Extreme Vortex States and the Hydrodynamic Blow-Up Problem

(Probing Fundamental Bounds in Hydrodynamics Using Variational Optimization Methods)

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Computational Time Provided by SHARCNET

Collaborators

- Charles Doering
  (University of Michigan)
- Dmitry Pelinovsky
  (McMaster University)

Agenda

- Navier-Stokes equation ($\Omega = [0, L]^d$, $d = 2, 3$)

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \boldsymbol{\nabla} p - \nu \Delta \mathbf{v} = \mathbf{0}, & \text{in } \Omega \times (0, T] \\ \boldsymbol{\nabla} \cdot \mathbf{v} = 0, & \text{in } \Omega \times (0, T] \\ \mathbf{v} = \mathbf{v}_0 & \text{in } \Omega \text{ at } t = 0 \\ \text{Boundary Condition} & \text{on } \Gamma \times (0, T] \end{cases}$$

- 2D Case
  - Existence Theory Complete — smooth and unique solutions exist for arbitrary times and arbitrarily large data
- 3D Case
  - Weak solutions (possibly nonsmooth) exist for arbitrary times
  - Classical (smooth) solutions (possibly nonsmooth) exist for *finite* times only
  - Possibility of "blow-up" (finite-time singularity formation)
  - One of the Clay Institute "Millennium Problems" ($ 1M!)

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What is known? — Available Estimates

- A Key Quantity — Enstrophy

$$\mathcal{E}(t) \triangleq \int_\Omega |\nabla \times \mathbf{v}|^2 \, d\Omega \qquad (= \|\nabla \mathbf{v}\|_2^2)$$

- Smoothness of Solutions $\Longleftrightarrow$ Bounded Enstrophy (Foias & Temam, 1989)

$$\max_{t\in[0,T]} \mathcal{E}(t) < \infty \quad ???$$

- Can estimate $\frac{d\mathcal{E}(t)}{dt}$ using the momentum equation, Sobolev's embeddings, Young and Cauchy-Schwartz inequalities, ...
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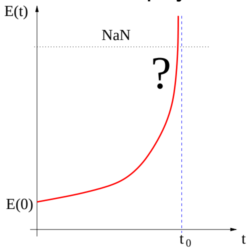

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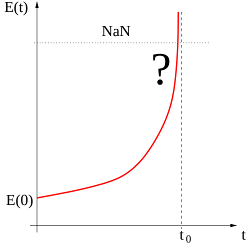

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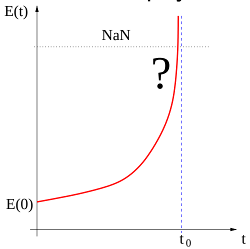

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$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$$

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- Can we actually find solutions which "saturate" a given estimate?
- Estimate $\frac{d\mathcal{E}(t)}{dt} \le c\mathcal{E}(t)^3$ at a *fixed* instant of time t

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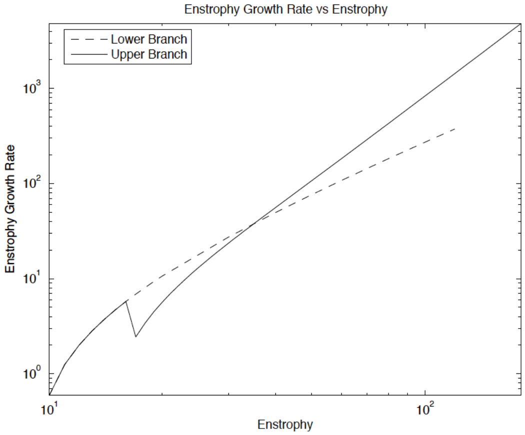

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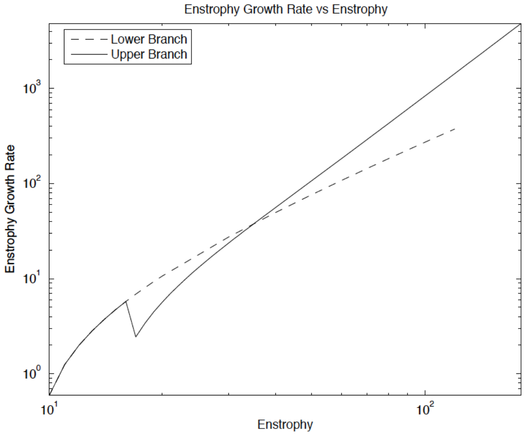

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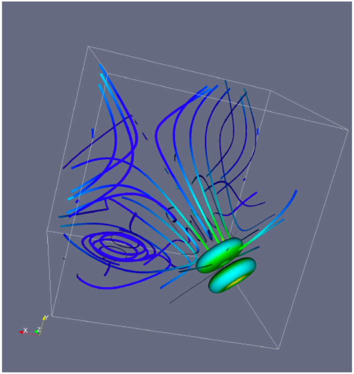

vorticity field (top branch)

- How about solutions which saturate $\frac{d\mathcal{E}(t)}{dt} \le c\mathcal{E}(t)^3$ over a *finite* time window $[0, T]$?

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Relevant Estimates

	Best Estimate	Sharp?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$	
1D Burgers finite–time	$\max_{t\in[0,T]}\mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3$	
2D Navier–Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}}\right)\mathcal{P}^2 + C_1\left(\frac{\mathcal{E}}{\nu}\right)\mathcal{P}$ $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu}\mathcal{K}^{1/2}\mathcal{P}^{3/2}$	
2D Navier–Stokes finite–time	$\max_{t>0}\mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$	
3D Navier–Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	Yes Lu & Doering (2008)
3D Navier–Stokes finite–time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1-4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$	

- QUESTION #1 ("SMALL")

  Sharpness of *instantaneous* estimates (at some *fixed* t)

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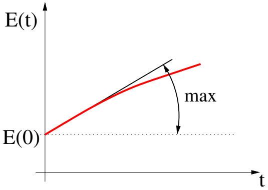
$$\max_{\mathbf{u}} \frac{d\mathcal{P}}{dt} \qquad (2D)$$



- QUESTION #2 ("BIG")

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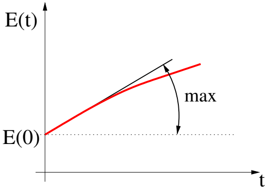

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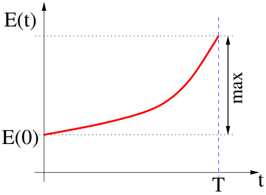

PROBLEM I

Instantaneous and Finite-Time Bounds for Growth of Enstrophy in 1D Burgers Problem

- Burgers equation ($\Omega = [0,1]$, $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{in } \Omega$$

$$u(x) = \phi(x) \qquad \text{at } t = 0$$

Periodic B.C.

- Solutions smooth for all times
- Questions of sharpness of enstrophy estimates still relevant

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3} \mathcal{E}(t)^{5/3}$$

- Best available finite-time estimate

$$\max_{t\in[0,T]} \mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2 \left(\frac{1}{\pi^2\nu}\right)^{4/3} \mathcal{E}_0\right]^3 \xrightarrow[\mathcal{E}_0\to\infty]{} C_2\mathcal{E}_0^3$$

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Spectral Properties of Solutions of Burgers Equation with Small Dissipation

Andrei Biryuk

Functional Analysis and Its Applications. Vol. **35**., no 1., 2001.

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This present paper concerns the initial value problem for the one dimensional ($\dim x = 1$) parabolic equation of Burgers type:

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = \delta u_{xx} \,, \tag{1.1}$$

with the initial state

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- Estimate $\frac{d\mathcal{E}(t)}{dt} \le c\mathcal{E}(t)^{5/3}$ at a *fixed* instant of time t

$$\max_{u \in H^1(\Omega)} \frac{d\mathcal{E}(t)}{dt}$$
$$\text{subject to} \;\; \mathcal{E}(t) = \mathcal{E}_0$$

  where

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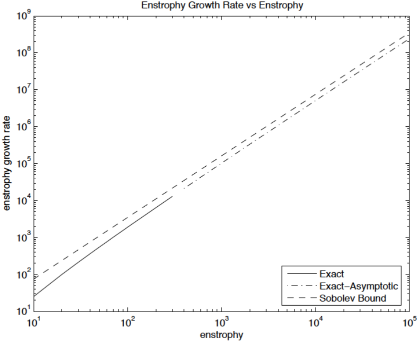

$$\left[\frac{d\mathcal{E}(t)}{dt}\right]_{max} = 0.2476\frac{\mathcal{E}_0^{5/3}}{\nu^{1/3}}$$

instantaneous estimate is sharp

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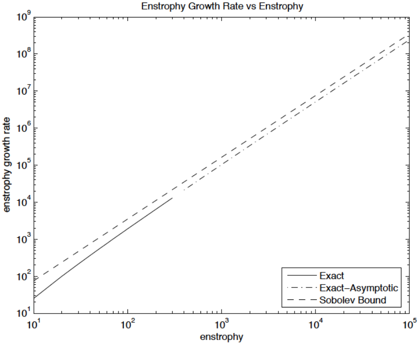

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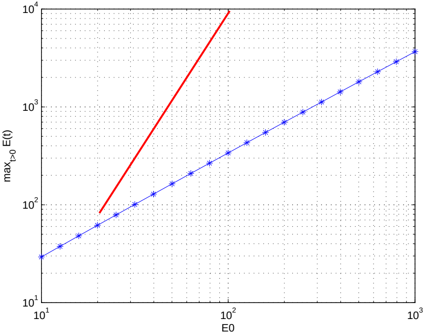

blue line: $\max_{t\in[0,T]} \mathcal{E}(t) \sim C\,\mathcal{E}_0^{1.048}$

red line: finite-time estimate (far from saturated)

Finite-Time Optimization Problem (I)

- Statement

$$\max_{\phi \in H^1(\Omega)} \mathcal{E}(T)$$

$$\text{subject to} \;\; \mathcal{E}(t) = \mathcal{E}_0$$

T, $\mathcal{E}_0$ — parameters

- Optimality Condition

$$\forall_{\phi' \in H^1} \qquad \mathcal{J}'_\lambda(\phi; \phi') = -\int_0^1 \frac{\partial^2 u}{\partial x^2}\Big|_{t=T} u'\Big|_{t=T} dx - \lambda \int_0^1 \frac{\partial^2 \phi}{\partial x^2}\Big|_{t=0} u'\Big|_{t=0} dx$$

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Finite-Time Optimization Problem (II)

- Gradient Descent

$$\begin{aligned} \phi^{(n+1)} &= \phi^{(n)} - \tau^{(n)} \nabla \mathcal{J}(\phi^{(n)}), \qquad n = 1, \dots, \\ \phi^{(0)} &= \phi_0, \end{aligned}$$

- Step size $\tau^{(n)}$ found via *arc minimization*

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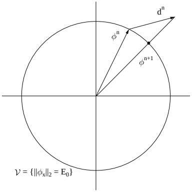

- Two parameters: T, $\mathcal{E}_0$ $(\nu = 10^{-3})$
- Optimal initial conditions corresponding to initial guess with wavenumber $m = 1$ (local maximizers)

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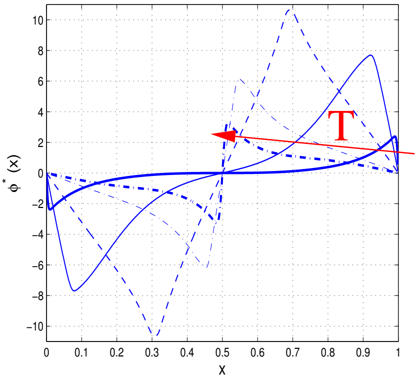

Fixed $\mathcal{E}_0 = 10^3$, different T

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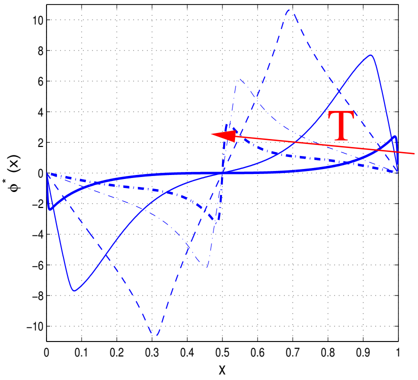

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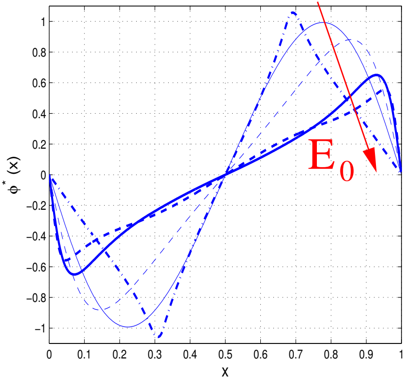
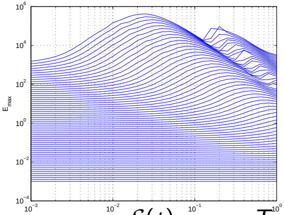
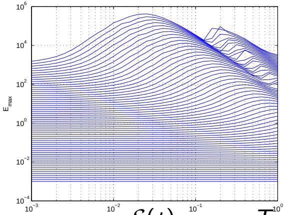


Fixed $T = 0.0316$, different $\mathcal{E}_0$



$\max_{t\in[0,T]} \mathcal{E}(t)$ versus T



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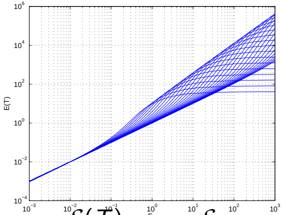

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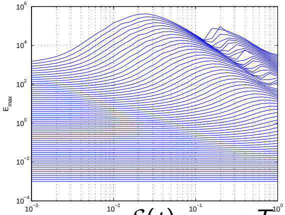

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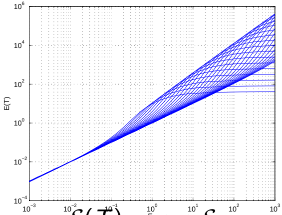

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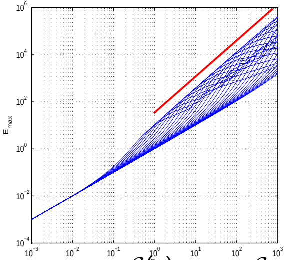

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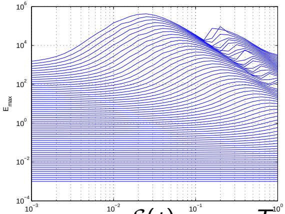

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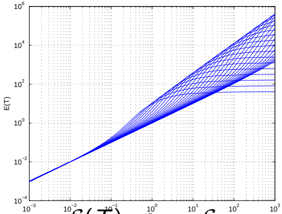

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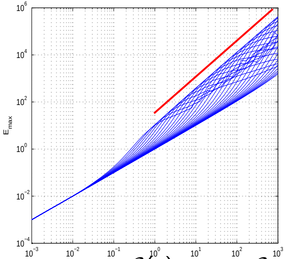
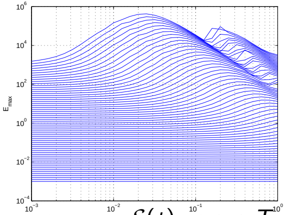


$\max_{t\in[0,T]} \mathcal{E}(t)$ versus $\mathcal{E}_0$

$$\max_{t\in[0,T]} \mathcal{E}(t) \sim C\mathcal{E}_0^{1.5}$$



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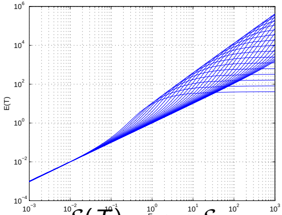

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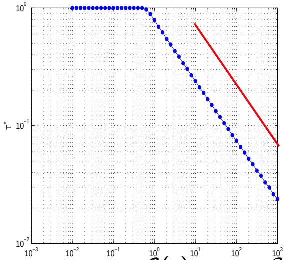
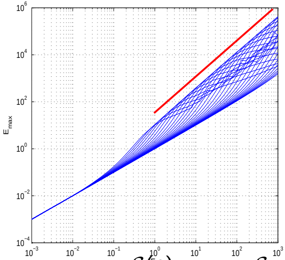


$\operatorname{argmax}_{t\in[0,T]} \mathcal{E}(t)$ versus $\mathcal{E}_0$



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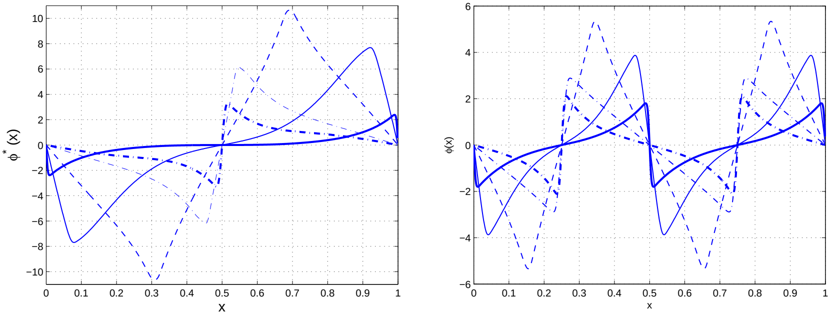
- Sol'ns found with initial guesses $\phi^{(m)}(x) = \sin(2\pi m x)$, $m = 1, 2, \ldots$



$m = 1$, $\mathcal{E}_0 = 10^3$ $\qquad$ $m = 2$, $\mathcal{E}_0 = 10^3$

- Change of variables leaving Burgers equation invariant ($L \in \mathbb{Z}^+$):

$$x = L\xi, \; (x \in [0,1], \; \xi \in [0, 1/L]), \qquad \tau = t/L^2$$

$$v(\tau, \xi) = Lu(x(\xi), t(\tau)), \qquad \mathcal{E}_v(\tau) = L^4 \mathcal{E}_u\left(\frac{t}{L^2}\right)$$

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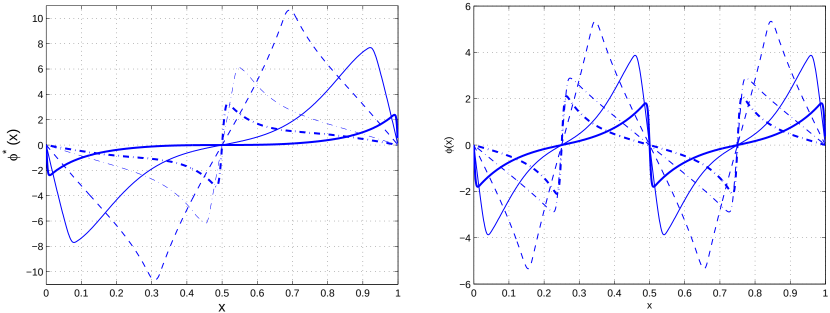

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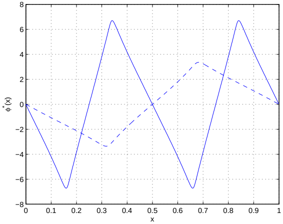
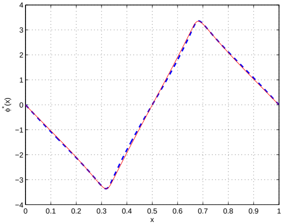
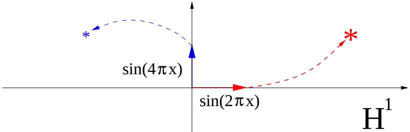
- Solutions for $m = 1$ and $m = 2$, after rescaling





- Using initial guess: $\phi^{(0)}(x) = \sin(2\pi m x)$, $m = 1$, or $m = 2$



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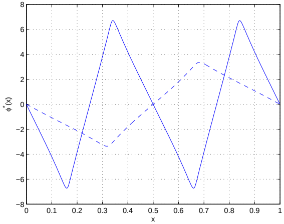
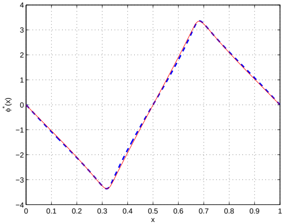
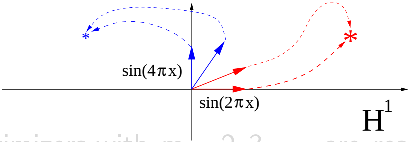




- Using initial guess:
  $$\phi^{(0)}(x) = \epsilon \sin(2\pi m x) + (1-\epsilon)\sin(2\pi n x), \; m \neq n, \; \epsilon > 0$$



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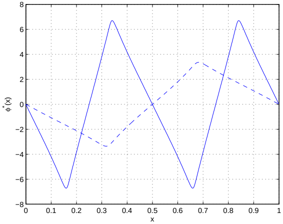
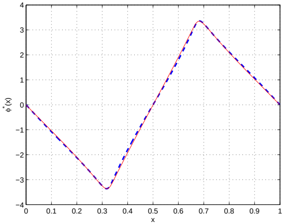
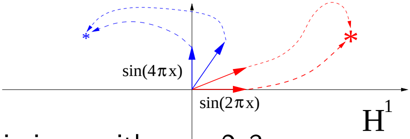




- Using initial guess:

$$\phi^{(0)}(x) = \epsilon \sin(2\pi m x) + (1-\epsilon)\sin(2\pi n x), \; m \neq n, \; \epsilon > 0$$



- All local maximizers with $m = 2, 3, \ldots$ are *rescaled copies* of the $m = 1$ maximizer

Relevant Estimates

	BEST ESTIMATE	SHARP?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$	YES Lu & Doering (2008)
1D Burgers finite–time	$\max_{t\in[0,T]}\mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3$	NO Ayala & P. (2011)
2D Navier–Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}}\right)\mathcal{P}^2 + C_1\left(\frac{\mathcal{E}}{\nu}\right)\mathcal{P}$ $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu}\mathcal{K}^{1/2}\mathcal{P}^{3/2}$	
2D Navier–Stokes finite–time	$\max_{t>0}\mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$	
3D Navier–Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	YES Lu & Doering (2008)
3D Navier–Stokes finite–time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1-4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$	

PROBLEM II

Instantaneous Bounds for Growth of Palinstrophy in 2D Navier-Stokes Problem

Relevant Estimates

	Best Estimate	Sharp?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$	Yes Lu & Doering (2008)
1D Burgers finite–time	$\max_{t\in[0,T]}\mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3$	No Ayala & P. (2011)
2D Navier–Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}}\right)\mathcal{P}^2 + C_1\left(\frac{\mathcal{E}}{\nu}\right)\mathcal{P}$ $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu}\mathcal{K}^{1/2}\mathcal{P}^{3/2}$	
2D Navier–Stokes finite–time	$\max_{t>0}\mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$	
3D Navier–Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	Yes Lu & Doering (2008)
3D Navier–Stokes finite–time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1-4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$	

Relevant Estimates

	Best Estimate	Sharp?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$	Yes Lu & Doering (2008)
1D Burgers finite–time	$\max_{t\in[0,T]}\mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3$	No Ayala & P. (2011)
2D Navier–Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}}\right)\mathcal{P}^2 + C_1\left(\frac{\mathcal{E}}{\nu}\right)\mathcal{P}$ $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu}\mathcal{K}^{1/2}\mathcal{P}^{3/2}$	
2D Navier–Stokes finite–time	$\max_{t>0}\mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$	
3D Navier–Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	Yes Lu & Doering (2008)
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- 2D VORTICITY EQUATION IN A PERIODIC BOX $(\omega = \mathbf{e}_z \cdot \boldsymbol{\omega})$

$$\frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu \Delta \omega \quad \text{where } J(f,g) = f_x g_y - f_y g_x$$

$$-\Delta \psi = \omega$$

- Enstrophy uninteresting in 2D flows (w/o boundaries)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \omega^2 \, d\Omega = -\nu \int_\Omega (\nabla \omega)^2 \, d\Omega < 0$$

- Evolution equation for the vorticity gradient $\nabla \omega$

$$\frac{\partial \nabla \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\nabla \omega = \nu \Delta \nabla \omega + \underbrace{\nabla \omega \cdot \nabla \mathbf{u}}_{\text{“STRETCHING” TERM}}$$

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- Palinstrophy

$$\mathcal{P}(t) \triangleq \int_\Omega (\boldsymbol{\nabla}\omega(t,\mathbf{x}))^2 \, d\Omega = \int_\Omega (\boldsymbol{\nabla}\Delta\psi(t,\mathbf{x}))^2 \, d\Omega$$

- Also of interest — Kinetic Energy

$$\mathcal{K}(t) \triangleq \int_\Omega \mathbf{u}(t,\mathbf{x})^2 \, d\Omega = \int_\Omega (\boldsymbol{\nabla}\psi(t,\mathbf{x}))^2 \, d\Omega$$

- Poincaré's inequality

$$\mathcal{K} \le (2\pi)^{-2}\,\mathcal{E} \le (2\pi)^{-2}\,\mathcal{P}$$

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- Estimates for the Rate of Growth of Palinstrophy

$$\frac{d\mathcal{P}(t)}{dt} = \int_\Omega J(\Delta\psi, \psi)\Delta^2\psi\, d\Omega - \nu \int_\Omega (\Delta^2\psi)^2\, d\Omega \quad \triangleq \mathcal{R}_\mathcal{P}(\psi)$$

- Using Poincaré's inequality (may not be sharp)

$$\frac{d\mathcal{P}(t)}{dt} \leq \frac{C}{\nu}\mathcal{P}^2,$$

- Bound on growth in finite time

$$\max_{t>0} \mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2 \qquad \text{(Ayala, 2012)}$$

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- Maximum Growth of $\frac{d\mathcal{P}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0, \mathcal{P}_0 > (2\pi)^2\mathcal{E}_0$

$$\max_{\psi \in \mathcal{S}_{\mathcal{P}_0,\mathcal{E}_0}} \quad \mathcal{R}_{\mathcal{P}_0}(\psi) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{P}_0,\mathcal{E}_0} = \left\{ \psi \in H^4(\Omega) : \begin{array}{c} \frac{1}{2}\int_\Omega (\boldsymbol{\nabla}\Delta\psi)^2\, d\Omega = \mathcal{P}_0 \\ \frac{1}{2}\int_\Omega (\Delta\psi)^2\, d\Omega = \mathcal{E}_0 \end{array} \right\}$$

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- Small Palinstrophy Limit: $\mathcal{P}_0 \to (2\pi)^2 \mathcal{E}_0$

$$\tilde{\varphi}_0 = \underset{\varphi \in \mathcal{S}_0}{\arg\max}\ \mathcal{R}_0(\varphi), \quad \mathcal{R}_0(\varphi) = -\nu \int_\Omega (\Delta^2 \varphi)^2 \, d\Omega,$$

$$\mathcal{S}_0 = \left\{ \varphi \in H^4(\Omega) : \frac{1}{2} \int_\Omega (\boldsymbol{\nabla} \Delta \psi)^2 \, d\Omega = \frac{(2\pi)^2}{2} \int_\Omega (\Delta \psi)^2 \, d\Omega \right\}$$

- Optimizers: Eigenfunctions of the Laplacian $(\tilde{\varphi}_0 \in \mathrm{Ker}(\Delta))$

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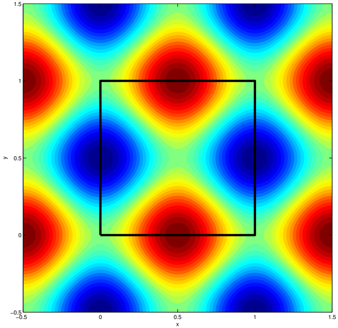
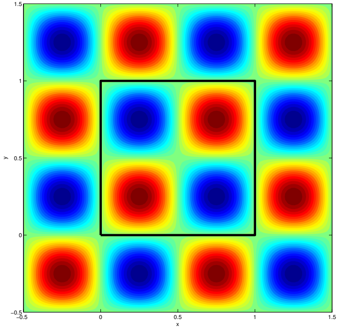


$\varphi(x, y) = \sin(\pi(y - x)) \sin(\pi(y + x))$



$\varphi(x, y) = \sin(2\pi x) \sin(2\pi y)$

Numerical Solution of Maximization Problem

- Discretization of Gradient Flow

$$\frac{d\psi}{d\tau} = -\nabla^{H^4}\mathcal{R}_\nu(\psi), \qquad \psi(0) = \psi_0$$

- Gradient in $H^4(\Omega)$ (via variational techniques)

$$\left[\mathrm{Id} - L^8\Delta^4\right]\nabla^{H^4}\mathcal{R}_\nu = \nabla^{L_2}\mathcal{R}_\nu \qquad \text{(Periodic BCs)}$$

$$\nabla^{L_2}\mathcal{R}_\nu(\psi) = \Delta^2 J(\Delta\psi,\psi) + \Delta J(\psi,\Delta^2\psi) + J(\Delta^2\psi,\Delta\psi) - 2\nu\Delta^4\psi$$

- Constraint satisfaction via arc minimization

Numerical Solution of Maximization Problem

- Discretization of Gradient Flow

$$\frac{d\psi}{d\tau} = -\nabla^{H^4}\mathcal{R}_\nu(\psi), \qquad \psi(0) = \psi_0$$

$$\psi^{(n+1)} = \psi^{(n)} - \Delta\tau^{(n)}\,\nabla^{H^4}\mathcal{R}_\nu(\psi^{(n)}), \qquad \psi^{(0)} = \psi_0$$

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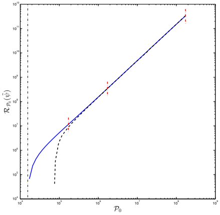
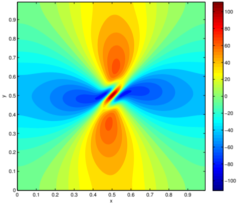
Maximizers with Fixed $(\mathcal{K}_0, \mathcal{P}_0)$

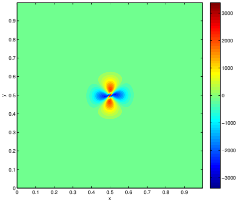
Estimate: $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu} \mathcal{K}_0^{1/2} \mathcal{P}_0^{3/2}$



max $\frac{d\mathcal{P}}{dt}$ vs. $\mathcal{P}_0$, $\mathcal{K}_0 = 10$



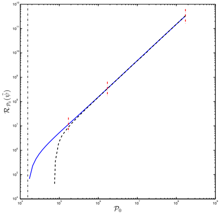
(a) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



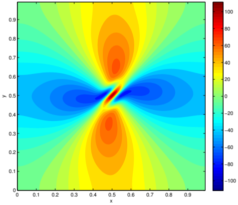
(c) $\mathcal{P}_0 \approx 10^4\mathcal{P}_c$

Maximizers with Fixed $(\mathcal{K}_0, \mathcal{P}_0)$

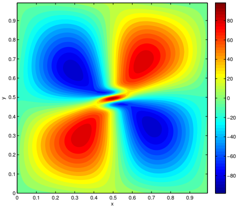
Estimate: $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu} \mathcal{K}_0^{1/2} \mathcal{P}_0^{3/2}$



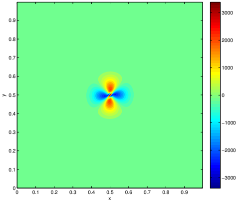
max $\frac{d\mathcal{P}}{dt}$ vs. $\mathcal{P}_0$, $\mathcal{K}_0 = 10$



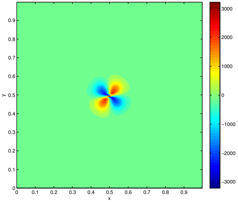
(a) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



(b) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



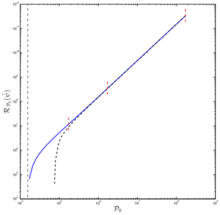
(c) $\mathcal{P}_0 \approx 10^4\mathcal{P}_c$



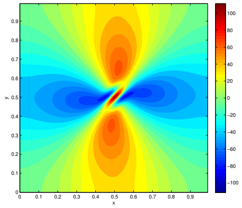
(d) $\mathcal{P}_0 \approx 10^4\mathcal{P}_c$

Maximizers with Fixed $(\mathcal{K}_0, \mathcal{P}_0)$

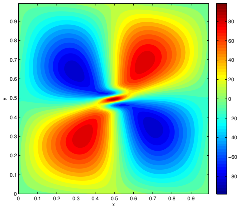
Estimate: $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu} \mathcal{K}_0^{1/2} \mathcal{P}_0^{3/2}$



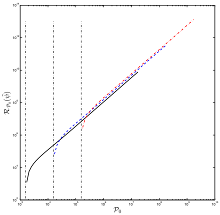
max $\frac{d\mathcal{P}}{dt}$ vs. $\mathcal{P}_0$, $\mathcal{K}_0 = 10$



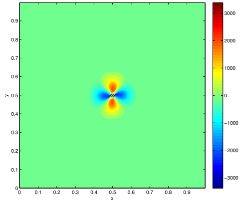
(a) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



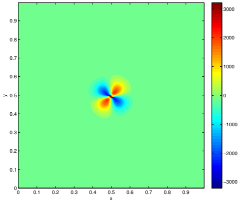
(b) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



max $\frac{d\mathcal{P}}{dt} \sim \mathcal{P}_0^{3/2}$ as $\mathcal{P}_0 \to \infty$



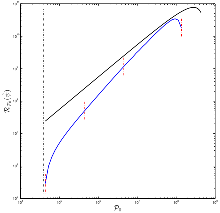
(c) $\mathcal{P}_0 \approx 10^4\mathcal{P}_c$



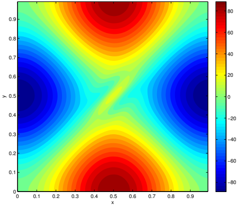
(d) $\mathcal{P}_0 \approx 10^4\mathcal{P}_c$

Maximizers with Fixed $(\mathcal{E}_0, \mathcal{P}_0)$

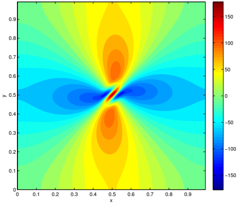
Estimate: $\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}_0}\right)\mathcal{P}_0^2 + C_1\left(\frac{\mathcal{E}_0}{\nu}\right)\mathcal{P}_0$



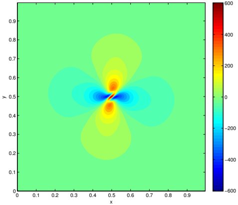
max $\frac{d\mathcal{P}}{dt}$ vs. $\mathcal{P}_0$, $\mathcal{E}_0 = 10^3$



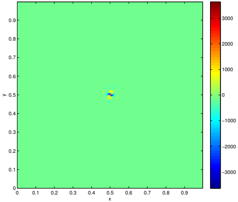
(a) $\mathcal{P}_0 \approx P_c$



(b) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



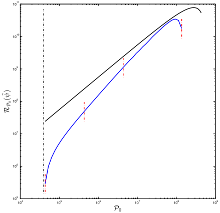
(c) $\mathcal{P}_0 \approx 10^2\mathcal{P}_c$



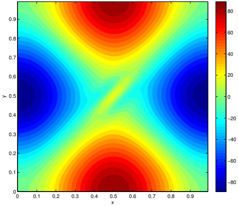
(d) $\mathcal{P}_0 \approx 10^{7/2}\mathcal{P}_c$

Maximizers with Fixed $(\mathcal{E}_0, \mathcal{P}_0)$

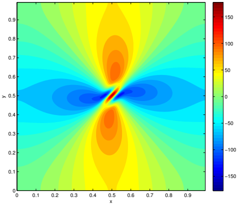
Estimate: $\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}_0}\right)\mathcal{P}_0^2 + C_1\left(\frac{\mathcal{E}_0}{\nu}\right)\mathcal{P}_0$



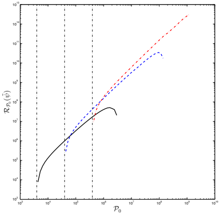
max $\frac{d\mathcal{P}}{dt}$ vs. $\mathcal{P}_0$, $\mathcal{E}_0 = 10^3$



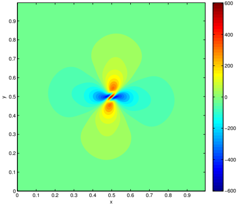
(a) $\mathcal{P}_0 \approx P_c$



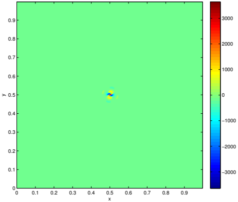
(b) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



max $\frac{d\mathcal{P}}{dt}$ vs. $\mathcal{P}_0$



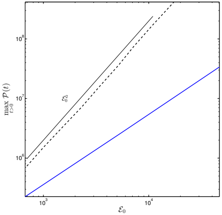
(c) $\mathcal{P}_0 \approx 10^2\mathcal{P}_c$



(d) $\mathcal{P}_0 \approx 10^{7/2}\mathcal{P}_c$

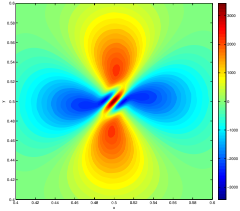
Maximizers with Fixed $(\mathcal{K}_0, \mathcal{P}_0)$

Finite-Time Estimate: $\max_{t>0} \mathcal{P}(t) \le \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$

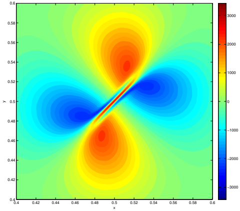


— $\mathcal{P}_0$-constraint

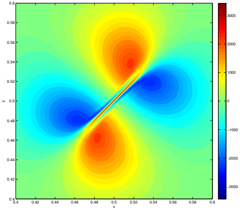
- - - $\{\mathcal{K}_0, \mathcal{P}_0\}$-constraint



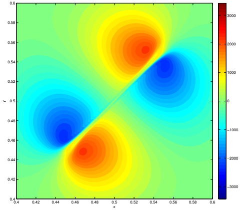
(a) $t = 0.000213$



(b) $t = 0.000458$



(c) $t = 0.000633$



(d) $t = 0.001265$

Relevant Estimates

	Best Estimate	Sharp?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \le \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$	Yes Lu & Doering (2008)
1D Burgers finite–time	$\max_{t\in[0,T]}\mathcal{E}(t) \le \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3$	No Ayala & P. (2011)
2D Navier–Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \le -\left(\frac{\nu}{\mathcal{E}}\right)\mathcal{P}^2 + C_1\left(\frac{\mathcal{E}}{\nu}\right)\mathcal{P}$ $\frac{d\mathcal{P}(t)}{dt} \le \frac{C_2}{\nu}\mathcal{K}^{1/2}\mathcal{P}^{3/2}$	[Yes] Ayala & P. (2013)
2D Navier–Stokes finite–time	$\max_{t>0}\mathcal{P}(t) \le \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$	[Yes] Ayala & P. (2013)
3D Navier–Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \le \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	Yes Lu & Doering (2008)
3D Navier–Stokes finite–time	$\mathcal{E}(t) \le \frac{\mathcal{E}(0)}{\sqrt{1-4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$	

Relevant Estimates

	Best Estimate	Sharp?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$	Yes Lu & Doering (2008)
1D Burgers finite–time	$\max_{t\in[0,T]}\mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3$	No Ayala & P. (2011)
2D Navier–Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}}\right)\mathcal{P}^2 + C_1\left(\frac{\mathcal{E}}{\nu}\right)\mathcal{P}$ $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu}\mathcal{K}^{1/2}\mathcal{P}^{3/2}$	[Yes] Ayala & P. (2013)
2D Navier–Stokes finite–time	$\max_{t>0}\mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$	[Yes] Ayala & P. (2013)
3D Navier–Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	Yes Lu & Doering (2008)
3D Navier–Stokes finite–time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1-4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$	???

PROBLEM III

Instantaneous Bounds for Growth of Enstrophy in 3D Navier-Stokes Problem

(preliminary results)

Relevant Estimates

	Best Estimate	Sharp?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$	Yes Lu & Doering (2008)
1D Burgers finite–time	$\max_{t\in[0,T]}\mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3$	No Ayala & P. (2011)
2D Navier–Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}}\right)\mathcal{P}^2 + C_1\left(\frac{\mathcal{E}}{\nu}\right)\mathcal{P}$ $\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu}\mathcal{K}^{1/2}\mathcal{P}^{3/2}$	[Yes] Ayala & P. (2013)
2D Navier–Stokes finite–time	$\max_{t>0}\mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$	[Yes] Ayala & P. (2013)
3D Navier–Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	Yes Lu & Doering (2008)
3D Navier–Stokes finite–time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1-4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$	???

- Rate of Growth of Enstrophy

$$\frac{d\mathcal{E}}{dt} = -\nu \int_\Omega |\Delta \mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega \mathbf{u} \cdot \nabla \mathbf{u} \cdot \Delta \mathbf{u} \, d\mathbf{x} \triangleq \mathcal{R}_{\mathcal{E}_0}(\mathbf{u})$$

- Best available instantaneous upper bound

$$\frac{d\mathcal{E}}{dt} \le \frac{C}{\nu^3} \mathcal{E}^3$$

- Finite-time estimates

- Rate of Growth of Enstrophy

$$\frac{d\mathcal{E}}{dt} = -\nu \int_\Omega |\Delta \mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega \mathbf{u} \cdot \nabla \mathbf{u} \cdot \Delta \mathbf{u} \, d\mathbf{x} \triangleq \mathcal{R}_{\mathcal{E}_0}(\mathbf{u})$$

- Best available instantaneous upper bound

$$\frac{d\mathcal{E}}{dt} \leq \frac{C}{\nu^3} \mathcal{E}^3$$

- Finite-time estimates

- Rate of Growth of Enstrophy

$$\frac{d\mathcal{E}}{dt} = -\nu \int_\Omega |\Delta \mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega \mathbf{u} \cdot \nabla \mathbf{u} \cdot \Delta \mathbf{u} \, d\mathbf{x} \triangleq \mathcal{R}_{\mathcal{E}_0}(\mathbf{u})$$

- Best available instantaneous upper bound

$$\frac{d\mathcal{E}}{dt} \leq \frac{C}{\nu^3} \mathcal{E}^3$$

- Finite-time estimates

$$\max_{t \geq 0} \mathcal{E}(t) \leq \frac{\mathcal{E}_0}{\sqrt{1 - \frac{4 C_3}{\nu^3} \mathcal{E}_0^2 t}}$$

- Rate of Growth of Enstrophy

$$\frac{d\mathcal{E}}{dt} = -\nu \int_\Omega |\Delta \mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega \mathbf{u} \cdot \nabla \mathbf{u} \cdot \Delta \mathbf{u} \, d\mathbf{x} \triangleq \mathcal{R}_{\mathcal{E}_0}(\mathbf{u})$$

- Best available instantaneous upper bound

$$\frac{d\mathcal{E}}{dt} \le \frac{C}{\nu^3}\mathcal{E}^3$$

- Finite-time estimates

$$\max_{t\ge 0} \mathcal{E}(t) \le \frac{\mathcal{E}_0}{\sqrt{1 - \frac{4C_3}{\nu^3}\mathcal{E}_0^2 t}}$$

$$\frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \le \frac{27}{(2\pi\nu)^4}\left[\mathcal{K}(0) - \mathcal{K}(t)\right]$$

- Single Constraint: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$

$$\max_{\mathbf{u}\in\mathcal{S}_{\mathcal{E}_0}} \quad \mathcal{R}_{\mathcal{E}_0}(\mathbf{u}) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{E}_0} = \left\{\mathbf{u} \in H^2(\Omega) : \nabla\cdot\mathbf{u} = 0,\ \mathcal{E}(\mathbf{u}) = \mathcal{E}_0\right\}$$

- Two Constraints: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$ and $\mathcal{K}_0 < (2\pi)^{-2}\mathcal{E}_0$

$$\max_{\mathbf{u}\in\mathcal{S}_{\mathcal{K}_0,\mathcal{E}_0}} \quad \mathcal{R}_{\mathcal{E}_0}(\mathbf{u}) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{K}_0,\mathcal{E}_0} = \left\{\mathbf{u} \in H^2(\Omega) : \nabla\cdot\mathbf{u} = 0,\ \mathcal{K}(\mathbf{u}) = \mathcal{K}_0,\ \mathcal{E}(\mathbf{u}) = \mathcal{E}_0\right\}$$

- Numerical solution via discretized gradient flow
(required resolutions up to 512^3)

- Single Constraint: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$

$$\max_{\mathbf{u}\in\mathcal{S}_{\mathcal{E}_0}} \quad \mathcal{R}_{\mathcal{E}_0}(\mathbf{u}) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{E}_0} \ = \left\{\mathbf{u} \in H^2(\Omega) : \nabla\cdot\mathbf{u} = 0,\ \mathcal{E}(\mathbf{u}) = \mathcal{E}_0\right\}$$

- Two Constraints: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$ and $\mathcal{K}_0 < (2\pi)^{-2}\mathcal{E}_0$

$$\max_{\mathbf{u}\in\mathcal{S}_{\mathcal{K}_0,\mathcal{E}_0}} \quad \mathcal{R}_{\mathcal{E}_0}(\mathbf{u}) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{K}_0,\mathcal{E}_0} \ = \left\{\mathbf{u} \in H^2(\Omega) : \nabla\cdot\mathbf{u} = 0,\ \mathcal{K}(\mathbf{u}) = \mathcal{K}_0,\ \mathcal{E}(\mathbf{u}) = \mathcal{E}_0\right\}$$

- Numerical solution via discretized gradient flow (required resolutions up to 512^3)

- Single Constraint: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$

$$\max_{\mathbf{u}\in\mathcal{S}_{\mathcal{E}_0}} \quad \mathcal{R}_{\mathcal{E}_0}(\mathbf{u}) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{E}_0} \; = \left\{ \mathbf{u} \in H^2(\Omega) : \nabla \cdot \mathbf{u} = 0, \; \mathcal{E}(\mathbf{u}) = \mathcal{E}_0 \right\}$$

- Two Constraints: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$ and $\mathcal{K}_0 < (2\pi)^{-2}\mathcal{E}_0$

$$\max_{\mathbf{u}\in\mathcal{S}_{\mathcal{K}_0,\mathcal{E}_0}} \quad \mathcal{R}_{\mathcal{E}_0}(\mathbf{u}) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{K}_0,\mathcal{E}_0} \; = \left\{ \mathbf{u} \in H^2(\Omega) : \nabla \cdot \mathbf{u} = 0, \; \mathcal{K}(\mathbf{u}) = \mathcal{K}_0, \; \mathcal{E}(\mathbf{u}) = \mathcal{E}_0 \right\}$$

- Numerical solution via discretized gradient flow
  (required resolutions up to 512^3)

Extreme Vortex States for $\mathcal{E}_0 \to 0$ (single constraint)

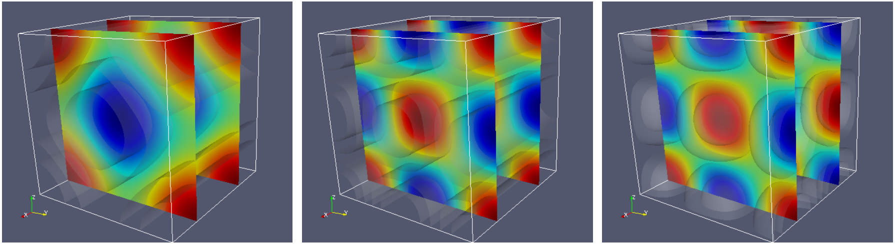
- In the limit $\mathcal{E}_0 \to 0$ optimal states found analytically
  $\Longrightarrow$ div-free eigenfunctions of vector Laplacian (3 branches)

- Case (a): Largest value of $d\mathcal{E}/dt$

- Case (c): Taylor-Green vortex (Taylor & Green 1937)

Extreme Vortex States for $\mathcal{E}_0 \to 0$ (single constraint)

- In the limit $\mathcal{E}_0 \to 0$ optimal states found analytically
  $\Longrightarrow$ div-free eigenfunctions of vector Laplacian (3 branches)



(a) $|\mathbf{k}|^2 = 1$ (b) $|\mathbf{k}|^2 = 2$ (c) $|\mathbf{k}|^2 = 3$

- Case (a): Largest value of $d\mathcal{E}/dt$
- Case (c): Taylor-Green vortex (Taylor & Green 1937)

Extreme Vortex States for $\mathcal{E}_0 \to 0$ (single constraint)

- In the limit $\mathcal{E}_0 \to 0$ optimal states found analytically
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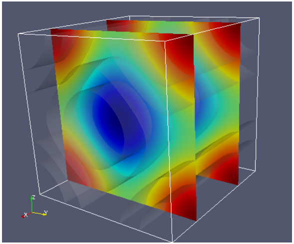
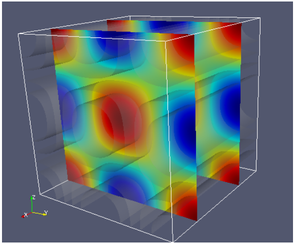
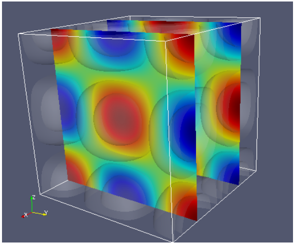

(a) $|\mathbf{k}|^2 = 1$

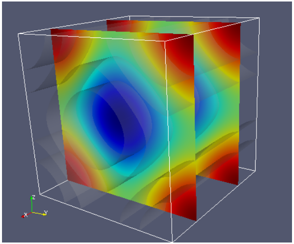

(b) $|\mathbf{k}|^2 = 2$


(c) $|\mathbf{k}|^2 = 3$

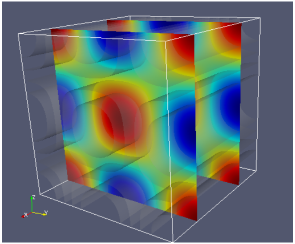
- Case (a): Largest value of $d\mathcal{E}/dt$
- Case (c): Taylor-Green vortex (Taylor & Green 1937)

Extreme Vortex States for $\mathcal{E}_0 \to 0$ (single constraint)

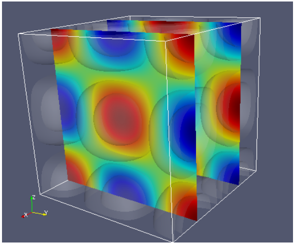
- In the limit $\mathcal{E}_0 \to 0$ optimal states found analytically
  $\Longrightarrow$ div-free eigenfunctions of vector Laplacian (3 branches)



(a) $|\mathbf{k}|^2 = 1$



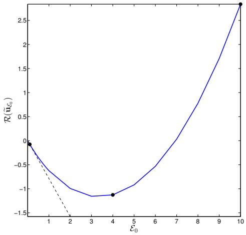
(b) $|\mathbf{k}|^2 = 2$

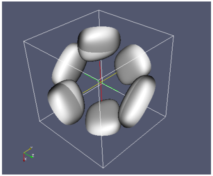


(c) $|\mathbf{k}|^2 = 3$

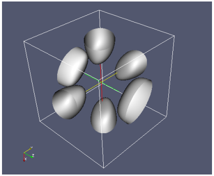
- Case (a): Largest value of $d\mathcal{E}/dt$
- Case (c): Taylor-Green vortex (Taylor & Green 1937)

Single-constraint maximizers $\tilde{\mathbf{u}}_{\mathcal{E}_0}$ (Lu & Doering 2008)

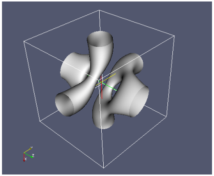




(a) $\mathcal{E}_0 = 1 \times 10^{-2}$



(b) $\mathcal{E}_0 = 4$



(c) $\mathcal{E}_0 = 10$

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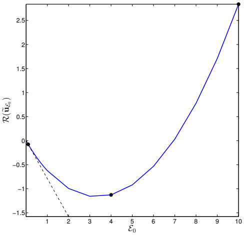

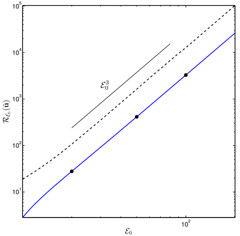

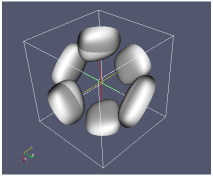

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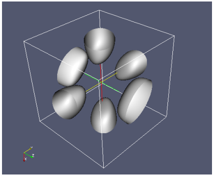

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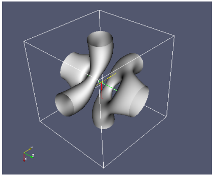

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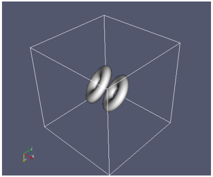
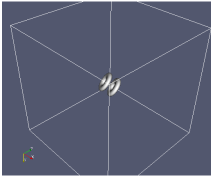
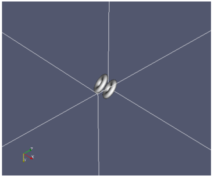


(d) $\mathcal{E}_0 = 20$

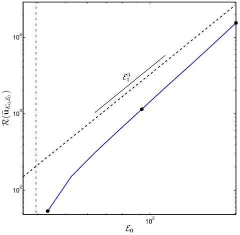


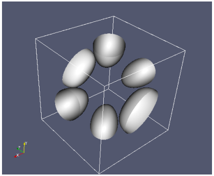
(e) $\mathcal{E}_0 = 50$



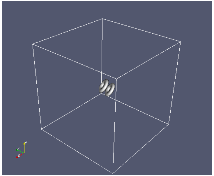
(f) $\mathcal{E}_0 = 100$

Two-constraint maximizers $\tilde{\mathbf{u}}_{\mathcal{K}_0,\mathcal{E}_0}$ $(\mathcal{K}_0 = 1)$

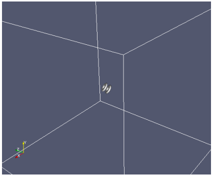




(a) $\mathcal{E}_0 \approx 40$

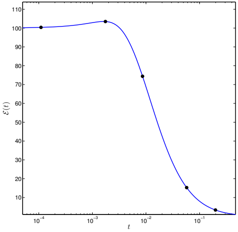


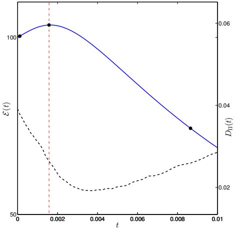
(b) $\mathcal{E}_0 \approx 90$

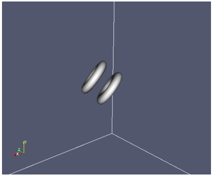


(c) $\mathcal{E}_0 \approx 200$

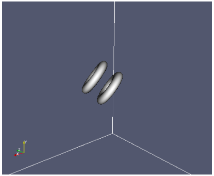
Time evolution of $\tilde{\mathbf{u}}_{\mathcal{E}_0}$ (single constraint: $\mathcal{E}_0 = 100$)



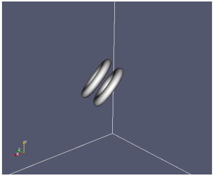




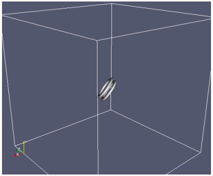
(a) $t = 0.0$



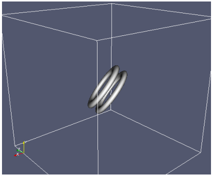
(b) $t = 1.1 \times 10^{-4}$



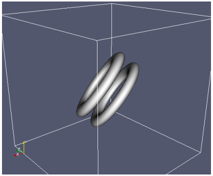
(c) $t = 1.75 \times 10^{-3}$



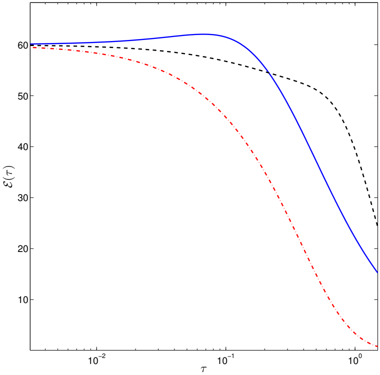
(d) $t = 8.63 \times 10^{-3}$



(e) $t = 5.74 \times 10^{-2}$

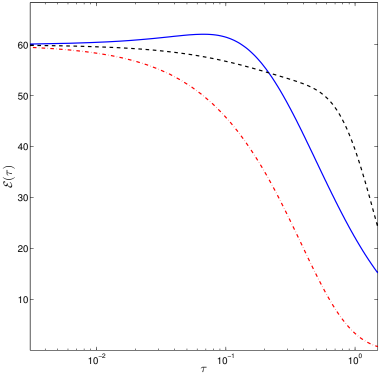


(f) $t = 0.198$



(a) single constraint ($\mathcal{E}_0 = 60$)

—— extreme (instantaneously optimal) states $\tilde{\mathbf{u}}_{\mathcal{E}_0}$,
- - - Taylor-Green vortex
_ . _ Kida-Pelz vortex



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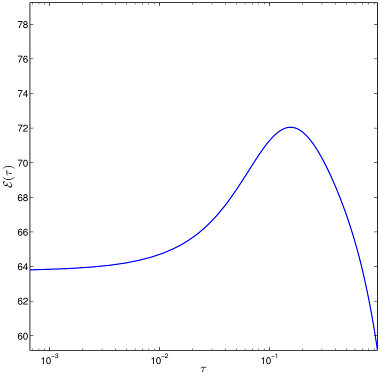

(b) two constraints ($\mathcal{K}_0 = 1$, $\mathcal{E}_0 = 64$)

—— extreme (instantaneously optimal) states $\tilde{\mathbf{u}}_{\mathcal{E}_0}$, $\tilde{\mathbf{u}}_{\mathcal{K}_0,\mathcal{E}_0}$
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$$\frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \leq C\,\left[\mathcal{K}(0) - \mathcal{K}(t)\right]$$

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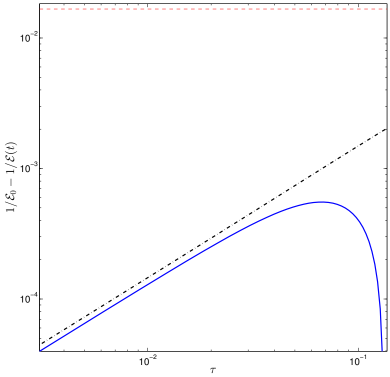

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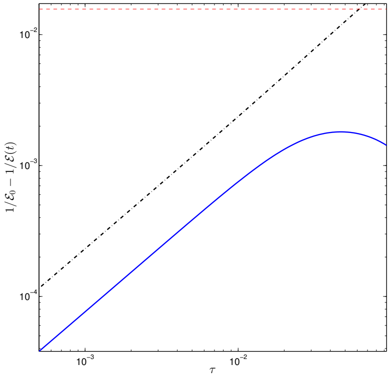

(b) two constraints ($\mathcal{K}_0 = 1$, $\mathcal{E}_0 = 64$)

—— LHS of the estimate, - - - RHS of the estimate

_ . _ $1/\mathcal{E}(0)$

$$\frac{d\mathcal{E}}{dt} \le C' \, \mathcal{E}^3 \quad \Longrightarrow \quad \frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \le C \, [\mathcal{K}(0) - \mathcal{K}(t)]$$

$$\Longrightarrow \quad \max_{t \ge 0} \mathcal{E}(t) \le \frac{\mathcal{E}(0)}{1 - C\,\mathcal{K}(0)\mathcal{E}(0)}, \quad C, C' - \text{numerical fit}$$

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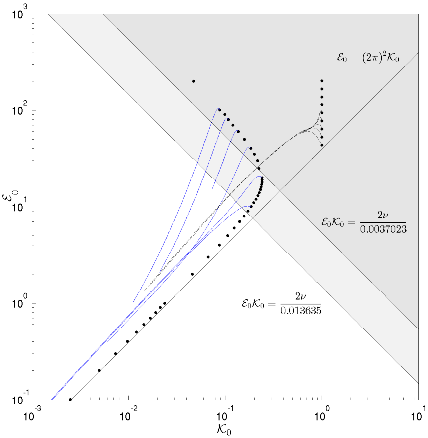

Conclusions

- Found extreme vortex states in 2D and 3D saturating the worst-case mathematical bounds (although, in contrast to some recent studies in 3D, e.g., Bustamante & Brachet 2012, Kerr et al. 2013, 2014, the Reynolds numbers are small).
- Identified regions in the initial data phase-space $\{\mathcal{K}_0, \mathcal{E}_0\}$ for which global regularity is guaranteed.
- So far, no evidence of blow-up in 3D, although due to small Reynolds numbers, the results are not conclusive.
- Need to solve the optimization problem on *finite* time windows, i.e.,

$$\max_{\mathbf{u}_0} \mathcal{E}(T)$$

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Open Questions and Open Problems

- Why do two-constraint optimizers exhibit a larger finite-time growth in 2D, but not in 3D? (finite *Re* effect?)
- Regularity problem for 3D Euler equation.
- Singularity formation in "active scalar" equations (fractional Burgers equation, surface quasi-geostrophic equation, etc.).
- Extreme behavior in the presence of noise.
- Extreme states with more complex structure: simultaneously maximize $\mathcal{R}_{\mathcal{E}_0}(\mathbf{u})$ and helicity $\mathcal{H}(\mathbf{u})$ (via multiobjective optimization)

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