\left[1 + \|F\|^{3/2+0}_{H^k} \|F\|^{3/2-0}_{H^{k-1}}\right] + \\ + \begin{cases} +C\|\partial^s F\|^2_{L^2} & \textit{for} \quad s'=0 \\ -\frac{1}{1+\kappa_+}\|\partial^s_{s'} F\|^2_{L^2} & \textit{else} \end{cases}$$

Proof. We shall first give the simple estimates for the relaxation operator

$$\begin{aligned} (1.3) \quad & |\int \partial^s_{s'} F \partial^s_{s'}\left[\sqrt{M}\int \sqrt{M'}F'\mathrm{d}v' - F\right]\mathrm{d}(v,x)| \leq \\ & \leq \int\int |\partial^s_{s'} F \partial_{s'}\sqrt{M}|\mathrm{d}v \int |\sqrt{M'}\partial^s F'|\mathrm{d}v'\mathrm{d}x - \|\partial^s_{s'}F\|^2_{L^2} \leq \\ & \leq \begin{cases} 0 & \text{for} \quad s'=0 \\ C\|\partial^s_{s'}F\|_{L^2}\|\partial^s F\|_{L^2} - \|\partial^s_{s'}F\|^2_{L^2} & \text{else} \end{cases} \end{aligned}$$

now we estimate the nonlinear part of the collision operator.

$$\begin{aligned} (1.4) \quad & \left|\int \partial^s_{s'} F \partial^s_{s'} B(F,F)\mathrm{d}(v,x)\right| = \\ & |\int \partial^s_{s'} F \int \sqrt{M'} \sum_{j'=0}^{s'}\sum_{j=0}^{s}\binom{s'}{j'}\binom{s}{j}\partial_{j'}(M'-M)\partial^j F' \partial^{s-j}_{s'-j'}F\mathrm{d}v'\mathrm{d}(v,x)| \end{aligned}$$

First we start with the terms in the sum that have derivatives only on one of the F's. For the term corresponding to $j = j^{'} = 0$ we use the L^∞ bound and completely neglect the negative part to get

$$|\int (\partial^s_{s'}F)^2 \int \sqrt{M'}(M'-M)F'\mathrm{d}v'\mathrm{d}(v,x)| \leq$$
$$\frac{\kappa_+}{1+\kappa_+}\|\partial^s_{s'}F\|^2_{L^2}\int \sqrt{M'}\mathrm{d}v' = \frac{\kappa_+}{1+\kappa_+}\|\partial^s_{s'}F\|^2_{L^2}\ .$$

For $j = 0,\ j^{'} \neq 0$ we have also by the L^∞ bound
(1.5)
$$|\int \partial^s_{s'}F\int \binom{s'}{j^{'}}\sqrt{M'}\partial_{j'}(M'-M)F'\partial^s_{s'-j'}F\mathrm{d}v'\mathrm{d}(v,x)| \leq C\|F\|_{H^k}\|F\|_{H^{k-|j'|}}\ .$$

Now we treat the last of the "extremal" cases, $|j| = k$. Note that this term only occurs in the sum if $s = j$ and thus $s' = 0$.
Thus we can estimate

$$|\int \partial^s F\int \sqrt{M'}(M'-M)\partial^s F' F\mathrm{d}v'\mathrm{d}(v,x)| \leq$$
$$\leq 2\max_v(M)\max_v\left(\frac{\kappa_+}{1+\kappa_+M}\right)\int \sqrt{M}|\partial^s F|\mathrm{d}v\int \sqrt{M'}|\partial^s F'|\mathrm{d}v'\mathrm{d}x \leq$$
$$\leq C\|\partial^s F\|^2_{L^2}\ .$$

In all the other cases we use the fact that some v derivatives go on the Maxwellian and (more important) the x derivative split on the product of the distribution function.

$$(1.6)\quad |\int \partial^s_{s'}F\int \sqrt{M'}\binom{s'}{j^{'}}\binom{s}{j}\partial_{j'}(M'-M)\partial^j F'\partial^{s-j}_{s'-j'}F\mathrm{d}v'\mathrm{d}(v,x)| \leq$$
$$\leq C\int\int |\partial^s_{s'}F\sqrt{M'}\partial^j F'\partial^{s-j}_{s'-j'}F|\mathrm{d}v'\mathrm{d}(v,x) \leq$$
$$\leq C\int \|\partial^s_{s'}F\|_{L^2_v}\|\partial^j F\|_{L^2_v}\|\partial^{s-j}_{s'-j'}F\|_{L^2_v}\mathrm{d}x$$

where C consists of the binomial coefficients and $2\max_v(M)$ or $\max_v(v^{j'}M)$ respectively. Now we split the integral in x in such a way that we can use Sobolev embedding to retain a product of L^2 norms.
First we treat the case $k > |j| \geq 2$. Here we estimate

$$\int \|\partial^s_{s'}F\|_{L^2_v}\|\partial^j F\|_{L^2_v}\|\partial^{s-j}_{s'-j'}F\|_{L^2_v}\mathrm{d}x \leq \|\partial^s_{s'}F\|_{L^2}\left\|\|\partial^{s-j}_{s'-j'}F\|_{L^2_v}\right\|_{L^\infty_x}\|\partial^j F\|_{L^2}\ ,$$

and use the compact embedding $H^0_\infty \subset H^{3/2+0}_{2,x}$ and the interpolation $H^{3/2+0}_{2,x} = \left[H^1_{2,x}, H^2_{2,x}\right]_{1/2+0}$ to derive

$$\int \|\partial^s_{s'} F\|_{L^2_v} \|\partial^j F\|_{L^2_v} \|\partial^{s-j}_{s'-j'} F\|_{L^2_v} \mathrm{d}x \le \|F\|^{3/2+0}_{H^k} \|F\|^{3/2-0}_{H^{k-1}} \,, \tag{1.7}$$

where we use that we are in the case $|s-j| + |s'-j^{'}| \le k-2$.
Now we are left with the case $|j| = 1$. But since we assumed $k \ge 3$ this means we can now embed and interpolate the term $\partial^j F'$ in the product yielding the same estimate as in equation (1.7).
Summing up the estimates for the single contributions in (1.4) and adding the contribution of the linear part of the collision operator (1.3) concludes the proof. □

The statement of the theorem shows that the collision operator has a damping effect on the velocity derivatives. This fact corresponds to the negative term of highest order in derivatives on the right hand side. Now we will show a similar result also in the case of second order derivatives.

1.2 Corollary. *Let the assumptions of Theorem 1.1 be valid with the exception that $k = 2$, then the following estimate holds:*

$$\begin{aligned} &\left|\int \partial^s_{s'} F \partial^s_{s'} \tilde{Q}_f(F) \mathrm{d}(v,x)\right| \\ &\qquad \le C\left[1 + \|F\|^{1/4}_{H^1_2} \|F\|_{H^1_4} \|F\|^{7/4}_{H^2_2}\right] + \begin{cases} +C\|\partial^s F\|^2_{L^2} & \textit{for} \quad s' = 0 \\ -\frac{1}{1+\kappa_+}\|\partial^s_{s'} F\|^2_{L^2} & \textit{else} \end{cases} \end{aligned}$$

Proof. The only point where we used the fact that $k \ge 3$ is to return to the L^2 norm in x by embedding when the derivatives split onto the product in the collision operator. Thus the estimates for the linear part of the collision operator as well as for the cases $j = 0$ and $|j| = k$ stay exactly the same. We are left with estimating (1.6) in a different way for $|j| = 1$ and start from the second line in (1.6).

$$\begin{aligned} &\int\int |\partial^s_{s'} F \sqrt{M'} \partial^j F' \partial^{s-j}_{s'-j'} F| \mathrm{d}v' \mathrm{d}(v,x) \le \\ &\qquad \le \|\partial^s_{s'} F\|_{L^2} \|\int \sqrt{M'} \partial^j F' \mathrm{d}v'\|_{L^4_x} \left\| \|\partial^{s-j}_{s'-j'} F\|_{L^2_v} \right\|_{L^4_x} \le \\ &\qquad \le \|\partial^s_{s'} F\|_{L^2} \left\| \|\partial^{s-j}_{s'-j'} F\|_{L^2_v} \right\|_{L^4_x} \|\partial^j F\|_{L^4} \le \|F\|^{7/4}_{H^2_2} \|F\|^{1/4}_{H^1_2} \|F\|_{H^1_4} \end{aligned}$$

where we used the embedding $H^0_4 \subset H^{3/4}_2 = [H^0_2, H^1_2]_{3/4}$ in the last inequality. □

4.2 Uniform regularity in the homogeneous setting

Now we will show how to derive uniform in time regularity in the spatially homogeneous setting from these bounds. The proof is based on a maximum principle and induction on the number of derivatives.

2.1 Theorem. *Consider the simple space homogeneous equation*

$$\partial_t F = \tilde{Q}_f(F)$$

with $F(t, v)$ only depending on time $t \in [0, \infty[$ and velocity v in $\mathbb{R}^3$, and suspect to initial conditions $F(t = 0) = F_0$.
If for $k \geq 3$ the initial data F_0 is in H^k and satisfies the bounds (0.2) *then the unique solution is in H^k uniformly in time, i.e. $F \in L^\infty([0, \infty[, H^k)$, with explicit bound.*

Proof. We already discussed the existence of a solution and the propagation of the bounds in the previous chapter.
Now we shall establish the bounds in H^k on this solution. We proceed by induction on the number of derivatives $|s'|$. For $s' = 0$ the uniform boundedness of the solution is clear.
For $|s'| = 1$ we use the differential equation to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{s'} F\|_{L^2}^2 = 2\int \partial_{s'} F \partial_{s'} \tilde{Q}_f(F)\mathrm{d}v = 2\int \partial_{s'} F \partial_{s'}(Q_r(F) + B(F, F))\mathrm{d}v \ .$$

Now we estimate

$$\int \partial_{s'} F \partial_{s'} Q_r(F)\mathrm{d}v \leq C\|F\|_{L^2}\|\partial_{s'} F\|_{L^2} - \|\partial_{s'} F\|_{L^2}^2 \ ,$$

and for the bilinear part

$$\int \partial_{s'} F \partial_{s'} B(F, F)\mathrm{d}v \leq \int \partial_{s'} F^2 \int \sqrt{M'}^3 F'\mathrm{d}v'\mathrm{d}v \leq \frac{\kappa_+}{1+\kappa_+}\|\partial_{s'} F\|_{L^2}^2 \ ,$$

where we discarded the negative term in the first estimate and used the bounds on F in the second one. Together these two estimates lead to the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{s'} F\|_{L^2}^2 \leq 2\left(C\|F\|_{L^2}\|\partial_{s'} F\|_{L^2} - \frac{1}{1+\kappa_+}\|\partial_{s'} F\|_{L^2}^2\right) \ .$$

Since we already have established a bound on $\|F\|_{L^2}$ this yields, by maximum principle, the desired bound on the first order derivatives. Before we proceed to the second order derivatives it is important to note that exactly the same argument also works for $\|\partial_{s'} F\|_{L^4}^4$ - yielding again a uniform bound since the initial data is in H_4^1 by Sobolev embedding.

Now for $|s'| = 2$ we can use the estimates from Corollary 1.2 and the fact that we already have uniform bounds on H_4^1 and H_2^1 to estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{s'} F\|_{L^2}^2 = 2\int \partial_{s'} F \partial_{s'} \tilde{Q}_f(F)\mathrm{d}v \leq C(1 + \|F\|_{H^2}^{7/4}) - \frac{1}{1+\kappa_+}\|\partial_{s'} F\|_{L^2}^2 \ .$$

Summing these estimates over all possible s' with $|s'| = 2$ and applying a maximum principle finishes the proof for $k = 2$. For $k = 3$ and higher the induction is straightforward by the estimates from Theorem 1.1, using only the quadratic norms. □

Remark: Observe that with the above rescaling due to the identity

$$\|F\|_{L^2}^2 = \|f\|_M^2$$

by bounding the norm of F we deduce a bound on f in exactly the space that we needed in the Entropy method - however only in the homogeneous setting.

While this result gives some regularizing property of the collision operator I have to confess, that its not at all conclusive in the non homogeneous case as can be seen from the following example that was pointed out to me by C. Villani.

Example: Consider the free transport equation

$$\partial_t f + v \cdot \nabla_x f = 0 \ ,$$

spatially confined in the torus, i.e. $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$, and subject to an initial condition $f(t = 0) = f_0$. In the spatially homogeneous case the equation of motion reduces to an ordinary differential equation and we expect the velocity-derivatives to be bounded uniformly. However in the x-dependent case, using the explicit representation of the solution obtained from tracking the characteristics

$$f(t, x, v) = f_0(x - vt, v) \ ,$$

we can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f\|^2 = 0 \ .$$

Still the velocity derivatives grow in time

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v f\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla_v f_0(x - vt, v)|^2 \mathrm{d}v\mathrm{d}x = 2t\|\nabla_x f_0\|^2 - 2\int \nabla_v f_0 \cdot \nabla_x f_0 \mathrm{d}v\mathrm{d}x$$

if the initial distribution is not uniform in x. This example is of course worse then the fermionic relaxation equation in the sense that it completely lacks a (regularizing) collision operator but it shows that observations from homogeneous settings can not be transformed to the non homogeneous situation straightforwardly. We tried (in collaboration with C. Mouhot and C. Villani) to use the transport operator to transfer some of the regularizing properties from the velocity to the space direction but did not succeed in the large. This was the motivation for the study that led to the results presented in the next part.

Part III

Coercivity in Sobolev spaces

Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus

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Abstract: For a general class of linear collisional kinetic models in the torus, including in particular the linearized Boltzmann equation for hard spheres, the linearized Landau equation with hard and moderately soft potentials and the semi-classical linearized fermionic and bosonic relaxation models, we prove explicit coercivity estimates on the associated integro-differential operator for some modified Sobolev norms. We deduce existence of classical solutions near equilibrium for the full non-linear models associated, with explicit regularity bounds, and we obtain explicit estimates on the rate of exponential convergence towards equilibrium in this perturbative setting. The proof is based on a linear energy method which combines the coercivity property of the collision operator in the velocity space with transport effects, in order to deduce coercivity estimates in the whole phase space.

Mathematics Subject Classification (2000): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05].

Keywords: collisional kinetic models, Boltzmann equation, Landau equation, relaxation, semi-classical relaxation, bosons, fermions, Fokker-Planck, weak external field, Poisson self-consistent potential, rate of convergence to equilibrium, explicit, energy method.

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5.1 Introduction

In this paper we study a general class of linear inhomogeneous kinetic equations in the torus (including linearized Boltzmann, Landau, classical and semi-classical relaxation, and Fokker-Planck equations). We then use the properties of their evolution semi-groups to gain insight into the behaviour of the full non-linear models in the cases of Boltzmann and semi-classical relaxation models. The main tool is an estimate of the following type. Assume that the linear (self-adjoint) collision operator L acting only on the velocity space is coercive for a certain norm, and that L has a structure "mixing part + coercive part". Then the integro-differential operator $T = L - v \cdot \nabla_x$ (taking into account transport effects) acting on phase space x and v, satisfies a coercivity property in x and v, which implies in particular the existence of a spectral gap estimate when L has a spectral in velocity (note that in general T is not sectorial). Moreover our proof shows how to compute the constant of coercivity of T according to the one of L in the velocity space (and thus the rate of exponential convergence to equilibrium when L has a spectral gap in velocity). Before we explain our method and results in more detail, let us introduce the models and problems in a precise way.

5.1.1 The Problem and its motivation

In order to study the convergence to equilibrium, a new quantitative method in the large, the "entropy-entropy-production" method (EEP-method), has developed from the beginning of the 1990 decade, see [**7, 8, 49**] in the spatially homogeneous setting, and [**16**] in the spatially inhomogeneous setting. It has provided new powerful and robust tools in the study of relaxation towards equilibrium (see for instance [**16, 17, 6, 21, 45**]), and seems to be the best suitable approach to deal with non-linear models in the large. However, it can be seen from these references that the method, while very robust with respect to nonlinearities and able to deal with external potentials at no increased difficulties, has two major shortcomings. First it relies on uniform in time estimates of regularity of the solution, that are usually hard to establish. Second, it seems to fail to give the optimal rate of convergence, in particular when L has a spectral gap in velocity it fails to give exponential rate of convergence to equilibrium.

In this article we provide a linear energy method, combined with some explicit coercivity estimates on the linearized collision operator in order to overcome these problems for linear models, or for non-linear models in the perturbative setting. The difference when compared to the EEP-method is that the convergence as well as the uniform in time regularity bounds are obtained in a single step. However this approach is linear and therefore limited to the perturbative setting near equilibrium for the non-linear models. A similar approach has recently been used in Guo's papers [**29, 30, 35, 31, 47**]. Previous works in this direction like [**27**] and [**28**] had the drawbacks of being non explicit and, similarly to the EEP-method, of giving only "almost exponential decay" (*i.e.* as $O(t^{-\infty})$) to equilibrium. Also another approach has been developed in [**33**] for the Fokker-Planck model in a confining external field, and recently generalized to the linear relaxation model in [**32**] (see also [**51**] for related recent results on the Fokker-Planck equation). Our results are partly included in these references, however our proof is simpler and more explicit, and also our viewpoint unifies previous scattered results.

Most of these articles deal with the Boltzmann equation that we address in section 5.5.4. Our main abstract theorem applies to the Boltzmann linearized collision operator for hard spheres (and hard potentials with Grad's cutoff assumption): smooth solutions with explicit regularity bounds and rate of convergence to equilibrium are constructed near equilibrium. A similar study has been performed in [**28**] in a non-constructive way for the Vlasov-Poisson-Boltzmann system near equilibrium and the Landau equation [**27**]. In [**29**] the same method was also applied to the Boltzmann equation for cutoff soft potentials.

Guo's argument relies, roughly speaking, on the coercivity of the linearized Boltzmann operator "in the mean" – *i.e.* if integrated over a time interval – for small perturbations of the Maxwellian. Due to this averaging in time the convergence that can be deduced from the results in [**28, 27**] is only almost exponential. Later Guo refined his approach in [**30**] to meet the needs of application to the Vlasov-Maxwell-Boltzmann system. The new idea is to use a norm that includes temporal derivatives. In this norm the fields are almost controlled by the deviation of the distribution from the Maxwellian – to be more precise they "loose" one derivative. This leads to *instantaneous* coercivity and thus global perturbative solutions for the Vlasov-Maxwell-Boltzmann system as well as exponential decay to equilibrium in the case of Vlasov-Poisson-Boltzmann, in these norms. The proof, while being very complicated, is constructive. More recently a farther refined method based also on norms including temporal derivatives has been used to show almost exponential decay for various of the mentioned models and the relativistic Landau Maxwell system in [**47**].

The EEP-method has been applied to the Boltzmann equation in [**17**] – again assuming uniform in time regularity bounds on the solution – but for a large class of collision operators and most important – without perturbative assumptions. All these works have been carried out on bounded domains, essentially the torus apart from [**17**] where various types of boundary conditions are considered. For completeness we also refer to a recent preprint dealing with the problem in whole space [**31**] and the article [**35**] for a related non-linear energy method.

Despite this vast amount of recent literature on the problem our proof has two advantages. First it is simpler than the ones in the articles quoted above, mostly due the fact that we take advantage of the mixing properties of the collision operator. Second our proof separates in a very clear way between linear effects in these energy estimates (transport + linear collision), which are expressed in a coercivity estimate on this linear part, and the problems arising from the small remaining bilinear part when considering solutions near the equilibrium. We are able to derive exponential convergence to equilibrium without resorting to norms including time derivatives and in a purely instantaneous manner.

5.1.2 The models

We will study initial value problems for equations of the form

$$\partial_t f + v \cdot \nabla_x f = Q(f), \qquad t \geq 0,\ x \in \Omega,\ v \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$ denotes the dimension of space. In this equation, f denotes the distribution of particles in phase space, therefore it is a time-dependent non-negative $L^1(\Omega \times \mathbb{R}^N)$ function. The operator Q models the collisional interactions of particles (either binary or between particles and a surrounding medium). It is local in x and t and it depends on the particular model of interaction chosen. The term $v \cdot \nabla_x f$ corresponds to the free flow of particles. For the spatial domain Ω, we shall consider here the periodic case, that is $\Omega = \mathbb{T}^N$. Hence $f = f(t, x, v)$ satisfies initial conditions $f(t = 0, x, v) = f_0(x, v)$ and periodic boundary conditions.

Under very general assumptions, equations of class (1.1) admit a unique equilibrium in the torus, which we shall denote by f_∞, and which is independent of t, x (this is trivial for relaxation models admitting only mass conservation, and for models admitting conservation of mass, momentum and energy, such as Boltzmann and Landau equations, this is shown easily inspiring from the arguments in [**17**] and [**14**]).

Then one can consider the linearization around the equilibrium. In order to reduce to a Hilbert space setting, one usually considers perturbations of the form $f = f_\infty + f_\infty^{1/2} h$. Discarding the bilinear term, it yields the following linearized equation on h

$$\partial_t h + v \cdot \nabla_x h = L(h), \tag{1.2}$$

where L depends on the precise form of the collision operator Q. The unknown h belongs to $L^2(\mathbb{T}^N \times \mathbb{R}^N)$.

Now let us give a general framework for linear collisional kinetic models. We shall denote as usual by L^2 the Lebesgue space of square integrable functions, and, for $k \in \mathbb{N}$, H^k the Sobolev space of L^2 functions with square integrable derivatives up to order k. We shall indicate as a subscript the variables (x or v) these functional space refers to. For the variable x, Sobolev spaces refer implicitly to *periodic* Sobolev spaces. When no subscript is used, these functional spaces always refer to x and v.

H1. (Structure) We consider a linear operator of the form

$$T = L - v \cdot \nabla_x, \tag{1.3}$$

where the operator L is a closed and self-adjoint operator on L^2_v, local in t, x. Moreover we assume that it writes

$$L = K - \Lambda \tag{1.4}$$

where Λ **is a coercive operator** in the sense: there is a norm $\|\cdot\|_{\Lambda_v}$ on $\mathbb{R}^N$ (the space of velocities), such that

$$\nu_0^\Lambda \, \|h\|^2_{L^2_v} \leq \nu_1^\Lambda \|h\|^2_{\Lambda_v} \leq \langle \Lambda(h), h \rangle_{L^2_v} \leq \nu_2^\Lambda \, \|h\|^2_{\Lambda_v}$$

and also

$$\langle \nabla_v \Lambda(h), \nabla_v h \rangle_{L^2_v} \geq \nu_3^\Lambda \, \|\nabla_v h\|^2_{\Lambda_v} - \nu_4^\Lambda \, \|h\|^2_{L^2_v}$$

for some constants $\nu_0^\Lambda, \nu_1^\Lambda, \nu_2^\Lambda, \nu_3^\Lambda, \nu_4^\Lambda > 0$.
Moreover we assume that L satisfies (for some constant $C^L > 0$)

$$\langle L(h), g \rangle_{L^2_v} \leq C^L \, \|h\|_{\Lambda_v} \, \|g\|_{\Lambda_v}.$$

To shorten the notation we also introduce the norm

$$\|\cdot\|_\Lambda := \| \|\cdot\|_{\Lambda_v} \|_{L^2_x} \ .$$

H2. (Mixing property in velocity) We assume that K **has a regularizing effect** in the following sense: for any $\delta > 0$, there is some explicit $C(\delta)$ such that for any $h \in H^1_v$,

$$\langle \nabla_v K(h), \nabla_v h \rangle_{L^2_v} \leq C(\delta) \, \|h\|^2_{L^2_v} + \delta \, \|\nabla_v h\|^2_{L^2_v} \ . \tag{1.5}$$

H3. (Relaxation to the local equilibrium) We assume that L, as an operator on L^2_v, has a finite dimensional kernel

$$N(L) = \mathrm{Span}\,\{\varphi_1, \dots, \varphi_n\}$$

and we denote by Π_l the orthogonal projection on $N(L)$ in L^2_v. We make the following **local coercivity assumption**: there is $\lambda > 0$ such that

$$\langle L(h), h\rangle_{L^2_v} \leq -\lambda \, \|h - \Pi_l(h)\|^2_{\Lambda_v}.$$

This together with **H1** implies in particular that L is non-positive and has a spectral gap in L^2_v, whose size is bounded from below by

$$\lambda_L = \left(\frac{\nu_0^\Lambda}{\nu_1^\Lambda}\right) \lambda > 0.$$

In the structure assumption **H1**, K shall typically stand for a multiplicative operator by a function ν (usually called the *collision frequency*) in the case of short-range interactions, or some diffusion operator in the case of long-range interactions. The norm $\|\cdot\|_\Lambda$ that we define can also loosely be seen as the norm of the graph of $\Lambda^{1/2}$. The coercivity of Λ is linked with the coercivity property of the whole linearized collision operator L in **H3** (where the word "local" refers to the position x and the fact that it pushes the dynamic towards local equilibrium). This coercivity property is crucial to ensure exponential decay towards equilibrium, even in the homogeneous setting. In the case where it is weakened, which happens for interactions with "weak collision effect" such as soft potentials (see e.g. in [**41**]), one expects convergence rates of the form e^{-t^τ} with $\tau < 1$ (see [**47**]).

The kernel of L in L^2_v, which is composed of functions belonging to $N(L)$ for any x, corresponds to the manifold of local equilibriums for the linearized kinetic models. Therefore when the x variable is added, Π_l is the projection on the "fluid part", and $(\mathrm{Id} - \Pi_l)$ is the projection on the "kinetic" part. It is defined by

$$\Pi_l(h) = \sum_{i=1}^n \left(\int_{\mathbb{R}^N} h\,\varphi_i \,\mathrm{d}v\right) \varphi_i.$$

It is trivial that any local equilibrium uniform in space is indeed a global equilibrium. Since L is self-adjoint, the $\varphi_1, \dots, \varphi_n$ belong to the kernel of its adjoint L^*, and thus by integrating in x and v we get

$$\forall\, i = 1, \dots, n, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^N \times \mathbb{R}^N} h\,\varphi_i \,\mathrm{d}x\,\mathrm{d}v = 0.$$

Hence we denote

$$\Pi_g(h) = \sum_{i=1}^{n} \left(\int_{\mathbb{T}^N \times \mathbb{R}^N} h\, \varphi_i \, \mathrm{d}x \, \mathrm{d}v \right) \varphi_i$$

which is time and space independent (this can be shown easily to be the orthogonal projection on $N(T)$ in $L^2_{x,v}$, directly or using the coercivity property of Theorem 1.1).

A detailed study of physical models satisfying assumptions **H1-H2-H3** shall be given in Section 5.5.

5.1.3 Main results

First we state the main result on the coervivity estimates on T and the consequence on its evolution semi-group.

1.1 Theorem. *Let T be a linear operator on $L^2_{x,v}$ satisfying assumptions* **H1-H2-H3**. *Then T generates a strongly continuous evolution semi-group e^{tT} on $H^1_{x,v}$, which satisfies*

$$\|e^{tT}(\mathrm{Id} - \Pi_g)\|_{H^1_{x,v}} \le C_T \, \exp\left[-\tau_T t\right]$$

for some explicit constants C_T, $\tau_T > 0$ depending only on the constants appearing in assumptions **H1-H2-H3**. *More precisely*

$$\forall\, h \in H^k, \quad \langle Th, h\rangle_{\mathcal{H}^1} \le -C'_T \left(\|h - \Pi_g(h)\|^2_\Lambda + \|\nabla_{x,v}(h - \Pi_g(h))\|^2_\Lambda \right)$$

for some (explicit) Hilbert norm $\mathcal{H}^1$ equivalent to the H^1 norm, and some explicit constant $C'_T > 0$.

Remarks:

1. The method does not rely on an abstract result from spectral theory such as Weyl's theorem, like Ukai's proof of the existence of a spectral gap for the Boltzmann equation for hard spheres in [**50**]. Hence we do not need the compactness property of K, although we require a regularizing property on K which is strongly related (see the discussion in [**42**]). Our method can be seen as a quantitative version of Ukai's result (in the case of the linearized Boltzmann equation). In particular it shows that apart from 0, the spectrum of T is included in

$$\{\xi \in \mathbb{C}\, ;\, \mathrm{Re}(\xi) \le -\tau_T\}.$$

The abstract setting emphasizes what is effectively required from the linearized collision operator to deduce exponential convergence, and it allows to apply the method to other models as well. Since for the linearized Boltzmann equation, it was proved in [**50**] that T is not sectorial (its essential spectrum is given by a half-plane), our work can also be seen as a method to prove exponential decay of the semi-group for a whole class of non-sectorial operators, which is in general quite tricky. In the particular case of Fokker-Planck type operators, other methods have been developed in [**33, 51**] to solve this question.

2. In order to obtain a completely quantitative result of convergence to equilibrium, one has to get estimates on the constant λ in assumption **H3**. This question had remained open for a long time for important physical models such as the linearized Boltzmann collision for hard spheres or the Landau collision operators for hard and moderately soft potentials. It has been solved recently in the works [**4, 41, 44**]. Therefore this theorem allows to compute rates of convergence explicitly for all the models we consider in Section 5.5 (except semi-classical relaxation for bosons).

Our second main result is for the nonlinear Boltzmann and Landau models.

1.2 Theorem. *Let us consider either the Boltzmann equation* (5.7) *for hard spheres or hard potentials with cutoff or the Landau equation* (5.19) *with* $\gamma \geq -2$, *in the torus. Let* $0 \leq f_0 \in L^1_2$ *be the initial datum. We denote by* f_∞ *the unique equilibrium associated to* f_0. *Let* k *be such that* $E(k/2) > N/2$ *(where* E *denotes the integer value) and let the initial datum satisfy*

$$\|f_\infty^{-1/2}\,(f_0 - f_\infty)\,\|_{H^k} \leq \varepsilon$$

for some $0 < \varepsilon \leq \varepsilon_0$ *where* ε_0 *depends explicitly on the collision operator.*
Then there exists a unique global solution $0 \leq f = f(t,x,v) \in \mathcal{C}([0,\infty[, H^k)$ *of the initial value problem* (1.1), *such that*

$$\forall\, t \geq 0, \qquad \|f_\infty^{-1/2}(f(t,\cdot,\cdot) - f_\infty)\|_{H^k} \leq C\,\exp\,[-\tau t]$$

for some explicit constants $C, \tau > 0$.
The conclusion also still holds true when a repulsive self-consistent Poisson potential is added (without smallness condition), in the case of the Boltzmann equation for hard spheres, or the Landau equation with $\gamma \geq -1$.

The next theorem deals with the semi-classical relaxation models.

1.3 Theorem. *Consider the semi-classical relaxation equation* (5.2) *in the torus for fermions (*$\epsilon = 1$*) or bosons (*$\epsilon = -1$*), with an initial datum* $0 \leq f_0 \in L^1$. *Let the equilibrium distribution be given by*

$$f_\infty = \frac{\kappa_\infty\,\mathcal{M}}{1 + \epsilon\,\kappa_\infty\,\mathcal{M}},$$

with $1 + \epsilon\,\kappa_\infty\,\mathcal{M} > 0$*, where* $\mathcal{M}$ *is the normalized Maxwellian and* κ_∞ *is defined by mass conservation. Let* k *be such that* $E(k/2) > N/2$ *(where* E *denotes the integer value) and let the initial datum satisfy*

$$\left\| (1 + \epsilon\,\kappa_\infty\,\mathcal{M})\,(\kappa_\infty\,\mathcal{M})^{-1/2}\,(f_0 - f_\infty) \right\|_{H^k} \leq \varepsilon$$

for some $0 < \varepsilon \leq \varepsilon_0$ *where* ε_0 *depends explicitly on the collision operator. Then there exists a unique global solution* $0 \leq f = f(t,x,v) \in \mathcal{C}([0,\infty[, H^k)$ *of the initial value problem* (1.1)*, such that*

$$\forall\, t \geq 0, \qquad \left\|(1 + \epsilon\,\kappa_\infty\,\mathcal{M})\,(\kappa_\infty\,\mathcal{M})^{-1/2}\,(f(t,\cdot,\cdot) - f_\infty)\right\|_{H^k} \leq C\,\exp\left[-\tau t\right]$$

for some explicit constants $C, \tau > 0$*.*

Remark: The condition on the form of the equilibrium distribution is in fact trivially fulfilled for the fermionic case. In the bosonic case it is a condition of smallness on κ_∞ and thus on the initial mass. Indeed it is equivalent to impose that the mass of the initial datum is small enough such that no condensation occurs. This is not for technical reasons but necessary to ensure exponential convergence as can be seen from the detailed asymptotics study in the spatially homogeneous case in [**19**].

5.1.4 Outline of the article

The article is structured as follows. In Section 5.2 we give the proof of Theorem 1.1, which turns out to be very short and simple. Then in Section 5.3 we expose several extensions of the method. In particular we show how to generalize Theorem 1.1 to higher-order Sobolev norms, and how to include a weak external field or a self-consistent Poisson potential in the study. Section 5.4 is devoted to the application of the previous study to the genuine non-linear problems of the form (1.1) near equilibrium, and we prove the abstract Theorem 4.1. Finally in Section 5.5, we prove the general assumptions of Theorem 1.1 and Theorem 4.1 for an extensive list of physical models: classical or semi-classical relaxation, Boltzmann equation for hard spheres or hard potentials with cutoff, Landau equation for hard potentials or moderately soft potentials, linear Fokker-Planck equation. Then Theorem 1.2 follows from this study together with the Theorem 4.1 (we also comment on the marginal differences between the proofs of Theorem 1.2 and Theorem 1.3).

5.2 Proof of Theorem 1.1

We divide the proof into several steps.

Step 1. Since by assumption **H3**, the operator L is non-positive, the $L^2_{x,v}$ norm is decreasing along the flow, and it is straightforward to deduce that T generates a strongly continuous contraction evolution semi-group on $L^2_{x,v}$. In order to estimate the semi-group in $H^1_{x,v}$, let us consider $h_0 \in H^1_{x,v} \cap \mathrm{Dom}(T) \cap N(T)^\perp$ and $h = h(t,x,v)$ the associated solution of the equation $\partial_t h = T(h)$. Let us study the evolution of the $H^1_{x,v}$ norm of h.

Step 2. We estimate the time evolution of the L^2 norm of h. Using the skew symmetry of the transport part together with assumption **H3**, we find immediately

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{L^2} \le -2\,\lambda\,\|h - \Pi_l h\|^2_\Lambda. \tag{2.1}$$

Step 3. We estimate the time derivative of the gradients in x and v.

- For the gradient with respect to x we obtain, thanks to assumption **H3**,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x h\|^2_{L^2} \le -2\,\lambda\,\|\nabla_x h - \Pi_l(\nabla_x h)\|^2_\Lambda. \tag{2.2}$$

- For the gradient with respect to v we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nabla_v h\|^2_{L^2} = 2\,\langle \nabla_v K(h), \nabla_v h\rangle_{L^2} - 2\,\langle \nabla_v \Lambda(h), \nabla_v h\rangle_{L^2} - 2\,\langle \nabla_x h, \nabla_v h\rangle_{L^2}.$$

Using assumption **H1** we have

$$-2\,\langle \nabla_v \Lambda(h), \nabla_v h\rangle_{L^2} \le -2\,\nu_3^\Lambda\,\|\nabla_v h\|^2_\Lambda + 2\,\nu_4^\Lambda\,\|h\|^2_{L^2}.$$

Assumption **H2** yields

$$\begin{aligned} 2\,\langle \nabla_v K(h), \nabla_v h\rangle_{L^2} \le \frac{\nu_1^\Lambda}{\nu_0^\Lambda}\frac{\nu_3^\Lambda}{2}\,\|\nabla_v h\|^2_{L^2} + 2\,C\left(\frac{\nu_1^\Lambda}{\nu_0^\Lambda}\frac{\nu_3^\Lambda}{4}\right)\|h\|^2_{L^2} \le \\ \frac{\nu_3^\Lambda}{2}\,\|\nabla_v h\|^2_\Lambda + 2\,C\left(\frac{\nu_1^\Lambda \nu_3^\Lambda}{4\nu_0^\Lambda}\right)\|h\|^2_{L^2}\,. \end{aligned}$$

Furthermore we split

$$\begin{aligned} 2\,\langle \nabla_x h, \nabla_v h\rangle_{L^2} &\le \frac{\nu_1^\Lambda}{\nu_0^\Lambda}\frac{\nu_3^\Lambda}{2}\,\|\nabla_v h\|^2_{L^2} + \frac{\nu_0^\Lambda}{\nu_1^\Lambda}\frac{2}{\nu_3^\Lambda}\,\|\nabla_x h\|^2_{L^2} \\ &\le \frac{\nu_3^\Lambda}{2}\,\|\nabla_v h\|^2_\Lambda + \frac{2\nu_0^\Lambda}{\nu_1^\Lambda \nu_3^\Lambda}\,\|\nabla_x h\|^2_{L^2}\,. \end{aligned}$$

Using the last three inequalities we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v h\|^2_{L^2} \leq \left[2C\left(\frac{\nu_1^\Lambda \nu_3^\Lambda}{4\nu_0^\Lambda}\right) + 2\,\nu_4^\Lambda\right] \|h\|^2_{L^2} + \frac{2\nu_0^\Lambda}{\nu_1^\Lambda \nu_3^\Lambda}\,\|\nabla_x h\|^2_{L^2} - \nu_3^\Lambda\,\|\nabla_v h\|^2_\Lambda\,.$$

Now we write

$$\|h\|^2_{L^2} \leq 2\,\|h - \Pi_l(h)\|^2_{L^2} + 2\,\|\Pi_l(h)\|^2_{L^2}\,.$$

Since $\Pi_g(h) = 0$ we deduce that $\Pi_l(h)$ has zero mean on the torus, and Poincaré's inequality in the torus yields (for a constant C_P only depending on the dimension N)

$$\|\Pi_l(h)\|^2_{L^2} \leq C_P\,\|\Pi_l(\nabla_x h)\|^2_{L^2} \leq C_P\,\|\nabla_x h\|^2_{L^2},$$

and thus we get for some explicit constants $C_1, C_2 > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v h\|^2_{L^2} \leq C_1\,\|h - \Pi_l(h)\|^2_\Lambda + C_2\,\|\nabla_x h\|^2_{L^2} - \nu_3^\Lambda\,\|\nabla_v h\|^2_\Lambda. \tag{2.3}$$

- For the mixed term we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \nabla_x h, \nabla_v h\rangle_{L^2} = -\|\nabla_x h\|^2_{L^2} + 2\,\langle \nabla_x L(h), \nabla_v h\rangle_{L^2}.$$

Then we write (using assumption **H1** and Cauchy-Schwarz's inequality in x)

$$\begin{aligned} 2\,\langle \nabla_x L(h), \nabla_v h\rangle_{L^2} &= 2\,\langle L\left(\nabla_x h - \Pi_l(\nabla_x h)\right), \nabla_v h\rangle_{L^2} \\ &\leq 2\,C^L\,\|\nabla_x h - \Pi_l(\nabla_x h)\|_\Lambda\,\|\nabla_v h\|_\Lambda \\ &\leq C^L\,\eta\,\|\nabla_x h - \Pi_l(\nabla_x h)\|^2_\Lambda + C^L\,\eta^{-1}\,\|\nabla_v h\|^2_\Lambda \end{aligned}$$

for any $\eta > 0$. Hence we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \nabla_x h, \nabla_v h\rangle \leq -\,\|\nabla_x h\|^2_{L^2} + C^L\,\eta\,\|\nabla_x h - \Pi_l(\nabla_x h)\|^2_\Lambda + C^L\,\eta^{-1}\,\|\nabla_v h\|^2_\Lambda\,. \tag{2.4}$$

Step 4. Now it remains to combine equations (2.1), (2.2), (2.3) and (2.4): we pick $A, \alpha, \beta, \gamma > 0$ and compute

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big[A\,\|h\|^2_{L^2} + \alpha\,\|\nabla_x h\|^2_{L^2} + \beta\,\|\nabla_v h\|^2_{L^2} + \gamma\,\langle \nabla_x h, \nabla_v h\rangle_{L^2}\Big] \\ &\quad\leq (\beta C_1 - 2A\lambda)\,\|h - \Pi_l(h)\|^2_\Lambda + \left(\eta\gamma C^L - 2\alpha\lambda\right)\,\|\nabla_x h - \Pi_l(\nabla_x h)\|^2_\Lambda \\ &\qquad + \left(\eta^{-1}\gamma C^L - \beta\nu_3^\Lambda\right)\,\|\nabla_v h\|^2_\Lambda + (C_2\beta - \gamma)\,\|\nabla_x h\|^2_{L^2}. \end{aligned}$$

For a given β with $\nu_3^\Lambda \beta > 1$, we first fix A big enough such that

$$(\beta C_1 - 2A\lambda) \leq -1,$$

then γ big enough such that

$$(C_2\beta - \gamma) \leq -1,$$

then η big enough such that

$$\left(\eta^{-1}\gamma C^L - \beta \nu_3^\Lambda\right) \leq -1,$$

then α big enough such that

$$\left(\eta\gamma C^L - 2\alpha\lambda\right) \leq -1$$

and such that $\gamma^2 < \alpha\beta$ and $\alpha \geq \beta$. For this choice we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big[A\,\|h\|_{L^2}^2 + \alpha\,\|\nabla_x h\|_{L^2}^2 + \beta\,\|\nabla_v h\|_{L^2}^2 + \gamma\,\langle\nabla_x h, \nabla_v h\rangle\Big] \\ &\qquad\qquad \leq -\left[\|\nabla_x h\|_{L^2}^2 + \|\nabla_v h\|_\Lambda^2 + \|h - \Pi_l(h)\|_\Lambda^2 + \|\nabla_x h - \Pi_l(\nabla_x h)\|_\Lambda\right]. \end{aligned}$$

The function

$$\mathcal{F}(t) = \Big[A\,\|h\|_{L^2}^2 + \alpha\,\|\nabla_x h\|_{L^2}^2 + \beta\,\|\nabla_v h\|_{L^2}^2 + \gamma\,\langle\nabla_x h, \nabla_v h\rangle\Big]$$

satisfies (remember that $\alpha \geq \beta$)

$$\begin{aligned} &A\,\|h\|_{L^2}^2 + (\beta/2)\,\left[\|\nabla_x h\|_{L^2}^2 + \|\nabla_v h\|_{L^2}^2\right] \leq \mathcal{F}(t) \\ &\qquad\qquad\qquad\qquad \leq A\,\|h\|_{L^2}^2 + (3\alpha/2)\,\left[\|\nabla_x h\|_{L^2}^2 + \|\nabla_v h\|_{L^2}^2\right]. \end{aligned}$$

Moreover by Poincaré's inequality we have (since $\Pi_l(h)$ has zero mean on the torus):

$$\|h\|_\Lambda^2 \leq 2\,\|h - \Pi_l(h)\|_\Lambda^2 + 2\,\|\Pi_l(h)\|_\Lambda^2 \leq C\,\left(\|h - \Pi_l(h)\|_\Lambda^2 + \|\nabla_x h\|_{L^2}^2\right)$$

for some explicit constant $C > 0$, and similarly

$$\begin{aligned} &\|\nabla_x h\|_\Lambda^2 \leq 2\,\|\nabla_x h - \Pi_l(\nabla_x h)\|_\Lambda^2 + 2\,\|\Pi_l(\nabla_x h)\|_\Lambda^2 \\ &\qquad\qquad\qquad\qquad \leq C'\,\left(\|\nabla_x h - \Pi_l(\nabla_x h)\|_\Lambda^2 + \|\nabla_x h\|_{L^2}^2\right) \end{aligned}$$

for $C' > 0$. Hence we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \leq -K\,\left(\|h\|_\Lambda^2 + \|\nabla_{x,v} h\|_\Lambda^2\right)$$

for some explicit $K > 0$, and that $\mathcal{F}(t)$ is equivalent to the square of the H^1 norm of h. We define the norm $\mathcal{H}^1$ by

$$\|\cdot\|_{\mathcal{H}^1} = \Big\{A\|\cdot\|_{L^2}^2 + \alpha\|\nabla_x\cdot\|_{L^2}^2 + \beta\|\nabla_v\cdot\|_{L^2}^2 + \gamma\,\langle\nabla_x\cdot, \nabla_v\cdot\rangle_{L^2}\Big\}^{1/2}.$$

This concludes the proof.

5.3 Some generalizations

5.3.1 Higher-order Sobolev spaces

In this section we show how to extend the previous method to higher-order Sobolev spaces. Let $k \geq 1$ denote the total number of derivatives. Then for two multi-indexes j and l such that $k = |j| + |l|$ we use the shorthand $\partial_l^j = \partial/\partial v_j \, \partial/\partial x_l$. For a multi-index j we shall denote by $c_i(j)$ the value of the i-th coordinate of j, for $i = 1, \dots, N$.

In order to treat the higher-order derivatives we shall strengthen assumptions **H1** and **H2** into

H1'. We assume **H1**. Moreover we assume that for any $k \geq 1$, for any multi-indexes j and l such that $k = |j| + |l|$ and $|j| \geq 1$, we have

$$\langle \partial_l^j \Lambda(h), \partial_l^j h \rangle_{L^2} \geq \nu_5^\Lambda \, \|\partial_l^j h\|_\Lambda^2 - \nu_6^\Lambda \, \|h\|_{H^{k-1}}^2$$

for some constants $\nu_5^\Lambda, \nu_6^\Lambda > 0$.

H2'. We assume that K **has a regularizing effect** in the sense: for any $k \geq 1$, for any multi-indexes j and l such that $k = |j| + |l|$ and $|j| \geq 1$, for any $\delta > 0$, there is some explicit $C(\delta)$ such that

$$\langle \partial_l^j K(h), \partial_l^j h \rangle_{L^2} \leq C(\delta) \, \|h\|_{H^{k-1}}^2 + \delta \, \|\partial_l^j h\|_{L^2}^2 \ . \tag{3.1}$$

Again these strengthened assumptions are satisfied by the physical models we discussed in the introduction, as we check in Section 5.5. Now we can formulate the coercivity estimate on T and the consequence on its semi-group.

3.1 Theorem. *Let T be an operator on L^2 satisfying assumptions* **H1'-H2'-H3**. *Then T generates a strongly continuous evolution semi-group e^{tT} on H^k, which satisfies*

$$\|e^{tT}(\mathrm{Id} - \Pi_g)\|_{H^k} \leq C_T \, \exp\left[-\tau_T t\right]$$

for some explicit constants C_T, $\tau_T > 0$ depending only on the constants appearing in **H1'-H2'-H3**. *More precisely*

$$\forall \, h \in H^k, \quad \langle Th, h \rangle_{\mathcal{H}^k} \leq -C_T' \left(\sum_{|j|+|l| \leq k} \|\partial_l^j (h - \Pi_g(h))\|_\Lambda^2 \right)$$

for some (explicit) Hilbert norm $\mathcal{H}^k$ equivalent to the H^k norm, and some explicit constant $C_T' > 0$.

Proof of Theorem 3.1. We write $\dot{H}^k$ for the homogeneous Sobolev semi-norm, *i.e.*

$$\|h\|_{\dot{H}^k}^2 = \sum_{|j|+|l|=k} \|\partial_l^j h\|_{L^2}^2.$$

Again we pick $h_0 \in H^k \cap N(T)^\perp \cap \mathrm{Dom}(B)$ and we observe that h will stay in $N(T)^\perp$ for all times. For $k = 1$ the result is given by Theorem 1.1. We proceed by induction on k.

First note that since the equation commutes with x-derivatives we have for the purely x-derivatives the analog of equation (2.2), namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_l^0 h\|_{L^2}^2 \leq -2\,\lambda\,\|\partial_l^0 h - \Pi_l(\partial_l^0 h)\|_\Lambda^2. \tag{3.2}$$

For derivatives including the v-component (i.e. $|j| \geq 1$), by means of **H2'** with $\delta \leq (\nu_5^\Lambda \nu_0^\Lambda)/(2\nu_1^\Lambda)$ we have the following estimate

(3.3)
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_l^j h\|^2 \leq -2 \sum_{i\,|\,c_i(j)>0} \langle \partial_l^j h, \partial_{l+\delta_i}^{j-\delta_i} h\rangle_{L^2} + \left(2C(\delta) + 2\nu_6^\Lambda\right)\|h\|_{H^{k-1}}^2 - \nu_5^\Lambda \|\partial_l^j h\|_\Lambda^2.$$

For all l with $|l| = k$ and $c_i(l) > 0$, we consider the mixed term

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h\rangle = -\|\partial_l^0 h\|_{L^2}^2 + 2\,\langle L\left(\partial_l^0 h - \Pi_l(\partial_l^0 h)\right), \partial_{l-\delta_i}^{\delta_i} h\rangle.$$

By means of **H1** we obtain

(3.4)
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h\rangle \leq -\|\partial_l^0 h\|_{L^2}^2 + C^L\,\eta\,\|\partial_l^0 h - \Pi_l(\partial_l^0 h)\|_\Lambda^2 + C^L\,\eta^{-1}\,\|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2.$$

For l with $|l| = k$ and i such that $c_i(l) > 0$, we define the following combination of derivatives of order 0 and 1 in v:

$$\mathcal{Q}_{l,i} := \alpha\,\|\partial_l^0 h\|_{L^2}^2 + \beta\,\|\partial_{l-\delta_i}^{\delta_i} h\|_{L^2}^2 + \gamma\,\langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l h\rangle. \tag{3.5}$$

By adjusting the constants $\alpha, \beta, \gamma > 0$ and using Poincaré's inequality in the same way as in the proof of Theorem 1.1 in Section 5.2 it is straightforward to obtain

$$c\left(\|\partial_l^0 h\|_{L^2}^2 + \|\partial_{l-\delta_i}^{\delta_i} h\|_{L^2}^2\right) \leq \mathcal{Q}_{l,i} \leq C\left(\|\partial_l^0 h\|_{L^2}^2 + \|\partial_{l-\delta_i}^{\delta_i} h\|_{L^2}^2\right) \tag{3.6}$$

for some explicit constants $c, C > 0$, and that the time derivative fulfills the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}_{l,i} \leq -K\,\left(\|\partial_l^0 h\|_\Lambda^2 + \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2\right) + C_0\,\|h\|_{H^{k-1}}^2 \tag{3.7}$$

for some explicit constants $K, C_0 > 0$.

Now we combine all the derivatives in the following way

$$(3.8)\quad \mathcal{F}_k(t) := \sum_{|l|+|j|=k,\, |j|\geq 2} \left(\frac{\nu_0}{2}\right)^{-2|l|} \|\partial_l^j h\|^2 + \frac{2}{K}\left(\frac{\nu_0}{2}\right)^{-2(k-1)} \sum_{|l|=k,\, i\,|\, c_i(l)>0} \mathcal{Q}_{l,i}$$

where $\nu_0 = \nu_0^\Lambda / \nu_1^\Lambda > 0$ is the constant such that (by assumption **H1**)

$$\|h\|_\Lambda \geq \nu_0 \, \|h\|_{L^2}.$$

By (3.6), $\mathcal{F}_k$ is equivalent to the square of the homogeneous Sobolev norm $\dot{H}^k$.

To estimate the mixed terms in the right-handside of (3.8) coming from (3.3), we write

$$\sum_{|l|+|j|=k,|l|=s} 2\left(\frac{\nu_0}{2}\right)^{-2s} \langle \partial_l^j h, \partial_{l+\delta_i}^{j-\delta_i} h\rangle_{L^2} \leq$$
$$\sum_{|l|+|j|=k,|l|=s} \left(\frac{\nu_0}{2}\right)^{-2s+1} \|\partial_l^j h\|^2 + \sum_{|l|+|j|=k,|l|=s} \left(\frac{\nu_0}{2}\right)^{-2s-1} \|\partial_{l+\delta_i}^{j-\delta_i} h\|^2 \leq$$
$$\frac{1}{2}\sum_{|l|+|j|=k,|l|=s} \left(\frac{\nu_0}{2}\right)^{-2s} \|\partial_l^j h\|_\Lambda^2 + \frac{1}{2}\sum_{|l|+|j|=k,|l|=s} \left(\frac{\nu_0}{2}\right)^{-2(s+1)} \|\partial_{l+\delta_i}^{j-\delta_i} h\|_\Lambda^2,$$

and we derive by combining (3.2), (3.3) and (3.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_k(t) \leq C_+ \, \|h\|_{H^{k-1}}^2 - \sum_{|j|\geq 2} \left(\frac{\nu_0}{2}\right)^{-2|l|} \|\partial_l^j h\|_\Lambda$$
$$- \left(\frac{\nu_0}{2}\right)^{-2(k-1)} \left(\sum_{|j|=1} \|\partial_l^j h\|_\Lambda^2 + 2\sum_{|l|=k} \|\partial_l h\|_\Lambda^2 \right).$$

By using (3.6) we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_k(t) \leq C_+ \, \|h\|_{H_{k-1}}^2 - K_- \left(\sum_{|j|+|l|=k} \|\partial_l^j h\|_\Lambda^2 \right)$$

for some explicit constants $C_+, K_- > 0$. Together with the induction assumption for $\mathcal{F}_1, \ldots, \mathcal{F}_{k-1}$ this concludes the proof of the step k by considering some combination of $\mathcal{F}_1, \ldots, \mathcal{F}_k$. □

5.3.2 Weak external potential

If the particles are submitted to an external force field, which is given as the gradient of the scalar potential V, the evolution equation on the distribution generalizes in the following way

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = Q(f, f). \tag{3.9}$$

We still consider for the spatial domain $\Omega = \mathbb{T}^N$. We assume that $V = V(x)$ is C^2. For simplicity we restrict to collision operators for which the problem without potential admits a *Maxwellian* equilibrium in this section and the next one. Moreover in this subsection we shall restrict further to collisional models admitting only mass conservation as a conservation law, *i.e.* we restrict to the relaxation model. The stationary solution is given by

$$f_\infty = e^{-V} \mathcal{M}$$

where $\mathcal{M}$ is the Maxwellian and the constant is determined by the conservation laws. We also assume that the external field is weak in the sense:

$$\|V\|_{C^2(\mathbb{T}^N)} \leq \varepsilon \tag{3.10}$$

for some ε depending on the collision operator.
Then we consider fluctuations around equilibrium of the form

$$f = f_\infty + \sqrt{f_\infty}\, h$$

and we compute the following linearized equation on h:

$$\partial_t h + v \cdot \nabla_x h - \nabla_x V \cdot \nabla_v h = L(h) \tag{3.11}$$

where L is the linearized operator associated with Q as before. We define the operator T by

$$T = L - v \cdot \nabla_x + \nabla_x V \cdot \nabla_v.$$

Let us assume that L satisfies assumptions **H1-H2-H3** and that the kernel of L in L^2_v is given by $\mathrm{Span}\{e^{-V/2}\mathcal{M}^{1/2}\}$ (this assumption is satisfied for the classical relaxation model). In this case the kernel of T in $L^2_{x,v}$ is trivially given by $\mathrm{Span}\{e^{-V/2}\mathcal{M}^{1/2}\}$.

We only sketch the proof of the following result:

3.2 Theorem. *Under the previous assumptions on T, there is $\varepsilon_0 > 0$ such that for any $V \in C^1$ satisfying* (3.10) *with $\varepsilon \leq \varepsilon_0$, the operator T satisfies the conclusion of Theorem 1.1. If moreover conditions* **H1'-H2'-H3'** *hold for L, and $V \in C^{k+1}$, then the operator T satisfies the conclusion of Theorem 3.1.*

Proof. First let us recall that the L^2 norm of h is decreasing using the non-positivity of L and the antisymmetry of $-v \cdot \nabla_x + \nabla_x V \cdot \nabla_v$ in this Hilbert space.

We will only show how to establish the bound on first-order derivatives. The generalization to higher-order is straightforward.

Let us consider $h \in N(T)^\perp \cap \mathrm{Dom}(T) \cap H^1_{x,v}$. The time evolution for the L^2 norm of h and the L^2 norm of its gradient in v are unchanged. For the gradient in x, equation (2.2) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla_x h\|^2_{L^2} \leq -2\,\lambda\, \|\nabla_x h - \Pi_l(\nabla_x h)\|^2_\Lambda + 2\,\varepsilon\, \|\nabla_v h\|^2_{L^2} \|\nabla_x h\|^2_{L^2}.$$

Finally for the mixed term we have to replace (2.4) by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla_x h, \nabla_v h\rangle \leq -\, \|\nabla_x h\|^2_{L^2} + C^L\, \eta\, \|\nabla_x h - \Pi_l(\nabla_x h)\|^2_\Lambda \\ + C^L\, \eta^{-1}\, \|\nabla_v h\|^2_\Lambda + \varepsilon \|\nabla_v h\|^2_{L^2}. \end{aligned}$$

Hence following the same arguments as in Section 5.2, we find the following differential inequality on the quadratic form $\mathcal{F}$:

$$\frac{\mathrm{d}\mathcal{F}(t)}{\mathrm{d}t} \leq -K(\|h\|^2_\Lambda + \|\nabla_{x,v} h\|^2_\Lambda) + a\,\varepsilon \left\{ \|\nabla_v h\|_{L^2} \|\nabla_x h\|_{L^2} + \|\nabla_v h\|^2_{L^2} \right\}$$

for some explicit constant $a, K > 0$. Therefore it concludes the proof for $\varepsilon > 0$ small enough. □

Remarks:

1. It is possible that for a spatial domain $\Omega = \mathbb{R}^N$ a modified version of this strategy could be applied, assuming additionally that V satisfies a log-Sobolev inequality on Ω.

2. This subsection about weak external fields illustrates the fact that our method is robust, since it is based on *a priori* estimates, which remain true up to a perturbation.

5.3.3 Self-consistent potential

Let us consider a collisional kinetic model for particles which interact through collisions and also through a self-consistent potential. In this subsection we exclude the semi-classical relaxation collision operators. For the potential we consider the physically most common case of Poisson interaction. More precisely

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \epsilon \nabla_x V \cdot \nabla_v f = Q(f,f) \\ \Delta_x V = \rho - \rho_0 \end{cases}$$

where the coupling is *via* the density $\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v$. The equation models particles interacting by binary collision or by scattering a background medium in thermal equilibrium, and at the same time interacting *via* electrostatic forces in the case $\epsilon = -1$ or by gravitational attraction in the case $\epsilon = +1$. Existence of global classical solutions in the large to the Vlasov-Poisson system in the torus has been proven in [**5**]. For the system including various collision operators, the existence of classical solutions has been established very recently in the articles discussed in the introduction.

We consider the previous equation in the torus $x \in \mathbb{T}^N$ for $\epsilon = -1$ (electrostatic interaction). It admits a global Maxwellian equilibrium f_∞, determined by the global conservation laws (for a more general discussion of the possible equilibria, we refer to [**15**] for instance).

Then we consider the linearization $f = f_\infty + \sqrt{f_\infty} h$ around this equilibrium. Discarding the bilinear terms yields

$$\begin{cases} \partial_t h + v \cdot \nabla_x h - (v \cdot \nabla_x V)\, f_\infty^{1/2} = L(h) \\ \Delta_x V = \left(\int_{R^N} h\, f_\infty^{1/2}\,\mathrm{d}v \right) \end{cases}$$

where L is the linearized collision operator associated with Q.

Now let us denote by T_p the operator on L^2 defined by

$$T_p = L(h) - v \cdot \nabla_x h + (v \cdot \nabla_x V(h))\, f_\infty^{1/2}. \tag{3.12}$$

Then (defining Π_g as before) we have the following theorem

3.3 Theorem. *Let L satisfies the assumptions* **H1-H2-H3**. *Then the operator T_p defined in* (3.12) *satisfies the conclusion of Theorem 1.1. If moreover L satisfies assumptions* **H1'-H2'-H3**, *then the operator T_p satisfies the conclusion of Theorem 3.1.*

Proof of Theorem 3.3. The proof is almost exactly the same as to the one of Theorem 1.1. Therefore we shall only indicate the differences in the estimates. Essentially the norm has to be modified in order to take into account the interaction energy. For the L^2 norm, one has by integration by parts and using that L is mass conserving

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\|h\|_{L^2}^2 + \|\nabla_x V\|_{L^2_x}^2 \right) \leq -2\,\lambda\, \|h - \Pi_l(h)\|_\Lambda^2.$$

Similarly one has on the gradient in x

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla_x h\|^2_{L^2} + \sum_{1\le i,j\le N} \left\| \frac{\partial^2 V}{\partial x_i \partial x_j} \right\|^2_{L^2_x} \right) \le -2\,\lambda\,\|\nabla_x h - \Pi_l(\nabla_x h)\|^2_\Lambda.$$

For the time evolution of the gradient in v one has the additional term

$$-2 \int \left(\nabla_x V \cdot \nabla_v f_\infty^{1/2}\right)\, h$$

and for the mixed term, one has the additional terms

$$-2 \int \Delta_x V\, h\, f_\infty^{1/2} + C \int \left(\sum_{i,j} v_i v_j \frac{\partial^2 V}{\partial x_i \partial x_j} \right) h\, f_\infty^{1/2}.$$

Since it is straightforward that all these additional terms as well as the gradient of V can be controlled by the L^2 norm of h, the end of the proof is straightforward as in Theorem 1.1. □

Remarks:

1. We need the interaction to be repulsive in order to have the right sign for the interaction energy. For gravitational self-interaction potentials, our method would work as well under an additional assumption of smallness of the potential, *i.e.* assuming $\epsilon > 0$ and $\epsilon \le \epsilon_0$ with ϵ_0 depending on the collision operator (the proof is similar to the case of a weak external potential).

2. As noticed in [**28**], the non-linear term arising from a self-interaction Poisson potential can be controlled by

$$C\,\|h\|_{L^2}\,\|h\langle v\rangle^{1/2}\|^2_{L^2}.$$

Therefore one can extend the construction of smooth solutions near equilibrium in Theorem 1.2 to the case when a self-consistent Poisson potential is added, as long as the coercivity norm Λ of the linearized problem is stronger than the norm $\|h\langle v\rangle^{1/2}\|_{L^2}$. This is case for instance for the Boltzmann equation for hard spheres (which was the case considered in [**28**]), and this is also the case for the Landau equation with $\gamma \ge -1$ (see Subsection 5.5.5).

5.4 Application to full non-linear models near equilibrium

In this section we prove existence and uniqueness of smooth global solutions near equilibrium thanks to the coercivity estimates on the linearized models.

This yields also explicit exponential rate of convergence to equilibrium. Obviously there is nothing else to prove for linear models (such as the classical relaxation or linear Fokker-Planck equation). Therefore let us assume that the collision operator is bilinear and let us denote the remaining term in the linearization process $f = f_\infty + f_\infty^{1/2} h$:

$$\Gamma(h,h) = f_\infty^{-1/2}\, Q(f_\infty h, f_\infty h).$$

In this section we shall consider an operator T satisfying assumptions **H1'-H2'-H3** (we make the additional assumptions **H1'-H2'** in order to get coercivity estimates in higher-order Sobolev spaces). Moreover we shall assume on the bilinear form Γ:

H4. There is $k_0 \in \mathbb{N}$ and $C_\Gamma > 0$ such that for $k \geq k_0$,

$$\|\Gamma(h,h)\|_{H^k} \leq C_\Gamma\, \|h\|_{H^k} \left(\sum_{|j|+|l| \leq k} \|\partial_l^j h\|_\Lambda^2 \right)^{1/2}.$$

Then we have

4.1 Theorem. *Let Q a (bilinear) collision operator such that*

(i) equation (1.1) *admits an equilibrium $0 \leq f_\infty \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$;*

(ii) the linearized operator $L = L(h)$ around f_∞ with the scaling $f = f_\infty + f_\infty^{1/2} h$ satisfies **H1'-H2'-H3***;*

(iii) the bilinear remaining term Γ in the linearization satisfies **H4***.*

Then for any $k \geq k_0$ (where k_0 is defined in **H4***), there is $\varepsilon_0 > 0$ such that for any distribution $0 \leq f_0 \in L^1$ with*

$$\|(f_0 - f_\infty)\, f_\infty^{-1/2}\|_{H^k} \leq \varepsilon_0$$

there exists a unique global smooth solution $0 \leq f = f(t,x,v)$ to equation (1.1)*, which satisfies*

$$\|(f_0 - f_\infty)\, f_\infty^{-1/2}\|_{H^k} \leq C_0\, \varepsilon_0\, e^{-\tau t}$$

for some explicit constant $C_0, \varepsilon_0, \tau > 0$, depending only on the constants appearing in **H1'-H2'-H3-H4***.*

Proof of Theorem 4.1. We explain the proof by *a priori* arguments, on a given smooth solution. The construction of positive solutions thanks to the estimates above is based on, by now standard, fixed point arguments (we refer the reader to [**28, 27**] for instance).
The function $h = (f - f_\infty)\, f_\infty^{-1/2}$ satisfies $\Pi_g(h) = 0$ and it solves

$$\partial_t h = T(h) + \Gamma(h,h).$$

Then we estimate the time evolution of the $\mathcal{H}^k$ norm, defined in Theorem 3.1:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}^k} = 2\langle Th, h\rangle_{\mathcal{H}^k} + 2\langle \Gamma(h,h), h\rangle_{\mathcal{H}^k}.$$

We deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}^k} \le -C_T \left(\sum_{|j|+|l|\le k} \|\partial^j_l h\|^2_\Lambda \right) + C_\varepsilon \,\|\Gamma(h,h)\|^2_{H^k} + \varepsilon\, \|h\|^2_{H^k}.$$

Hence we deduce taking ε small enough

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}^k} \le -\frac{C_T}{2} \left(\sum_{|j|+|l|\le k} \|\partial^j_l h\|^2_\Lambda \right) + C'_\Gamma\, \|h\|^2_{H^k} \left(\sum_{|j|+|l|\le k} \|\partial^j_l h\|^2_\Lambda \right).$$

This concludes the proof by maximum principle since the Λ norm controls the L^2 norm. □

5.5 Proof of the general assumptions for physical models

5.5.1 Linear relaxation

We consider the linear relaxation equation in the torus

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\kappa}\left[\left(\int_{\mathbb{R}^N} f(t,x,v_*)\,\mathrm{d}v_* \right) \mathcal{M}(v) - f \right], \tag{5.1}$$

for $x \in \mathbb{T}^N$ and $v \in \mathbb{R}^N$ ($N \ge 1$). Here $\kappa > 0$ denotes the *Knudsen number* and $\mathcal{M}$ denotes the normalized Maxwellian:

$$\mathcal{M}(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{N/2}}$$

with mass 1, momentum 0 and temperature 1. This equation preserves the total mass of the distribution

$$\forall\, t \geq 0, \qquad \int_{\mathbb{T}^N \times \mathbb{R}^N} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{T}^N \times \mathbb{R}^N} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

but admits no other conservation law. For a given initial datum $f_0 \geq 0$, it admits a unique global equilibrium $f_\infty = \rho_\infty \mathcal{M}$, where ρ_∞ is the total mass of f_0, defined by

$$\rho_\infty = \int_{\mathbb{T}^N \times \mathbb{R}^N} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Finally let us add that the Cauchy theory is straightforward for (5.1) since it is linear (see [**6**] for instance for more details on this equation).

We linearize the equation as

$$f = f_\infty + \sqrt{f_\infty} h = \rho\, \mathcal{M} + \rho^{1/2} M\, h,$$

where $M := \sqrt{\mathcal{M}}$. The equation for h reads

$$\partial_t h + v \cdot \nabla_x h = \frac{1}{\kappa} \left[\left(\int_{\mathbb{R}^N} h'\, M' \,\mathrm{d}v' \right) M - h \right] =: L(h)$$

where we have used the classical notation $h' = h(v')$. We split the operator L into

$$L = K - \Lambda, \quad K(h) = \frac{1}{\kappa} \left(\int_{\mathbb{R}^N} h'\, M' \,\mathrm{d}v' \right) M, \quad \Lambda(h) = \kappa^{-1}\, h.$$

Therefore L satisfies **H1** taking $\| \cdot \|_\Lambda = \| \cdot \|_{L^2_{x,v}}$, and assumption **H2** follows straightforwardly (with $C(\delta) = 0$) from

$$\nabla_v K(h) = \frac{1}{\kappa} \left(\int_{\mathbb{R}^N} h'\, M' \,\mathrm{d}v' \right) \nabla_v M.$$

Observe also that the strengthened assumptions **H1'** and **H2'**, that are necessary to ensure decay in higher-order Sobolev norms, are satisfied straightforwardly.

The operator L is local in x and t. When x is fixed, it is well-defined and bounded on L^2_v, and it is self-adjoint non-positive on this space. More precisely its Dirichlet form is given by

$$\langle L(h), h \rangle_{L^2_v} = -\frac{1}{2\kappa} \int_{\mathbb{R}^N \times \mathbb{R}^N} \left(\frac{h}{M} - \frac{h'}{M'} \right)^2 \mathcal{M}\, \mathcal{M}' \,\mathrm{d}v\,\mathrm{d}v'.$$

Therefore its kernel is $N(L) = \mathrm{Span}\,\{M\}$, and we define Π_l the (orthogonal) projection on this space in L^2_v:

$$\Pi_l(h) = \left(\int_{\mathbb{R}^N} h'\, M'\, \mathrm{d}v' \right) M = \kappa\, K(h).$$

Then it is straightforward that L has a spectral gap $\lambda_L = \kappa^{-1}$, since

$$\int_{\mathbb{R}^N} L(h)\, h\, \mathrm{d}v = -\frac{1}{\kappa}\, \|h - \Pi_l(h)\|^2_{L^2}.$$

Hence assumption **H3** is satisfied.

5.5.2 Semi-classical relaxation

Let us now modify the previous relaxation equation in order to take into account quantum effects associated with the Pauli's exclusion principle. We thus consider the following semi-classical model (weakly non-linear) for charged particles (here $\epsilon \in \{-1, 0, 1\}$):

$$(5.2)\quad \partial_t f + v \cdot \nabla_x f = \frac{1}{\kappa} \int_{\mathbb{R}^N} \left[\mathcal{M}(1 - \epsilon f) f' - \mathcal{M}'(1 - \epsilon f') f \right] \mathrm{d}v' := Q_\epsilon(f, f),$$

for $x \in \mathbb{T}^N$ and $v \in \mathbb{R}^N$, where $\mathcal{M}$ is the normalized Maxwellian as before, and we have used the shorthand $f' = f(v')$. Regardless of the choice of ϵ this scattering operator is mass preserving but admits no other conservation law. The case $\epsilon = 0$ is the standard linear relaxation model of the previous subsection.

In the case $\epsilon = 1$ this operator is probably the simplest model describing a gas of fermions relaxing towards the thermodynamic equilibrium for a perfect fermigas, that is the *Fermi-Dirac distribution*. The operator Q_{+1} describes the interaction of the fermions with a background medium at rest with constant temperature. The factors $(1-f)$ correspond to correlation of particles before and after collision due to Pauli's exclusion principle. Note that this modification of the standard relaxation mechanism is (at least for the usual range of temperatures and densities) necessary only in particular regimes, such as the one of a gas of electrons. Those are, due to their small mass, most likely to satisfy Sommerfelds degeneracy condition (see [**10**]). This equation can be seen as the scattering counterpart of the full fermion Boltzmann equation studied for example in [**18, 10**]. For a more detailed introduction to models describing scattering as well as binary collisions for fermions see [**39**]. For this model, a Cauchy theory can be obtained using maximum principle arguments to treat the (weak) nonlinearity (see [**45**]) assuming some bounds on the initial datum. The long-time behaviour of solutions to this equation has been studied by the EEP-method in [**45**] (leading to polynomial rates of convergence to equilibrium), however the necessary uniform regularity bounds on the solution were assumed.

In the case $\epsilon = -1$, Q_{-1} describes the interaction of bosons with background medium at rest with some constant temperature, and is probably the simplest model describing a gas of bosons relaxing towards the thermodynamic equilibrium for a perfect bosongas, that is the *Bose-Einstein distribution*. In this setting the existence of a boson in a velocity and space interval will increase the chance of another boson being scattered to this interval (see [**10**]). This mechanism leads to Bose-Einstein condensation for low temperatures and large densities. However since we linearize around the *regular* equilibrium we cannot describe this phenomenon but only the situation farther away from the critical mass for the phase transition. In the spatially homogeneous setting very precise asymptotics including optimal rates for the convergence above as well as below the critical mass have been given in [**19**]. The authors study a model for Compton scattering of photons against electrons. In the non condensate case their model - in an appropriate scaling - is analogous to the one we study here because in this case their cross-section is bounded away from 0. The authors prove exponential convergence in the non-condensate case, which is consistent with the result we obtain in the x-dependent situation. Moreover they derive an optimal rate for the convergence in the condensate case which is only polynomial. Thus is seems unavoidable to impose a bound on the initial mass to retain exponential convergence. The existence of solutions to a more elaborate collisional model (in the spatially homogeneous case) has been shown in [**36**] (see also [**37**]). In the same work the weak convergence to regular equilibrium states (similar to the ones that our simplified model admits) has been shown to hold true above a specific temperature, larger than the critical one, and for which an explicit bound is given.

The equation preserves the total mass of the distribution

$$\forall\, t \geq 0, \qquad \int_{\mathbb{T}^N \times \mathbb{R}^N} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{T}^N \times \mathbb{R}^N} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = \rho$$

and admits (recalling in the boson case the bound imposed on the initial datum in Theorem 1.3) a unique equilibrium

$$f_\infty = \frac{\kappa_\infty \mathcal{M}}{1 + \epsilon \kappa_\infty \mathcal{M}},$$

where κ_∞ is determined by the mass ρ of f_0.

We linearize the equation for a general scaling $f = f_\infty + m\,h$ (m is a given positive function). Discarding the bilinear term, the equation for h reads after straightforward computations

$$\partial_t h + v \cdot \nabla_x h = \frac{1}{\kappa} \int_{\mathbb{R}^N} \left[h' \frac{m'(1+\epsilon\kappa_\infty \mathcal{M}')}{m(1+\epsilon\kappa_\infty \mathcal{M})} \mathcal{M} - h \frac{1+\epsilon\kappa_\infty \mathcal{M}}{1+\epsilon\kappa_\infty \mathcal{M}'} \mathcal{M}' \right] \mathrm{d}v' =: L_m(h).$$

We make the choice $m = (1+\epsilon\kappa_\infty \mathcal{M})^{-1} M \sqrt{\kappa_\infty}$, where $M := \sqrt{\mathcal{M}}$. It yields $L = K - \Lambda$ with

$$K(h) = \frac{1}{\kappa} \left(\int_{\mathbb{R}^N} h' \, M' \, \mathrm{d}v' \right) M,$$

and Λ is the multiplicative operator by ν with

$$\nu(v) = \frac{1}{\kappa} (1+\epsilon\kappa_\infty \mathcal{M}) \left(\int_{\mathbb{R}^N} \frac{\mathcal{M}'}{1+\epsilon\kappa_\infty \mathcal{M}'} \, \mathrm{d}v' \right) = \frac{\rho}{\kappa\kappa_\infty} (1+\epsilon\kappa_\infty \mathcal{M}).$$

Therefore L satisfies **H1** taking $\| \cdot \|_\Lambda = \| \cdot \|_{L^2_{x,v}}$, and assumption **H2** follows straightforwardly (with $C(\delta) = 0$) from

$$\nabla_v K(h) = \frac{1}{\kappa} \left(\int_{\mathbb{R}^N} h' \, M' \, \mathrm{d}v' \right) \nabla_v M.$$

Again the strengthened assumptions **H1'** and **H2'** are also satisfied straightforwardly.

The resulting operator L is local in x and t. When x is fixed, it is well-defined and bounded on L^2_v, and it is self-adjoint non-positive on this space. More precisely its Dirichlet form is given by

$$\begin{aligned} \langle L(h), h \rangle_{L^2} = -\frac{1}{2\kappa} \int_{\mathbb{R}^N \times \mathbb{R}^N} & \left(\frac{h(1+\epsilon\kappa_\infty \mathcal{M})}{M} - \frac{h'(1+\epsilon\kappa_\infty \mathcal{M}')}{M'} \right)^2 \times \\ & (1+\epsilon\kappa_\infty \mathcal{M})^{-1} (1+\epsilon\kappa_\infty \mathcal{M}')^{-1} \mathcal{M} \, \mathcal{M}' \, \mathrm{d}v \, \mathrm{d}v' \\ = -\frac{1}{2\kappa} \int_{\mathbb{R}^N \times \mathbb{R}^N} & \left(\frac{hM}{f_\infty} - \frac{h'M'}{f'_\infty} \right)^2 f_\infty \, f'_\infty \, \mathrm{d}v \, \mathrm{d}v'. \end{aligned}$$

Therefore its kernel is

$$N(L) = \mathrm{Span} \left\{ \frac{f_\infty}{M} \right\}.$$

We define Π_l the (orthogonal) projection on this space in L^2_v:

$$\Pi_l(h) = \left(\int_{\mathbb{R}^N} h' \, \frac{f'_\infty}{M'} \, \mathrm{d}v' \right) \frac{f_\infty}{M}.$$

First let us explain how to show assumption **H3** by non-constructive approach. As $\nu \geq \underline{\nu}$ with $\underline{\nu} = \inf_{\mathbb{R}^N} \nu > 0$, we deduce that the (bounded) multiplicative operator Λ on L^2_v has its spectrum included in $(-\infty, -\underline{\nu}]$. Then the operator K is straightforwardly compact on L^2_v and thus by Weyl's theorem about compact perturbation of the essential spectrum of a self-adjoint operator in a Hilbert space, we deduce that the essential spectrum of L is included in $(-\infty, -\underline{\nu}]$. The remaining discrete spectrum lies in $\mathbb{R}_-$ because of the sign of the Dirichlet form, and as 0 is an isolated eigenvalue, we deduce that there is $\lambda_0 > 0$ such that the non-zero part of the spectrum of L lies in $(-\infty, -\lambda_0]$. This immediately shows **H3** with $\lambda_L = \lambda_0/\overline{\nu}$ where $\overline{\nu} = \sup_{\mathbb{R}^N} \nu < +\infty$.

It is interesting to note that if the mass in the boson case matches the critical one then ν is still nonnegative but it becomes zero for $v = 0$, which of course is the place where condensation happens. In this case our method breaks down because we lack the spectral gap (the essential spectrum reaches 0 and one only expect some polynomial rates of convergence to equilibrium, which is consistent with [**19**]).

Second we restrict to the fermionic case ($\epsilon = 1$) and we explain how to estimate explicitly λ_L. Let us consider some function h orthogonal to $M^{-1} f_\infty$. Then

$$\int_{\mathbb{R}^N} hM \,\mathrm{d}v = \int_{\mathbb{R}^N} h \left(M - \frac{f_\infty}{\kappa_\infty M} \right) \mathrm{d}v = \int_{\mathbb{R}^N} h \frac{\kappa_\infty M^3}{1 + \kappa_\infty \mathcal{M}} \,\mathrm{d}v.$$

Hence we deduce that

$$\begin{aligned} \left(\int_{\mathbb{R}^N} hM \,\mathrm{d}v \right)^2 \le \left(\int_{\mathbb{R}^N} h^2 \nu \,\mathrm{d}v \right) \left(\int_{\mathbb{R}^N} \nu^{-1} \frac{\kappa_\infty^2 M^6}{(1 + \kappa_\infty \mathcal{M})^2} \,\mathrm{d}v \right) \\ = \frac{\kappa}{\rho} \left(\int_{\mathbb{R}^N} h^2 \nu \,\mathrm{d}v \right) \left(\int_{\mathbb{R}^N} f_\infty^3 \,\mathrm{d}v \right). \end{aligned}$$

From the exact formula for f_∞ it is straightforward that $f_\infty^3 < f_\infty$ and thus

$$\frac{1}{\rho} \left(\int_{\mathbb{R}^N} f_\infty^3 \,\mathrm{d}v \right) < 1.$$

We get

$$\begin{aligned} \langle L(h), h \rangle_{L^2} = \left[\frac{1}{\kappa} \left(\int_{\mathbb{R}^N} hM \,\mathrm{d}v \right)^2 - \left(\int_{\mathbb{R}^N} h^2 \nu \,\mathrm{d}v \right) \right] \\ \le - \left[1 - \frac{1}{\rho} \left(\int_{\mathbb{R}^N} f_\infty^3 \,\mathrm{d}v \right) \right] \left(\int_{\mathbb{R}^N} h^2 \nu \,\mathrm{d}v \right). \end{aligned}$$

Since in the fermionic case $\nu \ge (\kappa \kappa_\infty)^{-1}$, we deduce that L has a spectral gap λ_0, with the explicit estimate:

$$\lambda_L \ge \frac{1}{\kappa \kappa_\infty} \left[1 - \frac{1}{\rho} \left(\int_{\mathbb{R}^N} f_\infty^3 \,\mathrm{d}v \right) \right].$$

Hence we deduce an explicit estimate on λ_L since $\lambda_L = \lambda_0/\overline{\nu}$ where $\overline{\nu} = \sup_{\mathbb{R}^N} \nu = (1+\kappa_\infty \mathcal{M}(0))/(\kappa\kappa_\infty)$ is explicit.
Now we want to establish the bound **H4** on the bilinear part. It is given by

$$\Gamma(h,h) = \frac{\epsilon \kappa_\infty^{1/2}}{\kappa}\, h \left(\int_{\mathbb{R}^N} h'\, M'\, \frac{(\mathcal{M}' - \mathcal{M})}{1+\kappa_\infty \mathcal{M}'}\, \mathrm{d}v' \right).$$

Therefore **H4** is immediately obtained by using Leibniz rule on higher-order derivatives, the trivial bound $L^2 \times L^2 \to L^2$ on Γ, and Sobolev embeddings (which requires that $E(k_0/2) > N/2$ where E denotes the entire part of a real number).

Remark: The scaling that we used to linearize the collision operator is not exactly the same as in Theorem 4.1 since we choose $m = f_\infty^{1/2}(1+\epsilon\kappa_\infty \mathcal{M})^{-1/2}$. However it is easy to see (following exactly the same proof) that the statement of Theorem 4.1 remains true also with the scaling $f = f_\infty + m\, h$ when the factors $f_\infty^{-1/2}$ are replaced by the some factors m^{-1} with the same decay at large velocities. This leads to the statement of Theorem 1.3.

5.5.3 The linear Fokker-Planck equation

We consider the linear Fokker-Planck equation in the torus

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + f v)\,, \tag{5.3}$$

for $x \in \mathbb{T}^N$ and $v \in \mathbb{R}^N$ ($N \geq 1$).
This equation preserves the total mass of the distribution

$$\forall\, t \geq 0, \quad \int_{\mathbb{T}^N \times \mathbb{R}^N} f(t,x,v)\, \mathrm{d}x\, \mathrm{d}v = \int_{\mathbb{T}^N \times \mathbb{R}^N} f_0(x,v)\, \mathrm{d}x\, \mathrm{d}v$$

but admits no other conservation law. For a given initial datum $f_0 \geq 0$, it admits a unique global equilibrium $f_\infty = \rho_\infty\, \mathcal{M}$, where ρ_∞ is the total mass of f_0, defined by

$$\rho_\infty = \int_{\mathbb{T}^N \times \mathbb{R}^N} f_0(x,v)\, \mathrm{d}x\, \mathrm{d}v,$$

and $\mathcal{M}$ is the normalized Maxwellian distribution

$$\mathcal{M}(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{N/2}}$$

with mass 1, mean 0 and temperature 1.

We study fluctuations around the equilibrium in the form

$$f = f_\infty + \sqrt{f_\infty} h = \rho_\infty \mathcal{M} + \rho_\infty^{1/2} M\, h,$$

where $M := \sqrt{\mathcal{M}}$. The equation for h reads

$$\partial_t h + v \cdot \nabla_x h = \Delta_v h + \left(\frac{N}{2} - \frac{|v|^2}{4} \right) h =: L(h) \tag{5.4}$$

We split the operator L into

$$L = K - \Lambda, \quad K = 0, \quad \Lambda = L$$

($K = 0$ is typical for a purely diffusive collisional model). Assumption **H2** is obviously fulfilled with $C(\delta) = 0$.

Let us prove that L satisfies **H1** taking

$$\|h\|_\Lambda = \left(\|vh\|_{L^2}^2 + \|\nabla_v h\|_{L^2}^2 \right)^{1/2}.$$

Indeed one can check by integration by part that this norm is stronger than the L^2 norm, and straightforward computations yields

$$\langle L(h), h \rangle_{L^2} \geq C_1 \, \|h\|_\Lambda^2 - C_2 \, \|h\|_{L^2}^2 \tag{5.5}$$

for explicit constants $C_1, C_2 > 0$. Moreover the operator L is local in x and t. When x is fixed, it is well-defined and bounded on L^2_v, and it is self-adjoint non-positive on this space. More precisely its Dirichlet form is given by

$$\langle L(h), h \rangle_{L^2_v} = - \int_{\mathbb{R}^N} \left\| \nabla_v h + \frac{v}{2} \, h \right\|^2 \, \mathrm{d}v.$$

Therefore its kernel is $N(L) = \mathrm{Span}\,\{M\}$, and we define Π_l the (orthogonal) projection on this space in L^2_v:

$$\Pi_l(h) = \left(\int_{\mathbb{R}^N} h' \, M' \, \mathrm{d}v' \right) M.$$

Classical computations based on Poincaré's inequality with measure $\mathcal{M}$ show that

$$\int_{\mathbb{R}^N} \left\| \nabla_v h + \frac{v}{2} \, h \right\|^2 \, \mathrm{d}v \geq 2 \, \|h\|_{L^2}^2. \tag{5.6}$$

Then combining (5.5,5.6) yields

$$\langle L(h), h \rangle_{L^2} \leq -\lambda \, \|h\|_\Lambda^2$$

for some explicit constant $\lambda > 0$.

Finally we have

$$\begin{aligned} \langle \nabla_v L(h), \nabla_v h \rangle_{L^2} &= \langle L(\nabla_v h), \nabla_v h \rangle_{L^2} - \langle (v/2) h, \nabla_v h \rangle_{L^2} \\ &= \langle L(\nabla_v h), \nabla_v h \rangle_{L^2} + \frac{N}{2} \, \|h\|_{L^2} \leq -\Lambda_L \, \|h\|_\Lambda^2 + \frac{N}{2} \, \|h\|_{L^2}. \end{aligned}$$

The two last inequalities conclude the proof of **H1** and **H3**.

5.5.4 The Boltzmann equation

Let us consider the Boltzmann equation (here $N \geq 2$)

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \quad t \geq 0,\ x \in \mathbb{T}^N,\ v \in \mathbb{R}^N \tag{5.7}$$

with a collision operator

$$Q(f,f) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(|v-v_*|, \cos\theta)\ (f' f'_* - f f_*)\ \mathrm{d}v_*\, \mathrm{d}\sigma.$$

We adopt the notations $f' = f(v')$, $f_* = f(v_*)$ and $f'_* = f(v'_*)$, where

$$v' = (v+v_*)/2 + (|v-v_*|/2)\,\sigma, \qquad v'_* = (v+v_*)/2 - (|v-v_*|/2)\,\sigma$$

stand for the pre-collisional velocities of particles which after collision have velocities v and v_*. Moreover $\theta \in [0,\pi]$ is the deviation angle between $v' - v'_*$ and $v - v_*$, and B is the Boltzmann collision kernel determined by physics (related to the cross-section $\Sigma(v-v_*,\sigma)$ by the formula $B = |v-v_*|\,\Sigma$). On physical grounds, it is assumed that $B \geq 0$ and B is a function of $|v-v_*|$ and $\cos\theta$.

Boltzmann's collision operator has the fundamental properties of conserving mass, momentum and energy

$$\int_{\mathbb{R}^N} Q(f,f)\,\phi(v)\,\mathrm{d}v = 0, \qquad \phi(v) = 1, v, |v|^2$$

and satisfying celebrated Boltzmann's H theorem, which writes formally

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} f \log f\,\mathrm{d}v = -\int_{\mathbb{R}^N} Q(f,f)\log(f)\,\mathrm{d}v \geq 0.$$

The equilibrium distribution is given by the a Maxwellian distribution

$$\mathcal{M}(\rho_\infty, u_\infty, T_\infty)(v) = \frac{\rho_\infty}{(2\pi\,T_\infty)^{N/2}} \exp\left(-\frac{|u_\infty - v|^2}{2\,T_\infty}\right),$$

where ρ_∞, u_∞, T_∞ are the density, mean velocity and temperature of the gas

$$\rho_\infty = \int_{\mathbb{T}^N \times \mathbb{R}^N} f(v)\,\mathrm{d}x\,\mathrm{d}v, \quad u = \frac{1}{\rho_\infty}\int_{\mathbb{T}^N \times \mathbb{R}^N} v\,f(v)\,\mathrm{d}x\,\mathrm{d}v,$$

$$T = \frac{1}{N\rho_\infty}\int_{\mathbb{T}^N \times \mathbb{R}^N} |u_\infty - v|^2\,f(v)\,\mathrm{d}x\,\mathrm{d}v$$

which are determined by the mass, momentum and energy of the initial datum thanks to the conservation properties.

The main physical case of application of this subsection is that of hard spheres in dimension $N = 3$, where (up to a normalization constant)

$$B(|v - v_*|, \cos\theta) = |v - v_*|. \tag{5.8}$$

More generally we shall make the following assumption on the collision kernel:

B1. We assume that B takes the product form

$$B(|v - v_*|, \cos\theta) = \Phi(|v - v_*|)\, b(\cos\theta), \tag{5.9}$$

with Φ and b non-negative and not identically equal to 0. This decoupling assumption is made for the sake of simplicity and could probably be relaxed at the price of technical complications.

B2. Concerning the kinetic part, we assume Φ to be given by

$$\Phi(z) = C_\Phi\, z^\gamma \tag{5.10}$$

with $\gamma \in [0, 1]$. It is customary in physics and mathematics to study the case when $\Phi(v - v_*)$ behaves like a power law $|v - v_*|^\gamma$, and one traditionally separates between hard potentials ($\gamma > 0$), Maxwellian potentials ($\gamma = 0$), and soft potentials ($\gamma < 0$). We assume here that we deal with **hard potentials** (or Maxwell molecules). This assumption is crucial since for soft potentials, the linearized operator has no spectral gap.

B3. Concerning the angular part, we assume that it is C^1 with the controls from above

$$\forall\, z \in [-1, 1], \quad b(z),\ b'(z) \le C_b. \tag{5.11}$$

This assumption implies in particular that B satisfies Grad's **angular cutoff** (see [**24**]). Note that the smoothness assumption on b is made for the sake of simplicity and could be relaxed by using truncations and mollifications in the proof.

When b is integrable on the sphere $\mathbb{S}^{N-1}$ (as here thanks to **B3**), we define

$$\ell_b := \|b\|_{L^1(\mathbb{S}^{N-1})} := \left|\mathbb{S}^{N-2}\right| \int_0^\pi b(\cos\theta) \sin^{N-2}\theta \, d\theta < +\infty.$$

Without loss of generality we set $\ell_b = 1$ in the sequel. Then one can split the collision operator in the following way

$$\begin{aligned} Q(g, f) &= Q^+(g, f) - Q^-(g, f) \\ Q^+(g, f) &= \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi(|v - v_*|)\, b(\cos\theta)\, g'_* f' \, dv_* \, d\sigma. \\ Q^-(g, f) &= \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi(|v - v_*|)\, b(\cos\theta)\, g_* f \, dv_* \, d\sigma = (\Phi * g)\, f. \end{aligned}$$

We introduce the so-called *collision frequency*

$$\nu(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi(|v - v_*|)\, b(\cos\theta)\, \mathcal{M}(v_*)\, \mathrm{d}v_*\, \mathrm{d}\sigma = (\Phi * \mathcal{M})(v), \tag{5.12}$$

and denote by $\nu_0 > 0$ the minimum value of ν.
Using the notation $M = \mathcal{M}^{1/2}$ the linearized collision operator is given by

$$\begin{aligned} L(h) &= M^{-1}\left[Q(Mh, \mathcal{M}) + Q(\mathcal{M}, Mh)\right] \\ &= M \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi(|v - v_*|)\, b(\cos\theta)\, \mathcal{M}_* \left[\frac{h'_*}{M'_*} + \frac{h'}{M'} - \frac{h_*}{M_*} - \frac{h}{M}\right] \mathrm{d}v_*\, \mathrm{d}\sigma. \end{aligned}$$

L is self-adjoint on the space L^2_v. It splits between a multiplicative part and a non-local part as follows

$$L(h) = K(h) - \Lambda(h) \quad \text{with} \quad \Lambda(h) = \nu(v)\, h$$

and

$$K(h) = L^+(h) - L^*(h) \quad \text{with} \quad L^*(h) = M\left[(hM) * \Phi\right)]$$

and

$$L^+(h) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi(|v - v_*|)\, b(\cos\theta)\, \left[h' M'_* + h'_* M'\right]\, M_*\, \mathrm{d}v_*\, \mathrm{d}\sigma.$$

K is bounded and compact in L^2_v, as proved in [**25**].
From the classical spectral theory of L it is well-known that with the usual changes of variables

$$\begin{aligned} \langle Lh, h\rangle_{L^2_v} = -\frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} & \Phi(|v - v_*|)\, b(\cos\theta) \\ & \left[\frac{h'_*}{M'_*} + \frac{h'}{M'} - \frac{h_*}{M_*} - \frac{h}{M}\right]^2 \mathcal{M}\,\mathcal{M}_*\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \le 0. \end{aligned}$$

This implies that the spectrum of L in L^2_v is included in $\mathbb{R}_-$. Moreover the null space of L is

$$N(L) = \mathrm{Span}\left\{M, v_1 M, \dots, v_N M, |v|^2 M\right\}. \tag{5.13}$$

Using the fact that

$$\nu_1^\Lambda\, (1 + |v|)^\gamma \le \nu(v) \le \nu_2^\Lambda\, (1 + |v|)^\gamma$$

for some explicit constants $\nu_1^\Lambda, \nu_2^\Lambda > 0$, and that $\nabla_v \nu \in L^\infty$ with explicit bound since $\gamma \in [0, 1]$, we deduce that assumption **H1** is satisfied with the norm

$$\|h\|_\Lambda = \|h(1 + |v|)^{\gamma/2}\|_{L^2}.$$

Now we want to prove that

$$(5.14) \qquad \forall\, h \perp N(L), \quad -\langle h, Lh\rangle_{L^2_v} \geq \lambda \,\|h\|^2_\Lambda.$$

Controls from below on the collision kernel are necessary to ensure the existence of a spectral gap for the linearized operator. Under our assumptions, the non-constructive proof of Grad shows that L has a spectral gap. Moreover explicit estimates on the spectral gap λ_L have recently been obtained in [**4**] and extended to explicit estimates of the form (5.14) in [**41**]. Following these results L satisfies **H3** (for the norm Λ) with explicit bound.

Now we fix some $\delta > 0$ and check that L satisfies assumption **H2**.

Concerning the part L^*, this accounts essentially to Young's inequality. We easily compute the kernel of the operator

$$L^* h(v) = \int_{V\in\mathbb{R}^N} h(v+V)\, k^*(v,V)\,\mathrm{d}V$$

with

$$k^*(v,V) = M(v)^{1/2}\,\Phi(|V|)\,M(v+V)^{1/2}.$$

We introduce the splitting $k^* = k^{*,s}_\varepsilon + k^{*,r}_\varepsilon$, with

$$k^{*,s}_\varepsilon(v,V) = \mathcal{I}_{\{|V|\geq\varepsilon\}}\,k^*(v,V)$$

where $\mathcal{I}$ denotes some mollified indicator function. This induces the corresponding decomposition $L^* = L^{*,s}_\varepsilon + L^{*,r}_\varepsilon$. It is straightforward that

$$\|L^{*,r}_\varepsilon\|_{L^2\to L^2} \xrightarrow{\varepsilon\to 0} 0$$

and

$$|\nabla_v k^{*,s}_\varepsilon|\,, \quad |\nabla_V k^{*,s}_\varepsilon|\,, \quad |\nabla_v k^{*,r}_\varepsilon| \leq C(\varepsilon)\,M(V)^{1/8}.$$

Hence we deduce

$$(5.15) \qquad \|\nabla_v L^{*,s}_\varepsilon h\|_{L^2} \leq C(\varepsilon)\,\|h\|_{L^2}$$

and

$$(5.16) \qquad \|\nabla_v L^{*,r}_\varepsilon h\|_{L^2} \leq \delta\,\|h\|_{H^1} + C(\varepsilon)\,\|h\|_{L^2}$$

if ε is small enough.

Now we turn to the part L^+. We follow Grad computations [**25**, Sections 2 and 3] (recalled also in [**9**, Chapter 7, Section 2]) to compute the kernel of L^+, and apply the same kind of estimates as in [**42**]. We make the changes the variables

- $\sigma \in \mathbb{S}^{N-1}$, $v_* \in \mathbb{R}^N \longrightarrow \omega = (v'-v)/|v'-v| \in \mathbb{S}^{N-1}$, $v_* \in \mathbb{R}^N$: the jacobian amounts to change b into
$$\tilde{b}(\theta) = 2^{N-1} \sin^{N-2}\theta/2\, b(\theta);$$
- then $\omega \in \mathbb{S}^{N-1}$, $v_* \in \mathbb{R}^N \longrightarrow \omega \in \mathbb{S}^{N-1}$, $u = v - v_* \in \mathbb{R}^N$: the jacobian is equal to 1;
- then keeping ω fixed, decompose orthogonally $u = u_0\omega + W$ with $u_0 \in \mathbb{R}$ and $W \in \omega^\perp$: the jacobian is equal to 1;
- finally keeping $W \in V^\perp$ fixed, $\omega \in \mathbb{S}^{N-1}$, $u_0 \in \mathbb{R} \longrightarrow V = u_0\omega \in \mathbb{R}^N$: the jacobian is $(1/2)|V|^{-(N-1)}$.

Thus we get
$$L^+h(v) = \int_{V\in\mathbb{R}^N} h(v+V)\, k^+(v,V)\, \mathrm{d}V$$
with
$$k^+(v,V) = \mathrm{cst}\, |V|^{-(N-1)} \int_{W\in V^\perp} \Phi(\sqrt{|V|^2+|W|^2})\, \tilde{b}\left(\frac{|W|^2-|V|^2}{|W|^2+|V|^2}\right) \times \\ M(v+W)^{1/2}\, M(v+V+W)^{1/2}\, \mathrm{d}W.$$

This kernel can be written as
$$k^+(v,V) = \mathrm{cst}\, M(V)^{1/4}\, |V|^{-(N-1)} \times \\ \int_{W\in V^\perp} \Phi(\sqrt{|V|^2+|W|^2})\, \tilde{b}\left(\frac{|W|^2-|V|^2}{|W|^2+|V|^2}\right) M(v+V+W/2)\, \mathrm{d}W.$$

Moreover it is shown in [**9**, Chapter 7, Section 2] that
$$\|\mathbf{1}_{|\cdot|\geq R} L^+\|_{L^2\to L^2} \xrightarrow{R\to\infty} 0.$$

We use this to perform the splitting
$$k^+ = k_\varepsilon^{+,s} + k_\varepsilon^{+,r}$$
with
$$k_\varepsilon^{+,s}(v,V) = \mathcal{I}_{\{|v|\leq\varepsilon^{-1}\}}\, \mathcal{I}_{\{|V|\geq\varepsilon\}}\, k^+(v,V),$$
where $\mathcal{I}$ denotes some mollified indicator function. The corresponding decomposition of L is denoted by
$$L^+ = L_\varepsilon^{+,s} + L_\varepsilon^{+,r}.$$

It is straightforward that

$$\|L_\varepsilon^{+,r}\|_{L^2\to L^2} \xrightarrow{\varepsilon\to 0} 0$$

and

$$\left|\nabla_v k_\varepsilon^{+,s}\right|, \quad \left|\nabla_V k_\varepsilon^{+,s}\right|, \quad \left|\nabla_v k_\varepsilon^{+,r}\right| \le C(\varepsilon)\, M(V)^{1/8}.$$

Hence we deduce

$$\left\|\nabla_v L_\varepsilon^{+,s} h\right\|_{L^2} \le C(\varepsilon)\,\|h\|_{L^2} \tag{5.17}$$

and

$$\left\|\nabla_v L_\varepsilon^{+,r} h\right\|_{L^2} \le \delta\,\|h\|_{H^1} + C(\varepsilon)\,\|h\|_{L^2} \tag{5.18}$$

as long as ε is small enough.
This concludes the proof by gathering (5.15), (5.16), (5.17) and (5.18).
Finally let us consider the bilinear part given by

$$\begin{aligned}\Gamma(h_1,h_2) &= M^{-1}\left[Q(Mh,Mh)+Q(Mh,Mh)\right] \\ &= \int_{\mathbb{R}^N\times\mathbb{S}^{N-1}} \Phi(|v-v_*|)\, b(\cos\theta)\, M_*\left[(h_1)'_*(h_2)' - (h_1)_*(h_2)\right]\,\mathrm{d}v_*\,\mathrm{d}\sigma \\ &= \int_{\mathbb{R}^N\times\mathbb{S}^{N-1}} \Phi(|u|)\, b(\cos\theta)\, M_*\left[(h_1)'_*(h_2)' - (h_1)_*(h_2)\right]\,\mathrm{d}u\,\mathrm{d}\sigma\end{aligned}$$

with the notation $u = v - v_*$. We estimate

$$\begin{aligned}&\int_{\mathbb{R}^N} \Gamma(h_1,h_2)\,\varphi\,\mathrm{d}v \\ &\le C\,\|\varphi\|_{L^2} \left(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N\times\mathbb{S}^{N-1}} |(h_1)'|\,|(h_2)'_*|\,M(v+u)\,\langle u\rangle^\gamma\,\mathrm{d}u\,\mathrm{d}\sigma\right|^2 \mathrm{d}v\right)^{1/2} \\ &\le C\,\|\varphi\|_{L^2}\left(\int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} |(h_1)'|^2\,|(h_2)'_*|^2\,\langle v\rangle^\gamma\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}\sigma\right) \\ &\le C\,\|\varphi\|_{L^2}\left(\|h_1\|_{L^2_v}\,\|h_2\|_{\Lambda_v} + \|h_1\|_{\Lambda_v}\,\|h_2\|_{L^2_v}\right),\end{aligned}$$

which implies

$$\|\Gamma(h_1,h_2)\|_{L^2_v} \le C\left(\|h_1\|_{L^2_v}\,\|h_2\|_{\Lambda_v} + \|h_1\|_{\Lambda_v}\,\|h_2\|_{L^2_v}\right).$$

Together with Leibniz formula to differentiate Γ according to v and x and Sobolev embeddings this concludes the proof of **H4** for $E(k_0/2) > N/2$.

5.5.5 The Landau equation

This subsection deals with the Landau equation (for $N \geq 2$)

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \qquad t \geq 0,\ x \in \mathbb{T}^N,\ v \in \mathbb{R}^N \tag{5.19}$$

which features the collision operator

$$Q(f,f)(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^N} \mathbf{A}(v - v_*) \left[f_* \left(\nabla f\right) - f \left(\nabla f\right)_* \right] \mathrm{d}v_* \right),$$

where $\mathbf{A}(z) = |z|^2\, \Phi(|z|)\, \mathbf{P}(z)$, Φ is a non-negative function, and $\mathbf{P}(z)$ is the orthogonal projection onto $z^\perp$, i.e

$$\left(\mathbf{P}(z)\right)_{i,j} = \delta_{i,j} - \frac{z_i z_j}{|z|^2}$$

In this brief notation f_* stands for $f(v_*)$. This operator is used for instance in models for plasma. In this case the interaction among the particles is via the Coulomb potential and $\Phi(|z|) = |z|^{-3}$ in dimension 3. For more details see [**53**, Chapter 1, Section 1.7] and the references therein. Indeed in this case the Boltzmann collision operator does not make sense anymore (see [**52**, Annex I, Appendix]).

Landau's collision operator has the fundamental properties of conserving mass, momentum and energy

$$\int_{\mathbb{R}^N} Q(f,f)\, \phi(v)\, \mathrm{d}v = 0, \qquad \phi(v) = 1, v, |v|^2$$

and satisfying celebrated Boltzmann's H theorem, which writes formally

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} f \log f \,\mathrm{d}v = - \int_{\mathbb{R}^N} Q(f,f) \log(f)\, \mathrm{d}v \geq 0.$$

The equilibrium distribution is given by the Maxwellian distribution

$$\mathcal{M}(\rho_\infty, u_\infty, T_\infty)(v) = \frac{\rho_\infty}{(2\pi\, T_\infty)^{N/2}} \exp\left(- \frac{|u_\infty - v|^2}{2\, T_\infty} \right),$$

where ρ_∞, u_∞, T_∞ are determined as in the Boltzmann case.

We make the following assumption on the collision kernel:

L1. We assume Φ to be given by

$$\Phi(z) = C_\Phi\, z^\gamma \tag{5.20}$$

with $\gamma \in [-2, 1]$. By analogy with the Boltzmann equation, one could say that this assumption covers hard and moderately soft potentials.

We consider fluctuations around equilibrium of the form $f = \mathcal{M} + Mh$. The linearized collision operator is given by

$$L(h) = M^{-1}\, \nabla_v \cdot \left(\int_{\mathbb{R}^N} \mathbf{A}(v - v_*) \left[\left(\frac{\nabla_v h}{M} \right) - \left(\frac{\nabla_v h}{M} \right)_* \right] \mathcal{M}\mathcal{M}_* \, \mathrm{d}v_* \right).$$

L is self-adjoint on the space L^2_v. It splits between a "convolution part" and a diffusive part

$$L(h) = K(h) - \Lambda(h)$$

with

$$K(h) = -M^{-1}\, \nabla_v \cdot \left(\int_{\mathbb{R}^N} \mathbf{A}(v - v_*) \left(\frac{\nabla_v h}{M} \right)_* \mathcal{M}\mathcal{M}_* \, \mathrm{d}v_* \right),$$

and

$$\Lambda(h) = -M^{-1}\, \nabla_v \cdot \left(\int_{\mathbb{R}^N} \mathbf{A}(v - v_*) \left(\frac{\nabla_v h}{M} \right) \mathcal{M}\mathcal{M}_* \, \mathrm{d}v_* \right).$$

Estimate **H2** on K is easily verified since

$$K(h) = \int_{\mathbb{R}^N} k(v, v_*)\, h_* \, \mathrm{d}v_*$$

where the kernel

$$k(v, v_*) = \left[(\nabla_v)^T \left(\frac{\mathbf{A}(v - v_*)\mathcal{M}\mathcal{M}_*}{MM_*} \right) (\nabla_v) \right]$$

belongs straightforwardly to $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ and also to $H^1(\mathbb{R}^N \times \mathbb{R}^N)$ except possibly for a small region $(v - v_*) \sim 0$ which can split as in the Boltzmann case.

It is well-known from the classical spectral theory of L that with the usual changes of variables we have

$$\langle h, Lh \rangle_{L^2_v} = -\frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \Phi(|v - v_*|)\, |v - v_*|^2 \left\| \mathbf{P} \left[\left(\frac{\nabla_v h}{M} \right) - \left(\frac{\nabla_v h}{M} \right)_* \right] \right\|^2 \mathcal{M}\,\mathcal{M}_* \, \mathrm{d}v \, \mathrm{d}v_* \leq 0.$$

This implies that the spectrum of L in L^2_v is included in $\mathbb{R}_-$. Moreover the null space of L is

$$N(L) = \mathrm{Span}\left\{ M, v_1 M, \ldots, v_N M, |v|^2 M \right\}. \tag{5.21}$$

Now we can draw on the technical estimates proved in [**27**]. First we define the norm

$$\|h\|^2_{\Lambda_v} = \|h\langle v\rangle^{1+\gamma/2}\|^2_{L^2_v} + \left\| (\mathbf{P}(v)\nabla_v h)\, \langle v\rangle^{1+\gamma/2} \right\|^2_{L^2_v} + \left\| ((1 - \mathbf{P}(v))\nabla_v h)\, \langle v\rangle^{\gamma/2} \right\|^2_{L^2_v}$$

which is stronger than L^2_v as soon as $\gamma \geq -2$. In [**27**, Section 2] it is proven that

$$\langle \Lambda h, h \rangle_{L^2_v} \geq C \, \|h\|^2_{\Lambda_v}$$

with explicit constant. For the bilinear term Theorem 3 from [**27**] together with Sobolev embeddings yields **H4** in the norm Λ with explicit constant as long as $E(k_0/2) > N/2$. Again the stronger assumptions **H1'-H2'** are deduced straightforwardly with the same arguments. Finally assumption **H3** is proved in [**27**, Section 2, Lemma 5] by non-constructive arguments (and an explicit proof is given in the work in progress [**44**]). This concludes the proof.

5.5.6 Remarks on other models

Linear models of radiative transfert in the torus enter straightforwardly our abstract framework. It is likely that linear scattering Boltzmann models or semi-conductors collisional models also do so. Also it is easy to see on the linear relaxation models (as well as on more general linear scattering models) that one could add with very few changes in our proof some scattering rate $\Sigma = \Sigma(x)$ depending on x in front of the collision operator: assuming that $\Sigma \in C^\infty$ and

$$\forall \, x \in \mathbb{T}^N, \quad 0 < \Sigma_- \leq \Sigma(x) \leq \Sigma_+ < +\infty$$

for some constants $\Sigma_-, \Sigma_+ > 0$, the conclusion of Theorem 1.1 still holds.

For the Boltzmann equation with soft potentials and Grad's angular cut-off, smooth solutions near the Maxwellian equilibrium have been built in Guo [**29**]: by including polynomial weight in v depending on the order of the derivatives in the energy estimates, it is likely that one can adapt our proof to build a norm which is decreasing along the flow inspiring from [**29**]. However in this case the integro-differential operator T is not coercive for this norm, instead it satisfies degenerated coercivity estimates for some weaker norms. This is enough to built smooth solution, but does not yield exponential convergence towards equilibrium. Nevertheless as noticed in [**47**] one can deduce from it polynomial rates of decay to equilibrium by interpolating between a ladder of norms.

Our analysis works at the linear level for the linearized Boltzmann equation for hard potentials without Grad's angular cutoff assumption (in this case explicit spectral gap estimates on L can be found in [**4**]). However at now it is not known how to control the non-linear term in terms of a coercivity norm Λ adapted to the linearized operator. In [**41**] (see also [**44**]), it is shown how to write the linearized collision operator in the form $K - \Lambda$ with some regularizing K and some coercive Λ, and how to obtain coercivity estimates on L. The functional space of these coercivity estimates is a *local* Sobolev space with the right fractional order, but which does not seem sufficient to control the non-linear term.

Bibliography

[1] N. Ben Abdallah and J. Dolbeault. Relative entropies for the Vlasov-Poisson system in bounded domains. *C. R. Acad. Sci. Paris Ser I Math.*, 330:867–872, 2000.

[2] A. Arnold et al. Entropies and equilibria of many-particle systems: An essay on recent research. *Monatshefte für Mathematik*, 142:35–43, 2004.

[3] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for fokker-planck type equations. *Comm. Partial Differential Equations*, 26:43–100, 2001.

[4] C. Baranger and C. Mouhot. Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials. to appear in *Rev. Mat. Iberoamericana*.

[5] J. Batt and G. Rein. Global classical solutions of the periodic Vlasov-Poisson system in three dimensions. *C. R. Acad. Sc. Paris*, pages 411–416, 1991.

[6] M. J. Cáceres, J. A. Carrillo, and T. Goudon. Equilibration rate for the linear inhomogeneous relaxation-time Boltzmann equation for charged particles. *Comm. Partial differential Equations*, 28:969–989, 2003.

[7] E. A. Carlen and M. C. Carvalho. Strict entropy production bounds and stability of the rate of convergence to equilibrium for the Boltzmann equation. *J. Statist. Phys. 67*, pages 575–608, 1992.

[8] E. A. Carlen and M. C. Carvalho. Entropy production estimates for Boltzmann equations with physically realistic collision kernels. *J. Statist. Phys.*, 74:743–782, 1994.

[9] C. Cercignani, R. Illner, and M. Pulvirenti. *The mathematical theory of dilute gases*. Springer-Verlag, New York, 1994.

[10] S. Chapman and T. G. Cowling. *The mathematical theory of non-uniform gases.* Cambridge University Press, 1990. 3'rd Edition.

[11] I. Choquet, P. Degond, and C. Schmeiser. Energy-transport models for charge carriers involving impact ionization in semiconductors. *Transp. Th. and Stat. Phys.*, 32:99–132, 2003.

[12] R. Dautray and J. L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology*, volume 6. Springer, New York, 1993.

[13] P. Degond, T. Goudon, and F. Poupaud. Diffusion limit for non homogeneous and non-micro-reversible processes. *Indiana University Mathematics Journal*, 49:1175–1197, 2000.

[14] L. Desvillettes. Convergence to equilibrium in large time for Boltzmann and B.G.K. equations. *Arch. Rational Mech. Anal.*, 110:73–91, 1990.

[15] L. Desvillettes and J. Dolbeault. On long time asymptotics of the Vlasov-Poisson-Boltzmann equation. *Comm. Partial Diff. Eq.*, 16:451–489, 1991.

[16] L. Desvillettes and C. Villani. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: The linear fokker-planck equation. *Comm. Pure Appl. Math.*, 54:1–42, 2001.

[17] L. Desvillettes and C. Villani. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.*, 159:245–316, 2005.

[18] J. Dolbeault. Kinetic models and quantum effects: a modified Boltzmann equation for Fermi-Dirac particles. *Archives of Rational Mechanics and Analysis*, 127:101–131, 1994.

[19] M. Escobedo, S. Mischler, and J.J.L. Velazquez. Asymptotic description of Dirac mass formation in kinetic equations for quantum particles. *J. Diff. Equ.*, 202:208–230, 2004.

[20] Lawrence C. Evans. *Partial Differential Equations.* AMS, Providence, 1998.

[21] K. Fellner, L. Neumann, and C. Schmeiser. Convergence to global equilibrium for spacially inhomogeneous kinetic models of non-micro-reversible processes. *Monatshefte ür Mathematik*, 141:289–299, 2004.

[22] E. Gabetta, P. A. Markowich, and A. Unterreiter. A note on the entropy production of the radiative transfer equation. *Applied Mathematics Letters*, 12:111–116, 1999.

[23] F. Golse and F. Poupaud. Limite fluide des equations de Boltzmann des semi-conducteurs pour une statistique de Fermi-Dirac. *J. on Asymptotic Analysis*, 6:135–160, 1992.

[24] H. Grad. Principles of the kinetic theory of gases. In *Flügge's Handbuch der Physik*, volume XII, pages 205–294. Springer-Verlag, 1958.

[25] H. Grad. Asymptotic theory of the Boltzmann equation. II. In *Rarefied Gas Dynamics*, volume I, pages 26–59, 1963. Proc. 3rd Internat. Sympos., Palais de l'UNESCO Paris (1962).

[26] H. Grad. Asymptotic equivalence of the Navier-Stokes and the nonlinear Boltzmann equations. In R. Finn, editor, *Proceedings of Symposia in Applied Mathematics*, volume 17, pages 154–183. AMS, 1965.

[27] Y. Guo. The Landau equation in a periodic box. *Comm. Math. Phys.*, 231:391–434, 2002.

[28] Y. Guo. The Vlasov-Poisson-Boltzmann system near Maxwellians. *Comm. Pure Appl. Math.*, 55:1104–1135, 2002.

[29] Y. Guo. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Rat. Mech. Anal.*, 169:305–353, 2003.

[30] Y. Guo. The Vlasov-Maxwell-Boltzmann system near Maxwellians. *Invent. Math.*, 153:593–630, 2003.

[31] Y. Guo. The Boltzmann equation in the whole space. *Indiana Math. J.*, 53:1081–1094, 2004.

[32] F. Hérau. Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation. Preprint 2005.

[33] F. Hérau and F. Nier. Isotropic hypoellipticity and trend to the equilibrium for the Fokker-Planck equation with high degree potential. *Arch. Rat. Mech. An.*, 171:151–218, 2004.

[34] H. J. Hwang. Regularity for the vlasov-poisson system in a convex domain. *SIAM J. on Math. Anal.*, 36:121–171, 2004.

[35] T.-P. Liu, T. Yang, and S.-H. Yu. Energy method for the Boltzmann equation. *Physica D*, pages 178–192, 2004.

[36] X. Lu. A modified Boltzmann equation for Bose-Einstein particles: Isotropic solutions and long-time behavior. *J. Stat. Phys.*, 98:1335–1394, 2000.

[37] X. Lu. On isotropic distributional solutions to the Boltzmann equation for Bose-Einstein particles. *J. Statist. Phys.*, 116:1597–1649, 2004.

[38] P. Markowich, F. Poupaud, and C. Schmeiser. Diffusion approximation of nonlinear electron-phonon collision mechanisms. *Math. Mod. and Numer. Anal.*, 29:857–869, 1995.

[39] P.A. Markowich, C.A. Ringhofer, and C. Schmeiser. *Semiconductor equations*. Springer-Verlag, Vienna, 1990.

[40] A. Mellet. Diffusion limit of a nonlinear kinetic model without the detailed balance principle. *Monatshefte Mathematik*, 134:305–329, 2002.

[41] C. Mouhot. Explicit coercivity estimates for the Boltzmann and Landau operators. Preprint 2004.

[42] C. Mouhot. Rate of convergence for the spatially homogeneous Boltzmann equation for hard potentials. to appear in *Comm. Math. Phys.*

[43] C. Mouhot and L. Neumann. Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. Preprint.

[44] C. Mouhot and R. Strain. Explicit spectral gap and coercivity estimates for soft potential Boltzmann and Landau collision operators without cutoff. Work in progress.

[45] L. Neumann and C. Schmeiser. Convergence to global equilibrium for a kinetic model for fermions. *SIAM J. Math. Anal.*, 36:1652–1663, 2005.

[46] F. Poupaud and C. Schmeiser. Charge transport in semiconductors with degeneracy effects. *Math. Meth. in the Appl. Sci.*, 14:301–318, 1991.

[47] R. M. Strain and Y. Guo. Almost exponential decay near Maxwellians. Preprint 2004.

[48] M. Taylor. *Partial Differential Equations*, volume 1-3. Springer, New York, 1996.

[49] G. Toscani and C. Villani. Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. *Comm. Math. Phys.*, 203:667–706, 1999.

[50] S. Ukai. On the existence of global solutions of the mixed problem for non-linear Boltzmann equation. *Proc. Japan Acad.*, 50:179–184, 1974.

[51] C. Villani. Hypocoercivity part I: Hörmander operators in Hilbert spaces. Work in progress.

[52] C. Villani. Contribution à l'étude mathématique des collisions en théorie cinétique, 2000. Univ. Paris-Dauphine France.

[53] C. Villani. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics*, volume I, pages 71–305. North-Holland, 2002.

[54] C. Villani. Convergence to equilibrium: Entropy production and hypocoercivity, 2004. Harold Grad lecture, available at `http://www.umpa.ens-lyon.fr/~cvillani`.

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