Kinetic models for supply chains

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Workshop on Kinetic Models

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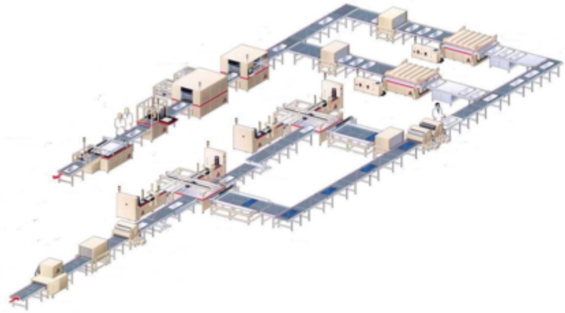
Motivation

Simulation of production processes in production lines and networks



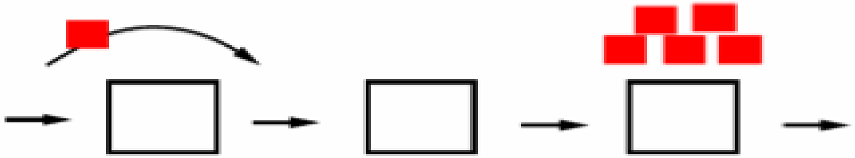
Main assumption: Reasonable to assume a continuous flow of products

Examples for mass volume production facilities are semi-conductor industries and Coca-Cola bottles

References will be given later

Basic modeling assumptions

- Single product flow (extension see below)
- Machines have a processing velocity $v > 0$ and a processing capacity $\mu > 0$
- Dynamics: Parts arrive and are processed according to the capacity
- Typically modeled by discrete event simulations (rigorous derivation of continuous model from DES possible)



Simplest conservation law

- Single product flow
- Production line described by $x \in \mathcal{R}$
- Product density in machine x described by $\rho(x,t)$
- Evolution: Free transport with velocity $v(x)$ up to the maximal capacity $\mu(x)$

  ... implies

$$\partial_t \rho + \partial_x \min\{\mu(x), v(x)\rho\} = 0$$

- Problem: Equation gives rise to δ-distributions as soon as $v\rho$ exceeds μ
- Problem: Extension to network structures?

Extension to networks

- Assumption of finite size machines implies $\mu(x) = \sum i = 1^N \mu_i \chi_{I_i}$ with capacities $\mu_i \in \mathcal{R}$
- Think of the production line as a network of machines with *constant* capacities. This implies a simple transport equation for each machine and suitable coupling conditions taking care of the possibly different capacities
- ρ_i is the product density in machine i:

$$\partial_t \rho_i + \partial_x v \rho_i = 0$$

- Coupling?

Coupling through buffering queues

- There is a buffering queue between two machines - the time evolution of the number of parts in the queue in front machine i is recorded by $q_i(t)$
- Dynamics of the system + boundary conditions at $x = 0$ and $f_i = \min\{\mu_i, v\rho_i\}$

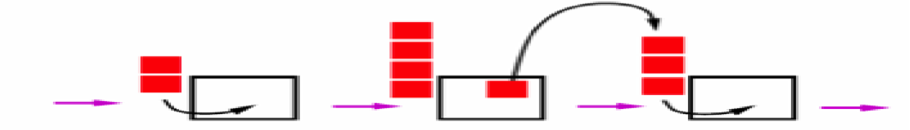
$$\partial_t q_i = f_{i-1} - f_i, \qquad \partial_t \rho_i + \partial_x v\rho_i = 0$$

- Consider a discretization of $\partial_t \rho + \partial_x \min\{\mu, v\rho\} = 0$ with μ discontinuous at $x = 0$
  Let $\rho(x, 0) = \rho_0(t)$ be the discretized density and write a Godunov discretization of the pde

$$\partial_t(\Delta x \rho_0) = f^{-\frac{1}{2}} - f^{\frac{1}{2}}$$

  Problem: Only if $v\rho_{-1} > \mu(0+)$ we have the possibility of a δ-distribution
  Hence, at $x = 0$ we define $q = \Delta x \rho_0$ and since μ is constant for $x <> 0$ we can use the Upwind discretization

Analytical results for the coupled network model



- Equations:

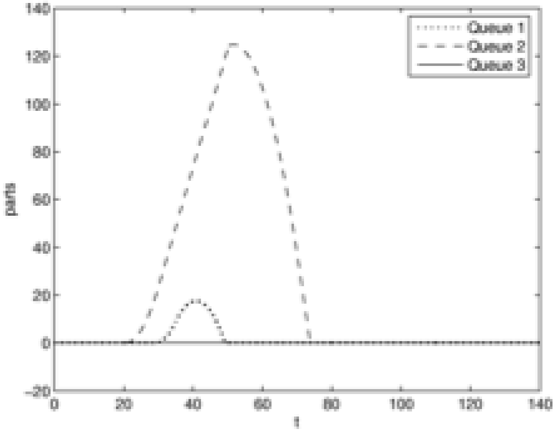
$$\partial_t q_i = f_{i-1} - f_i, \qquad \partial_t \rho_i + \partial_x v\rho_i = 0$$

- Boundary conditions:

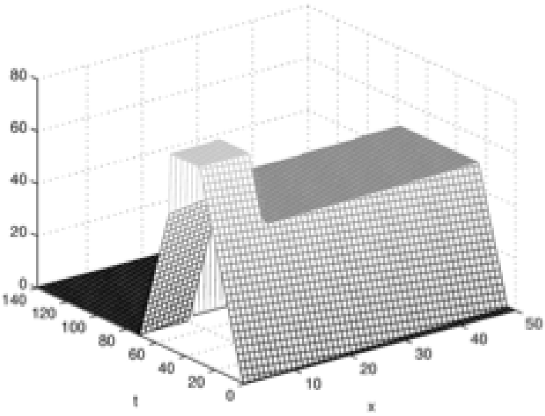
$$f_i = \min\{\mu_i, v\rho_i\}, \qquad v\rho_i(0,t) = f_i$$

- *Existence results*
  For initial data with suitable small BV–norm ρ_i^0 and networks without closed loops there exists a weak solution to the coupled system of transport equations and ordinary differential equations such that $\rho_i \in C^{0,1}(0,T,L^1(0,1))$ and $q_i \in W^{1,1}(0,T)$.

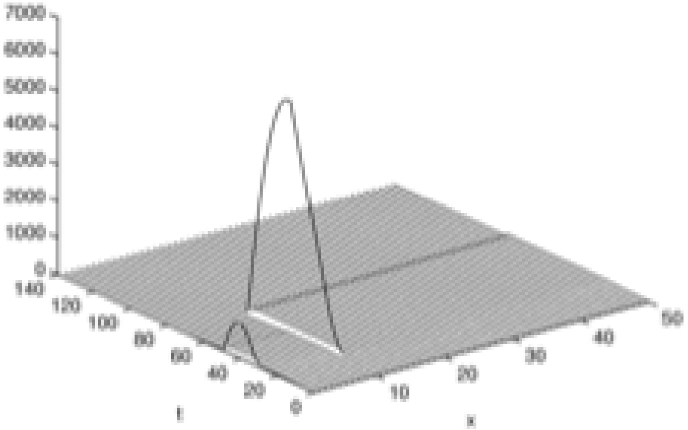
Production line with three different capacities $\mu_0 > \mu_1 > \mu_2$. Evolution of the buffers.



Production line with three different capacities $\mu_0 > \mu_1 > \mu_2$. Evolution of the production densities.

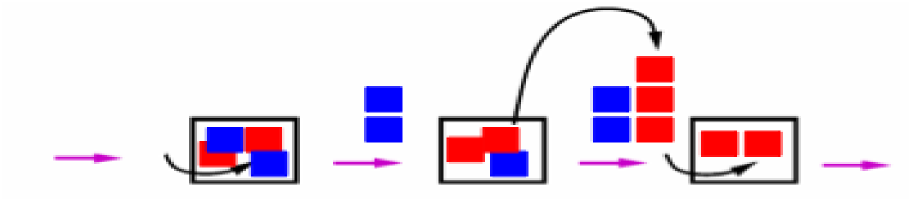


Simulation of the original model with discontinuous $\mu(x)$ in the same setting. Oberseve the numerical δ peaks at the transition from μ_0 to μ_1 and to μ_2.



Multiple policies through kinetic models

- Consider a product flow with two different products (blue, red)
- Each machine can process both products
- Each machine assigns production capacity (up to the maximal capacity) according to the priority of the products
- Example. Red more important than blue and capacities are decreasing along the line $\mu_i = 5 - i$



Kinetic model for flow of products with priorities (due to A-D-R)

- $f(t,x,y)$ products at time t and position x and priority y (lower value of y corresponds to a higher priority)
- Model: *Products with higher priority move faster through the supply chain*
- Products with priority less or equal than y are moved with maximal velocity. The number of products with priority less than y is $\int_{-\infty}^{y} f(t,x,y')dy'$ and the flux is $\beta = \int_{-\infty}^{y} f(t,x,y')v(x,y')dy'$
- We have a maximal production capacity of μ. If the flux is larger than μ, then the actual processing velocity is zero, below it is $v(x,y')f(t,x,y')$. Hence the actual velocity is

$$v(x,y)H\left(\mu(x) - \int_{-\infty}^{y} f(t,x,y')v(x,y')dy'\right)$$

Kinetic model for a production line with priorities

- Kinetic model as introduced by Armbruster-Degond-Ringhofer

$$\partial_t f + \partial_x \left(H(\mu(x) - \beta(x,y,t)\right) v(x,y) f(x,y,t) = 0$$

- Moment equations of the type $\partial_t m_j + \partial_x F_j = 0$ are obtained for $m_j = \int y^j f dy$ and the system is closed (formally by)

$$f^e = \sum_{k=1}^{K} \rho_k \delta(y - Y_k)$$

  for a finite set of priorities Y_k with densities ρ_k
- Moment equations allow for $\delta-$distributions as solutions!
- Extension to networks possible?

Example for macroscopic equations in case of two priorities

$$\partial_t \rho_k + \partial_x q_k = 0, \qquad \partial_t \rho_k Y_k + \partial_x q_k Y_k = 0$$

The flux q_k is defined as follows

- If $\mu < \rho_1 v_1$, then $q_1 = mu$ and $q_2 = 0$
- If $\rho_1 v_1 < \mu < \rho_1 v_1 + \rho_2 v_2$, then $q_1 = v\rho_1$ and $q_2 = \mu - q_1$
- If $\rho_1 v_1 + \rho_2 v_2 \leq \mu$, then $q_k = \rho_k v$

In the case of a single product we recover the previous dynamics

Network formulation

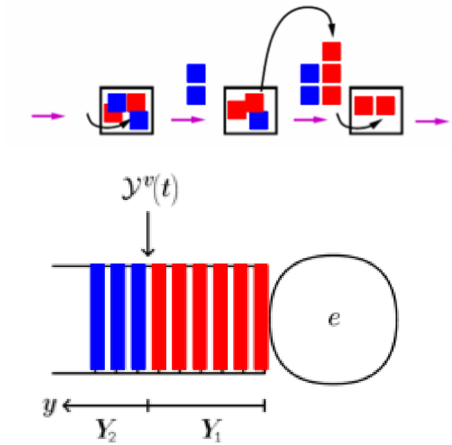
- Introduce finite size machines to obtain a network formulation by setting
$$\mu(x) = \sum \mu_i \chi_{I_i}$$
- Leads to kinetic model for the density of parts f_i on arc i
$$\partial_t f_i + \partial_x \left(H(\mu_i - \beta) \, v_i f_i = 0 \right.$$
- Suitable coupling conditions? Introduce buffering queue depending on the priority and buffering exceeding demands and supplies:
$$\partial_t Q_i(y,t) = \Phi_{i-1} - \Phi_i$$
where Φ is the flux $\Phi = H\left(\mu_i - \beta(y,t)\right) v_i f_i$ and $\beta = \int_{-\infty}^{y} v' f dv'$

Suitable inflow boundary conditions

$$\partial_t Q_i(y,t) = \Phi_{i-1} - \Phi_i, \qquad \Phi = H\left(\mu_i - \beta(y,t)\right) v_i f_i$$

- Condition is not sufficient to determine inflow boundary condition for f^i. it is necessary to prescribe treatment of products at the vertex.
- As in the dynamics of the PDE the products have different priority and are passed through the vertex according to their priority
- Since the connected processor might have less capacity than the connected processor we need to introduce a *pointer* variable Y to indicate the priority still being processed

Example with previous model and network model using a pointer variable



Suitable inflow boundary conditions - part II

$$\partial_t Q_i(y,t) = \Phi_{i-1} - \Phi_i, \qquad \Phi = H\left(\mu_i - \beta(y,t)\right) v_i f_i$$

- Equation for the pointer: Assume at time t_n the pointer is such that all incoming parts are being processed. At time t_{n+1} two cases have to distinguished.
  - The inflow $\Phi^{e-1}(y, t_{n+1})$ parts with priority $y < Y$ exceeed the maximal capacity
    $\Longrightarrow$ need to decrease the pointer – Y determined by
    $$\int_{-\infty}^{Y(t_{n+1})} \Phi^{e-1}(y, t_n) dy = \mu^e$$
  - The inflow $\Phi^{e-1}(y, t_{n+1})$ parts with priority $y < Y$ does *not* exceeed the maximal capacity
    $\Longrightarrow$ need to increase the pointer such that more parts with lower priority are processed. The remaining capacity of
    $\mu^e - \int_{-\infty}^{Y(t_n)} \Phi^{e-1}(y, t_n) dy = \mu^e$ (maximal capacity - processed parts) will be used such that lower priority parts are processed:

Suitable inflow boundary conditions - part III

$\partial_t Q_i(y,t) = \Phi_{i-1} - \Phi_i, \qquad \Phi = H\left(\mu_i - \beta(y,t)\right) v_i f_i$

- Pointer dynamics for $Y(t_n)$
  higher priority parts arrive: $\int_{-\infty}^{Y(t_{n+1})} \Phi^{e-1}(y,t_n)dy = \mu^e$
  lower priority parts arrive: $\int_{Y(t_n)}^{Y(t_{n+1})} (\Delta t \Phi^{e-1}(y,t_{n+1}) + Q(t_{n,y})dy =$
  $(\mu^e - \int_{-\infty}^{Y(t_n)} \Phi^{e-1}(y,t_n)dy)\Delta t$
- Dynamics in the limit $\Delta t \to 0$:

$$\text{if } \partial_t Y < 0 : \qquad \int_{-\infty}^{Y(t)} \Phi^{e-1}(y,t)dy = \mu^e$$

$$\text{if } \partial_t Y > 0 : \qquad Q(t,Y)\partial_t Y = (\mu^e - \int_{-\infty}^{Y(t)} \Phi^{e-1}(y,t)dy)\Delta t$$

- In both cases the total outflow of the buffer is
  $\Phi^e(y,t) = \Phi^{e-1}(y,t)H(Y-y) + \left(\mu - \int_{-\infty}^{Y} \Phi^{e-1}(y,t)dy\right)\delta(Y-y)$

Remarks and further steps

- Dynamics of a kinetic model for products with priority consists of a transport equation for the parts with transport according to the priority combined with buffering queues and a suitable pointer dynamics

$$\partial_t f^e + \partial_x \left(H(\mu - \beta) v f^e \right) = 0$$

- Pointer dynamics can be understood as (proof available)

$$Y = \min\{\min\{Y : Q(Y,t) \neq 0\}, Y : \int_{-\infty}^{Y} \Phi^{e-1}(y,t)dy = \mu^e\}$$

- Obviously: Kinetic dynamic to complex for reasonable studies $\rightarrow$ moment equations for the coupled model

$$m_j^e = \int y^j f^e dy, \qquad F_j^e y^j (H(\mu^e - \beta^e) v f^e dy$$

- Apply equilibrium closure $f^e = \sum_{k=1}^{K} \rho^e \delta(y - Y_k)$

Moment equations

Macroscopic equations:

$$\partial_t \rho_k^e + \partial_x q_k^e = 0, \qquad \partial_t q_k^e + \partial_x q_k^e Y_k^e = 0$$

- Equilibrium closure relations imply that the pointer only attains values in the finite set
$$Y \in \{Y_1, \ldots, Y_K, +\infty\}$$
- Integration of the kinetic coupling condition gives macroscopic coupling conditions. We give some examples.
- $Y = +\infty : Y_k(x^e, t) = Y_k(x^{e-1}, t)$ and $q_k^e = q_k^{e-1}$ (all products pass)
- $Y = Y_\kappa : Y_k(x^e, t) = Y_k(x^{e-1}, t)$ and $q_k^e = q_k^{e-1}$ for $k \leq \kappa - 1$ and $q_\kappa = \mu^e - \sum_{k=1}^{\kappa-1} q_k^{e-1}$ (only products with priority less than κ pass)

Moment equations – Queue

Macroscopic equations:

$$\partial_t \rho_k^e + \partial_x q_k^e = 0, \qquad \partial_t q_k^e + \partial_x q_k^e Y_k^e = 0$$

- Integration of the equation for the queue and closure yields a moment equations for queues in terms of the moments Y_k

$$\partial_t \pi_k^e = q_k^{e-1} - q_k^e$$

- Summary:

$$\begin{aligned}
\partial_t \rho_k^e + \partial_x q_k^e = 0, \qquad \partial_t q_k^e + \partial_x q_k^e Y_k^e = 0 \\
\partial_t \pi_k^e = q_k^{e-1} - q_k^e \\
Y = Y_\kappa : \ Y_k(x^e, t) = Y_k(x^{e-1}, t) \\
q_k^e = q_k^{e-1}, \ k \le \kappa - 1, \qquad q_\kappa = \mu^e - \sum_{k=1}^{\kappa-1} q_k^{e-1} \\
\kappa = \min\{\min\{k : \pi_k \neq 0\}, \qquad k : \sum^{k-1} q_k^{e-1} \le \mu^e \le \sum^{k} q_k^{e-1}\}
\end{aligned}$$

Final remarks

- In the case $K = 1$ we obtain the simple model for single product flow as before
- In the case $K = 2$ we obtain the following dynamics for the pointer and the queues

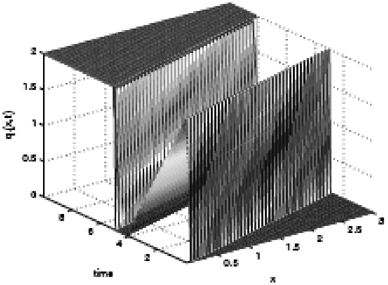
$$q_1^e = q_1^{e-1} + \delta(Y_1 - Y)(\mu^e - q_1^{e-1}),$$

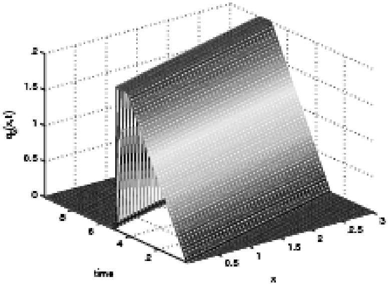
$$q_2^e = q_2^{e-1} - \delta(Y_1 - Y)(q_2^{e-1}) + \delta(Y_2 - Y)(\mu^e - q_1^{e-1} - q_2^{e-1})$$

$$Y :_k^e = Y_k^{e-1}$$

$$Y = \begin{array}{l} Y_1 \text{ if } \pi_1^e \neq 0, q_1^{e-1} > \mu^e \\ Y_2 \text{ if } \pi_1^e = 0, \pi_2^e \neq 0, q_1^{e-1} + q_2^{e-1} > \mu^e > q_1^{e-1} \\ +\infty \text{ if } \pi_1^e = 0 = \pi_2^e, q_1^{e-1} + q_2^{e-1} < \mu^e \end{array}$$

Single processor with dynamic priorities. Mass fluxes q_1 and q_2 shown – production of q_2 stopped due to higher priority of parts 1.





Two processors connected by queues. Time evolution of queues and pointer variable (2). Parts with subindex one have higher priority. For $t < 1$ and $t > 8$ all parts are processed and inbetween only priority one parts.

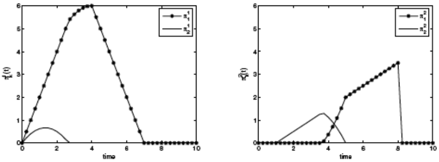


FIG. 4.6. *Amount of parts in the queues for processor* 1 *(left) and* 2 *(right).*

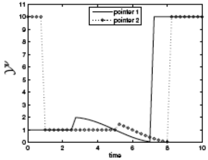
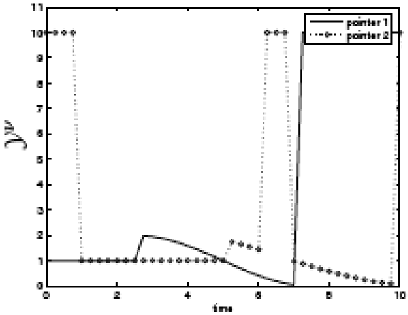


FIG. 4.7. *Movement of the pointers* $\mathcal{Y}^\nu$ *for* $\nu = 1, 2$.

Two processors connected by queues. Attribute dependent velocity. Time evolution of queues and pointer variable (2). Parts with subindex one have higher priority and higher processing velocity For $t < 1$ and $t > 6$ (compared with $t > 8$ in the previous example) all parts are processed and inbetween only priority one parts.



Thank you for your attention.

References

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- Extension to network model: Göttlich, Herty, Klar, CMS 2007
- Analysis of the coupled PDE–ODE model: Herty, Klar, Piccoli, SIMA 2007
- Model for different properties: Armbruster, Degond, Ringhofer, SIAP 2007