Imperial College London

Single-point second-moment turbulence models – why, where and where not

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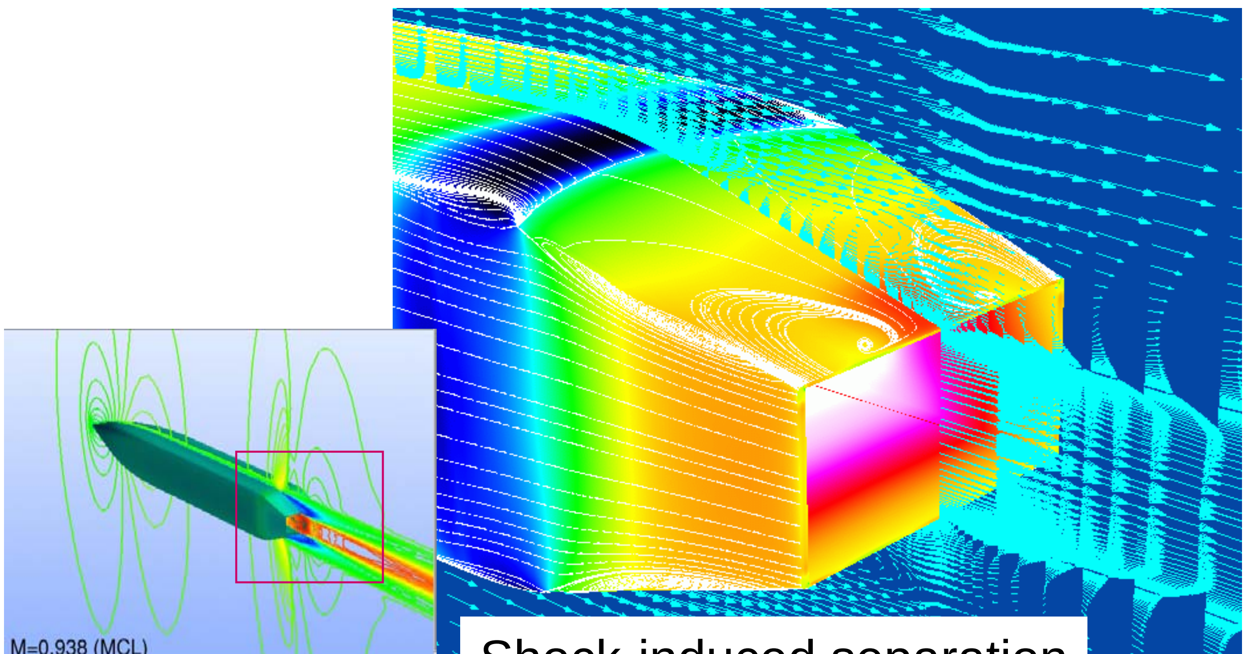
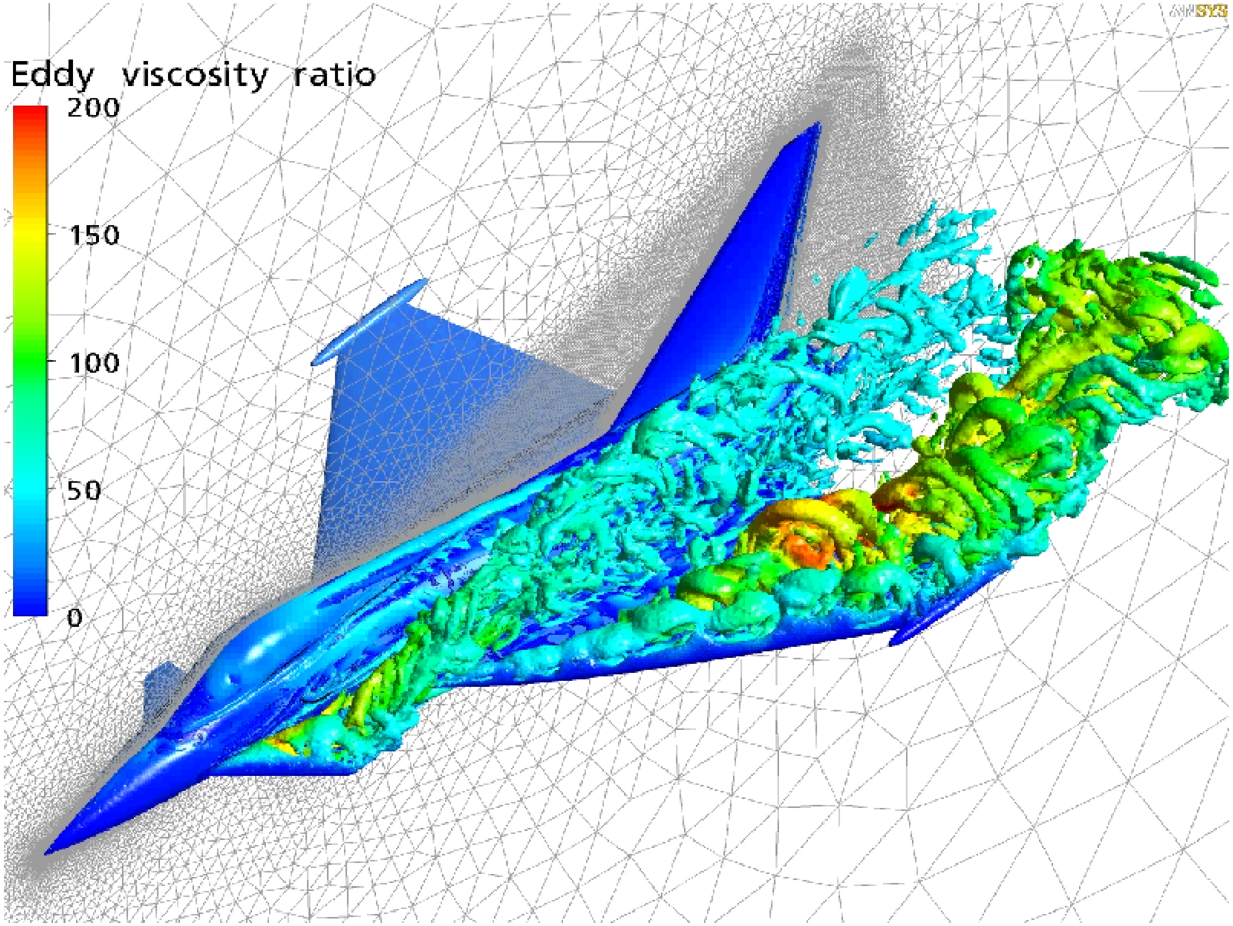


The holy grail

- We are promised a 'model-free' CFD world

A Boeing 747 is not a homogeneous square box!



Hybrid LES-RANS

Courtesy: ANSYS, Germany

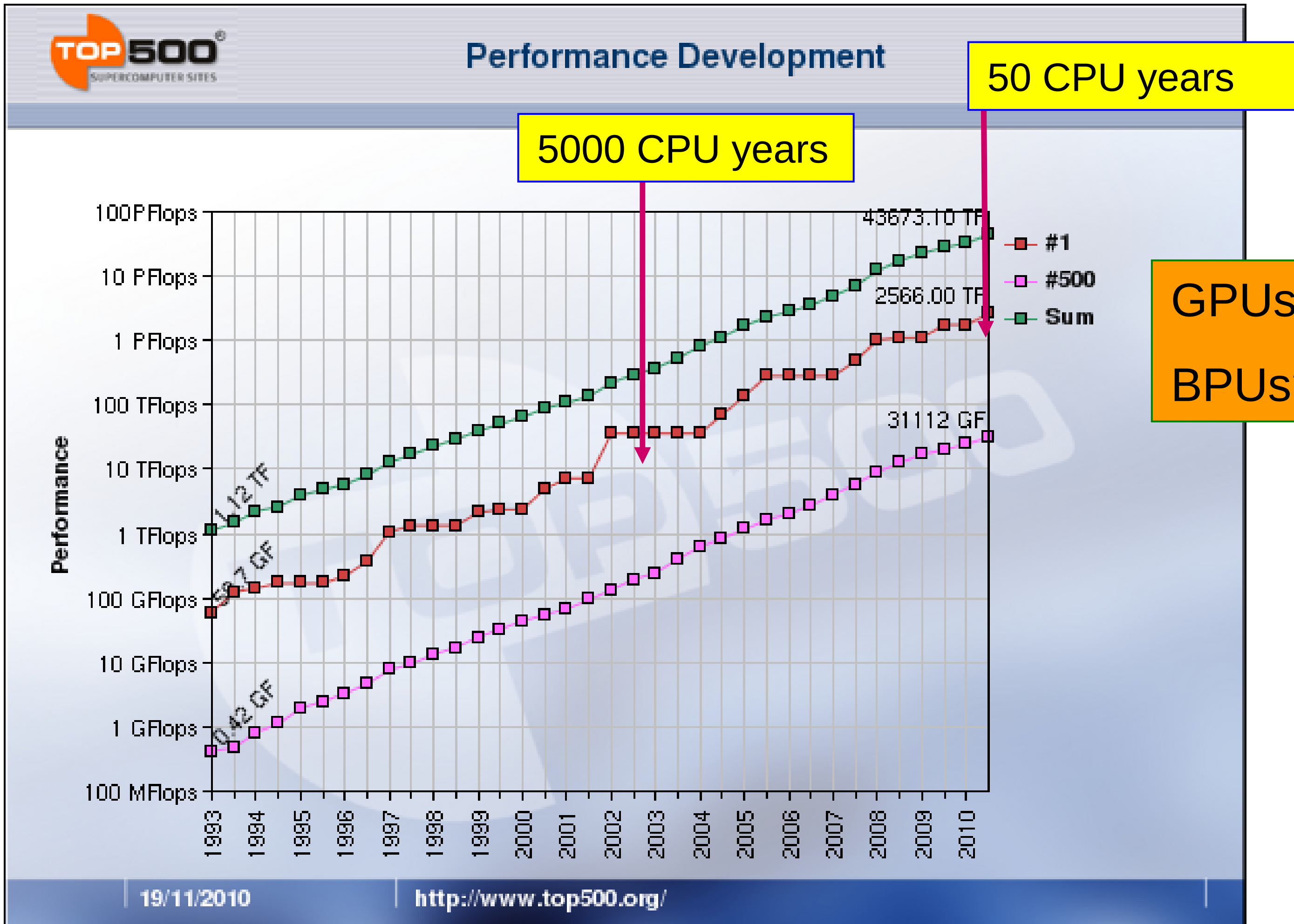
Some scales and estimates

- Mean-flow scales: t, ℓ
- Kolmogorov scales: $\tau, \ \eta$
- Ratios: $t / \tau \sim Re^{1/2}, \ell / \eta \sim Re^{3/4}$
- Grid: $N_\eta \sim Re^{9/4}$

- Aircraft: $Re \sim 10^8$
- Nodes: $N_\eta \sim 10^{19}$
- Time steps: $N_\tau \sim 10^6 - 10^7$
- Current estimate of time of realisation: 2080
- Current estimate for LES: 2045 (based on resolution at Taylor scale)
- Current capability: RANS and RANS-LES hybrids
- 95%+ of all engineering CFD is based on RANS

The cost

- Mesh: 10^{19}
- Cost: 5000 CPU years per 1 second of flying at 1Tflop throughput



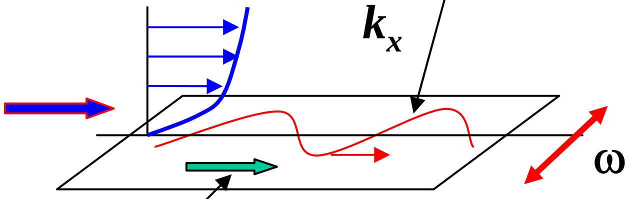
GPUs?

BPUs?

DNS - Status

- Model-free DNS used to
  - investigate fundamental physics;
  - examine subgrid-scale models (a-priori testing)
  - Develop, calibrate and validate RANS models
- Largest channel-flow DNS: $Re_\tau = 964$, 2.7x10^9 nodes (Del Alamo et al, 2004)

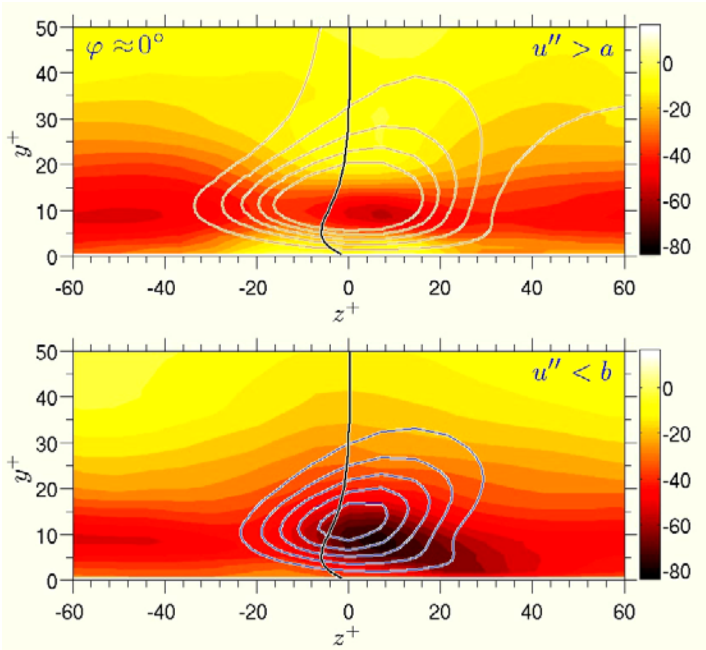
- Example: insight into origin of drag reduction by spanwise wall oscillation (Touber & Leschziner, 2010)
  - $Re_\tau = 500\ (\rightarrow 1000)$
  - Drag reduction up to 40%
  - 0.5x10^9 nodes, 1M CPU hours



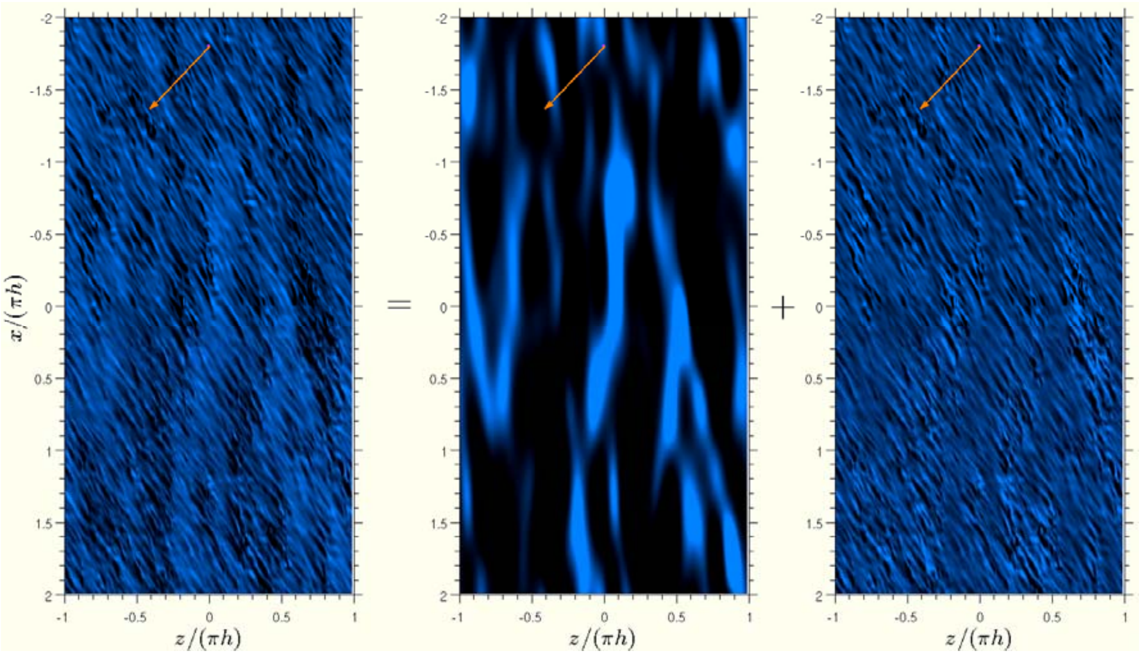
Fundamental mechanism of streak response

- Streak formation and re-orientation mechanisms
- Conditional sampling and averaging
- Decomposition of small streaks/super-streaks
- Modulation mechanisms
- Linear analysis (GOP)

Reduction of wall-normal fluctuations around streaks in % due to actuation



Streak decay, regeneration, reorientation and modulation



The "RANS" equations

- Time-averaged framework:

$$\frac{\partial \rho \overline{U}_i \overline{U}_j}{\partial x_j} = -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\mu\left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\rho\overline{u_i u_j} + \overline{B}_i$$

- Unsteady – URANS framework
  - Triple decomposition $U = \underbrace{\underbrace{\overline{U}}_{\text{Mean}} + \underbrace{\tilde{u}}_{\text{Periodic}}}_{\text{Phase-average}} + \underbrace{u'}_{\text{Stochastic}}$

"coherent"

$\overline{U} \Longrightarrow$
$$\frac{\partial \rho \overline{U}_j \overline{U}_i}{\partial x_j} = \frac{\partial \overline{P}}{\partial x_i} + \mu\frac{\partial^2 \overline{U}_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\rho\left(-\overline{\tilde{u}_i\tilde{u}_j} - \overline{u_i' u_j'}\right)$$

$\tilde{u} \Longrightarrow$
$$\frac{\partial \rho \tilde{u}_i}{\partial t} + \frac{\partial \rho \overline{U}_j \tilde{u}_i}{\partial x_j} = \frac{\partial \tilde{p}}{\partial x_i} + \mu\frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\rho\left(\overline{\tilde{u}_i\tilde{u}_j} - \tilde{u}_i\tilde{u}_j\right)$$
$$+\frac{\partial}{\partial x_j}\rho\left(\overline{u_i' u_j'} - < u_i' u_j' >\right) - \frac{\partial \rho \overline{U}_i \tilde{u}_j}{\partial x_j}$$

  - Requires closure equations for the periodic and stochastic terms: too complex in practice – URANS use RANS models + $\partial / \partial t$

Nature of Modelling

- Reynolds stresses related to known or determinable quantities:

$$\overline{u_i u_j} = f_{ij}\begin{pmatrix} S_{ij}, \Omega_{ij}, S_{kl}S_{kl}, \Omega_{kl}\Omega_{kl}, \\ \text{length-scale surrogates} \\ \text{turbulence invariates} \end{pmatrix}$$

S_{ij} Strain tensor

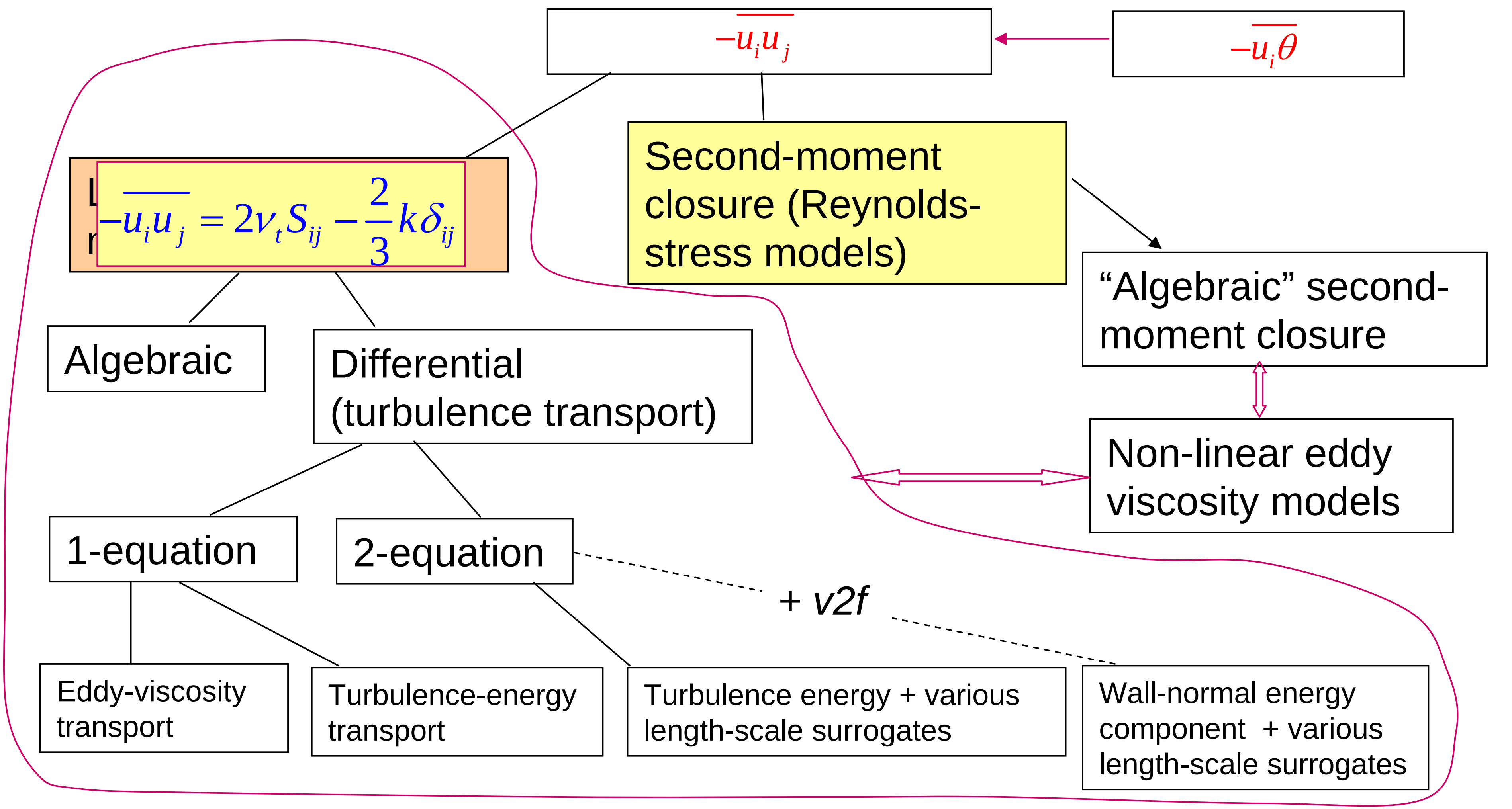
Ω_{ij} Vorticity tensor

- Ultimately, need to relate to stresses and mean velocity.

- Modelling principles – not only "ad-hoc curve fitting"
  - strong fundamental foundation;
  - resolution of anisotropy;
  - correct response to shear and normal straining;
  - correct response to curvature and body forces;
  - frame-invariance ("objectivity");
  - realisability;
  - correct approach to 2-component turbulence at wall and fluid-fluid interfaces;
  - satisfactory numerical stability;
  - economy.

Model types – basic classification

- About 150 models & major variations, many meant for restricted flow classes



Defects of linear eddy-viscosity models

- Linear EVM:
  - Well suited to thin shear flow
  - Much less well suited to separated and highly 3d flow
  - No resolution of anisotropy
  - Wrong sensitivity to flow curvature, rotation, normal straining and body forces
  - Reliant on ad-hoc corrections
- Defects are rooted in
  - Inapplicability of linear stress-strain relations
  - Isotropic nature of viscosity, relating to scalar turbulence properties
  - Calibration by reference to simple, near-equilibrium flows
  - Excessive extrapolation to complex condition.
- Only fundamentally credible alternative
  - Modelling based on exact equations for the Reynolds stresses
  - Strong resistance from engineering community - complexity

Reynolds-Stress-Transport Modelling

- Introduce the Reynolds decomposition $U_i = \overline{U}_i + u_i$ etc. into the NS equations.
- Subtract from this the corresponding RANS equation.
- Repeating the above, but with the indices i and j interchanged.
- Add the two equations.
- Time-averaging the result:

$$\underbrace{\frac{D\overline{u_i u_j}}{Dt}}_{C_{ij}} = \underbrace{-\left\{\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right\}}_{P_{ij}} + \underbrace{\left(\overline{f_i u_j} + \overline{f_j u_i}\right)}_{F_{ij}} \underbrace{- 2\nu\,\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}}_{\varepsilon_{ij}}$$

$$\underbrace{+\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\Phi_{ij}} \underbrace{- \frac{\partial}{\partial x_k}\left\{\overline{u_i u_j u_k} + \frac{\overline{p u_j}}{\rho}\delta_{ik} + \frac{\overline{p u_i}}{\rho}\delta_{jk} - \nu\frac{\partial \overline{u_i u_j}}{\partial x_k}\right\}}_{d_{ij}}$$

Pressure-velocity

- $C_{ij}, P_{ij}, F_{ij}, \Phi_{ij}, \varepsilon_{ij}$ and d_{ij} represent, respectively, stress convection, production by strain, production by body forces (e.g. buoyancy), dissipation, pressure-strain redistribution and diffusion

The Argument for resolving anisotropy

- Production is a key process: it drives the stresses.
- It requires no approximations if stresses and velocity are known
- It is reasonable to assume, tentatively:

  Stress = Production x Time (*capital = interest rate x time*)

- Exact equations imply complex stress-strain linkage

$$\rho\overline{u_i u_j} \longleftrightarrow -\tau \underbrace{\left\{ \overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} \right\}}_{P_{ij}} + \tau \times \textit{Body-force production}$$

- Hence, simple EVM stress-strain linkage is inapplicable
- Analogous linkage between scalar fluxes and production

$$\rho\overline{u_i \varphi} \longleftrightarrow -\tau_\varphi \underbrace{\left\{ \overline{u_i u_k}\frac{\partial \Phi}{\partial x_k} + \overline{u_i \varphi}\frac{\partial U_i}{\partial x_k} \right\}}_{P_{\overline{u_i\varphi}}} + \tau_\varphi \times \textit{Body-force production}$$

- Hence, Fourier-Fick law (eddy-diffusivity approximation) $\rho\overline{u_i\varphi} = -\frac{\mu_t}{\sigma_\varphi}\frac{\partial \Phi}{\partial x_i}$ not valid

The equations for thin shear flow

- Only one shear strain, only one shear stress

$$\frac{D\overline{uv}}{Dt} = -\overline{v^2}\frac{\partial \overline{U}}{\partial y} + \overline{\frac{p}{\rho}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)} - \frac{\partial}{\partial y}\left(\overline{uv^2} + \frac{\overline{pu}}{\rho}\right) + \frac{\mu}{\rho}\frac{\partial \overline{uv}}{\partial y} - \varepsilon_{12}$$

$$\frac{D\overline{u^2}}{Dt} = -2\overline{uv}\frac{\partial \overline{U}}{\partial y} + 2\overline{\frac{p}{\rho}\frac{\partial u}{\partial x}} - \frac{\partial}{\partial y}\left(\overline{u^2 v}\right) + 2\frac{\mu}{\rho}\frac{\partial \overline{u^2}}{\partial y} - \varepsilon_{11}$$

$$\frac{D\overline{v^2}}{Dt} = 0 + 2\overline{\frac{p}{\rho}\frac{\partial v}{\partial y}} - \frac{\partial}{\partial y}\left(\overline{v^3} + \frac{2\overline{pv}}{\rho}\right) + 2\frac{\mu}{\rho}\frac{\partial \overline{v^2}}{\partial y} - \varepsilon_{22}$$

$$\frac{D\overline{w^2}}{Dt} = 0 + 2\overline{\frac{p}{\rho}\frac{\partial w}{\partial z}} - \frac{\partial}{\partial x}\left(\overline{vw^2}\right) + 2\frac{\mu}{\rho}\frac{\partial \overline{w^2}}{\partial y} - \varepsilon_{33}$$

$\sum = k - equation$

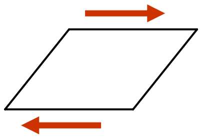
Anisotropy

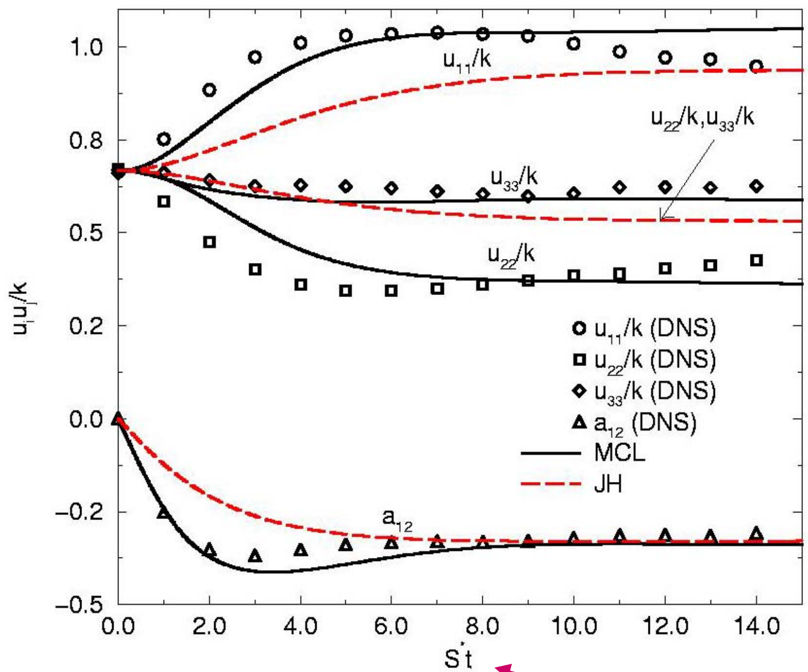
$\sum = 0$

$\sum = 2\varepsilon$

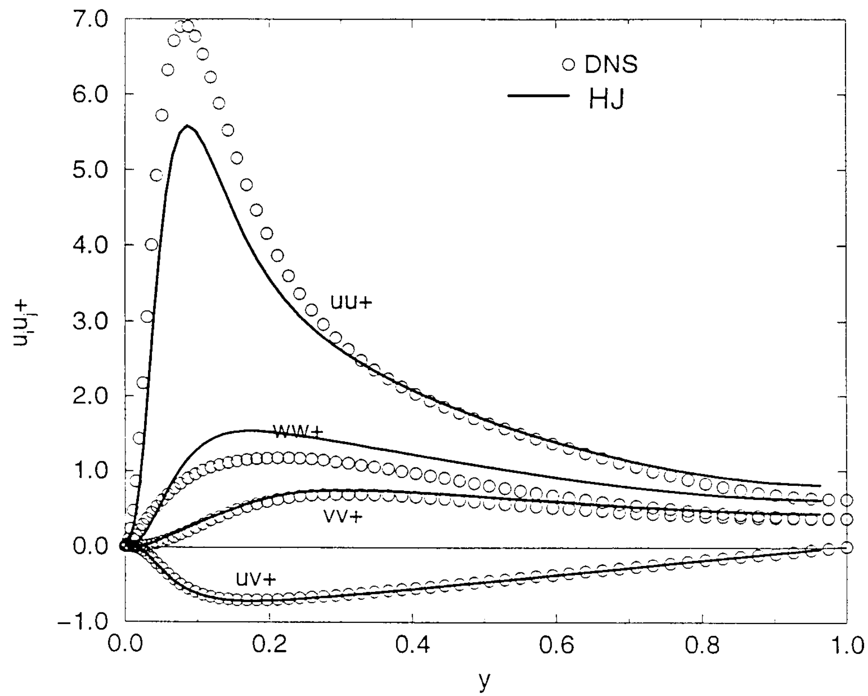
Anisotropy in simple shear

- Homogeneous shear
  - Development in time of stresses normalized by k



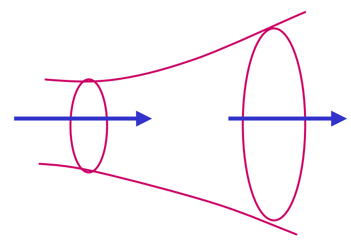


- Channel flow
  - Normal and shear stresses



Strain rate x time

The importance of anisotropy: expansion (deceleration)



Positive generation

$$\frac{D\overline{u^2}}{Dt} = -2\overline{u^2}\frac{\partial U}{\partial x} + ...$$

Negative generation

$$\frac{D\overline{v^2}}{Dt} = \overline{v^2}\frac{\partial U}{\partial x} + ...$$

$$\frac{D\overline{w^2}}{Dt} = \overline{w^2}\frac{\partial U}{\partial x} + ...$$

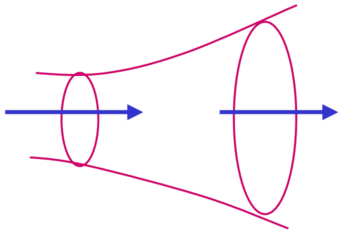
$$\frac{Dk}{Dt} = -\frac{1}{2}(2\overline{u^2} - \overline{v^2} - \overline{w^2})\frac{\partial U}{\partial x} + ...$$

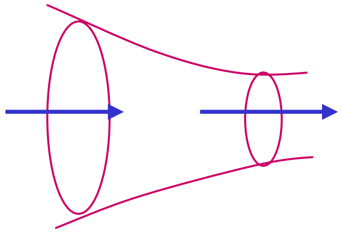
Eddy viscosity

$$2C_\mu \frac{k^2}{\varepsilon}\left(\frac{\partial U}{\partial x}\right)^2$$

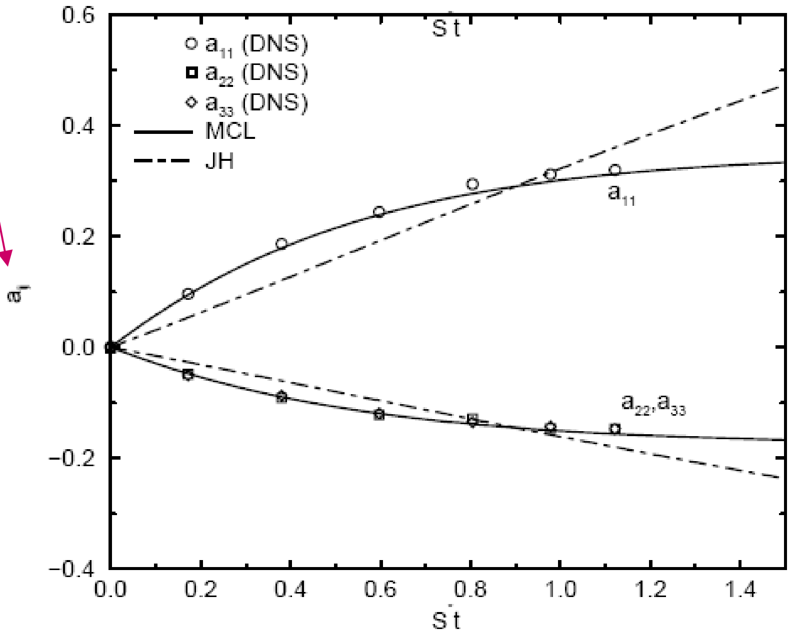
- Low or negative k-production, relative to very high EVM production

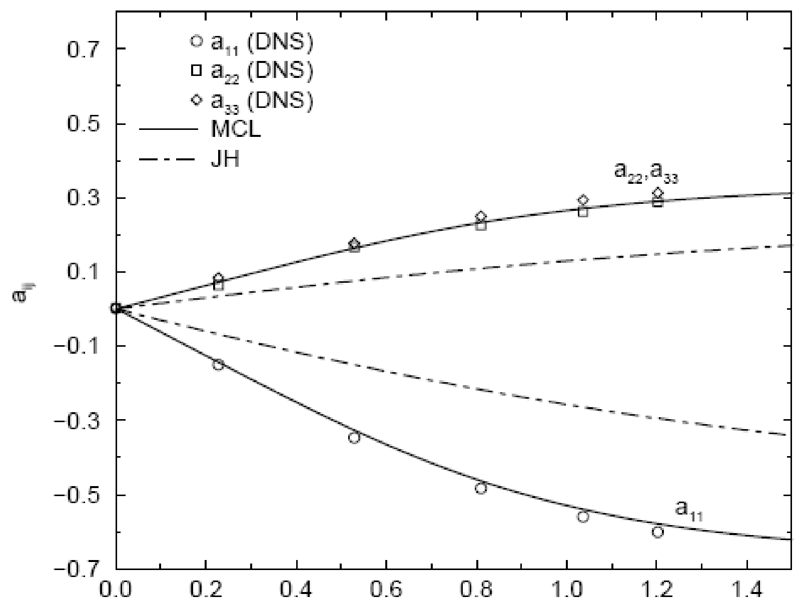
Anisotropy in expansion and contraction





$$a_{ij} \equiv \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}$$

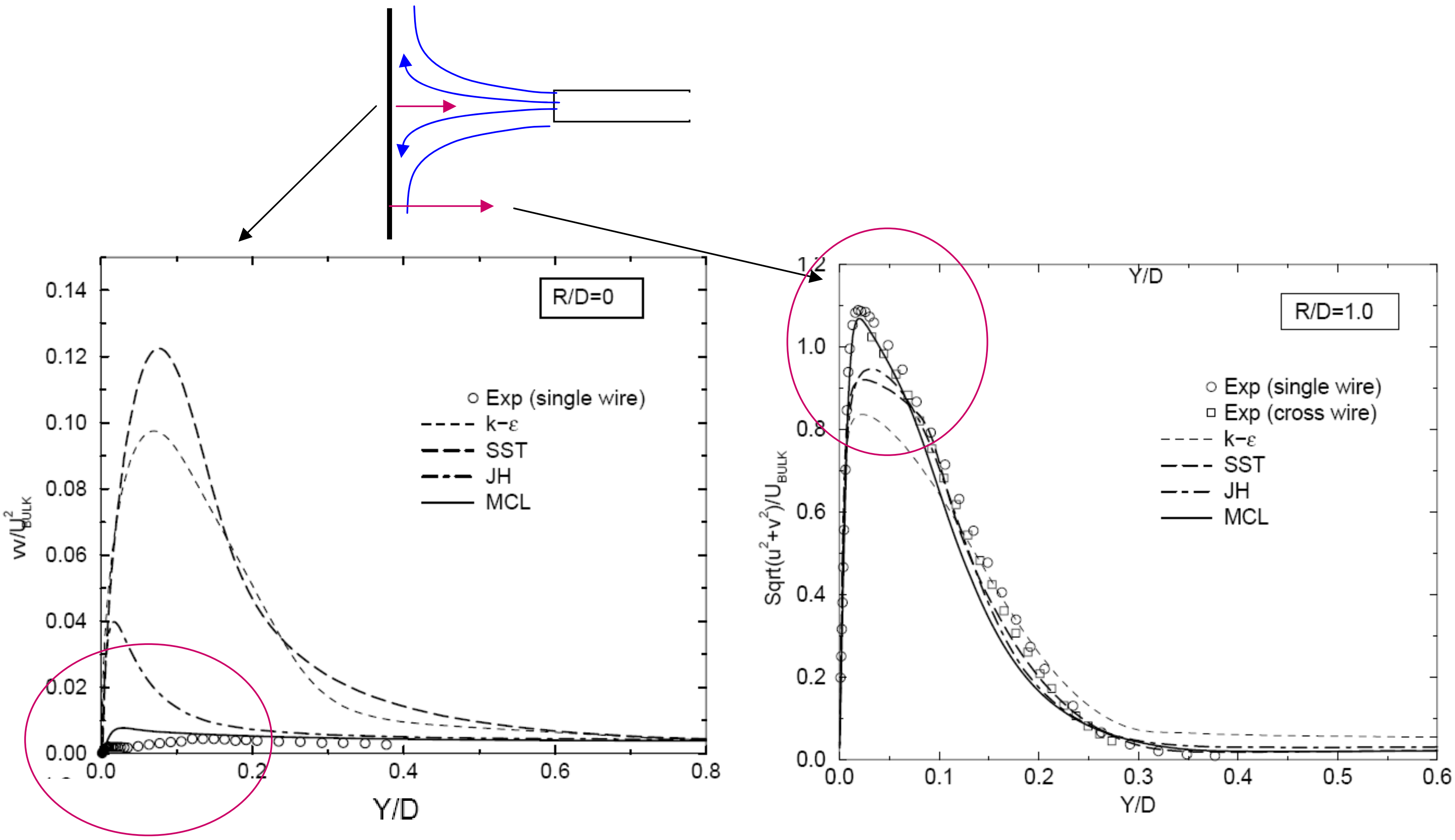




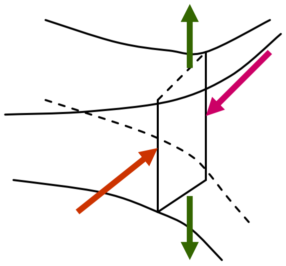
$$S^* = k/\varepsilon\sqrt{S_{ij}S_{ij}/2}$$

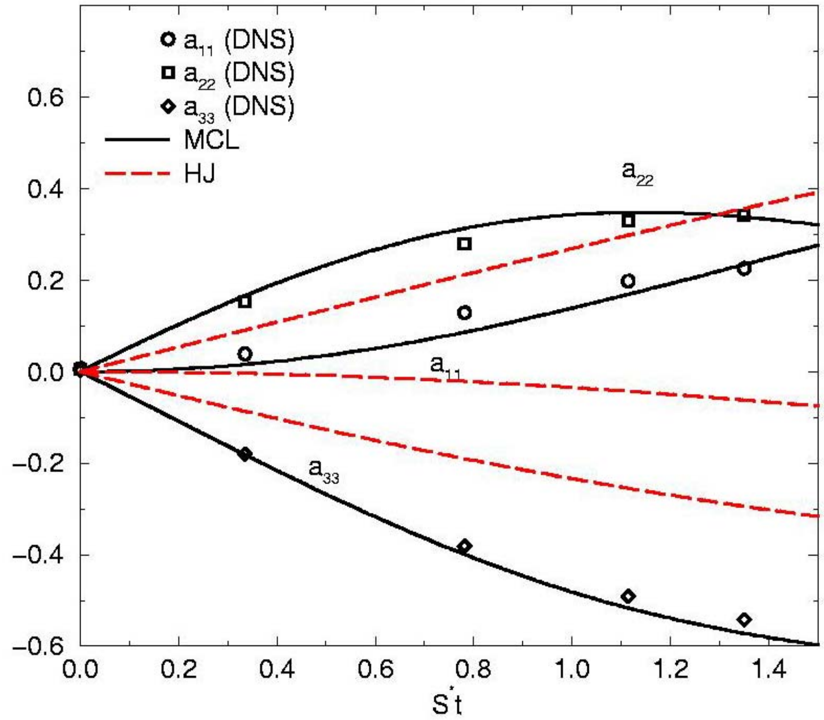
Round impinging jet

- Wall-normal stress and mean velocity



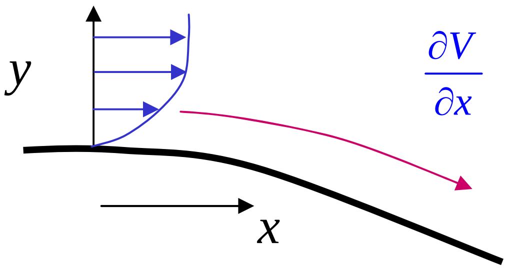
Anisotropy in plain strain



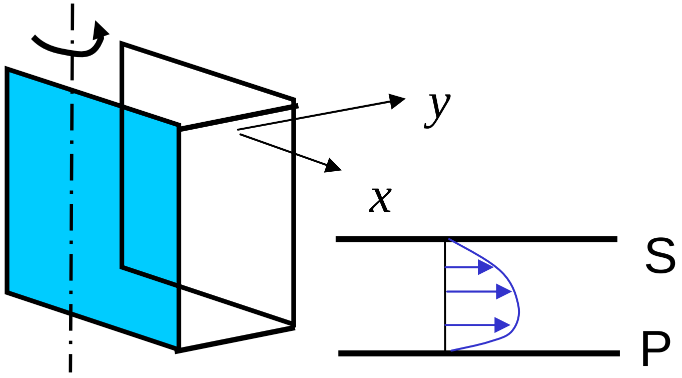


Other sensitivities

- Strong effect of curvature on anisotropy and shear stress.



- Strong effects of rotation on anisotropy and shear stress



- Inapplicability of Fourier-Fick law in scalar transport
  - Production of flux vector:

$$P_{u_i\phi} = -\overline{u_i u_k}\frac{\partial \Phi}{\partial x_k} - \overline{u_i \varphi}\frac{\partial U_i}{\partial x_k}$$

Reynolds-Stress-Transport Modelling

- Closure of exact stress-transport equations

$$\underbrace{\frac{D\overline{u_i u_j}}{Dt}}_{C_{ij}=Advective Transport} = \underbrace{-\left\{ \overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} \right\}}_{P_{ij}=Production} + (Pressure - velocity)$$

$$+ Diffusion - Dissipation$$

- Pressure-velocity, dissipation and diffusion require approximation
- About 10-15 major closures forms
- Modern closure aims at realisability, 2-component limit, coping with strong inhomogeneity and compressibility
- Additional equations for dissipation tensor ε_{ij}
- At least 7 pde's in 3D (up to 17 in heat/scalar transport)
- Numerically difficult in complex geometries and flow
- Can be costly
- Dissipation and pressure-velocity are major sources of error

The exact dissipation-rate equation

$$\frac{\mathbf{D}\varepsilon}{\mathbf{D}t} = \underbrace{\frac{\partial \varepsilon}{\partial t}}_{L_\varepsilon} + \underbrace{\frac{\partial U_k \varepsilon}{\partial x_k}}_{C_\varepsilon}$$

$$= \boxed{\underbrace{-2\nu\left(\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} + \overline{\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}}\right)\frac{\partial U_i}{\partial x_k}}_{P_\varepsilon^1 + P_\varepsilon^2}} \quad \boxed{\underbrace{-2\nu \overline{u_k \frac{\partial u_i}{\partial x_l}}\frac{\partial^2 U_i}{\partial x_k \partial x_l}}_{P_\varepsilon^3}}$$

$$\boxed{\underbrace{-2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}}}_{P_\varepsilon^4}} \quad - \boxed{\underbrace{2\overline{\left(\nu \frac{\partial^2 u_i}{\partial x_k \partial x_l}\right)^2}}_{Y}}$$

$$\underbrace{\underbrace{+\frac{\partial}{\partial x_k}\left(\nu \frac{\partial \varepsilon}{\partial x_k}\right)}_{\mathcal{D}_\varepsilon^\nu} + \boxed{\underbrace{\frac{\partial}{\partial x_k}\left(-\overline{u_k \varepsilon}\right)}_{\mathcal{D}_\varepsilon^t}} + \boxed{\underbrace{\frac{\partial}{\partial x_k}\left(-\frac{2\nu}{\rho}\overline{\frac{\partial p}{\partial x_i}\frac{\partial u_k}{\partial x_i}}\right)}_{\mathcal{D}_\varepsilon^p}}}_{\mathcal{D}_\varepsilon}$$

Modelled dissipation-rate equation

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_k}\left[\left(\nu\delta_{kl} + C_\varepsilon \frac{k}{\varepsilon}\overline{u_k u_l}\right)\frac{\partial \varepsilon}{\partial x_l}\right] - C_{\varepsilon 1}\frac{k}{\varepsilon}\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon\tilde{\varepsilon}}{k}$$

$$+ \text{ "special" model fragments}$$

Generalised gradient-diffusion approximation

$$\frac{\varepsilon}{k}\{C_{\varepsilon 1}(\text{Production of } k) - C_{\varepsilon 2} f_\varepsilon (\text{Dissipation of } k)\}$$

- In energy equilibrium, $P_k = \varepsilon$, and the imbalance is absorbs by diffusion
- Transport equations for ε_{ij} are too complex as basis for modelling
- Anisotropy in dissipation – algebraic approximations of the form:

$$\varepsilon_{ij} = f_e \frac{2}{3}\varepsilon\delta_{ij} + (1 - f_e)\frac{\overline{u_i u_j}}{k}\varepsilon$$

- In most models, $f_e = 1$ reflecting assumption of small-scale isotropy

Closure – stress diffusion

- Regarded as least influential (suggested by DNS/LES).
- Represented as gradient-diffusion with tensorial diffusivity.
- Simplest model:

$$Diff_{ij} = -\frac{\partial}{\partial x_k}\left\{ c_d \frac{k}{\varepsilon} \overline{u_k u_m} \frac{\partial \overline{u_j u_j}}{\partial x_m} \right\}$$

- Based on observation that the most important fragment in the exact diffusion term is $\overline{u_k u_i u_j}$.
- It can be shown, via transport equations for triple correlation, $\overline{u_k u_i u_j}$, that the production of these triple correlations is by gradients of stresses of the form

$$P_{ijk} = -\overline{u_k u_m} \frac{\partial \overline{u_j u_j}}{\partial x_m} + \ldots..$$

- Suggests (also on dimensional grounds)

$$Diff_{ij} = -\frac{\partial}{\partial x_k}\left\{ c(\text{time scale}) \times (\text{production}_{\text{ijk}}) \right\}$$

Closure – pressure-strain / velocity

- Extremely important: responsible for redistribution among normal stresses. Regarded as the hardest term to model
- Pressure-velocity dictates energy transfer and hence $\overline{v^2}$
- But $\overline{v^2}$ dictates $\overline{uv}$

$$\frac{D\overline{uv}}{Dt} = -\overline{v^2}\frac{\partial \overline{U}}{\partial y} + \overline{\frac{p}{\rho}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)} - \frac{\partial}{\partial y}\left(\overline{uv^2}+\frac{\overline{pu}}{\rho}\right) + \frac{\mu}{\rho}\frac{\partial \overline{uv}}{\partial y} - \varepsilon_{12}$$

$$\frac{D\overline{u^2}}{Dt} = -2\overline{uv}\frac{\partial \overline{U}}{\partial y} + 2\overline{\frac{p}{\rho}\frac{\partial u}{\partial x}} - \frac{\partial}{\partial y}\left(\overline{u^2 v}\right) + 2\frac{\mu}{\rho}\frac{\partial \overline{u^2}}{\partial y} - \varepsilon_{11}$$

$$\frac{D\overline{v^2}}{Dt} = 0 + 2\overline{\frac{p}{\rho}\frac{\partial v}{\partial y}} - \frac{\partial}{\partial y}\left(\overline{v^3}+\frac{2\overline{pv}}{\rho}\right) + 2\frac{\mu}{\rho}\frac{\partial \overline{v^2}}{\partial y} - \varepsilon_{22}$$

$$\frac{D\overline{w^2}}{Dt} = 0 + 2\overline{\frac{p}{\rho}\frac{\partial w}{\partial z}} - \frac{\partial}{\partial x}\left(\overline{vw^2}\right) + 2\frac{\mu}{\rho}\frac{\partial \overline{w^2}}{\partial y} - \varepsilon_{33}$$

Closure – pressure-strain

- Subject to constrains:
  - Isotropisation: transfer of energy from largest stress to lower ones
  - Inhibition of isotropisation at walls/interfaces (splatting, reflection)
  - shear stresses have to decline as isotropisation progresses
- Guidance provided by 'exact' integration for pressure-fluctuations and substitution in pressure-velocity correlation

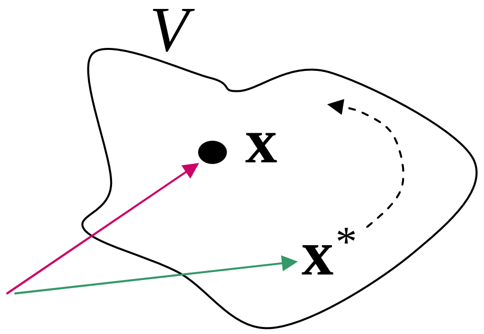
$$\Phi_{ij} = \overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

$$= \frac{1}{4\pi}\int_V \left\{ \overline{\left(\frac{\partial^2 u_l u_m}{\partial x_l \partial x_m}\right)^* \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \right\} \frac{dV(\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|}$$

$$+ \frac{1}{4\pi}\int_V \left\{ 2\overline{\left(\frac{\partial u_m}{\partial x_l}\right)^* \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \left(\frac{\partial U_l}{\partial x_m}\right)^* \right\} \frac{dV(\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|}$$

A_{ij} (first integrand term), B_{ijkl} (bracketed term in second integrand)

+ body-force and surface terms



Closure – pressure-strain

- Suggests the general Ansatz:

$$\Phi_{ij} = \varepsilon\, A_{ij}\,\{a_{ij}\} + k\, B_{ijkl}\,\{a_{ij}\}\frac{\partial U_k}{\partial x_l} \qquad \left\{ a_{ij} \equiv \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} \right\}$$

(+body-force and wall terms)

- Most complex model is cubic ⇒
- Much more popular is the quasi-linear form

$$\Phi_{ij} = -C_1 \frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij} k\right) - C_2\left(P_{ij} - \frac{2}{3}\delta_{ij} P_k\right)$$

(+body-force and wall terms)

- This is a sink term in the second-moment equations, depressing anisotropy in proportion to anisotropy of stresses and productions
- Ensures that anisotropy in stresses and productions drives energy from above-average normal stresses to below-average ones
- Coefficients sensitized to anisotropy invariants, turbulence Reynolds number…..in lieu of non-linear expansions

Closure – pressure-strain

$$\phi_{ij}^{*} = \phi_{ij1}^{*} + \phi_{ij2}^{*} + \phi_{ij1}^{\text{inh}} + \phi_{ij2}^{\text{inh}} \tag{6}$$

with

$$\phi_{ij1}^{*} = -c_1 \bar{\rho}\tilde{\epsilon}^{*}\left[a_{ij} + c_1'\left(a_{ik}a_{kj} - \frac{1}{3}A_2\delta_{ij}\right)\right] - \bar{\rho}\tilde{\epsilon}^{*} A^{\frac{1}{2}} a_{ij}$$

$$\phi_{ij2}^{*} = -0.6\left(P_{ij} - \frac{1}{3}\delta_{ij}P_{kk}\right) + 0.3 a_{ij}P_{kk}$$

$$- \frac{0.2\bar{\rho}}{\tilde{k}}\left[\widetilde{u_k''u_j''}\,\widetilde{u_l''u_i''}\left(\frac{\partial \tilde{u}_k}{\partial x_l} + \frac{\partial \tilde{u}_l}{\partial x_k}\right) - \widetilde{u_l''u_k''}\right.$$

$$\left.\times\left(\widetilde{u_i''u_k''}\frac{\partial \tilde{u}_j}{\partial x_l} + \widetilde{u_j''u_k''}\frac{\partial \tilde{u}_i}{\partial x_l}\right)\right] - c_2[A_2(P_{ij} - D_{ij}) + 3a_{mi}a_{nj}$$

$$\times (P_{mn} - D_{mn})] + c_2'\left\{\left(\frac{7}{15} - \frac{A_2}{4}\right)\left(P_{ij} - \frac{1}{3}\delta_{ij}P_{kk}\right)\right\}$$

$$+ 0.1\left[a_{ij} - \frac{1}{2}\left(a_{ik}a_{kj} - \frac{1}{3}\delta_{ij}A_2\right)\right]P_{kk} - 0.05 a_{ij}a_{lk}P_{kl}$$

$$+ \frac{0.1}{\tilde{k}}\left[\left(\widetilde{u_i''u_m''}P_{mj} + \widetilde{u_j''u_m''}P_{mi}\right) - \frac{2}{3}\delta_{ij}\,\widetilde{u_l''u_m''}P_{ml}\right]$$

$$+ \frac{0.1}{\tilde{k}^2}\left[\widetilde{u_i''u_l''}\,\widetilde{u_k''u_j''} - \frac{1}{3}\delta_{ij}\,\widetilde{u_l''u_m''}\,\widetilde{u_k''u_m''}\right]$$

$$\times\left[6D_{lk} + 13\bar{\rho}\tilde{k}\left(\frac{\partial \tilde{u}_l}{\partial x_k} + \frac{\partial \tilde{u}_k}{\partial x_l}\right)\right] + \frac{0.2}{\tilde{k}^2}\,\widetilde{u_l''u_i''}\,\widetilde{u_k''u_j''}(D_{lk} - P_{lk})\}$$

$$\phi_{ij1}^{\text{inh}} = f_{w1}\frac{\bar{\rho}\tilde{\epsilon}}{\tilde{k}}\left(\widetilde{u_l''u_k''}d_l^A d_k^A \delta_{ij} - \frac{3}{2}\widetilde{u_i''u_k''}d_j^A d_k^A - \frac{3}{2}\widetilde{u_j''u_k''}d_i^A d_k^A\right)$$

$$+ f_{w2}\frac{\bar{\rho}\tilde{\epsilon}}{\tilde{k}^2}\left(\widetilde{u_m''u_n''}\,\widetilde{u_m''u_l''}d_n^A d_l^A \delta_{ij} - \frac{3}{2}\widetilde{u_i''u_m''}\,\widetilde{u_m''u_l''}d_j^A d_l^A\right.$$

$$\left.- \frac{3}{2}\widetilde{u_j''u_m''}\,\widetilde{u_m''u_l''}d_i^A d_l^A\right)$$

$$\phi_{ij2}^{\text{inh}} = f_I\bar{\rho}\tilde{k}\frac{\partial u_l}{\partial x_n}d_l d_n\left(d_i d_j - \frac{1}{3}d_k d_k \delta_{ij}\right)$$

with

$$P_{ij} = -\left(\bar{\rho}\,\widetilde{u_i''u_k''}\frac{\partial \tilde{u}_j}{\partial x_k} + \bar{\rho}\,\widetilde{u_j''u_k''}\frac{\partial \tilde{u}_i}{\partial x_k}\right)$$

$$D_{ij} = -\left(\bar{\rho}\,\widetilde{u_i''u_k''}\frac{\partial \tilde{u}_k}{\partial x_j} + \bar{\rho}\,\widetilde{u_j''u_k''}\frac{\partial \tilde{u}_k}{\partial x_i}\right)$$

$$c_1 = 3.2 f_A A_2^{\frac{1}{2}}\min\left[(Re_t/160)^2, 1\right], \qquad c_1' = 1.1$$

$$c_2 = \min\{0.55[1 - \exp(-A^{\frac{3}{2}}Re_t/100)], 3.2A/(1+S)\}$$

$$c_2' = \min[0.6, A^{\frac{1}{2}}] + 3.5(S - \Omega)/(3 + S + \Omega) - 4\min[S_I, 0]$$

$$S = \frac{\tilde{k}}{\tilde{\epsilon}}\sqrt{\frac{1}{2}S_{ij}S_{ij}}, \qquad \Omega = \frac{\tilde{k}}{\tilde{\epsilon}}\sqrt{\frac{1}{2}\Omega_{ij}\Omega_{ij}}$$

$$S_I = \sqrt{6}S_{ij}S_{jk}S_{ki}/(S_{ln}S_{ln})^{\frac{3}{2}}$$

$$S_{ij} = \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}, \qquad \Omega_{ij} = \frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i}$$

$$f_{w1} = 2.35(1 - A^{\frac{1}{2}})\min[1, \max(0, 1 - (Re_t - 55)/70)]$$

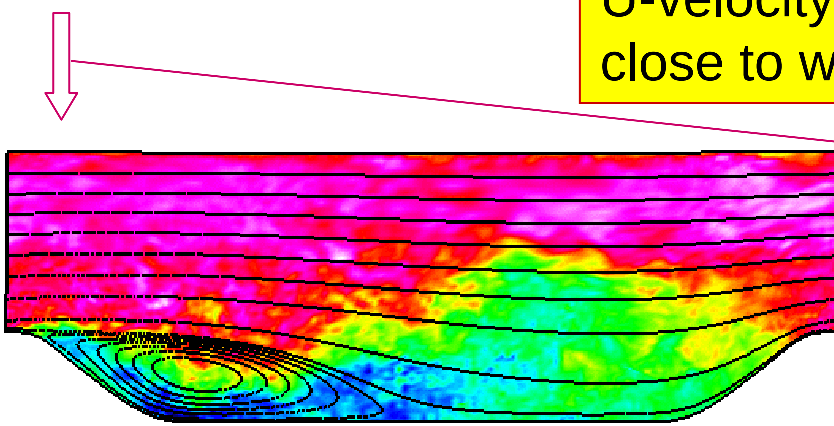
$$f_{w2} = 0.6A_2(1 - A^{\frac{1}{2}})\min[1, \max(0, 1 - (Re_t - 50)/85)] + 0.1$$

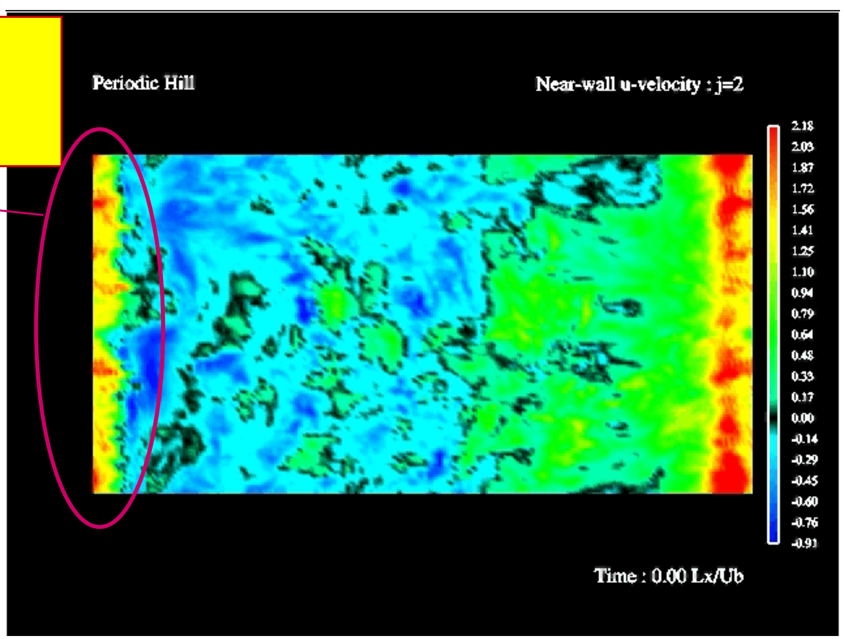
$$f_I = 3f_A, \qquad f_A = \begin{cases} (A/14)^{\frac{1}{2}} & A < 0.05 \\ A/0.7^{\frac{1}{2}} & 0.05 < A < 0.7 \\ A^{\frac{1}{2}} & A > 0.7 \end{cases}$$

Model Performance

- Construction and calibration rely heavily on highly-resolved experimental & simulation data
- Done mostly by reference to thin-shear-flow data
- Models work well for many flows
- Notable exception: flow separating from curved surfaces (2d & 3d)
- Associated with dynamics of highly unsteady separation (& pre-separation)

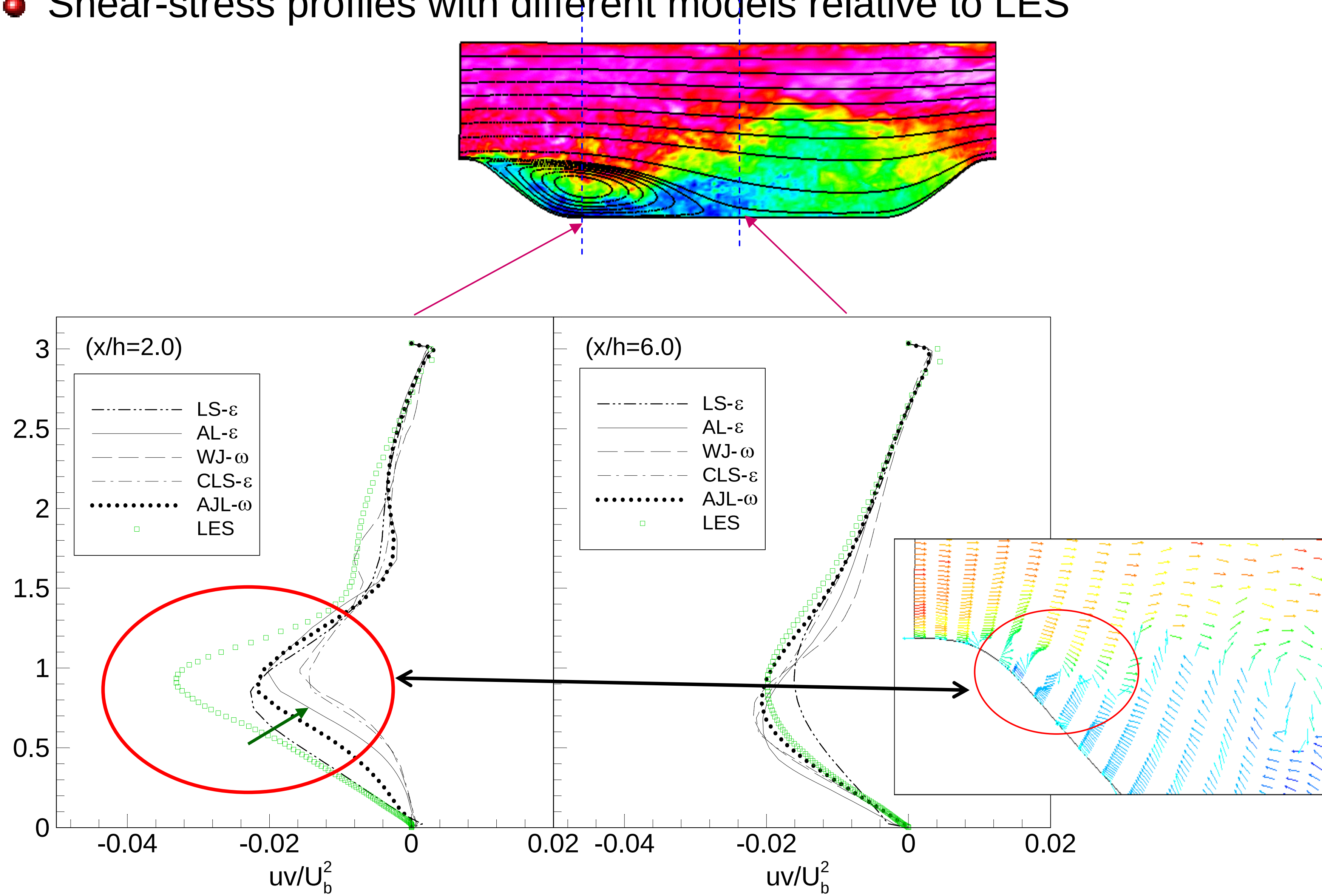
U-velocity close to wall





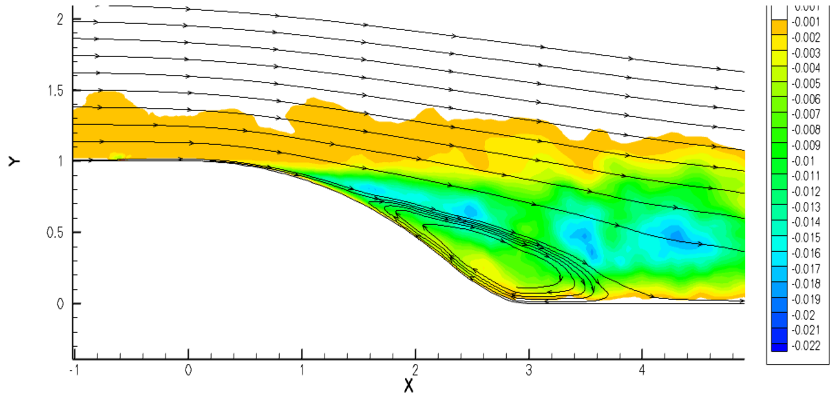
Separation from curved surface

- Shear-stress profiles with different models relative to LES

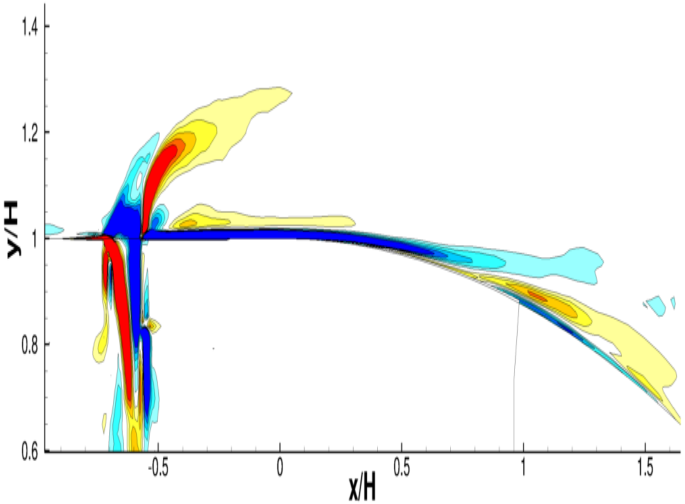


Model developments

- Model defects are difficult to cure, but efforts are ongoing
- Example: re-examination of dissipation and pressure-velocity interaction terms in separation from curved ramp
- Foundation: highly-resolved simulation – near DNS, 25M nodes

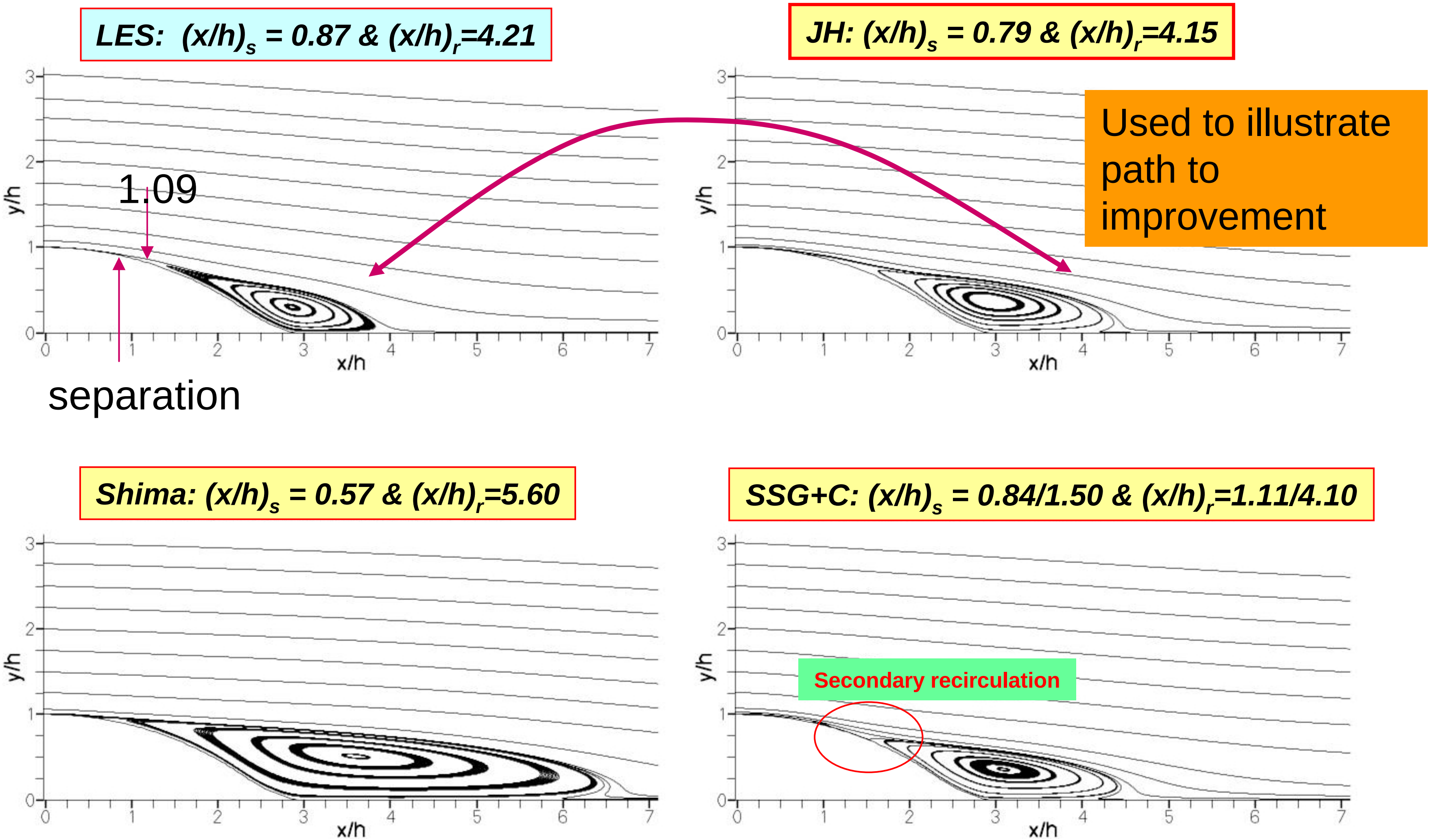


- Re_H=13700; Re_Θ = 1150
- Second moments, invariants, budgets of all second moments….
- Part of larger study on separation control with synthetic jets
- Experimental data



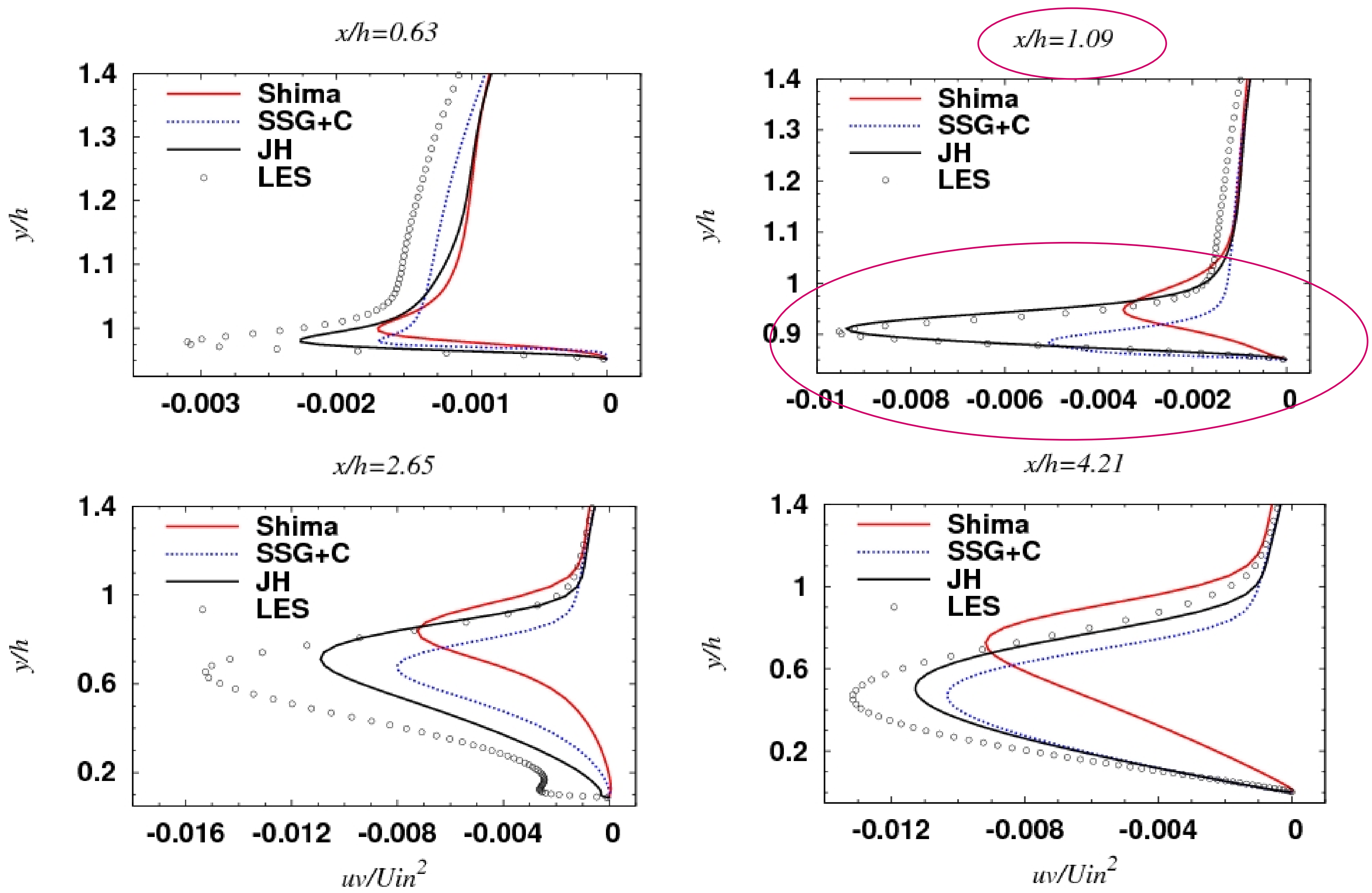
Starting point

- Choice of basic model, based on full computation



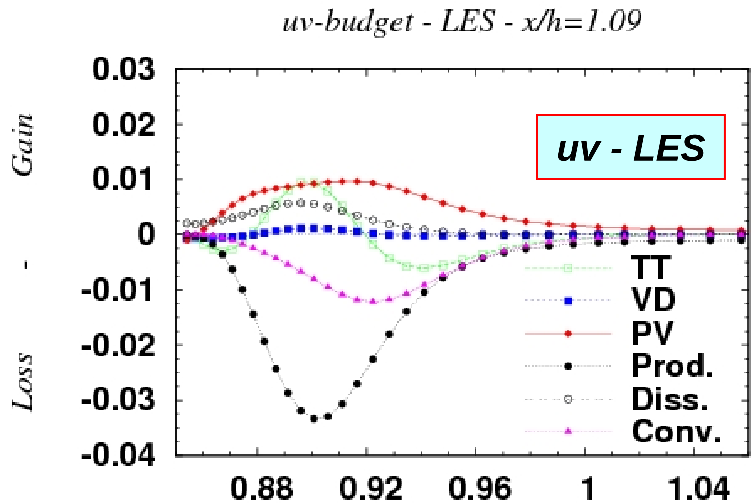
Defect identification

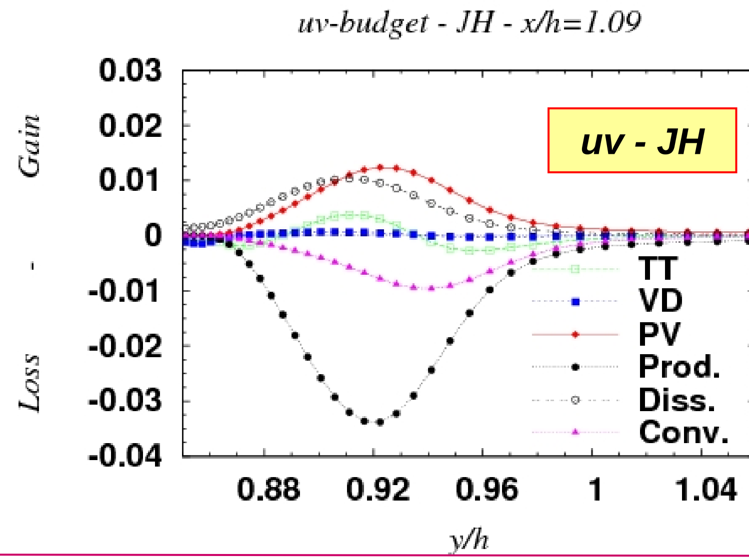
- Focus on shear stress in separated shear layer

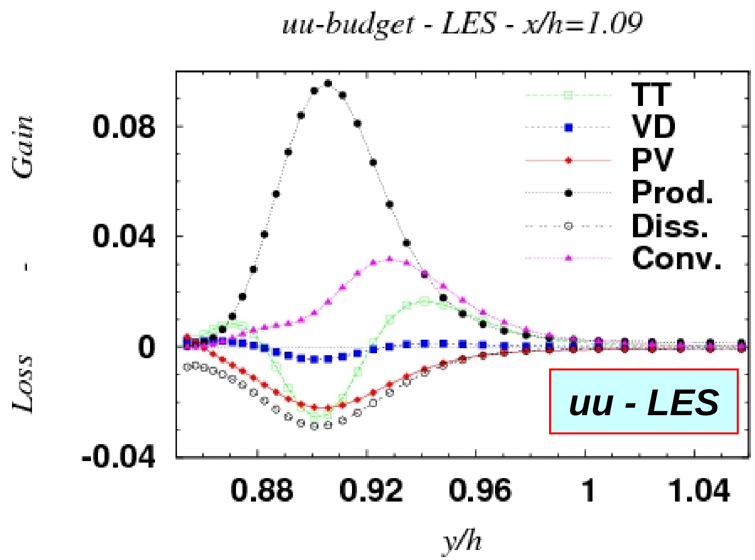


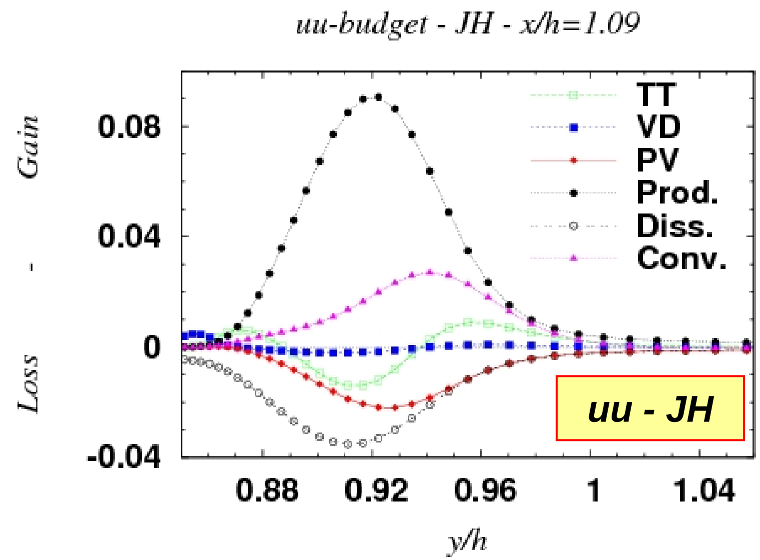
Defect identification

- Budgets for $\overline{uv}$ and $\overline{uu}$



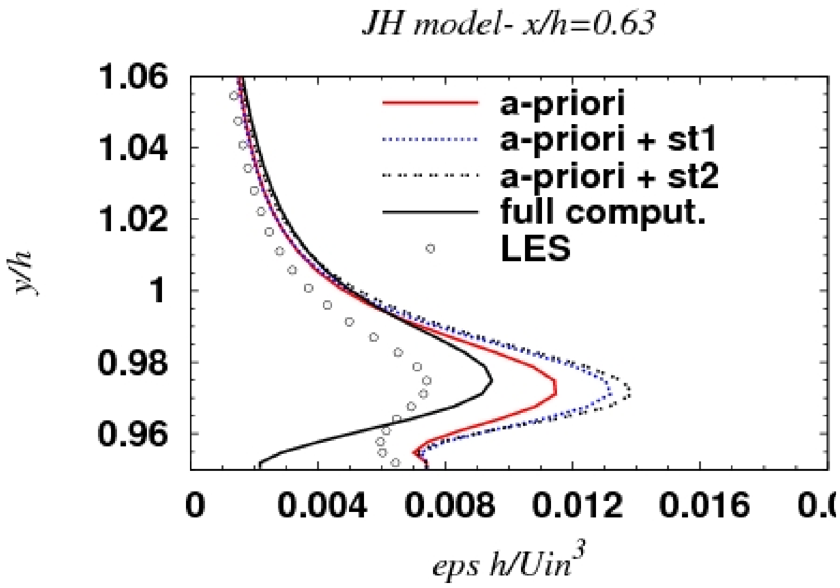


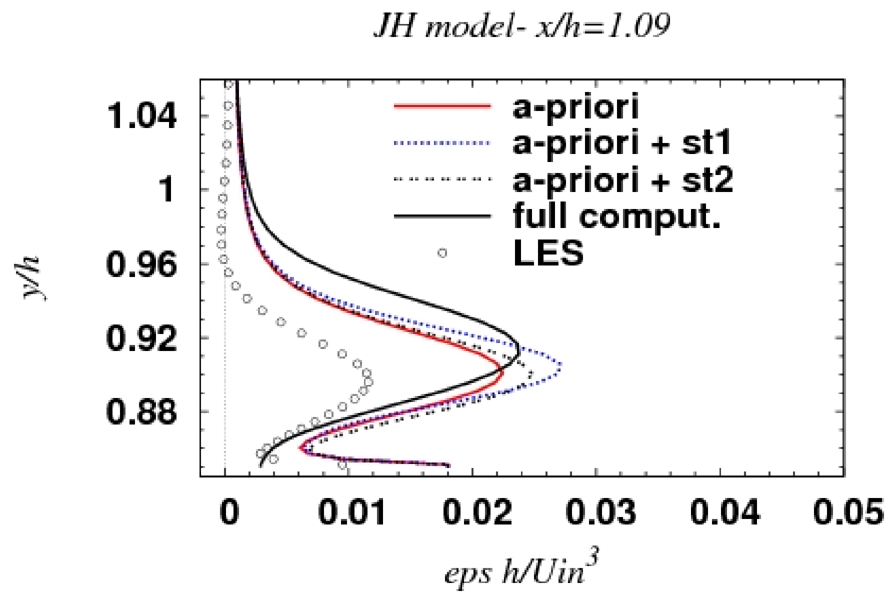




Model fragmentation - dissipation

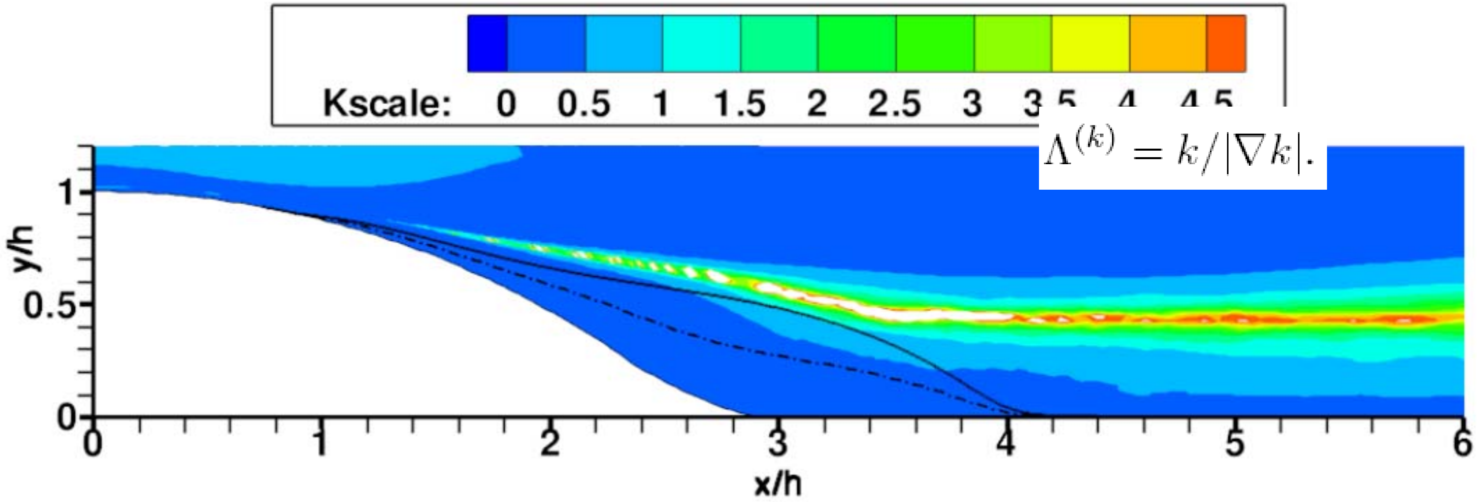
- A-priori study of dissipation-rate equation
- Isolated solution of equation
- LES strains and stresses input into equation
- Only output is dissipation
- Examination of a range of corrections in efforts to procure agreement with LES data for dissipation rate

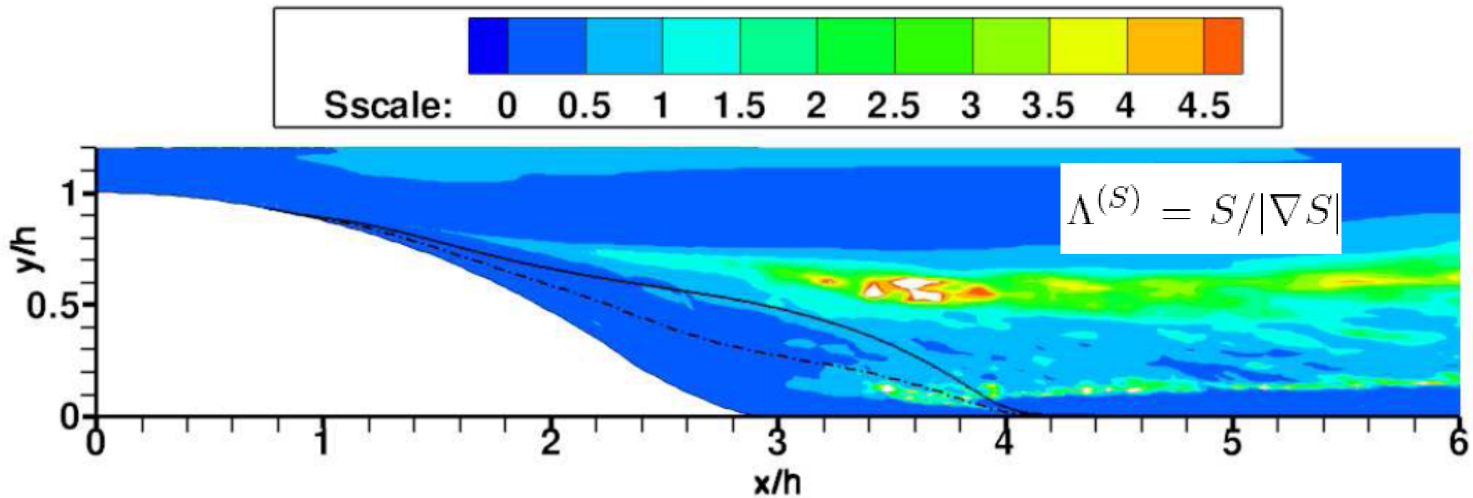




Model fragmentation - dissipation

- Ongoing efforts to sensitize dissipation to mean-flow/turbulence length scales





Model fragmentation – dissipation components

- A-priori study of dissipation anisotropy – stresses and ε from LES into

$$\varepsilon_{ij} = f_s \varepsilon_{ij}^* + (1 - f_s)\frac{2}{3}\delta_{ij}\varepsilon$$

$$\varepsilon_{ij}^* = \frac{\varepsilon}{k}\frac{\overline{u_i u_j} + (\overline{u_i u_j} n_j n_k + \overline{u_j u_k} n_i n_k + \overline{u_k u_l} n_j n_k n_i n_j) f_d}{1 + \frac{3}{2}\frac{\overline{u_p u_q}}{k} n_p n_q f_d}$$

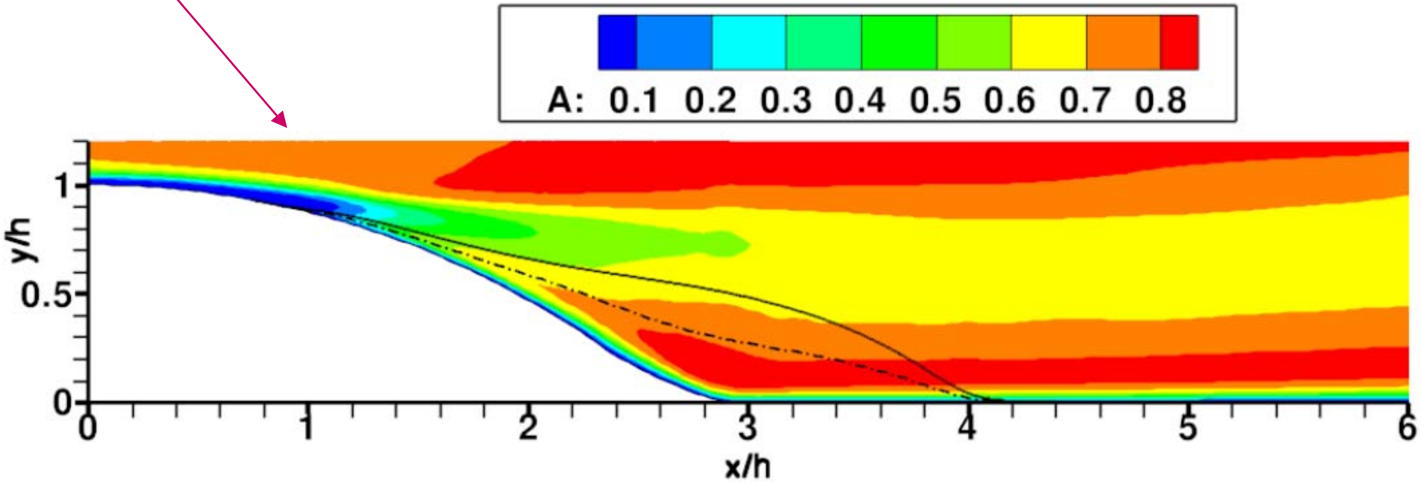
Various proposals

$$f_s = 1 - \sqrt{A}E$$

$$f_d = (1 + 0.1\mathrm{Re}_t)^{-1}$$

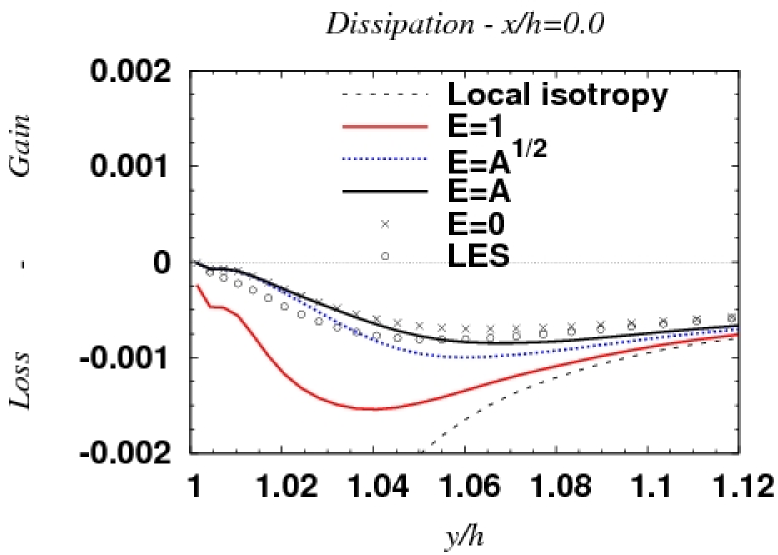
- Weighting function sensitized to anisotropy invariant

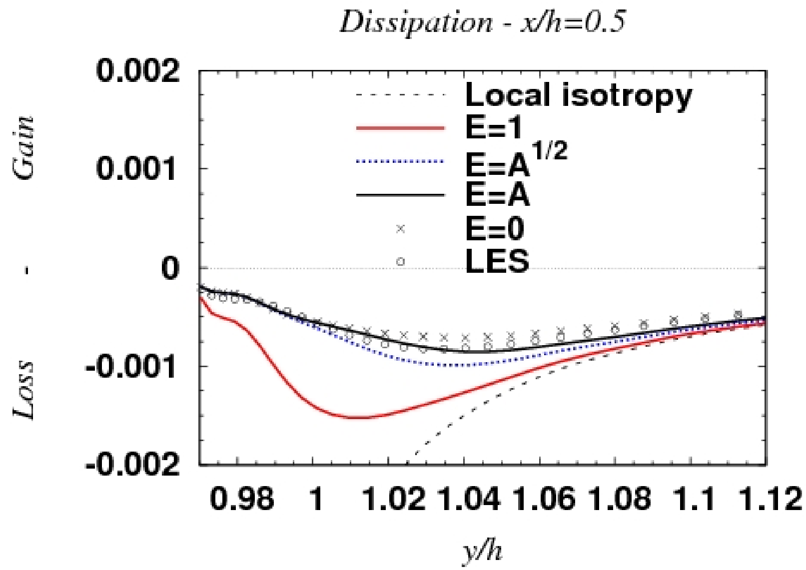
$$A = 1 - \tfrac{9}{8}(A_2 - A_3) \qquad A_2 = a_{ij}a_{ij}; \quad A_3 = a_{ij}a_{jk}a_{ki}; \quad a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}$$

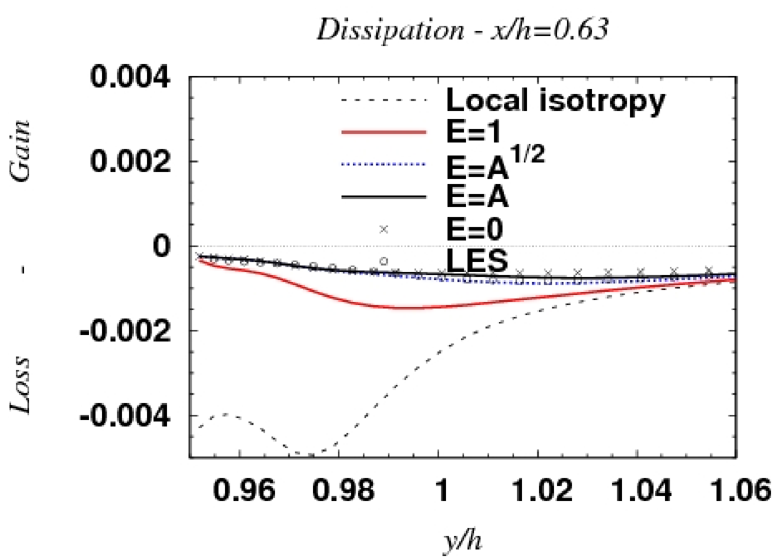


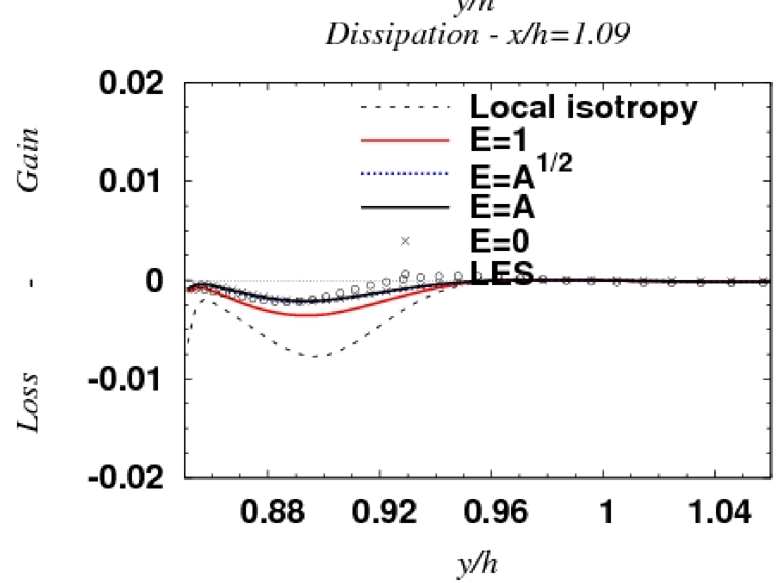
Model fragmentation – dissipation components

- Component ε_{22}









Model fragmentation – pressure-velocity

- Quasi-linear approximation

$$\Phi_{ij} = -C_1 \frac{\varepsilon}{k}\left(u_i u_j - \frac{2}{3}\delta_{ij} k\right) - C_2\left(P_{ij} - \frac{2}{3}\delta_{ij} P_k\right) \quad \text{(+wall-reflection terms)}$$

- Coefficients sensitized to anisotropy invariants, in compensation to the omission of high-order fragments

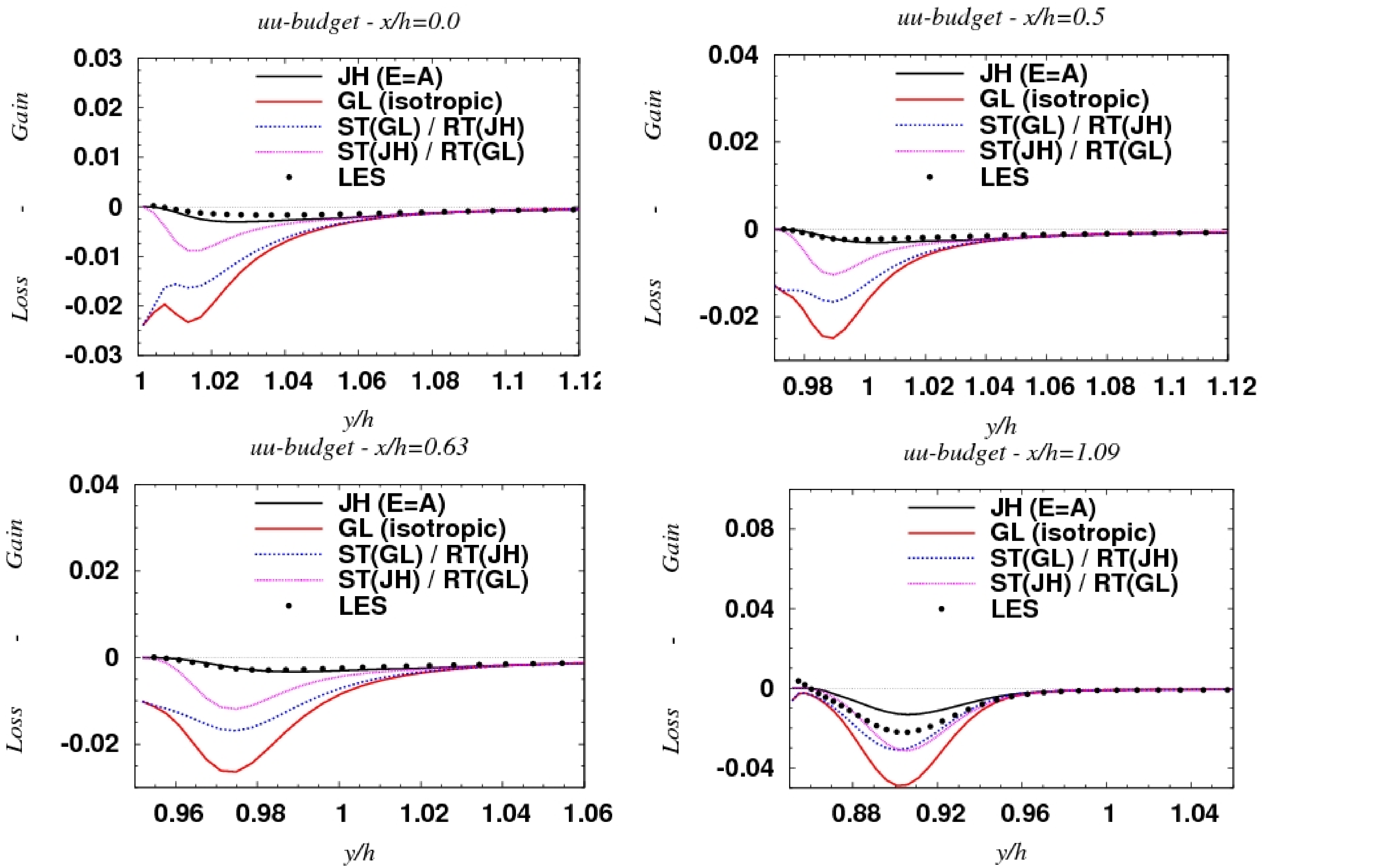
$$C_1 = C + \sqrt{AE} \qquad C = 2.5A[\min\{0.6, A_2\}]^{1/4} f$$

$$f = \min\left\{\left(\frac{\mathrm{Re}_t}{150}\right)^{3/2}, 1\right\}$$

$$C_2 = 0.8A^{1/2}$$

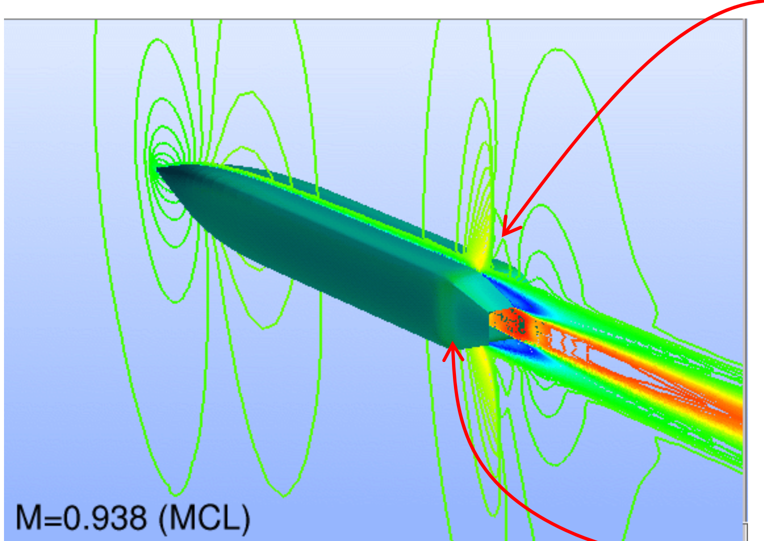
Model fragmentation – pressure-velocity

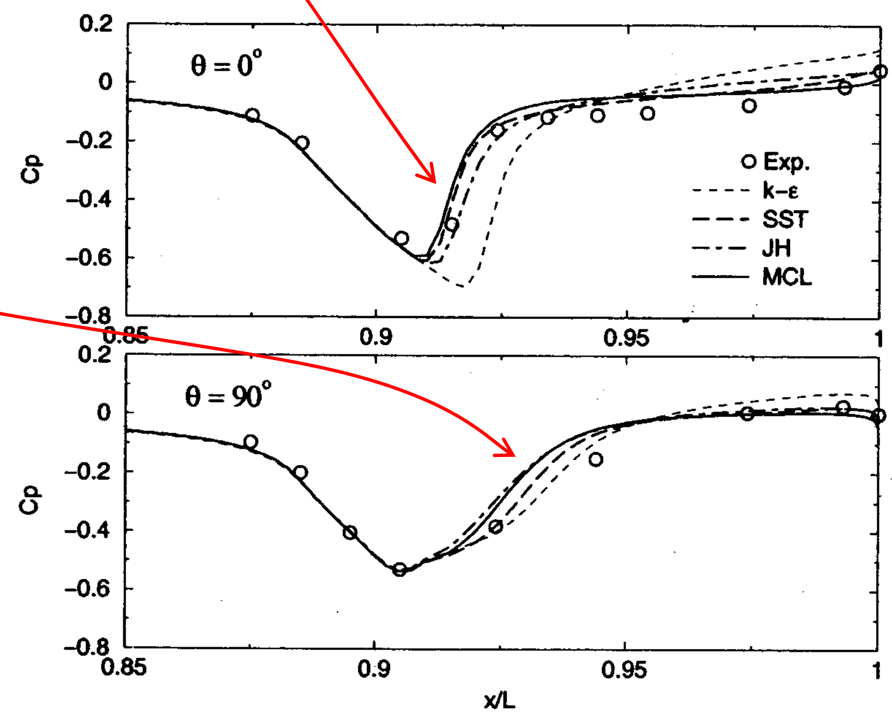
- Sensitivity of coefficients to pressure-velocity interaction of $\overline{uu}$



Shock-induced Separation on 3D Jet-Afterbody - RSTM

- General view and surface-pressure coefficient





Re_L =2x10^7

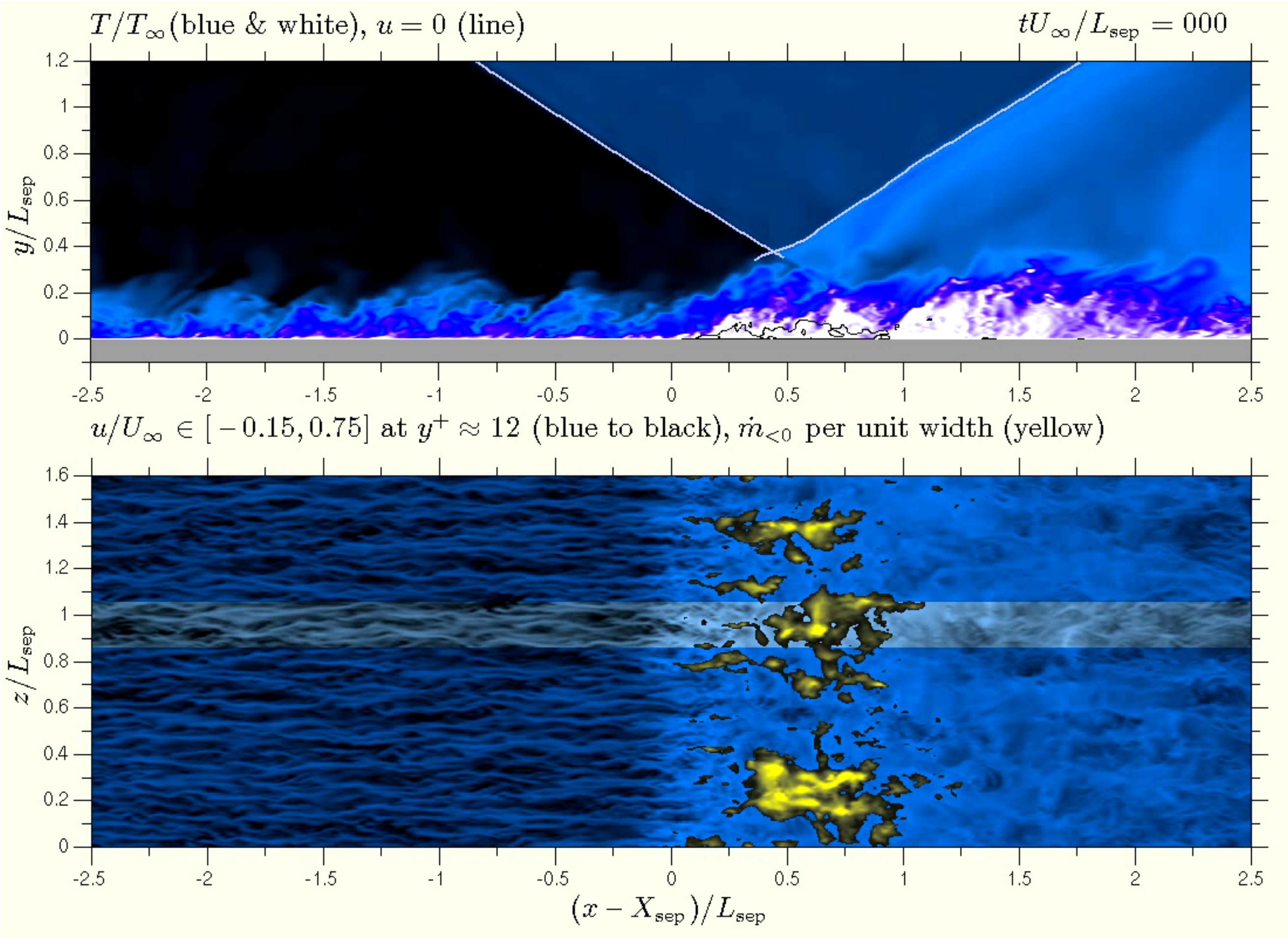
0.5 M nodes

CFL=O(1000)

Desktop workstation, a few CPU hours

Shock-induced Separation on flat plate – LES

- Touber and Sandham, 2010
- Re_τ =3000, M=2.3
- 20 M nodes, 240,000 CPU hours



Concluding remarks

- Fundamentally, Second-moment closure is far superior to eddy-viscosity modelling.
- In reality, closure is extremely challenging, because the anisotropy is an extremely influential model element and is difficult to approximate.
- Redistribution and dissipation are especially influential.
- Many ways of construction models, but all involve calibration.
- Does involve "curve-fitting", but is based on rational principles and physically tenable assumptions.
- Little used, because of "the-simpler-the-better" attitude.
- Second-moment closure is inappropriately complex in (most) thin shear flows, but the only fundamentally solid approach in complex strain.