Contemporary Mathematics

A dichotomy for finitely generated subgroups of word hyperbolic groups

Goulnara N. Arzhantseva

Abstract. Given $L > 0$ elements in a word hyperbolic group G, there exists a number $M = M(G, L) > 0$ such that at least one of the assertions is true: (i) these elements generate a free and quasiconvex subgroup of G; (ii) they are Nielsen equivalent to a system of L elements containing an element of length at most M up to conjugation in G. The constant M is given explicitly. The result is generalized to groups acting by isometries on Gromov hyperbolic spaces. For proof we use a graph method to represent finitely generated subgroups of a group.

1. Introduction

Let H be a subgroup of a word hyperbolic group G. It is known that either H is elementary (that is, it contains a cyclic subgroup of finite index) or H contains a non-abelian free subgroup of rank two. In the case G is torsion-free, there are, up to conjugacy, finitely many Nielsen equivalence classes of non-free subgroups of G generated by two elements [**3**].

Our main result gives a sufficient condition for H to be free and quasiconvex in G. It is an improvement of a result due to Gromov [**4**, 5.3.A].

Theorem 1. *For any $\delta \geq 0$ and an integer $L > 0$ there exists a number $M = M(\delta, L) > 0$ with the following property.*

Let G be a δ-hyperbolic group with respect to a finite generating set $\mathcal{X}$ and H be a subgroup of G generated by $h_1, \cdots, h_L$. Then at least one of the following assertions is true.

(i) *H is free on $h_1, \cdots, h_L$ and quasiconvex in G;*
(ii) *The tuple $(h_1, \ldots, h_L)$ is Nielsen equivalent to an L-tuple $(h'_1, \ldots, h'_L)$ with h'_1 conjugate to an element in G of word length at most M with respect to $\mathcal{X}$.*

2000 *Mathematics Subject Classification.* 20F67;20E07.

Key words and phrases. Word hyperbolic groups, quasigeodesics, Nielsen equivalence.

The work has been supported in part by the Swiss National Science Foundation, No. PP002-68627.

The constant $M = M(\delta, L)$ can be calculated explicitly. Then, as an immediate consequence of Theorem 1 we obtain Gromov's result. Let

$$\mathsf{Rad}\ H = 1/2 \inf_{h \in H \setminus \{\mathsf{id}\}} |[h]|,$$

where $|[h]| = \inf_{h'} |h'|$ for all $h' \in G$ conjugate to h and $|h|$ is the word length of h. This is called the injectivity radius of H.

THEOREM 2. (cf. [**4**, 5.3.A]) Let G be a δ-hyperbolic group and H be a subgroup of G generated by L elements. If

$$\mathsf{Rad}\ H \geq \frac{1}{2}\left(\frac{14400(\delta+1)}{\ln 2} L^2\right)^{1+1/2},$$

then H is free and quasiconvex[1] in G.

The following result is a natural generalization of Theorem 1 to groups acting on hyperbolic spaces.

THEOREM 3. *For any $\delta \geq 0$ and an integer $L > 0$ there exists a number $M = M(\delta, L) > 0$ with the following property.*

Let G be a group on L generators $g_1, \cdots, g_L$ acting on a δ-hyperbolic space (X, d) by isometries. Then at least one of the following assertions is true.

(i) *G is free on $g_1, \cdots, g_L$ and for every $x \in X$ the map $G \to X$ which assigns to each $g \in G$ the element $gx \in X$ is a quasi-isometric embedding;*
(ii) *The tuple $(g_1, \ldots, g_L)$ is Nielsen equivalent to an L-tuple $(g'_1, \ldots, g'_L)$ with $d(g'_1 y, y) < M$ for some $y \in X$.*

The constant $M = M(\delta, L)$ can be calculated explicitly.

For proof of Theorem 1 a representation of finitely generated subgroups of a group by labelled graphs is used [**1**]. The technique is of independent interest. In particular, transformations of a labelled graph defined below can be viewed as a generalization of free reductions and Nielsen reductions of tuples of group elements.

Theorem 3 is obtained with essentially the same methods, except that, in our arguments, instead of the word length metric, we refer to the metric on G induced from the group action.

I was informed that Kapovich and Weidmann showed both results (without an explicit estimate on the constant) independently of my work and by different methods [**6**].

Acknowledgments This work has been done during my visit to University Luis Pasteur, Strasbourg, May 2000. I thank Thomas Delzant for suggesting the problem and hospitality. I also thank the referee for useful comments.

2. Auxiliary information

Let G be a group and $\mathcal{X}$ be a finite set of generators for G. We fix both for the rest of the paper. All words are assumed to be in the alphabet $\mathcal{X}^{\pm 1}$. We shall make no essential distinction between words and elements of G. If w and v are words then the notation $w =_G v$ means that they represent the same group element.

[1]This lower bound is not optimal. For example, one can take $\frac{1}{2}\left(\frac{14400(\delta+1)}{\ln 2} L^2\right)^{1+\varepsilon}$ with $\varepsilon > 0$ or a better constant satisfying condition (4) below.

2.1. Graphs representing subgroups. Let Γ be a graph. By an edge of Γ we mean a directed edge, i.e., an edge of Γ in the usual sense with any of its two possible directions. If e is an edge of Γ then e^{-1} denotes the edge with the opposite direction. A map ψ from the edges of Γ to $\mathcal{X}^{\pm 1}$ is called a *labelling function* on Γ if it satisfies $\psi(e^{-1}) = (\psi(e))^{-1}$ for any edge e. By the label $\psi(p)$ of a path $p = e_1 e_2 \dots e_k$ of length k in Γ we mean the word $\psi(e_1)\psi(e_2)\dots\psi(e_k)$. The label of a path of length 0 (which by definition is identified with a vertex of Γ) is the empty word.

A *labelled graph* is a finite connected graph Γ with a labelling function ψ and a distinguished vertex O. Any labelled graph Γ represents a subgroup $H(\Gamma)$ of a free group $F = F(\mathcal{X})$, which is the image of the fundamental group $\pi_1(\Gamma, O)$ under the homomorphism induced by ψ. In other words, $x \in H(\Gamma)$ if and only if x may be represented by a word which can be read on a circuit at O.

It is easy to see that any finitely generated subgroup $H \leq F$ may be represented by a labelled graph. To do this, we first take words $h_1, h_2, \cdots, h_k$ in the alphabet $\mathcal{X}^{\pm 1}$ that represent generators of H. Next we take a rose of k circles attached to a point O and make each of the circles a circuit labelled h_i, $1 \leq i \leq k$. For the resulting labelled graph Γ, we obviously have $H(\Gamma) = H$.

We define two types of transformations of a labelled graph Γ, which preserve the subgroup $H(\Gamma)$ and which we call *reductions*. A transformation of the first type is identification of two edges with the same label and the same initial vertex. A transformation of the second type is removal of a vertex of degree 1 other than O, together with the incident edge.

A labelled graph Γ is said to be *reduced* if it admits no reductions, that is, it has no pair of edges with the same label and initial vertex and no vertices of degree 1 with the possible exception of the distinguished vertex O.

Starting from a labelled graph Γ with $H(\Gamma) = H$ and performing all possible reductions, we reach a reduced labelled graph which represents the subgroup H. It is known [**10, 8**] that a reduced labelled graph representing a subgroup $H \leq F$ is unique up to graph isomorphism (that is, it does not depend on the order of reductions, the choice of the initial graph Γ, and the choice of generators for H).

If Γ is a reduced labelled graph then it is easy to see that a reduced word w represents an element of $H(\Gamma)$ if and only if w is the label of a reduced circuit at O in Γ. It follows in particular that the label of a path p in Γ starting at O represents an element of $H(\Gamma)$ only if O is also the terminal vertex of p.

A finitely generated subgroup H of G can also be presented by a labelled graph Γ. It suffices to consider the graph obtained from a lift of the subgroup generators under the natural homomorphism $F \to G$. However the reduced form of Γ is not unique in this case. For Γ representing $H \leq G$ we introduce a transformation of the third type as well, see [**1**].

Denote by p_- (p_+) the initial (terminal) vertex of a path p. By an *arc* we mean a path p all of whose vertices except p_- and p_+ have degree 2 and are distinct from the distinguished vertex O.

— (*arc reduction*) Let vertices O_1 and O_2 in Γ be joined by a path p so that $\psi(p) \equiv w$ and the word w is equal to some word v in G. Let q' be an arc in Γ which is a subpath of p. First let us add to Γ a new graph formed by a single arc q with label $\psi(q) \equiv v$ such that $O_1 = q_-$ and $O_2 = q_+$

are the only common points of the new graph and Γ. Then let us remove from Γ all edges and vertices of q' except q'_- and q'_+.

Remark. We remove an arc whenever we add another one. Hence the transformations preserve the Euler characteristic of Γ which is the number of vertices minus the number of edges. Thus if the fundamental group $\pi_1(\Gamma)$ of Γ is generated by L loops, then for any graph Γ' obtained from Γ by the transformations, $\pi_1(\Gamma')$ is L-generated as well.

Lemma 4. [**1**, Lemma 1] *If a labelled graph Γ' is carried into a labelled graph Γ by transformations of types 1–3 or their inverses, then it represents the same subgroup of G as Γ does.*

2.2. Nielsen equivalence.

Definition 5. Let $\mathcal{U} = (u_1, \ldots, u_k)$ and $\tilde{\mathcal{U}} = (\tilde{u}_1, \ldots, \tilde{u}_k)$ be k-tuples of elements of G. They are *Nielsen equivalent* if $\mathcal{U}$ can be carried into $\tilde{\mathcal{U}}$ by a finitely many *regular elementary Nielsen transformations* defined as follows:

(t1) replace u_i by u_i^{-1} for some i;
(t2) replace u_i by $u_i u_j$ for some $i \neq j$;

In both cases u_t remains unchanged if $t \neq i$.

Remark. If $w_1, \cdots, w_k$ are words we shall consider $(w_1, \ldots, w_k)$ as the k-tuple of group elements represented by these words.

Note that the regular elementary Nielsen transformations generate a group containing every permutation of the u_i. It is obvious that if a k-tuple $\mathcal{U}$ is carried by a regular elementary Nielsen transformation into a k-tuple $\tilde{\mathcal{U}}$, then they generate the same subgroup of G. Proceeding by induction we see that Nielsen equivalent tuples of group elements generate the same subgroup of G. Moreover, in a free group any two bases of a finitely generated subgroup are Nielsen equivalent. Here by a *basis* of a subgroup we mean a set of group elements freely generating the subgroup. In general, this is not true for a non-free group G. However we have the following

Lemma 6. *Let H be a subgroup of G generated by elements represented by words $h_1, \cdots, h_k$. Let Γ be a graph obtained from a rose Ω of k circuits at a vertex O labelled by $h_1, \cdots, h_k$ by transformations of 1-3 types or their inverses. Then for any basis $l_1, \cdots, l_k$ of $\pi_1(\Gamma, O)$ the k-tuple $(\psi(l_1), \ldots, \psi(l_k))$ is Nielsen equivalent to $(h_1, \ldots, h_k)$.*

Proof. Let Γ be obtained from Ω by transformations of types 1–3 or their inverses. Hence there is a sequence of graphs $\Gamma_0 = \Omega, \ldots, \Gamma_j, \ldots, \Gamma_f = \Gamma$, so that Γ_i is carried into Γ_{i+1} by only one transformation. If $f = 0$, i.e. Γ coincides with Ω, the lemma obviously holds. Suppose that $f > 0$. We *claim* that for each basis $s_1, \cdots, s_k$ of $\pi_1(\Gamma_{j+1}, O)$ there exists a basis $t_1, \cdots, t_k$ of $\pi_1(\Gamma_j, O)$ such that $(\psi(t_1), \ldots, \psi(t_k))$ is Nielsen equivalent to $(\psi(s_1), \ldots, \psi(s_k))$. This is obvious if the performed transformation is a reduction. For t_i we take a loop in Γ_j which were carried by the reduction into s_i, $1 \leq i \leq k$ (possibly t_i and s_i coincide). Let's now consider an arc reduction. Let q be the added arc and q' the removed arc. By definition, there is a path $p \in \Gamma_j$ with the same endpoints as $q \in \Gamma_{j+1}$ such that p contains q' as a subpath and $\psi(p) =_G \psi(q)$ in G. Let's define $t_i \in \Gamma_j$. We start with $s_i \in \Gamma_{j+1}$. We replace each entry of q in s_i by p. In such a way we obtain loops t_i at O in Γ_j, $1 \leq i \leq k$. Clearly, it will be a basis of $\pi_1(\Gamma_j, O)$. The k-tuple of elements represented by labels of these loops is Nielsen equivalent to $(\psi(s_1), \dots, \psi(s_k))$. Indeed, we have $\psi(s_i) =_G \psi(t_i)$ as $\psi(p) =_G \psi(q)$ in G.

The claim is true for any $1 \leq j \leq f$. Thus, for any basis $l_1, \cdots, l_k$ of $\pi_1(\Gamma, O)$ there exists a basis $l'_1, \cdots, l'_k$ of $\pi_1(\Omega, O)$ such that $(\psi(l_1), \dots, \psi(l_k))$ is Nielsen equivalent to $(\psi(l'_1), \dots, \psi(l'_k))$. As it was mentioned above the labels of any two bases of $\pi_1(\Omega, O)$ represent Nielsen equivalent k-tuples. Hence $(\psi(l'_1), \dots, \psi(l'_k))$ is Nielsen equivalent to $(h_1, \dots, h_k)$. This completes the proof. □

2.3. Word hyperbolic groups. For a background on word hyperbolic groups we refer to [**4, 5**], and [**2**].

Let $C(G)$ be the *Cayley graph* of G with respect to $\mathcal{X}$. It is a graph whose set of vertices is G and whose set of edges is $G \times (\mathcal{X}^{\pm 1})$. An edge (h, x) starts at the vertex $h \in G$ and ends at the vertex hx. We consider an edge (h, x) of $C(G)$ as labelled by the letter x. The label $\varphi(\rho)$ of a path $\rho = e_1 e_2 \dots e_n$ in $C(G)$ is the word $\varphi(e_1)\varphi(e_2)\dots\varphi(e_n)$ where $\varphi(e_i)$ is the label of the edge e_i. We regard $\varphi(\rho)$ as an element of G. We endow $C(G)$ with a metric by assigning to each edge the metric of the unit segment $[0, 1]$ and then defining the distance $|g - h|$ to be the length of a shortest path between g and h. Thus $C(G)$ becomes a geodesic metric space. Obviously, this metric on $C(G)$ is invariant under the natural left action of G. For any $g \in G$, we write $|g|$ for the length of a shortest path from the unit element to g. In particular, $|g - h| = |g^{-1}h|$.

Definition 7. [**4**, 6.3.B] The *Gromov product* of points x and y of a metric space $\mathcal{M}$ with respect to a point $z \in \mathcal{M}$ is defined to be

$$(x|y)_z = \frac{1}{2}(|x - z| + |y - z| - |y - x|)$$

where $|x - y|$ denotes the distance between x and y.

A geodesic metric space $\mathcal{M}$ is called *δ-hyperbolic*, for $\delta \geq 0$, if

$$(x|y)_w \geq \min\{(x|z)_w, (z|y)_w\} - \delta$$

for any $x, y, z, w \in \mathcal{M}$.

A group G is *δ-hyperbolic* with respect to a finite generating set $\mathcal{X}$ if the Cayley graph $C(G)$ with respect to $\mathcal{X}$ is a δ-hyperbolic space. The group G is called *word hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$ and $\mathcal{X}$.

It turns out that the word hyperbolicity of a group is independent of the finite generating set chosen [**4**, 2.3.E].

Lemma 8. [**9**, Lemma 21] *Let $c \geq 14\delta$ and $c_1 > 12(c + \delta)$, and suppose that a geodesic n-gon $[x_1, \dots, x_n]$ in a δ-hyperbolic metric space satisfies the conditions $|x_{i-1} - x_i| > c_1$ for $i = 2, \dots, n$ and $(x_{i-2}|x_i)_{x_{i-1}} < c$ for $i = 3, \dots, n$. Then the polygonal line $\rho = [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{n-1}, x_n]$ is contained in the $2c$-neighbourhood of the side $\tau = [x_n, x_1]$, and the side τ is contained in the 14δ-neighbourhood of ρ.*

Corollary 9. *Under the assumptions of the previous lemma there is a constant $\lambda = \lambda(c, c_1) > 0$ such that*

$$|\tau| \geq \lambda \|\rho\|,$$

where $\|\rho\|$ is the length of the path ρ.

PROOF. By the lemma, there exist points $v_1, \ldots, v_n$ on $[x_n, x_1]$ such that $|x_i - v_i| \leq 2c$, $1 \leq i \leq n$. The hypothesis of the lemma imply easily that v_{i-1} is located between v_i and v_{i-2}. So,

$$|\tau| = \sum_{i=2}^{n} |v_{i-1} - v_i| \geq \sum_{i=2}^{n} (|x_{i-1} - x_i| - 4c) = \|\rho\| - (n-1)4c.$$

Taking $\lambda = c/c_1$ and using $\|\rho\| > (n-1)c_1$ we obtain $|\tau|/\|\rho\| \geq \lambda$. $\square$

The following lemma is obvious as any side of a geodesic triangle in a δ-hyperbolic space belongs to the 4δ-neighbourhood of the union of the other two sides [**5**, Ch.2, Pr. 21].

LEMMA 10. (cp. [**4**, § 7]) Let ρ be a path in a δ-hyperbolic space. Then for every point A on a geodesic segment with the same endpoints as ρ

$$\inf_{B\in\rho} |A - B| \leq 4\delta \log_2 \|\rho\| + 1.$$

For a word w, the length $\|w\|$ is the length of a path in $C(G)$ labelled by w and $|w|$ is the length of a geodesic in $C(G)$ between the same endpoints. If $\|w\| = |w|$ the word w is called a *geodesic word*. The notation $w \equiv xy$ means that w can be decomposed, as a word, into a product of two words which represent elements $x, y \in G$. The following fact is known (see the proof of $(P1, \delta) \Rightarrow (P2, 4\delta)$ in [**5**, Ch.2, Pr. 21]). If G is δ-hyperbolic then

(H) for any two geodesic words u and v, if $u \equiv u_1u_2$, $v \equiv v_1v_2$ and $|u_1| = |v_1| \leq \frac{1}{2}(|u| + |v| - |u^{-1}v|)$ then $|u_1^{-1}v_1| \leq 4\delta$.

Notice that if G is δ-hyperbolic then it is also δ'-hyperbolic for any $\delta' > \delta$. So we can always assume $\delta \geq 1$.

3. Proof of Theorem 1

From now on, we assume G to be word hyperbolic and fix a number $\delta \geq 1$ such that (H) *holds.*

The following two lemmas are the main technical tools for the proof of our theorems.

LEMMA 11. *Let $\delta \geq 1$, $K > 16\delta + 1$. Let x be a word with $|x| < K \log_2 \|x\|$ and $\|x\| \geq 2^4$. Then there exists a subword y of x such that*

$$\tfrac{1}{2}\|x\| \leq \|y\| < \|x\| \quad \textit{and} \quad |y| < K \log_2 \|y\|.$$

PROOF. We denote by η a path starting at the unit vertex of the Cayley graph $C(G)$ and labelled by x. Let ρ be a geodesic path in $C(G)$ with the same endpoints as η and z be the label of ρ. Note that $\|z\| = |x|$. We take a middle point A on ρ so that $z = z_1z_2$, where A is a terminal vertex of a subpath labelled by z_1 and $\|z_1\| = \left[\frac{\|z\|}{2}\right]$.

Suppose that $\|z\| < 9\delta \log_2 \|x\|$. Then, for a desired subword y we take x without its terminal letter, i.e. $\|y\| = \|x\| - 1$. The assumption on $\|x\|$, $|x|$, and K implies easily the needed inequalities on $\|y\|$ and $|y|$.

The remaining case is $\|z\| \geq 9\delta \log_2 \|x\|$. By Lemma 10, there is a point B on η such that $|A-B| \leq 4\delta \log_2 \|\eta\|+1 = 4\delta \log_2 \|x\|+1$. Then B gives a decomposition of η into two subpaths labelled by words x_1 and x_2 with $x = x_1x_2$ and

$$|x_i| \leq \frac{K}{2} \log_2 \|x\| + 4\delta \log_2 \|x\| + 1 = \left(\frac{K}{2} + 4\delta\right) \log_2 \|x\| + 1.$$

The words x_1 and x_2 are nontrivial which easily follows from the assumption on $\|z\|$ and the bound on $|A-B|$. Hence $\frac{\|x\|}{2} \leq \|x_i\| < \|x\|$ for $i = 1$ or $i = 2$. Without loss of generality, we assume that $\|x_1\| \geq \|x_2\|$. Since $\|x\| \geq 2^4$ and $K > 16\delta + 1$ we have

$$|x_i| \leq \left(\frac{K}{2} + 4\delta\right) \log_2 \|x\| + 1 < K \log_2 \frac{\|x\|}{2} \leq K \log_2 \|x_1\|.$$

Thus, we can take x_1 for a desired subword y of x. □

Corollary 12. *Let $\delta \geq 1$, $K > 16\delta + 1$. Let $D \geq 2^4$ and x be a word with $|x| < K \log_2 \|x\|$ and $\|x\| \geq D$. Then there is a subword y of x such that*

$$\tfrac{D}{2} \leq \|y\| < D \quad \textit{and} \quad |y| < K \log_2 D.$$

Proof. By Lemma 11, there is a subword y of x with

$$\|y\| \geq \frac{D}{2} \text{ and } |y| < K \log_2 \|y\|. \tag{1}$$

We take such a y of the minimal possible length. We have $\|y\| < D$ for otherwise using the previous lemma we could find a subword y' of y satisfying (1) with $\|y'\| < \|y\|$. Hence $|y| < K \log_2 \|y\| < K \log_2 D$. □

Lemma 13. *Let $\delta \geq 1$. For any $T > 2^{\delta+7}$ there are numbers $\lambda = \lambda(T,\delta) > 0$ and $D_1 = D_1(T,\delta) > 0$ with the following property:*
Let x be a word with $\|x\| \geq D_1$. If $|y| \geq 20 \left(\frac{\delta T}{\log_2 T}\right)^{1/2} \log_2 \|y\|$ for any subword y of x with $\|y\| \geq T$ then $|x| \geq \lambda\|x\|$.

Proof. Set $c = T$, $c_1 = 12(c+\delta)+2$, and $D_1 = 2^{\frac{c_1}{K}}$, where $K = 20 \left(\frac{\delta T}{\log_2 T}\right)^{1/2}$. Take $\lambda = \lambda(c, c_1)$ by Corollary 9.

Suppose the lemma does not hold. Then there is a word x with $\|x\| \geq D_1$ and $|x| < \lambda\|x\|$ such that for any subword y of x with $\|y\| \geq T$ the inequality $|y| \geq K \log_2 \|y\|$ holds for $K = 20 \left(\frac{\delta T}{\log_2 T}\right)^{1/2}$.

We take any decomposition $x \equiv x_1x_2 \dots x_s$ where $D_1 \leq \|x_i\| \leq 2D_1$ for $1 \leq i \leq s$. For each x_i, we choose a shortest word z_i representing the same element of G. It is easy to see that $D_1 > T$. Then, by our assumption we have $\|z_i\| = |x_i| \geq K \log_2 \|x_i\| \geq K \log_2 D_1 \geq c_1$.

Let ρ be a path in $C(G)$ labelled with $x_1x_2 \dots x_s$. Each z_i labels a geodesic path with the same endpoints as the subpath of ρ labelled with x_i. By Lemma 8 applied to the $(s+1)$-gon in $C(G)$ formed by the endpoints of the paths labelled with x_i, for some i we have

$$|z_iz_{i+1}| < \|z_i\| + \|z_{i+1}\| - 2c. \tag{2}$$

Let us decompose z_i and z_{i+1} so that $z_i \equiv y_i z_i'$ and $z_{i+1} \equiv z_{i+1}' y_{i+1}$ with $\|z_i'\| = \|z_{i+1}'\| = c$ for some words y_i, y_{i+1}. By (2) and (H), $|z_i' z_{i+1}'| \le 4\delta$. By Lemma 10 we find a terminal subword x_i' of x_i and an initial subword x_{i+1}' of x_{i+1} such that $|x_i'(z_i')^{-1}| \le 4\delta \log_2 \|x_i\| + 1$ and $|(z_{i+1}')^{-1} x_{i+1}'| \le 4\delta \log_2 \|x_{i+1}\| + 1$. Thus,

$$|x_i' x_{i+1}'| \le 8\delta \log_2 2D_1 + 2 + 4\delta.$$

Since $\|x_i'\|, \|x_{i+1}'\| \ge c - 4\delta \log_2 2D_1 - 1$ we have

$$\|x_i' x_{i+1}'\| \ge 2c - 8\delta \log_2 2D_1 - 2.$$

It is easy to check that for $T > 2^{\delta+7}$ we have $20\left(\frac{\delta T}{\log_2 T}\right)^{1/2} \ge 104\delta$ which implies $c \ge 8\delta\left(1 + \frac{c_1}{K}\right) + 2$. From the last inequality we deduce that $2c - 8\delta \log_2 2D_1 - 2 \ge T$.

Now we prove that $8\delta \log_2 2D_1 + 2 + 4\delta < K \log_2(2c - 8\delta \log_2 2D_1 - 2)$. Indeed, since for $K \ge 104\delta$ we have $2c - 8\delta\left(1 + \frac{c_1}{K}\right) - 2 \ge c$, it suffices to verify that

$$8\delta\left(1 + \frac{c_1}{K}\right) + 2 + 4\delta < K \log_2 c.$$

Using $K \ge 104\delta$ and $c_1 = 12(c+\delta)+2$ we obtain $8\delta\left(1 + \frac{c_1}{K}\right) + 2 + 4\delta \le 96\delta\frac{c}{K} + 13\delta + 3$. The latter is less or equal to $112\delta\frac{c}{K}$ as $c > K$ and $\delta \ge 1$. Now by the choice of c and K, $112\delta\frac{c}{K} < K \log_2 c$.

Thus we have found a subword $y \equiv x_i' x_{i+1}'$ of x such that $\|y\| \ge T$ and $|y| < K \log_2 \|y\|$. This contradicts the assumption. $\square$

Proof of Theorem 1. Given $\delta \ge 1$, $L > 0$, and any $K > 16\delta + 1$, we choose $D = D(\delta, L)$ by the inequality

$$\frac{D}{\log_2 D} > 6KL. \tag{3}$$

Let G be a δ-hyperbolic group, H be a subgroup of G generated by L elements represented by words $h_1, \cdots, h_L$. Let Ω be a rose of L circuits at a vertex O labelled by $h_1, \cdots, h_L$. Let $\Gamma = \Gamma(H)$ be a graph representing H which is obtained from Ω by transformations of types 1–3 and has the minimal possible number of edges. In particular, Γ is reduced and $\pi_1(\Gamma)$ is L-generated. By Lemma 6, for any basis of $\pi_1(\Gamma, O)$, its image in G under the labelling function is Nielsen equivalent to the tuple $(h_1, \ldots, h_L)$.

Suppose that H is not free on generators represented by $h_1, \cdots, h_L$. Then there is a closed reduced path p in Γ starting at O labelled by a nontrivial word x representing the identity element of G, i.e. $x =_G 1$. We take such a p of minimal length. There are two cases.

First suppose $\|x\| < D$. Then p contains a simple circuit ν of length $< D$ as a subpath (possibly, $\nu = p$). Let μ be any reduced path starting at O and ending at a vertex v on ν. Then the label of $\mu\nu\mu^{-1}$ represents an element $h_1' \in H$. Obviously, h_1' is conjugate to an element of length less then D that is represented by the label of ν. Moreover, h_1' can be included in a system of generators of H. Indeed, suppose that $\nu = \nu_1 e \nu_2$, where ν_1 starts at v and e is an edge of ν. Then the tripod rooted at v consisting of tree branches μ, ν_1, and ν_2, can be included in a maximal tree spanning Γ. Hence h_1' is the label of one of L generators of $\pi_1(\Gamma, O)$ given by this maximal tree. Thus, by Lemma 6, $(h_1, \ldots, h_L)$ is Nielsen equivalent to an L-tuple $(h_1', \ldots, h_L')$ and the conclusion (ii) of the theorem holds.

The remaining case is $\|x\| \geq D$. Since $x =_G 1$, we have $0 = |x| < K \log_2 \|x\|$. By Corollary 12, there is a subword y of x with $\frac{D}{2} \leq \|y\| < D$ and $|y| < K \log_2 D$. We may assume that y labels a simple path γ. Otherwise, γ contains a simple circuit ν of length $< D$ as a subpath and we proceed as above. The number of arcs in Γ is less then $3L$. Since $\|y\| \geq \frac{D}{2}$, there is a subword u of y of length at least $\frac{D}{6L}$ which labels an arc. Using a transformation of Γ of the third type, we remove this arc and add a new arc of length $|y|$ with the same endpoints as γ. We label this arc by a shortest word representing the same group element as y. By (3) and the choice of y, the length of the new arc is less then $\frac{D}{6L}$. So, the number of edges in the obtained graph is less than one in Γ. This contradicts the choice of Γ.

Suppose that H is free on $h_1, \cdots, h_L$ but not quasiconvex. We are going to find a constant $T = T(\delta, L)$ such that $(h_1, \ldots, h_L)$ is Nielsen equivalent to an L-tuple containing an element conjugate to an element of length at most T. Take any $T > 2^{\delta+7}$ satisfying

$$\frac{T}{\log_2 T} > 6KL, \tag{4}$$

where $K = 20 \left(\frac{\delta T}{\log_2 T} \right)^{1/2}$. Choose $\lambda = \lambda(T, \delta) > 0$ and $D_1 = D_1(T, \delta) > 0$ by Lemma 13.

Since H is supposed to be free but non-quasiconvex there are a reduced circuit at O in Γ labelled by a word z and a subword x of z such that either x represents the identity element of G or $\|x\| \geq D_1$ and $|x| < \lambda \|x\|$.

We repeat the proof as above in the case when $x =_G 1$ slightly modifying the subcase $\|x\| < D_1$. Namely, if x labels a simple path then we identify the endpoints of this path removing an arc of this path. Since $x =_G 1$ we obtain a new labelled graph representing H with less number of edges than one in Γ. This is a contradiction. If the path labelled by x is not simple it contains a circuit of length less than D_1.

In the second case, by Lemma 13, there is a subword y of x such that $\|y\| \geq T$ and $|y| < K \log_2 \|y\|$. Using Corollary 12 we can assume that $T/2 \leq \|y\| < T$. Then, arguing as above we conclude that either there is a circuit of length less than T in Γ or we can use a transformation of the third type reducing the number of edges in Γ.

We take $M = \max\{D, T\}$ finishing the proof. □

Thus, $M = \max\{D, T\}$ with constants defined by (3) and (4). It is now a routine to check that Theorem 2 is a straightforward consequence of Theorem 1.

Theorem 3 can be shown by mimic arguments above where the word length is replaced by the length function induced by the group action. Namely, if $x_0 \in X$ is arbitrary, then the length of an element $g \in G$ is defined by $\ell(g) := d(gx_0, x_0)$ and the left-invariant distance between group elements g and h is given by $\ell(g^{-1}h)$.

References

[1] G.N.Arzhantseva and A.Yu.Ol'shanskii, *The class of groups all of whose subgroups with lesser number of generators are free is generic*, Mat. Zametki, **59(4)**(1996), 489-496 (in Russian), English translation in Math. Notes, **59(4)**(1996), 350-355.

[2] M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov*, Lecture Notes in Mathematics, 1441, Springer-Verlag, 1990.

[3] T. Delzant, *Sous-groupes deux générateurs des groupes hyperboliques*, Group theory from a geometrical viewpoint (Trieste, 1990), 177–189, World Sci. Publishing, River Edge, NJ, 1991.
[4] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., 8, Springer, New York-Berlin, 1987, 75–263.
[5] E. Ghys and P. de la Harpe (editors), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, 83, Birkhauser, 1990.
[6] I. Kapovich and R. Weidmann, *Nielsen methods and groups acting on hyperbolic spaces,* Geom. Dedicata 98 (2003), 95–121.
[7] R. Lyndon and P. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.
[8] S. W. Margolis and J. C. Meakin, *Free inverse monoids and graph immersions*, Int. J. Algebra and Comput., **3(1)** (1993), 79–99.
[9] A. Yu. Ol'shanskii, *Periodic quotients of hyperbolic groups,* Mat. Zbornik, **182(4)**(1991), 543-567 (in Russian), English translation in Math. USSR Sbornik **72**(2) (1992).
[10] J. Stallings, *Topology of finite graphs*, Inv.Math., **71** (1983), 552–565.

Université de Genève, Section de mathématiques, 2-4, rue du Lièvre, CH-1211 Genève 24

E-mail address: Goulnara.Arjantseva@math.unige.ch