Some parabolic models for chemotaxis in 2D

8th November 2004

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PhD thesis under the direction of Benoît Perthame and Philip K. Maini.

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1. The Patlak, Keller & Segel model

The Keller & Segel model for chemotaxis consists of two coupled parabolic equations :

- an advection-diffusion equation for the evolution of cell density $n(t, x)$,
- a reaction-diffusion equation for the evolution of chemical concentration $c(t, x)$.

Several variants of the following system have been studied

$$\begin{cases} \partial_t n + \nabla \cdot (-\nabla n + \chi n \nabla c) = 0 \quad t \geq 0\,,\ x \in \Omega \subset \mathbb{R}^2 \\ \Gamma \partial_t c - \Delta c = n - \alpha c \end{cases} .$$

Particularly the degenerate case under the assumption of high diffusion of chemical species [Jäger & Luckhaus]

$$\begin{cases} \partial_t n + \nabla \cdot (-\nabla n + \chi n \nabla c) = 0 \quad t \geq 0\,,\; x \in \Omega \\ -\Delta c = n - \fint n \end{cases} .$$

The first task is to study whether or not solutions of these coupled equations blow-up (in finite time).
The main result is the following.

Theorem 1 *There exists a constant C^* such that if $\chi M < C^*$ then the system admits global in time solution.*

At least two distinct approaches can be useful in order to prove this theorem.

1.1. *A priori* estimates

One can derive *a priori* estimates based on the following computation

$$\frac{d}{dt}\int \Phi(n)dx = \int -\Phi''(n)|\nabla n|^2 dx + \chi \int n\psi(n)dx,$$

with $\psi'(x) = x\Phi''(x)$
$\Phi(x)$ is a convex function growing faster than x near infinity, typically $\Phi(x) = x\ln x$.
It is possible to estimate the balance between the two terms, corresponding respectively to diffusion and aggregation of cells, thanks to a Gagliardo-Nirenberg-Sobolev inequality

$$\int n^2 \leq \mathcal{C}_{\mathcal{GNS}} M \int |\nabla\sqrt{n}|^2.$$

If dimension $d = 2$, the total mass of cells $M = \|n\|_{L^1}$ appears naturally from this inequality.

Consequently, the equi-integrability allows controls of the L^pnorms of the cell density n, by another computation with $\Phi(x) = (x-k)_+^p$.

$$\frac{d}{dt}\int_\Omega (n-k)_+^p + C\Big\{1 - C\int_\Omega (n-k)_+\Big\}\int_\Omega |\nabla (n-k)_+^{p/2}|^2$$
$$\leq Ck\int_\Omega (n-k)_+^p + Ck^2\Big(\int_\Omega (n-k)_+^p\Big)^{1-1/(p-1)}$$

1.2. The energy of the system in the case of Ω bounded

There is an energy for the previous system

$$\left\{ \begin{array}{rcl} \partial_t n & = & \nabla \cdot \{ n \nabla (\ln n - \chi c) \} \quad t \geq 0 \, , \; x \in \Omega \subset \mathbb{R}^2 \\ -\Delta c & = & n - ⨍ n \end{array} \right. ,$$

which is of the following type

$$\mathcal{E}(t) = \int n \ln n - \frac{\chi}{2} \int nc \, , \qquad \frac{d\mathcal{E}}{dt} = - \int n |\nabla (\ln n - \chi c)|^2 \leq 0.$$

Introducing the stationnary states of the system, it is possible to show that $\int |\nabla c|^2$ remains bounded.
As a consequence so does $\int n \ln n$.

A Sobolev-type inequality is used in te critical case of the imbedding

$$H^1(\Omega) \hookrightarrow L_A(\Omega)$$

where L_A is the Orlicz space associated to the convex function $A(s) = \exp\left(s^2\right)$.

Lemma 1 (Trudinger & Moser) *If* $u \in H^1(\Omega)$ *and* $\int u = 0$ *(*Neumann Boundary Conditions *) then*

$$\int e^u \leq C \exp\left(\frac{1}{8\pi} \int |\nabla u|^2\right)$$

2. Model for angiogenesis

Another very studied model for cell movement is angiogenesis. In its simplest form, the system is

$$\begin{cases} \partial_t n = \Delta n - \nabla \cdot (n\chi(c)\nabla c) & t \geq 0\,,\ x \in \mathbb{R}^2 \\ \partial_t c = -nc & \end{cases}.$$

This system also admits an energy, given by

$$\mathcal{E}(t) = \int n \ln n + \frac{1}{2}\int |\nabla \Phi(c)|^2, \qquad \frac{d\mathcal{E}}{dt} \leq 0$$

provided $\inf_{c\geq 0}\left\{\frac{c\chi'}{\chi} + 1\right\} \geq 0\,$; where Φ is defined by the differential equation

$$\Phi'(c) = \sqrt{\frac{\chi(c)}{c}}.$$

This estimation reveals that the family $\{n(t)\ln n(t)\}$ is equi-integrable.

Nevertheless, in order to control the L^p norms of n as in the previous section, another strategy has to be stated. For instance, it is possible to transform the first equation into a divergence form

$$\partial_t \left(\frac{n}{\phi(c)} \right) = \frac{1}{\phi(c)} \nabla \cdot \left\{ \phi(c) \nabla \left(\frac{n}{\phi(c)} \right) \right\} + \left(\frac{n}{\phi(c)} \right)^2 \phi(c) \chi(c) c,$$

where $\phi(c)$ is defined by another differential equation

$$\phi'(c) = \phi(c)\chi(c).$$

It is then possible to reproduce and adapt computations of $\frac{d}{dt} \int f \left(\frac{n}{\phi(c)} \right) \phi(c)$ and to apply similarly the Gagliardo-Nirenberg-Sobolev inequalities in the case of $f(x) = (x-k)_+^p$.

3. The generalized Keller & Segel model

To further studying different chemotactic models, we have chosen a generalization of the Keller & Segel model.
It has been proposed by Tyson & Murray for the modelisation of spatial organisation in bacterial population.

$$\left\{ \begin{array}{rcl} \partial_t n &=& \Delta n - \nabla \cdot (n \chi \nabla c) \quad t \geq 0\,,\ x \in \mathbb{R}^2 \\ -\Delta c &=& nf \\ \partial_t f &=& -nf \end{array} \right. .$$

Assumption of an additional chemical species : the stimulant f is necessary to produce the chemoattractant c.
And f is only consumed by the cells.
It renders an account of short and long range effects because of the diffusion of the chemical c, contrary to the local effect of the stimulant f.

Unfortunately, we know no energy structure for this system of three coupled equations, which makes it dramatically different from the previous ones.

We present here a first draft to understand the behavior of this system.

Indeed, if we simply reproduce the first method presented above, based on the *a priori* estimation

$$\frac{d}{dt}\int n \ln n = -4\int |\nabla\sqrt{n}|^2 + \chi\|f\|_\infty \int n^2,$$

we can't hope gaining anything but the condition $\chi\|f\|_\infty M < \mathcal{C}^*$.
This condition is not satisfying : it doesn't bring anything new by comparison to the classical Keller & Segel model ; and it doesn't capture the feature of the additional equation $\partial_t f = -nf$.

Another approach consists of finding a combinaison of the following type

$$\mathcal{W}(t) = \int n \ln n + \beta \int n f^{\gamma} + \alpha \frac{1}{2} \int |\nabla f^{\delta}|^2,$$

which is decreasing for well-chosen values of α and β, and under some conditions involving $\chi \|f\|_{\infty}$ and M. We first compute $\frac{d}{dt}\mathcal{W}$

$$\begin{aligned}
\frac{d}{dt} \int n \ln n \; &= \; -4 \int |\nabla \sqrt{n}|^2 + \chi \int n^2 f, \\
\beta \frac{d}{dt} \int n f^{\gamma} \; &= \; -\beta \int \nabla n \cdot \nabla f^{\gamma} + \chi \beta \int n \nabla c \cdot \nabla f^{\gamma} - \gamma \beta \int n^2 f^{\gamma}, \\
\alpha \frac{d}{dt} \frac{1}{2} \int |\nabla f^{\delta}|^2 \; &= \; -\frac{\delta}{2} \alpha \int \nabla n \cdot \nabla f^{2\delta} - \delta \alpha \int n |\nabla f^{\delta}|^2.
\end{aligned}$$

In order to eliminate the bad contribution of the no-sign terms and the positive one, we'll associate them with negative ones in two ways. The first group includes

$$-4\int|\nabla\sqrt{n}|^2-\left\{\begin{array}{c}\beta\int\nabla n\cdot\nabla f^\gamma\\-\frac{\delta}{2}\alpha\int\nabla n\cdot\nabla f^{2\delta}\end{array}\right\}-\delta\alpha\int n|\nabla f^\delta|^2,$$

and the second one includes

$$\chi\int n^2f-\gamma\beta\int n^2f^\gamma.$$

The unfriendly term $\chi\beta\int n\nabla c\cdot\nabla f^\gamma$ plays an ambivalent role in this description.

3.1. The first association

We force a remarkable square to appear thanks to the extrem terms. One can easily be convinced that we have to set $\delta \leq \gamma$.
Under this assumption, we are able to dominate

$$-4 \int |\nabla \sqrt{n}|^2 + 2\beta \frac{\gamma}{\delta} \|f\|_\infty^{\gamma-\delta} \int |\nabla \sqrt{n}| \cdot |\sqrt{n} \nabla f^\delta| - \delta\alpha \int n |\nabla f^\delta|^2.$$

A first condition appears for the homogeneity of α and β, for this expression to be non-positive.

$$\left(\beta \|f\|_\infty^{\gamma-\delta} \frac{\gamma}{\delta}\right)^2 \equiv \alpha\delta.$$

The same computation arises for the other term of the same type $-\frac{\delta}{2}\alpha \int \nabla n \cdot \nabla f^{2\delta}$ and we get another homogeneity condition

$$\delta\alpha \|f\|_\infty^{2\delta} \equiv 1.$$

3.2. What about $\int n\nabla c \cdot \nabla f^\gamma$?

We can combine this no-sign term in a general way

$$\begin{aligned}\int n|\nabla c\cdot\nabla f^\gamma| &= \frac{\gamma}{\gamma-\xi}\int nf^\xi|\nabla c\cdot\nabla f^{\gamma-\xi}| \\ &\le \left(\frac{\gamma}{\gamma-\xi}\right)^2\frac{K}{2}\int n|\nabla f^{\gamma-\xi}|^2+\frac{1}{2K}\int nf^{2\xi}|\nabla c|^2,\end{aligned}$$

with a homogeneity constant K which has to be determined.
We associate the first r.h.s term with $-\delta\alpha\int n|\nabla f^\delta|^2$. We set $\delta\le\gamma-\xi$ for this purpose.
It follows

$$\chi\frac{K}{2}\beta\left(\frac{\gamma}{\delta}\right)^2\|f\|_\infty^{2(\gamma-\xi-\delta)}\int n|\nabla f^\delta|^2-\alpha\delta\int n|\nabla f^\delta|^2,$$

and we get an additional homogeneity condition for K

$$\chi\beta\|f\|_\infty^{2(\gamma-\xi-\delta)}\left(\frac{\gamma}{\delta}\right)^2K\equiv\alpha\delta.$$

The second r.h.s term will be eliminated thanks to a combination of Sobolev and Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla c\|_4^4 \leq \mathcal{C}_{\mathcal{S}} \|nf\|_{4/3}^4,$$

$$\left(\int n^{4/3} \right)^3 \leq \mathcal{C}_{\mathcal{GNS}} M^3 \int |\nabla \sqrt{n}|^2.$$

So that

$$\int n f^{2\xi} |\nabla c|^2 \leq \frac{L}{2} \int n^2 f^{\omega} + \frac{1}{2L} \int f^{\theta} |\nabla c|^4,$$

with the relation $\omega + \theta = 4\xi$, and also

$$\int f^{\theta} |\nabla c|^4 \leq \mathcal{C}^* \|f\|_{\infty}^{4+\theta} M^3 \int |\nabla \sqrt{n}|^2.$$

Finally we have to deal with the last remaining terms, namely $\int n^2 f^{\omega}$ and $\int n^2 f$.

3.3. The second association

In order to eliminate those two terms, we of course associate them with $\int n^2 f^\gamma$. Only the case $\gamma \geq \max(1, \omega)$ is able to keep the homogeneity of the computations.
We use the following majoration which makes the distinction between high and low values of f

$$X^\omega \leq \kappa^{-\omega} c_\nu + \kappa^{\gamma-\omega} X^\gamma \,, \quad \omega\nu = \gamma,$$

with the constant $E_\nu = c_\nu^{\nu-1} = \frac{(\nu-1)^{(\nu-1)}}{\nu^\nu}$.

For each term $\int n^2 f^\omega$ and $\int n^2 f$ we get two new terms involving $\int n^2$ and $\int n^2 f^\gamma$.
We can use the first cited G.N.S. inequality to estimate $\int n^2$: we have determinated all the homogeneity constants introduced

$$\chi\beta K^{-1} L^{-1} \|f\|_\infty^{\theta+4} M^3 \equiv 1,$$

$$\chi \left(c_\gamma \chi M\right)^{\gamma-1} \equiv \gamma\beta,$$

$$\chi\beta K^{-1} L \left(\chi\beta K^{-1} L c_\nu M\right)^{\nu-1} \equiv \gamma\beta.$$

3.4. Consequences of the homogeneity relations and conclusion

Using these six homogeneity conditions, we can eliminate all the intermediate parameters, and finally we get two different consequences of these relations

$$E_\gamma \chi^\gamma \|f\|_\infty^\gamma M^{\gamma-1} \equiv \delta,$$

and

$$E_\nu \chi^{4\nu} \|f\|_\infty^{4\nu} M^{4\nu-1} \equiv \delta.$$

Consequently we assume $\gamma = 4\nu$ to unify these two relations, which forces $\gamma \geq 4$ and $\delta \leq \gamma - \xi \leq \gamma - \frac{1}{4}(\omega + \theta) \leq \gamma - 1$.

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