GROWTH TIGHT ACTIONS

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Abstract. We introduce and systematically study the concept of a growth tight action. This generalizes growth tightness for word metrics as initiated by Grigorchuk and de la Harpe. Given a finitely generated, non-elementary group G acting on a G–space $\mathcal{X}$, we prove that if G contains a strongly contracting element and if G is not too badly distorted in $\mathcal{X}$, then the action of G on $\mathcal{X}$ is a growth tight action. It follows that if $\mathcal{X}$ is a cocompact, relatively hyperbolic G–space, then the action of G on $\mathcal{X}$ is a growth tight action. This generalizes all previously known results for growth tightness of cocompact actions: every already known example of a group that admits a growth tight action and has some infinite, infinite index normal subgroups is relatively hyperbolic, and, conversely, relatively hyperbolic groups admit growth tight actions. This also allows us to prove that certain CAT(0) graphs of groups and snowflake groups admit cocompact, growth tight actions. These provide first examples of non-relatively hyperbolic groups admitting interesting growth tight actions. Our main result applies as well to cusp uniform actions on hyperbolic spaces and to the action of the mapping class group on Teichmüller space with the Teichmüller metric. Towards the proof of our main result, we give equivalent characterizations of strongly contracting elements and produce new examples of group actions with strongly contracting elements.

0. Introduction

The growth exponent of a set A with respect to a pseudo-metric d is

$$\delta_{A,d} := \limsup_{r\to\infty} \frac{\log \#\{a \in A \mid d(o,a) \leqslant r\}}{r}$$

where $\#$ denotes cardinality and $o \in A$ is some basepoint. The limit is independent of the choice of basepoint.

Let G be a finitely generated group. A left invariant pseudo-metric d on G induces a left invariant pseudo-metric $\bar{d}$ on any quotient G/Γ of G by $\bar{d}(g\Gamma, g'\Gamma) := d(g\Gamma, g'\Gamma)$.

Definition 0.1. G is *growth tight* with respect to d if $\delta_{G,d} > \delta_{G/\Gamma,\bar{d}}$ for every infinite normal subgroup $\Gamma \trianglelefteqslant G$.

One natural way to put a left invariant metric on a finitely generated group is to choose a finite generating set and consider the word metric. More generally, pseudo-metrics on a group are provided by actions of the group on metric spaces. Let $\mathcal{X}$ be a G–space, that is, a proper, geodesic metric space with a properly discontinuous, isometric G–action $G \curvearrowright \mathcal{X}$. The choice of a basepoint $o \in \mathcal{X}$ induces a left invariant pseudo-metric on G by $d_G(g, g') := d_{\mathcal{X}}(g.o, g'.o)$.

Define the growth exponent δ_G of G with respect to $\mathcal{X}$ to be the growth exponent of G with respect to an induced pseudo-metric d_G. This depends only on the G–space $\mathcal{X}$, since a different choice of basepoint in $\mathcal{X}$ defines a pseudo-metric that differs from

Date: December 31, 2013.
2010 *Mathematics Subject Classification.* 20F67, 20F65, 37C35, 20E06, 57Mxx.
Key words and phrases. Growth tight, Hopf property, entropy, relatively hyperbolic groups, mapping class groups, snowflake groups, rank 1 isometries, CAT(0) groups.

d_G by an additive constant. Likewise, let $\delta_{G/\Gamma}$ denote the growth exponent of G/Γ with respect to a pseudo-metric on G/Γ induced by $d_{\mathcal{X}}$.

Definition 0.2. $G \curvearrowright \mathcal{X}$ is a *growth tight action* if $\delta_G > \delta_{G/\Gamma}$ for every infinite normal subgroup $\Gamma \trianglelefteqslant G$.

Some groups admit growth tight actions for the simple reason that they lack any infinite, infinite index normal subgroups. For such a group G, every action on a G–space will be growth tight. Exponentially growing simple groups are examples, as, by the Margulis Normal Subgroup Theorem [32], are irreducible lattices in higher rank semi-simple Lie groups.

Growth tightness[1] for word metrics was studied by Grigorchuk and de la Harpe [24], who showed, for example, that a finite rank free group equipped with the word metric from a free generating set is growth tight. On the other hand, they showed that the product of a free group with itself, generated by free generating sets of the factors, is not growth tight. Together with the Normal Subgroup Theorem, these results suggest that for interesting examples of growth tightness we should examine 'rank 1' type behavior. Further evidence for this idea comes from the work of Sambusetti and collaborators, who in a series of papers [41, 42, 43, 18] prove growth tightness for the action of the fundamental group of a negatively curved Riemannian manifold on its Riemannian universal cover.

In the study of non-positively curved, or CAT(0), spaces there is a well established idea that a space may be non-positively curved but have some specific directions that look negatively curved. More precisely:

Definition 0.3 ([5])**.** A hyperbolic isometry of a proper CAT(0) space is *rank 1* if it has an axis that does not bound a half-flat.

In Definition 1.8, we introduce the notion for an element of G to be *strongly contracting* with respect to $G \curvearrowright \mathcal{X}$. In the case that $\mathcal{X}$ is a CAT(0) G–space, the strongly contracting elements of G are precisely those that act as rank 1 isometries of $\mathcal{X}$ (see Theorem 8.2).

In addition to having a strongly contracting element, we will assume that the orbit of G in $\mathcal{X}$ is not too badly distorted. There are two different ways to make this precise.

We say a G–space is *Q–quasi-convex* if there exists a Q–quasi-convex G–orbit (see Definition 1.1 and Definition 1.2). This means that it is possible to travel along geodesics joining points in the orbit of G without leaving a neighborhood of the orbit.

Theorem (Theorem 5.4)**.** *Let G be a finitely generated, non-elementary group. Let $\mathcal{X}$ be a quasi-convex G–space. If G contains a strongly contracting element then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

Alternatively, we can assume that the growth rate of the number of orbit points that can be reached by geodesics lying entirely, except near the endpoints, outside a neighborhood of the orbit is strictly smaller than the growth rate of the group:

[1] Grigorchuck and de la Harpe define growth tightness in terms of 'growth rate', which is just the exponentiation of our growth exponent. The growth exponent definition is analogous to the notion of 'volume entropy' familiar in Riemannian geometry, and is more compatible with the Poincaré series in Section 1.1.

Theorem (Theorem 5.3)**.** *Let G be a finitely generated, non-elementary group. Let $\mathcal{X}$ be a G–space. If G contains a strongly contracting element and there exists a $Q \geqslant 0$ such that the Q–complementary growth exponent of G is strictly less than the growth exponent of G, then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

(See Definition 5.2 for the notion of the Q–complementary growth exponent.)

The proof of Theorem 5.4 is a special case of the proof of Theorem 5.3.

Using Theorem 5.4, we prove:

Theorem (Theorem 7.6)**.** *If $\mathcal{X}$ is a quasi-convex, relatively hyperbolic G–space and G does not coarsely fix a peripheral subspace then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

This generalizes all previously known results for growth tightness of cocompact actions: every already known example of a group that admits a growth tight action and has some infinite, infinite index normal subgroups is relatively hyperbolic, and, conversely, relatively hyperbolic groups admit growth tight actions [2, 40, 52, 42, 39, 18].

We also use Theorem 5.4 to prove growth tightness for actions on non-relatively hyperbolic spaces. For instance, we give examples of non-relatively hyperbolic groups acting on CAT(0) spaces with rank 1 isometries, and we prove that these actions are growth tight:

Theorem (Theorem 8.3)**.** *If G is a finitely generated, non-elementary group and $\mathcal{X}$ is a quasi-convex, CAT(0) G–space such that G contains an element that acts as a rank 1 isometry on $\mathcal{X}$, then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

We even exhibit an infinite family of non-relatively hyperbolic, non-CAT(0) groups that admit cocompact, growth tight actions. These are a subfamily of the Brady-Bridson snowflake groups, and are explored in Section 10.

We prove growth tightness for interesting non-quasi-convex actions using Theorem 5.3. We generalize a theorem of Dal'bo, Peigné, Picaud, and Sambusetti [18] for Kleinian groups satisfying an additional Parabolic Gap Condition, see Definition 7.10, to cusp-uniform actions on arbitrary hyperbolic spaces satisfying the Parabolic Gap Condition:

Theorem (Theorem 7.11)**.** *Let G be a finitely generated, non-elementary group. Let $G \curvearrowright \mathcal{X}$ be a cusp uniform action on a hyperbolic space. Suppose that G satisfies the Parabolic Gap Condition. Then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

Once again, our theorems extend beyond actions on relatively hyperbolic spaces, as we use Theorem 5.3 to prove:

Theorem (Theorem 9.2)**.** *The action of the mapping class group of a hyperbolic surface on its Teichmüller space with the Teichmüller metric is a growth tight action.*

Mapping class groups, barring exceptional low complexity cases, are neither relatively hyperbolic nor CAT(0).

In Part 1 of this paper we prove our main results, Theorem 5.3 and Theorem 5.4.

In Part 2 we give equivalent characterizations of strongly contracting elements (Section 6), and produce new examples of group actions with strongly contracting elements. These include groups acting on relatively hyperbolic metric spaces (Section 7), certain CAT(0) groups (Section 8), mapping class groups (Section 9), and snowflake groups (Section 10). Our main theorems imply that all these groups admit growth tight actions. These are first examples of growth tight actions and groups which do not come from and are not relatively hyperbolic groups.

0.1. **Invariance.** Growth tightness is a delicate condition. A construction of Dal'bo, Otal, and Peigné [17], see Observation 7.9, shows that there exist groups G and non-cocompact, hyperbolic, equivariantly quasi-isometric G–spaces $\mathcal{X}$ and $\mathcal{X}'$ such that $G \curvearrowright \mathcal{X}$ is growth tight and $G \curvearrowright \mathcal{X}'$ is not. For cocompact G–spaces, the question is open:

Question 1. *Is growth tightness invariant among cocompact G–spaces?*

It would be interesting to have a condition to exclude growth tightness. For instance, Coulon [15] has shown that for any non-elementary torsion free hyperbolic group and any finite generating set there exists a sequence of quotients whose growth exponents limit to that of the group, so there is no hope of establishing a uniform gap between the growth exponent of a group and those of all of its quotients. At present, growth tightness can only be excluded for a particular action by exhibiting a quotient of the group by an infinite normal subgroup whose growth exponent is equal to that of the group. Even the following question of Grigorchuk and de la Harpe is still open [24]:

Question 2. *Does there exist a word metric for which $F_2 \times F_2$ is growth tight?*

Recall that $F_2 \times F_2$ is not growth tight with respect to a generating set that is a union of free generating sets of the two factors.

0.2. **The Hopf Property.** A group G is *Hopfian* if there is no proper quotient of G isomorphic to G.

Let $\mathfrak{D}$ be a set of pseudo-metrics on G that is *quotient-closed*, in the sense that if Γ is a normal subgroup of G such that there exists an isomorphism $\phi\colon G \to G/\Gamma$, then for every $d \in \mathfrak{D}$, the pseudo-metric on G obtained by pulling back via ϕ the pseudo-metric on G/Γ induced by d is also in $\mathfrak{D}$. For example, the set of word metrics on G coming from finite generating sets is quotient-closed.

Suppose further that $\mathfrak{D}$ contains a minimal growth pseudo-metric d_0, ie, $\delta_{G,d_0} = \inf_{d\in\mathfrak{D}} \delta_{G,d}$, and that G is growth tight with respect to d_0.

Proposition 0.4. *Let G be a finitely generated group with a bound on the cardinalities of its finite normal subgroups. Suppose that there exists a quotient-closed set $\mathfrak{D}$ of pseudo-metrics on G that contains a growth tight, infimal growth element d_0 as above. Then G is Hopfian.*

The hypothesis on bounded cardinalities of finite normal subgroups holds for all groups of interest in this paper, see Theorem 1.15.

Proof. Suppose that Γ is a normal subgroup of G such that $G \cong G/\Gamma$. Let d be the pseudo-metric on G obtained from pulling back the pseudo-metric on G/Γ induced by d_0. Since $\mathfrak{D}$ is quotient-closed, $d \in \mathfrak{D}$. By minimality, $\delta_{G,d_0} \leqslant \delta_{G,d}$, but by growth tightness, $\delta_{G,d} \leqslant \delta_{G,d_0}$, with equality only if Γ is finite. Thus, the only normal subgroups Γ for which we could have $G \cong G/\Gamma$ are finite. However, if $G \cong G/\Gamma$ for some finite Γ then G has arbitrarily large finite normal subgroups, contrary to hypothesis. $\square$

Grigorchuk and de la Harpe [24] suggested this as a possible approach to the question of whether a non-elementary Gromov hyperbolic group is Hopfian, in the particular case that $\mathfrak{D}$ is the set of word metrics on G. Arzhantseva and Lysenok [2] proved that every word metric on a non-elementary hyperbolic group is growth tight. They conjectured that the growth exponent of such a group achieves its infimum on some generating set and proved a step towards this conjecture [3]. Sambusetti [40] gave an examples of a (non-hyperbolic) group for which the set of word metrics does not realize its infimal growth exponent. In general it is difficult to determine whether a given group has a generating set that realizes the infimal growth exponent among word metrics. Part of our motivation for studying growth tight actions is to open new possibilities for the set $\mathfrak{D}$ of pseudo-metrics considered above.

Remark. Torsion free hyperbolic groups are Hopfian by a theorem of Sela [45]. Reinfeldt and Weidmann [38] have announced a generalization of Sela's techniques to hyperbolic groups with torsion, and concluded that all hyperbolic groups are Hopfian.

0.3. **The Rank Rigidity Conjecture.** The Rank Rigidity Conjecture [14, 6] asserts that if $\mathcal{X}$ is a locally compact, irreducible, geodesically complete CAT(0) space, and G is an infinite discrete group acting properly and cocompactly on $\mathcal{X}$, then one of the following holds:

(1) $\mathcal{X}$ is a higher rank symmetric space.
(2) $\mathcal{X}$ is a Euclidean building of dimension at least 2.
(3) G contains a rank 1 isometry.

In case (1), the Margulis Normal Subgroup Theorem implies that G is trivially growth tight, since it has no infinite, infinite index normal subgroups. Conjecturally, the Margulis Normal Subgroup Theorem also holds in case (2). Theorem 8.3 says that $G \curvearrowright \mathcal{X}$ is a growth tight action in case (3). Thus, a non-growth tight action of a non-elementary group on an irreducible CAT(0) space as above would provide a counterexample either to the Rank Rigidity Conjecture or to the conjecture that the Margulis Normal Subgroup Theorem applies to Euclidean buildings.

It is unclear when growth tightness holds if $\mathcal{X}$ is reducible. It is not hard to show that a direct product of groups acting on a product space with the l_1 metric fails to be growth tight. However, there are also examples [13] of infinite simple groups acting cocompactly on products of trees.

0.4. **Outline of the Proof of the Main Theorem.** Sambusetti [40] proved that a non-elementary free product of non-trivial groups has a greater growth exponent than that of either factor. Thus, a strategy to prove growth tightness is to find a subset of the orbit $\mathcal{G} = G.o$ that looks like a free product, with one factor that grows like the quotient group we are interested in. Specifically:

(1) Find a subset $\mathcal{A} \subset \mathcal{G} \subset \mathcal{X}$ such that $\delta_{\mathcal{A}} = \delta_{G/\Gamma}$. We will obtain $\mathcal{A} = A.o$ as a coarsely dense subset of a minimal section A of the quotient map $G \to G/\Gamma$ (see Definition 3.3).
(2) Construct an embedding from the free product set $\mathcal{A} * \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathcal{X}$. The existence of a strongly contracting element $h \in \Gamma$ is used in the construction of this embedding (see Proposition 4.1).
(3) Show that $\delta_{\mathcal{A},d_{\mathcal{X}}} < \delta_{G/\Gamma * \mathbb{Z}/2\mathbb{Z},d_{\mathcal{X}}}$. In this step it is crucial that $\mathcal{A}$ is divergent (see Definition 1.3 and Lemma 5.1).

This outline, due to Sambusetti, is nowadays standard. Typically step (2) is accomplished by a Ping-Pong argument, making use of fine control on the geometry of the space $\mathcal{X}$. Our methods are coarser than such a standard approach, and therefore can be applied to a wider variety of spaces. We use, in particular, a technique of Bestvina, Bromberg, and Fujiwara [7] to construct an action of G on a quasi-tree. Verifying that the map from the free product set into $\mathcal{X}$ is an embedding amounts to showing that elements in A do not cross certain coarse edges of the quasi-tree.

Part 1. Growth Tight Actions

1. Preliminaries

Fix a G–space $\mathcal{X}$. From now on, d is used to denote the metric on $\mathcal{X}$ as well as the induced pseudo-metric on G and G/Γ. Since there will be no possibility of confusion, we suppress d from the growth exponent notation.

We denote by $\mathcal{B}_r(x)$ the open ball of radius r about the point x and by $\mathcal{N}_r(\mathcal{A}) := \cup_{x\in\mathcal{A}}\mathcal{B}_r(x)$ the open r–neighborhood about the set $\mathcal{A}$. The closed r–ball and closed r–neighborhood are denoted $\overline{\mathcal{B}_r(x)}$ and $\overline{\mathcal{N}_r(\mathcal{A})}$, respectively.

All of the following definitions may be written without specifying C to indicate that some such C exists: Two subsets $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{X}$ are *C–coarsely equivalent* if $\mathcal{A}\subset\overline{\mathcal{N}_C(\mathcal{A}')}$ and $\mathcal{A}'\subset\overline{\mathcal{N}_C(\mathcal{A})}$. A map ϕ is *C–coarsely well defined* if the image of every point is C–coarsely equivalent to a point. Two maps ϕ and ϕ' with the same domain and codomain are *C–coarsely equivalent* or *C–coarsely agree* if $\phi(x)$ is C–coarsely equivalent to $\phi'(x)$ for every x in the domain. A subset $\mathcal{A}$ of $\mathcal{X}$ is *C–coarsely connected* if for every a and a' in $\mathcal{A}$ there exists a chain $a=a_0, a_1,\dots, a_n=a'$ of points in $\mathcal{A}$ with $d(a_i,a_{i+1})\leqslant C$.

Definition 1.1. A subset $\mathcal{A}\subset\mathcal{X}$ is *C–quasi-convex* if for every a_0, $a_1\in\mathcal{A}$ there exists a geodesic γ between a_0 and a_1 such that $\gamma\subset\overline{\mathcal{N}_C(\mathcal{A})}$.

Definition 1.2. A G–space $\mathcal{X}$ is *C–quasi-convex* if it contains a C–quasi-convex G–orbit.

For convenience, if $\mathcal{X}$ is a quasi-convex G–space we will assume we have chosen a basepoint $o\in\mathcal{X}$ such that $G.o$ is quasi-convex.

A group is *elementary* if it has a finite index cyclic subgroup.

We will use notation to simplify some calculations. Let C be a 'universal constant'. For us this will usually mean a constant that depends on $G\curvearrowright\mathcal{X}$ and a choice of $o\in\mathcal{X}$, but not on the point in $\mathcal{X}$ at which quantities a and b are calculated. Then

- for $a\leqslant Cb$ we write $a\overset{*}{\prec} b$,
- for $\frac{1}{C}b\leqslant a\leqslant Cb$ we write $a\overset{*}{\asymp} b$,
- for $a\leqslant b+C$ we write $a\overset{+}{\prec} b$, and
- for $b-C\leqslant a\leqslant b+C$ we write $a\overset{+}{\asymp} b$.

1.1. Poincaré Series and Growth.

Let $(\mathcal{X},o,d)$ be a based pseudo-metric space. Let $|x|:=d(o,x)$ be the induced semi-norm. Define the *Poincaré series* of $\mathcal{A}\subset\mathcal{X}$ to be

$$\Theta_{\mathcal{A}}(s):=\sum_{a\in\mathcal{A}}\exp(-s|a|)$$

Another related series is:

$$\Theta'_{\mathcal{A}}(s):=\sum_{n=0}^{\infty}\#(\overline{B}_n\cap\mathcal{A})\cdot\exp(-sn)$$

The series Θ and Θ' have the same convergence behavior, since $\Theta_{\mathcal{A}}(s)=\Theta'_{\mathcal{A}}(s)\cdot(1-\exp(-s))$. It follows that the growth exponent of $\mathcal{A}$ is a *critical exponent* for

Θ' and Θ: the series converge for s greater than the critical exponent and diverge for s less than the critical exponent.

Definition 1.3. $\mathcal{A} \subset \mathcal{X}$ is *divergent* if $\Theta_{\mathcal{A}}$ diverges at its critical exponent.

1.2. **Path Systems and Contracting Elements.** We define path systems and contracting elements following Sisto [47].

Definition 1.4. A *path system* in $\mathcal{X}$ is a transitive collection of uniform quasi-geodesics that is closed under taking subpaths.

Definition 1.5. Let $\mathcal{PS}$ be a path system in $\mathcal{X}$. Let $\mathcal{A}$ be a subset of $\mathcal{X}$. A map $\pi_{\mathcal{A}} \colon \mathcal{X} \to \mathcal{A}$ is a *C–$\mathcal{PS}$–contracting projection* if:

(1) $\pi_{\mathcal{A}}$ and $\mathrm{Id}_{\mathcal{A}}$ are C–coarsely equivalent on $\mathcal{A}$, and
(2) For every $\mathcal{P} \in \mathcal{PS}$ with endpoints x_0 and x_1, if $d(\pi_{\mathcal{A}}(x_0), \pi_{\mathcal{A}}(x_1)) > C$ then $d(\pi_{\mathcal{A}}(x_i), \mathcal{P}) \leqslant C$ for both $i \in \{0, 1\}$.

Definition 1.6. Let $\mathcal{PS}$ be an equivariant path system. An element $h \in G$ is a *$\mathcal{PS}$–contracting element* if for some (hence, any) choice of basepoint $o \in \mathcal{X}$:

(1) $i \mapsto h^i.o$ is a quasi-geodesic, and
(2) there exists a constant C such that for every $\mathcal{P} \in \mathcal{PS}$ with endpoints in $\langle h \rangle .o$ there is a C–$\mathcal{PS}$–contracting projection $\pi_{\mathcal{P}} \colon \mathcal{X} \to \mathcal{P}$.

Definition 1.7. A path system $\mathcal{PS}$ is *minimizing* if it contains a geodesic between each pair of points in $\mathcal{X}$.

Definition 1.8. An element $h \in G$ is a *strongly contracting element* if there exists a minimizing path system $\mathcal{PS}$ in $\mathcal{X}$ for which h is a $\mathcal{PS}$–contracting element.

1.3. **Axes for Contracting Elements.**

Definition 1.9. The *elementary closure* $E(h)$ of an element $h \in G$ is the unique maximal virtually cyclic subgroup of G containing h, if such a subgroup exists.

Lemma 1.10. *Let $h \in G$ be a $\mathcal{PS}$–contracting element.*

$$E(h) = \{g \in G \mid \langle h \rangle .o \textit{ and } g \langle h \rangle .o \textit{ are coarsely equivalent}\}$$

Proof. This follows from [16, Lemma 6.5]. ∎

Definition 1.11. If h is a contracting element, the *(quasi)-axis* of h, with respect to the basepoint o, is $\mathcal{H} := E(h).o \subset \mathcal{X}$.

A quasi-geodesic γ is *Morse* if for every λ there exists a C_λ such that every (λ, λ)–quasi-geodesic with endpoints on γ is contained in $\overline{\mathcal{N}_{C_\lambda}(\gamma)}$.

Lemma 1.12 (Morse, [47, Lemma 2.10])**.** *Let $h \in G$ be a $\mathcal{PS}$–contracting element. Then $i \to h^i.o$ is a Morse quasi-geodesic in $\mathcal{X}$ and $i \to h^i$ is a Morse quasi-geodesic in G with respect to any word metric on G.*

Dahmani, Guirardel, and Osin [16] define the concept of a *hyperbolically embedded subgroup.* This is a generalization of a peripheral subgroup of a relatively hyperbolic group. We will not state the definition, but we will quote some of their results.

Theorem 1.13 ([47, Theorem 5.6])**.** *Let $h \in G$ be a $\mathcal{PS}$–contracting element. Then $E(h)$ is hyperbolically embedded.*

Lemma 1.14. *If h is a $\mathcal{PS}$–contracting element then there exists a $\mathcal{PS}$–contracting projection $\pi_{\mathcal{H}}\colon \mathcal{X} \to \mathcal{H}$. Moreover, $\pi_{\mathcal{H}}$ can be chosen so that $g.\pi_{\mathcal{H}}(x) = \pi_{\mathcal{H}}(g.x)$ for all $x \in \mathcal{X}$ and for all $g \in E(h)$.*

The second part of the lemma occurs in the proof of [47, Theorem 5.6]. The first part is implicit there. We give a brief sketch of the proof:

Proof. Given $x \in \mathcal{X}$ let $M(C, x)$ denote the set of paths $\mathcal{P} \in \mathcal{PS}$ such that $\mathcal{P}\colon [0, T_{\mathcal{P}}] \to \mathcal{X}$ with $\mathcal{P}_0, \mathcal{P}_{T_{\mathcal{P}}} \in \langle h \rangle .o$ such that $d(\pi_{\mathcal{P}}(x), \{\mathcal{P}_0, \mathcal{P}_{T_{\mathcal{P}}}\}) > C$.

Using the Morse property it can be shown that there are constants K_0 and K_1 such that for all $x \in \mathcal{X}$ there exists a point $\pi(x) \in \mathcal{H}$ such that

$$\{\pi_{\mathcal{P}}(x) \mid \mathcal{P} \in M(K_0, x)\} \subset \overline{\mathcal{N}_{K_1}(\pi(x))}$$

It is easy to see that π defines a $\mathcal{PS}$–contracting projection.

Now define $\pi_{\mathcal{H}}(x) := \bigcup_{g \in E(h)} g.\pi(g^{-1}.x)$. □

From the projection $\pi_{\mathcal{H}}$ we can also define $\mathcal{PS}$–contracting projections onto each translate of $\mathcal{H}$ by $\pi_{g\mathcal{H}}\colon \mathcal{X} \to g\mathcal{H} : x \mapsto g.\pi_{\mathcal{H}}(g^{-1}.x)$.

Let us mention two additional results about hyperbolically embedded subgroups that are relevant.

Theorem 1.15 ([16, Theorem 2.23])**.** *If G has a hyperbolically embedded subgroup then G has a maximal finite normal subgroup.*

Recall that this theorem guarantees one of the hypotheses of Proposition 0.4.

Theorem 1.16 ([34])**.** *Let G be a non-elementary group acting minimally on a simplicial tree $\mathcal{T}$ with at least 3 vertices. Suppose that G does not fix a point in $\partial\mathcal{T}$ and that there exist vertices u and v of $\mathcal{T}$ such that the pointwise stabilizer of $\{u, v\}$ is finite. Then G contains an infinite order element h such that $E(h)$ is hyperbolically embedded in G.*

In particular, this theorem can be applied when $\mathcal{T}$ is the Bass-Serre tree corresponding to a non-trivial splitting of G as a graph of groups [46].

Corollary 1.17 ([34])**.** *Let G be a finitely generated, non-elementary group that splits non-trivially as a graph of groups and is not an ascending HNN-extension. Suppose that there exist two edges of the corresponding Bass-Serre tree whose stabilizers have finite intersection. Then G contains an infinite order element h such that $E(h)$ is hyperbolically embedded in G.*

1.4. **The Projection Complex.** We now follow the method of Bestvina, Bromberg, and Fujiwara [7] to build a projection complex $\mathcal{P}_K(\mathbb{Y})$ and a blown-up projection complex $\mathcal{Y}$. Let $\mathbb{Y}$ be the collection of coarse equivalence classes of G–translates of $\mathcal{H}$. For each $Y \in \mathbb{Y}$ let π_Y be the projection map defined in the previous section, and let $d^{\pi}_Y(x, z) := \text{diameter}\{\pi_Y(x), \pi_Y(z)\}$.

This choice of $\mathbb{Y}$ and projection maps satisfy Bestvina, Bromberg, and Fujiwara's projection axioms.

Definition 1.18 (Projection Axioms)**.** A set $\mathbb{Y}$ and projection distances d^{π}_Y satisfy the projection axioms if there exist ξ and η such that for all distinct $W, X, Y, Z \in \mathbb{Y}$:

(0) $d^{\pi}_Y(X) \leqslant \xi$
(1) $d^{\pi}_Y(X, Z) = d^{\pi}_Y(Z, X)$
(2) $d^{\pi}_Y(X, W) \leqslant d^{\pi}_Y(X, Z) + d^{\pi}_Y(Z, W)$
(3) $\min\{d^{\pi}_X(Y, Z), d^{\pi}_Y(X, Z)\} \leqslant \xi$

(4) $|\{V \in \mathbb{Y} \mid d^\pi_V(X,Y) > \eta\}| < \infty$

Lemma 1.19. *The set of coarse equivalence classes of G–translates of $\mathcal{H}$ and projection distance defined above satisfy the projection axioms.*

Proof. This is part of the proof of [47, Theorem 5.6]. □

From any $\mathbb{Y}$ and projection distances satisfying axioms (1)–(4), one can build a *projection complex* $\mathcal{P}_K(\mathbb{Y})$ whose vertices are the elements of $\mathbb{Y}$, and two vertices X, Z are joined by an edge if $d_Y(X,Z) < K$ for all Y (up to a small perturbation. For details, see [7].)

A *quasi-tree* is a geodesic metric space that is quasi-isometric to a simplicial tree. Manning [31] gave a characterization of quasi-trees as spaces satisfying a 'bottleneck' property. We use an equivalent formulation:

Definition 1.20 (Bottleneck Property)**.** A geodesic metric space satisfies the *bottleneck property* if there exists a number Δ such that for all x and y in $\mathcal{X}$, and for any point m on a geodesic segment from x to y, every path from x to y passes through the Δ–neighborhood of m.

Theorem 1.21 ([31, Theorem 4.6])**.** *A geodesic metric space is a quasi-tree if and only if it satisfies the bottleneck property.*

Theorem 1.22 ([7, Theorem D])**.** *For sufficiently large K the projection complex $\mathcal{P}_K(\mathbb{Y})$ is a quasi-tree.*

Moreover, we can blow-up each vertex of $\mathcal{P}_K(\mathbb{Y})$ to be a copy of a Cayley graph of $E(h)$. Specifically, choose the generating set of $E(h)$ consisting of h and all elements $g \in E(h)$ such that $d(o, g.o) \leqslant \operatorname{diam}(\langle h \rangle \backslash \mathcal{H})$. Let $\mathbb{Y}$ denote the set of copies of $E(h)$, one for each vertex of $\mathcal{P}_K(\mathbb{Y})$. Let Y_0 be the copy of $E(h)$ corresponding to $\mathcal{H}$, and fix a basepoint $\star \in Y_0$. Define projection maps to agree with the projection maps in $\mathcal{X}$. For each pair $X, Z \in \mathbb{Y}$ that come from adjacent vertices in $\mathcal{P}_K(\mathbb{Y})$, attach an edge of length K' from every point in $\pi_X(Z)$ to every point of $\pi_Z(X)$. The resulting graph we call $\mathcal{Y}$. As before, we define $d^\pi_Y(x,y) := d_{\mathcal{Y}}(\pi_Y(x), \pi_Y(z))$.

Lemma 1.23 ([7, Lemma 3.1])**.** *The constant K' can be chosen sufficiently large with respect to K so that $d_{\mathcal{Y}}(x,z) \geqslant d^\pi_Y(x,z)$ for each $Y \in \mathbb{Y}$, with equality if and only if both x and z are in Y. In particular, Y is totally geodesically embedded.*

Theorem 1.24 ([7, Theorem 3.10])**.** *$\mathcal{Y}$ is a quasi-tree.*

Lemma 1.25. *Let Δ be the bottleneck constant for $\mathcal{Y}$. There is a number N such that for all $n \geqslant N$ the points $h^n.\star$ are all contained in a single component of $\mathcal{Y} \setminus \mathcal{B}_\Delta(\star)$ and the points $h^{-n}.\star$ are all contained in a different single component of $\mathcal{Y} \setminus \mathcal{B}_\Delta(\star)$.*

Proof. This follows directly from the bottleneck property, since $\langle h \rangle .o$ lies within bounded Hausdorff distance from a geodesic in $\mathcal{Y}$. □

Let us call the component of $\mathcal{Y} \setminus \mathcal{B}_\Delta(\star)$ containing the large positive powers of h the 'h^∞ component', and the component containing the large negative powers the '$h^{-\infty}$ component'.

Corollary 1.26. *There exists a number K such that if $d^\pi_{\mathcal{H}}(o, g.o) > K$ then the closest point of $\langle h \rangle .\star \in \mathcal{Y}$ to $g.\star$ is $h^{\epsilon n}.\star$ for $\epsilon \in \{\pm 1\}$ and $n > 0$, and $g.\star$ is in the $h^{\epsilon\infty}$ component of $\mathcal{Y} \setminus \mathcal{B}_\Delta(\star)$.*

Proof. Since $i \to h^i.o$ is a quasi-geodesic, the map that sends $x \in \mathcal{H}$ to the set of vertices of Y_0 corresponding to elements $\{g \in E(h) \mid g.o = x\}$ is a quasi-isometry between $\mathcal{H}$ with its induced metric in $\mathcal{X}$ and Y_0 with its induced metric in $\mathcal{Y}$. Also, $i \to h^i.\star$ is a quasi-geodesic. Thus, for large enough $d^\pi_{\mathcal{H}}(o, g.o)$, if $h^{\epsilon n}.\star$ is a closest point to $g.\star$ in $\langle h \rangle .\star$ we may assume that $n > N$ from Lemma 1.25 and $d_{\mathcal{Y}}(\star, h^{\epsilon n}.\star) > 2\Delta + 1$. Thus, no geodesic from $g.\star$ to $h^{\epsilon n}.\star$ in $\langle h \rangle .\star$ can enter $\mathcal{B}_\Delta(\star)$, which, together with Lemma 1.25, implies that $g.\star$ is in the same complementary component of $\mathcal{B}_\Delta(\star)$ as the points $h^{\epsilon n'}.\star$ for all $n' \geqslant N$. □

2. Abundance of Contracting Elements and Quotients

Lemma 2.1. *If G contains a $\mathcal{PS}$–contracting element then so does every infinite normal subgroup of G. In particular, if $\mathcal{PS}$ is minimizing then every infinite normal subgroup of G contains a strongly contracting element.*

Proof. Let $\mathcal{PS}$ be an equivariant path system in $\mathcal{X}$, and let h be a $\mathcal{PS}$–contracting element. Every non-trivial power of h is also a $\mathcal{PS}$–contracting element. Let Γ be an infinite normal subgroup of G. If $\Gamma < E(h)$ then, since Γ is infinite and $\langle h \rangle$ is finite index in $E(h)$, some power of h is in Γ, and we are done. Otherwise, choose an element $g \in \Gamma$ such that $g \notin E(h)$. We claim that for sufficiently large n the element $gh^n g^{-1} h^{-n} \in \Gamma$ is $\mathcal{PS}$–contracting. This can be proven with an argument similar to [47, Lemma 4.1]. □

We also note that the existence of a $\mathcal{PS}$–contracting element implies that the action is interesting from the point of view of growth tightness: the growth exponent is positive and there exist infinite, infinite index normal subgroups.

Theorem 2.2 ([16])**.** *If G contains a $\mathcal{PS}$–contracting element then G has an infinite, infinite index normal subgroup and positive growth exponent.*

Proof. Let h be a $\mathcal{PS}$–contracting element. By Theorem 1.13, $E(h)$ is hyperbolically embedded. By [16, Theorem 5.15], for a sufficiently large n, the normal closure of $\langle h^n \rangle$ in G is the free product of the conjugates of $\langle h^n \rangle$. Since G contains a non-abelian free group it will have positive growth exponent. □

3. The Minimal Section

Let $\mathcal{X}$ be a G–space with basepoint o. Let Γ be an infinite normal subgroup of G. Let $\mathcal{PS}$ be some minimizing, equivariant path system in $\mathcal{X}$ such that there exists an $h \in \Gamma$ that is C–$\mathcal{PS}$–contracting. Let $\pi_{\mathcal{H}}\colon \mathcal{X} \to \mathcal{H} = E(h).o$ be the projection defined in Lemma 1.14.

Definition 3.1. For each element $g\Gamma \in G/\Gamma$ choose an element $\bar{g} \in g\Gamma$ such that $d(o, \bar{g}.o) = d(o, g\Gamma.o) = d(\Gamma.o, g\Gamma.o)$. Let $\bar{G} := \{\bar{g} \mid g\Gamma \in G/\Gamma\}$. We call $\bar{G}$ a *minimal section*, and let $\bar{\mathcal{G}}$ denote $\bar{G}.o$.

Observe that $\Theta_{G/\Gamma}(s) = \Theta_{\bar{\mathcal{G}}}(s)$.

Lemma 3.2. *There exists a constant K such that for every $\bar{g} \in \bar{G}$ and for every $f \in G$ we have $d^\pi_{f\mathcal{H}}(o, \bar{g}.o) \leqslant K$.*

Proof. If D is the diameter of $\langle h \rangle \backslash \mathcal{H}$ then $K := 6C + D$ will suffice. Suppose $d^\pi_{f\mathcal{H}}(o, \bar{g}.o) > K \geqslant 6C + D$. Then $d(\pi_{f\mathcal{H}}(o), \pi_{f\mathcal{H}}(\bar{g}.o)) > 4C + D$, so there exists an

$n \neq 0$ such that $d(\pi_{f\mathcal{H}}(o), fh^n f^{-1}.\pi_{f\mathcal{H}}(\bar{g}.o)) \leqslant D$. Thus:

$$\begin{aligned} d(o, fh^n f^{-1}\bar{g}.o) &< d(o, \pi_{f\mathcal{H}}(o)) + d(\pi_{f\mathcal{H}}(o), \pi_{f\mathcal{H}}(\bar{g}.o)) - 4C + d(\pi_{f\mathcal{H}}(\bar{g}.o), \bar{g}.o) \\ &\leqslant d(o, \bar{g}.o) \end{aligned}$$

The second inequality comes from the fact that $f\mathcal{H}$ is $\mathcal{PS}$–contracting for a minimizing $\mathcal{PS}$, so some geodesic from o to $\bar{g}.o$ passes through the C–neighborhoods of $\pi_{f\mathcal{H}}(o)$ and $\pi_{f\mathcal{H}}(\bar{g}.o)$. However, this contradicts the fact that $\bar{G}$ is a minimal section, since $fh^n f^{-1}\bar{g} \in \bar{g}\Gamma$. □

It will be convenient to consider a subset of $\bar{\mathcal{G}}$:

Definition 3.3. Choose η large enough so that if $\bar{g}\mathcal{H} = \bar{g}'\mathcal{H}$ then $d(\bar{g}.o, \bar{g}'.o) < \eta$. Choose a maximal subset A of $\bar{G}$ such that $1 \in A$ and $d_X(a.o, a'.o) \geqslant \eta$ for all a, $a' \in A$. Let $\mathcal{A} := A.o$.

By maximality, for every $\bar{g} \in \bar{G}$ there is some $a \in A$ such that $d(a.o, \bar{g}.o) < \eta$. There are boundedly many points of $\bar{\mathcal{G}}$ in a ball of radius η, so $\Theta_{\bar{\mathcal{G}}}(s)$ is bounded below by $\Theta_{\mathcal{A}}(s)$ and above by a constant multiple of $\Theta_{\mathcal{A}}(s)$. In particular, $\Theta_{\mathcal{A}}(s)$ has the same convergence behavior as $\Theta_{\bar{\mathcal{G}}}(s)$.

4. Embedding a Free Product Set

Let $A \subset \bar{G}$ be as in the previous section, and let $A^* := A \setminus \{1\}$. Consider the free product set $A^* * \mathbb{Z}_2 := \cup_{n=1}^{\infty}\{(a_1, \dots, a_n) \mid a_i \in A^*\}$. For any $N > 0$ we can map the free product set into $\mathcal{X}$ by $(a_1, \dots, a_n) \mapsto a_1 h^N a_2 h^N \cdots a_n h^N.o$. Our goal is to show that for sufficiently large N this map is an injection.

Proposition 4.1. *The map $A^* * \mathbb{Z}_2 \to \mathcal{X} : (a_1, \dots, a_n) \mapsto a_1 h^N \cdots a_n h^N.o$ is an injection for all sufficiently large N.*

The map is an injection because we have an action of G on the quasi-tree $\mathcal{Y}$, and for large enough N we have "quasi-edges" of the form $[y, yh^N]$. We have set things up so that the a's do not backtrack across such edges. See Figure 1. We make this precise:

Proof. By Lemma 3.2, there is a K such that for every $f \in G$ and every $\bar{g} \in \bar{G}$ we have $d^{\pi}_{f\mathcal{H}}(o, \bar{g}.o) \leqslant K$. Let Δ be the bottleneck constant for $\mathcal{Y}$, and let ξ be the constant from the projection axioms. Suppose that the map $Y_0 \to \mathcal{H} : g.\star \to g.o$ is a (λ, η)–quasi-isometry. Choose N large enough so that $d_{\mathcal{Y}}(\star, h^{N/2}.\star) > 2\lambda(\lambda\Delta + K + \eta + \xi)$.

Claim 4.1.1. For any $\epsilon_0, \epsilon_1 \in \{\pm 1\}$ and any $a_0, a_1 \in A$ the two separating balls $\mathcal{B}_{\Delta}(a_0 h^{\epsilon_0 N/2}.\star)$ and $\mathcal{B}_{\Delta}(a_1 h^{\epsilon_1 N/2}.\star)$ are disjoint unless $a_0 = a_1$ and $\epsilon_0 = \epsilon_1$.

Assume the claim is true. By our choice of N, for each $a \in A$ and both $\epsilon \in \{\pm 1\}$ the points $\star$ and $a.\star$ are in the same complementary component of $\mathcal{B}_{\Delta}(ah^{\epsilon N/2}.\star)$, and the points $a.\star$ and $ah^{\epsilon N}.\star$ are in different complementary components of $\mathcal{B}_{\Delta}(ah^{\epsilon N/2}.\star)$. This fact, together with the claim, implies that for any $a_1 h^N \dots a_n h^N \in G$, with $a_i \in A^*$, the following statements hold:

- The balls $\mathcal{B}_{\Delta}(a_1 h^{N/2}.\star)$, $\mathcal{B}_{\Delta}(a_1 h^N a_2 h^{N/2}.\star)$,..., $\mathcal{B}_{\Delta}(a_1 h^N \dots a_n h^{N/2}.\star)$ are pairwise disjoint.
- For each $1 \leqslant i \leqslant n$, the points $\star$, and $a_1 h^N \cdots a_{j-1} h^N a_j.\star$ for $j \leqslant i$ are contained in a common component of $\mathcal{B}_{\Delta}(a_1 h^N \cdots a_i h^{N/2}.\star)$.

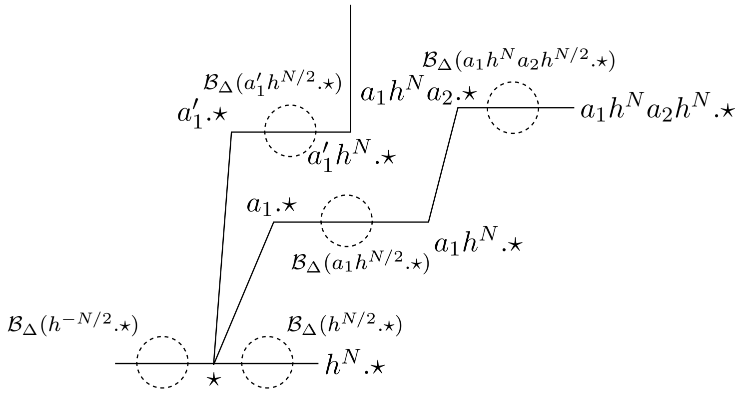

Figure 1. A does not cross h^N quasi-edges

- For each $1 \leqslant i \leqslant n$, the points $a_1h^N \cdots a_jh^N.\star$ for $j \geqslant i$ are contained in a common component of $\mathcal{B}_\Delta(a_1h^N \cdots a_ih^{N/2}.\star)$, distinct from the complementary component containing $\star$.

This implies $a_1h^N \cdots a_nh^N$ is non-trivial. Now suppose that $a_1'h^Na_2'h^N \cdots a_m'h^N$ is the image of another element of the free product set. If $a_1 \neq a_1'$, then $\star$, $a_1h^N \cdots a_nh^N.\star$, and $a_1'h^N \cdots a_m'h^N.\star$ are in three different complementary components of the union of the disjoint separating balls $\mathcal{B}_\Delta(a_1h^n \cdots a_{n-1}h^Na_nh^{N/2}.\star)$ and $\mathcal{B}_\Delta(a_1'h^n \cdots a_{m-1}'h^Na_m'h^{N/2}.\star)$. Thus, $a_1h^N \cdots a_nh^N \neq a_1'h^N \cdots a_m'h^N$. If $a_1 = a_1'$ we repeat the argument with $a_2h^N \cdots a_nh^N$ and $a_2'h^N \cdots a_m'h^N$.

It remains only to prove the claim. We wish to show that $\mathcal{B}_\Delta(a_0h^{\epsilon_0 N/2}.\star)$ and $\mathcal{B}_\Delta(a_1h^{\epsilon_1 N/2}.\star)$ are disjoint. Clearly $\mathcal{B}_\Delta(ah^{N/2}.\star)$ and $\mathcal{B}_\Delta(ah^{-N/2}.\star)$ are disjoint for any $a \in A$, so suppose $a_0 \neq a_1$. By our choice of A, this means that $a_0\mathcal{H}$ and $a_1\mathcal{H}$ are different axes. By Lemma 3.2, $d^\pi_{a_i\mathcal{H}}(o, a_j.o) \leqslant K$ if $i \neq j$, so $d^\pi_{a_i\mathcal{H}}(a_j.o, a_i.o) \leqslant 2K$, and $d^\pi_{a_iY_0}(a_j.\star, a_i.\star) \leqslant 2\lambda K + \eta$. The point now is that we have chosen N large enough so that the ball $\mathcal{B}_\Delta(a_ih^{\epsilon_i N/2}.\star)$ is far from $a_i.\star$ along a_iY_0. By Projection Axiom (3), both the ball and $a_i.\star$ project close to $a_j.\star$ on a_jY_0. Therefore, the projections to a_jY_0 of $\mathcal{B}_\Delta(a_ih^{\epsilon_i N/2}.\star)$ and $\mathcal{B}_\Delta(a_jh^{\epsilon_j N/2}.\star)$ are disjoint, which implies the balls are disjoint. □

5. Growth Gap

Lemma 5.1 ([18, Criterion 2.4],[40, Proposition 2.3])**.** *If the map*

$$A^* * \mathbb{Z}_2 \to G : (a_1, \ldots, a_n) \mapsto a_1h^N \cdots a_nh^N.o$$

is an injection, and if $\exp(|h^N| \cdot \delta_\mathcal{A}) < \Theta_\mathcal{A}(\delta_\mathcal{A})$*, then* $\delta_{A^* * \mathbb{Z}_2} > \delta_\mathcal{A}$*. In particular, the second condition is true if* $\bar{\mathcal{G}}$ *(hence,* $\mathcal{A}$*) is divergent.*

Definition 5.2. Let $Comp^G_{Q,r} \subset G.o$ be the set of points $g.o$ such that there exists a geodesic $[x, y]$ of length r with $x \in \overline{\mathcal{B}_Q(o)}$ and $y \in \overline{\mathcal{B}_Q(g.o)}$ whose interior is contained in $\mathcal{X} \setminus \overline{\mathcal{N}_Q(G.o)}$.

Define the *Q–complementary growth exponent* of G to be:

$$\delta^c_G := \limsup_{r\to\infty} \frac{\log \#Comp^G_{Q,r}}{r}$$

Theorem 5.3. *Let G be a finitely generated, non-elementary group. Let $\mathcal{X}$ be a G–space. If G contains a strongly contracting element and there exists a $Q \geqslant 0$*

such that the Q–complementary growth exponent of G is strictly less than the growth exponent of G, then $G \curvearrowright \mathcal{X}$ is a growth tight action.

Remark. The proof of Theorem 5.3 follows in part the proof of [18, Theorem 1.4] for geometrically finite Kleinian groups. For the divergence part of the proof, the Kleinian group ingredients of [18, Theorem 1.4] are inessential, and our changes are mostly cosmetic. The real generalization is in the use of Proposition 4.1 instead of a Ping-Pong argument.

Proof. Let Γ be an infinite, infinite index normal subgroup of G. By Lemma 2.1, there is a strongly contracting element in Γ. Let $\bar{G}$ be a minimal section of G/Γ. If $\delta_{\bar{G}} \leqslant \delta^c_G$ then we are done, since $\delta^c_G < \delta_G$, so suppose $\delta_{\bar{G}} > \delta^c_G$.

Claim 5.3.1. $\bar{G}$ is divergent.

Assume the claim, and let A be a maximal separated set in $\bar{G}$ as in Definition 3.3. Then A and $\bar{G}$ have the same critical exponent, and are both divergent. By Proposition 4.1, $A^* * \mathbb{Z}_2$ injects into G, so $\delta_{A^* * \mathbb{Z}_2} \leqslant \delta_G$. By Lemma 5.1, $\delta_A < \delta_{A^* * \mathbb{Z}}$. Thus, $\delta_{G/\Gamma} = \delta_A < \delta_{A^* * \mathbb{Z}} \leqslant \delta_G$, as desired.

It remains to prove the claim.

Let $r > 0$, and suppose $d(o, \bar{g}.o) = r$. Let $0 \leqslant M_0 \leqslant r$ and $M_1 = r - M_0$. Choose a geodesic $[o, \bar{g}.o]$ from o to $\bar{g}.o$, and let $[o, \bar{g}.o](M_0)$ denote the point of $[o, \bar{g}.o]$ at distance M_0 from o.

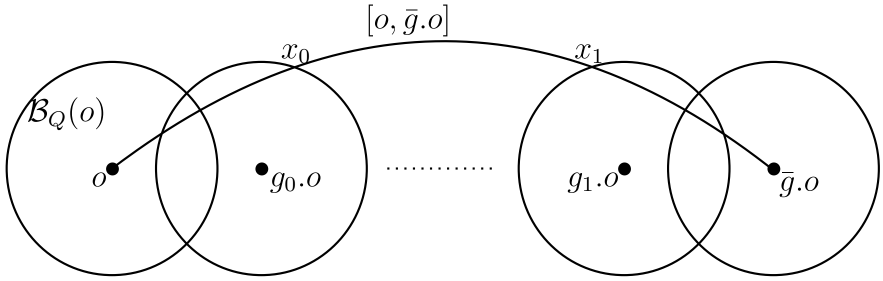

FIGURE 2. Splitting a geodesic into three subsegments

First, we suppose that $[o, \bar{g}.o](M_0) \in \mathcal{X} \setminus \overline{\mathcal{N}_Q(G.o)}$. Let $[x_0, x_1] \subset [o, \bar{g}.o]$ be the largest subsegment containing $[o, \bar{g}.o](M_0)$ such that $(x_0, x_1) \subset \mathcal{X} \setminus \overline{\mathcal{N}_Q(G.o)}$. Let $m_0 = d(o, x_0)$, and let $m_1 = d(x_1, \bar{g}.o)$. There exist group elements $g_i \in G$ such that $d(g_i.o, x_i) \leqslant Q$. See Figure 2. We have $\bar{g}.o = g_0 \cdot \underline{g_0^{-1} g_1} \cdot g_1^{-1}\bar{g}.o$. Now $m_0 - Q \leqslant d(o, \bar{g}_0.o) \leqslant d(o, g_0.o) \leqslant m_0 + Q$, and $m_1 - Q \leqslant d(o, \overline{g_1^{-1}\bar{g}}.o) \leqslant d(o, g_1^{-1}\bar{g}.o) \leqslant m_1 + Q$. Furthermore, $g_0^{-1} g_1 \in Comp^G_{Q, r-(m_0+m_1)}$. Thus, the point $\bar{g}.o$ can be expressed as a product of an element of $\bar{G}$ of length $m_0 \pm Q$, an element of $\bar{G}$ of length $m_1 \pm Q$, and the quotient of an element of $Comp^G_{Q, r-(m_0+m_1)}$.

(†) The same is also true if $[o, \bar{g}.o](M_0) \in \overline{\mathcal{N}_Q(G.o)}$, in which case we can take $m_0 = M_0$ and $m_1 = r - m_0$. Then choose $g_0 = g_1$ so that the contribution from $Comp^G_{Q, r-(m_0+m_1)}$ is trivial.

Let $V_{r,Q} := \# \left(\bar{G}.o \cap \overline{\mathcal{N}_{r+Q}(o)} \setminus \mathcal{B}_{r-Q}(o) \right)$. For every $M_0 + M_1 = r$ we have:

$$V_{r,Q} \overset{*}{\prec} \sum_{m_0=0}^{M_0} \sum_{m_1=0}^{M_1} V_{m_0,Q} \cdot V_{m_1,Q} \cdot \# Comp^G_{Q, r-(m_0+m_1)}$$

Choose $\xi > 0$ such that $\delta_{\bar{G}} \geqslant 2\xi + \delta^c_{\bar{G}}$. Since $\#Comp^G_{Q,r-(m_0+m_1)} \overset{*}{\prec} \exp((r-(m_0+m_1))(\delta_{\bar{G}}-\xi))$ whenever $r-(m_0+m_1)$ is sufficiently large, it follows that:

$$(1) \quad V_{r,Q} \cdot \exp(-r(\delta_{\bar{G}}-\xi)) \overset{*}{\prec} \left(\sum_{m_0=0}^{M_0} V_{m_0,Q} \cdot \exp(-m_0(\delta_{\bar{G}}-\xi))\right) \cdot \left(\sum_{m_1=0}^{M_1} V_{m_1,Q} \cdot \exp(-m_1(\delta_{\bar{G}}-\xi))\right)$$

Set $w_i := V_{i,Q} \cdot \exp(-i(\delta_{\bar{G}}-\xi))$ and $W_i := \sum_{j=1}^{i} w_i$. Then (1) and [18, Lemma 4.3] imply that $\sum_i w_i \cdot \exp(-is)$ diverges at its critical exponent, which is:

$$\limsup_i \frac{\log w_i}{i} = \left(\limsup_i \frac{\log V_{i,Q}}{i}\right) - (\delta_{\bar{G}}-\xi) = \xi$$

So $\infty = \sum_i w_i \cdot \exp(i\xi) = \sum_i V_{i,Q} \cdot \exp(-i\delta_{\bar{G}}) \overset{*}{\asymp} \Theta_{\bar{G}}(\delta_{\bar{G}})$. ☐

Theorem 5.4. *Let G be a finitely generated, non-elementary group. Let $\mathcal{X}$ be a quasi-convex G–space. If G contains a strongly contracting element then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

Proof. The proof is an easier special case of the proof of Theorem 5.3. If $\mathcal{X}$ is Q–quasi-convex then we can always choose to be in case (†) of the proof. ☐

Part 2. Strongly Contracting Elements

6. Contraction and Rank 1 Isometries

In this section we relate path system projections to rank 1 isometries.

Definition 6.1. $\pi\colon \mathcal{X} \to \mathcal{A}$ is (C, D, E)*–contracting* if $d(x_0, x_1) < \frac{1}{E} d(x_0, \mathcal{A}) - D$ implies $d(\pi(x_0), \pi(x_1)) \leqslant C$. We say π is *strongly contracting* if it is $(C, D, 1)$–contracting for some C, $D \geqslant 0$.

Some remarks are in order concerning overuse of the term 'contracting'. The term 'stongly contracting' is not standard. Indeed, in most of the literature what we are calling 'strongly contracting' is just called 'contracting'. This is the case, for example, in the literature on rank 1 isometries of CAT(0) spaces. However, Masur and Minsky [33] use 'contracting' in the sense of Definition 6.1, as do subsequent papers on the geometry of the mappping class group. In the present paper, we choose the term 'strongly contracting' to emphasize that we want $E = 1$.

Recall that in Part 1 we also have Sisto's $\mathcal{PS}$–contracting and related definitions of contracting and strongly contracting elements. For a $\mathcal{PS}$–contracting element h, Lemma 1.14 tells us that there is a $\mathcal{PS}$–contracting projection to an axis of h. We will see in Lemma 6.4 that this projection is contracting. Similarly, we will see in Proposition 6.6 that an element h is strongly contracting in the sense of Definition 1.8 if and only if closest point projection to the axis of h is coarsely well defined and strongly contracting in the sense of Definition 6.1.

Definition 6.2. $\pi\colon \mathcal{X} \to \mathcal{A}$ has the *Bounded Geodesic Image Property* if there are constants C and D such that for every geodesic γ, if $\gamma \cap \overline{\mathcal{N}_D(\mathcal{A})} = \emptyset$ then $\operatorname{diam}(\pi(\gamma)) \leqslant C$.

Lemma 6.3. *The Bounded Geodesic Image Property for $\pi\colon \mathcal{X} \to \mathcal{A}$ implies π is strongly contracting. The converse is true if π is coarsely equivalent to closest point projection.*

Proof. First, assume that π has the Bounded Geodesic Image Property, with constants C and D as in the definition. Let x be any point in $\mathcal{X} \setminus \overline{\mathcal{N}_D(\mathcal{A})}$. For any y such that $d(x,y) < d(x,\mathcal{A}) - D$, any geodesic $[x,y]$ remains outside $\overline{\mathcal{N}_D(\mathcal{A})}$, so its projection has diameter at most C. This proves strong contraction.

For the converse, suppose that π is C–coarsely equivalent to closest point projection and $(C,D,1)$–contracting. Enlarging C if necessary, we may assume $C \geqslant 1$.

Let $\gamma\colon [0,T] \to \mathcal{X}$ be a geodesic. Let $\gamma_t = \gamma(t)$. Assume that $d(\pi(\gamma_0),\pi(\gamma_T)) > C$. Suppose that γ stays outside of $\overline{\mathcal{N}_{D+2C}(\mathcal{A})}$.

There exists a minimal t_i such that $d(\gamma_0,\gamma_{t_i}) \geqslant d(\gamma_0,\mathcal{A}) - D$, otherwise γ would have C–bounded projection. Likewise, there is a maximal t_f such that $d(\gamma_{t_f},\gamma_T) \geqslant d(\gamma_T,\mathcal{A}) - D$. This implies $d(\pi(\gamma_0),\pi(\gamma_{t_i}))$ and $d(\pi(\gamma_{t_f}),\pi(\gamma_T))$ are at most C.

Let $s_j := t_i + 4Cj$. Let n be the greatest integer less than $\frac{t_f - t_i}{4C}$.

$$\begin{aligned} d(\pi(\gamma_{t_i}),\pi(\gamma_{t_f})) &\leqslant \sum_{j=1}^{n} d(\pi(\gamma_{s_j}),\pi(\gamma_{s_{j-1}})) + d(\pi(\gamma_{s_n}),\pi(\gamma_{t_f})) \\ &\leqslant 2C(n+1) < 2C + \frac{t_f - t_i}{2} \end{aligned}$$

But γ is a geodesic, so:

$$\begin{aligned} t_f - t_i &= d(\gamma_{t_i},\gamma_{t_f}) \\ &\leqslant d(\gamma_{t_i},\pi(\gamma_{t_i})) + d(\pi(\gamma_{t_i}),\pi(\gamma_{t_f})) + d(\gamma_{t_f},\pi(\gamma_{t_f})) \\ &\leqslant D + 3C + 2C + \frac{t_f - t_i}{2} + D + 3C \end{aligned}$$

So $t_f - t_i \leqslant 16C + 4D$, which implies $d(\pi(\gamma_{t_i}),\pi(\gamma_{t_f})) \leqslant 10C + 2D$.

Thus, if γ stays outside $\overline{\mathcal{N}_{D+2C}(\mathcal{A})}$ then $d(\pi(\gamma_0),\pi(\gamma_T)) \leqslant 12C + D$, so π has the Bounded Geodesic Image Property. □

Lemma 6.4 (cf. [47, Lemma 2.4])**.** *Let $\mathcal{PS}$ be a path system. Let $\pi\colon \mathcal{X} \to \mathcal{A}$ be a C–$\mathcal{PS}$–contracting projection. Suppose there exists a (λ,ϵ)–quasi-geodesic connecting x to y in $\mathcal{PS}$, then there is a constant D depending on λ, ϵ, and C such that:*

(1) *(coarse Lipschitz) $d(\pi(x),\pi(y)) \leqslant \lambda d(x,y) + D$.*
(2) *(contracting) If $d(x,y) < \frac{1}{\lambda} d(x,\mathcal{A}) - D$ then $d(\pi(x),\pi(y)) \leqslant C$.*

Suppose there exists a (λ,ϵ)–quasi-geodesic connecting x to $\pi(x)$ in $\mathcal{PS}$, then there is a constant D depending on λ, ϵ, and C such that:

(3) *(closest point projection) $d(x,\pi(x)) \leqslant \lambda d(x,\mathcal{A}) + D$.*

Lemma 6.4 is just a restatement of [47, Lemma 2.4], without uniformizing the constants. The proofs are identical.

Lemma 6.5. *If $\pi\colon \mathcal{X} \to \mathcal{A}$ is a $\mathcal{PS}$–contracting projection and $\mathcal{PS}$ is minimizing then π is strongly contracting and coarsely agrees with closest point projection to $\mathcal{A}$.*

Proof. Strong contraction comes from Lemma 6.4 (2), since we can take each $\lambda = 1$.

Let $x \in \mathcal{X}$ and let $a \in \mathcal{A}$ such that $d(x,\mathcal{A}) = d(x,a)$. Suppose that $d(\pi(x),a) > 2C$. Let $\mathcal{P} \in \mathcal{PS}$ be a geodesic from x to a. Since $d(a,\pi(a)) \leqslant C$, we have $d(\pi(x),\pi(a)) > C$, so $\mathcal{P}$ enters $\overline{\mathcal{N}_C(\pi(x))}$. Thus:

$$d(x,a) \geqslant d(x,\pi(x)) + d(\pi(x),a) - 2C > d(x,\pi(x))$$

This contradicts the choice of a. □

Proposition 6.6. *$\pi\colon \mathcal{X}\to\mathcal{A}$. The following are equivalent:*

1. *π is a contracting projection for the path system of all geodesics.*
2. *π is a $\mathcal{PS}$–contracting projection for some minimizing path system $\mathcal{PS}$.*
3. *π is strongly contracting and coarsely agrees with closest point projection to $\mathcal{A}$.*
4. *π has the Bounded Geodesic Image Property and coarsely agrees with closest point projection to $\mathcal{A}$.*

Proof. (1) $\implies$ (2) is clear. (2) $\implies$ (3) is Lemma 6.5. (3) and (4) are equivalent by Lemma 6.3. We need only show (4) $\implies$ (1).

Assume (4), so $\pi\colon \mathcal{X}\to\mathcal{A}$ is a map such that:

- $d(x,\pi(x))\leqslant d(x,\mathcal{A})+C$ for all $x\in\mathcal{X}$, and
- for every geodesic γ if $\gamma\cap\overline{\mathcal{N}_D(\mathcal{A})}=\emptyset$ then $\operatorname{diam}(\pi(\gamma))\leqslant C$.

Let $\mathcal{PS}$ be the path system consisting of all geodesics. Let $\gamma\colon [0,T]\to X$ be a geodesic with projection larger than C. Let t_0 and t_1 be the first and last times, respectively, such that $\gamma_t\in N_D(\mathcal{A})$. Then:

$$d(\pi(\gamma_0),\gamma_{t_0})\leqslant d(\pi(\gamma_0),\pi(\gamma_{t_0}))+d(\pi(\gamma_{t_0}),\gamma_{t_0})\leqslant 2C+D$$

Similarly, $d(\pi(\gamma_T),\gamma_{t_1})\leqslant 2C+D$. Thus, π is a $(2C+D)$–$\mathcal{PS}$–contraction. This completes the proof of Proposition 6.6. ☐

7. Actions on Relatively Hyperbolic Spaces

Dal'bo, Peigné, Picaud, and Sambusetti [18] proved growth tightness for geometrically finite Kleinian groups. Using our main theorems, Theorem 5.3 and Theorem 5.4, we generalize their results to all groups acting on relatively hyperbolic metric spaces.

7.1. Relatively Hyperbolic Metric Spaces.

Definition 7.1 (cf. [19, 48])**.** Let $\mathcal{X}$ be a geodesic metric space and let $\underline{\mathcal{P}}$ be a collection of uniformly coarsely connected subsets of $\mathcal{X}$. We say $\mathcal{X}$ is *hyperbolic relative to the peripheral sets* $\underline{\mathcal{P}}$ if the following conditions are satisfied:

1. For each A there exists a B such that $\operatorname{diam}(\overline{\mathcal{N}_A(\mathcal{P}_0)}\cap\overline{\mathcal{N}_A(\mathcal{P}_1)})\leqslant B$ for distinct $\mathcal{P}_0$, $\mathcal{P}_1\in\underline{\mathcal{P}}$.
2. The exists an $\epsilon\in(0,\frac{1}{2})$ and $M\geqslant 0$ such that if x_0, $x_1\in\mathcal{X}$ are points such that for some $\mathcal{P}\in\underline{\mathcal{P}}$ we have $d(x_i,\mathcal{P})\leqslant\epsilon\cdot d(x_0,x_1)$ for each i, then every geodesic from x_0 to x_1 intersects $\overline{\mathcal{N}_M(\mathcal{P})}$.
3. There exist σ and δ so that for every geodesic triangle either:
   (a) there exists a ball of radius σ intersecting all three sides, or
   (b) there exists a $\mathcal{P}\in\underline{\mathcal{P}}$ such that $\overline{\mathcal{N}_\sigma(\mathcal{P})}$ intersects all three sides and for each corner of the triangle, the points of the outgoing geodesics from that corner which first enter $\overline{\mathcal{N}_\sigma(\mathcal{P})}$ are distance at most δ apart.

We say $\mathcal{X}$ is *hyperbolic* if it hyperbolic relative to $\underline{\mathcal{P}}=\emptyset$.

Definition 7.2. A group G is *hyperbolic relative to a collection of finitely generated peripheral subgroups* if a Cayley graph of G is hyperbolic relative to the cosets of the peripheral subgroups.

Definition 7.3 (cf. [26])**.** Let $\mathcal{X}$ be a connected graph with edges of length bounded below. A *combinatorial horoball* based on X with parameter $a>0$ is a graph whose vertex set is $\operatorname{Vert}(\mathcal{X})\times(\{0\}\cup\mathbb{N})$, contains an edge of length 1 between (v,n) and

$(v, n+1)$ for all $v \in \mathrm{Vert}(\mathcal{X})$ and all $n \in \{0\} \cup \mathbb{N}$, and for each edge $[v, w] \in \mathcal{X}$ contains an edge $[(v, n), (w, n)]$ of length $e^{-an} \cdot \mathrm{length}([v, w])$.

Let $\mathcal{X}$ be hyperbolic relative to $\underline{\mathcal{P}}$. An *augmented space* is a space obtained from $\mathcal{X}$ as follows. By definition, there exists a constant C such that each $\mathcal{P} \in \underline{\mathcal{P}}$ is C–coarsely connected. For each $\mathcal{P} \in \underline{\mathcal{P}}$ choose a maximal subset of points that pairwise have distance at least C from one another. Let these points be the vertex set of a graph. For edges, choose a geodesic connecting each pair of vertices at distance at most $2C$ from each other. Use this graph as the base of a combinatorial horoball with parameter $a_{\mathcal{P}} > 0$. The augmented space is the space obtained from the union of $\mathcal{X}$ with horoballs $\mathcal{X}_{\mathcal{P}}$ for each $\mathcal{P} \in \underline{\mathcal{P}}$ by identifying the base of $\mathcal{X}_{\mathcal{P}}$ with the graph constructed in $\mathcal{P}$.

Definition 7.4. Let $\mathcal{X}$ be a hyperbolic G–space, and let $\underline{P}$ be the collection of maximal parabolic subgroups of G. Suppose there exists a G–equivariant collection of disjoint open horoballs centered at the points fixed by the parabolic subgroups. The *truncated space* is $\mathcal{X}$ minus the union of these open horoballs. We say $G \curvearrowright \mathcal{X}$ is *cusp uniform* if G acts cocompactly on the truncated space.

If G acts cocompactly on a G–space $\mathcal{X}'$ that is hyperbolic relative to a G–invariant peripheral system $\underline{\mathcal{P}}$, then an augmented space $\mathcal{X}$ can be built G–equivariantly, and $G \curvearrowright X$ will be a cusp uniform action.

Several different versions of the following theorem occur in the literature on relatively hyperbolic groups:

Theorem 7.5 ([9, 27, 48])**.** *If $\mathcal{X}$ is hyperbolic relative to $\underline{\mathcal{P}}$ then any augmented space with horoball parameters bounded below is hyperbolic.*

If $G \curvearrowright \mathcal{X}$ is a cusp uniform action then G is hyperbolic relative to the maximal parabolic subgroups and the truncated space is hyperbolic relative to boundaries of the deleted horoballs.

7.2. **Quasi-convex Actions.**

Theorem 7.6. *If $\mathcal{X}$ is a quasi-convex, relatively hyperbolic G–space and G does not coarsely fix a peripheral subspace then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

Proof. It follows from [48, Lemma 5.4] that every infinite order element of G that does not coarsely fix a peripheral subspace is contracting for the path system consisting of all geodesics. We conclude by Theorem 5.4. □

As we have mentioned, our preceding theorem encompasses much of what was already known about growth tight actions. The theorem was known when $\mathcal{X}$ is the Cayley graph of a hyperbolic group [2], or an arbitrary cocompact hyperbolic G–space [39]. Gerasimov [22] has shown that a relatively hyperbolic group has a non-trivial Floyd boundary. Thus, the case when $\mathcal{X}$ is the Cayley graph of a relatively hyperbolic group is a consequence of, an a priori more general[2], theorem of Yang, which also follows from our main results:

Theorem 7.7 ([52, Theorem 1.2])**.** *If G is a finitely generated group with a non-trivial Floyd boundary and $\mathcal{X}$ is a Cayley graph of G then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

[2]A construction of a non relatively hyperbolic group with a non-trivial Floyd boundary is currently an open problem.

The hypothesis that G has a non-trivial Floyd boundary means that every Floyd boundary of every Cayley graph of G consists of at least three points. See [52] for details.

Proof. Let $\mathcal{X}$ be a Cayley graph of G, and fix some Floyd boundary of $\mathcal{X}$. Gerasimov and Potyagailo [23, Proposition 8.2.4] prove that any element of G that fixes exactly two points in a Floyd boundary of $\mathcal{X}$ has a quasi-axis in $\mathcal{X}$ such that closest point projection to the quasi-axis satisfies the Bounded Geodesic Image Property. By Proposition 6.6, such an element is strongly contracting. The theorem then follows from Theorem 5.4. □

Corollary 7.8. *The action of a finitely generated group G with infinitely many ends on any one of its Cayley graphs is growth tight.*

Proof. Stallings' Theorem [50] says that G splits over a finite subgroup. G is hyperbolic relative to the factor groups of this splitting. The result then follows from Theorem 7.6. □

This corollary generalizes a result of Sambusetti [40, Theorem 1.4], who proved growth tightness for groups that split over a finite subgroup provided the factor groups satisfy additional hypotheses.

7.3. **Cusp Uniform Actions.** Theorem 7.6 and Theorem 7.5 show that if $G \curvearrowright \mathcal{X}$ is a cusp uniform action on a hyperbolic space then the action of G on the truncated space is a growth tight action. A natural question is whether $G \curvearrowright \mathcal{X}$ is a growth tight action. This action is not quasi-convex if the parabolic subgroups are infinite, as geodesics in $\mathcal{X}$ will travel deeply into horoballs, and, indeed, an example of Dal'bo, Otal, and Peigné [17] shows $G \curvearrowright \mathcal{X}$ need not be growth tight.

To see how growth tightness can fail, consider the combinatorial horoball from Definition 7.3. If $\mathcal{X}$ is, say, the Cayley graph of some group and we build the combinatorial horoball with parameter $a > 0$ based on $\mathcal{X}$, then the r ball about a basepoint $o \in \mathcal{X}$ in the horoball metric intersected with $\mathcal{X} \times \{0\}$ contains the ball of radius $C \cdot \exp(\frac{ar}{2})$ in the $\mathcal{X}$–metric, for a constant C depending only on a. Thus, if the number of vertices of balls in $\mathcal{X}$ grows faster than polynomially in the radius, then the growth exponent with respect to the horoball metric will be infinite. Furthermore, even if growth in $\mathcal{X}$ is polynomial we can make the growth exponent in the horoball be as large as we like by taking a to be sufficiently large. Dal'bo, Otal, and Peigné construct non-growth tight examples of relatively hyperbolic groups with two cusps by taking one of the horoball parameters to be large enough so that the corresponding parabolic subgroup dominates the growth of the group; that is, the growth exponent of the parabolic subgroup is equal to the growth exponent of the whole group. Quotienting by the second parabolic subgroup then does not decrease the growth exponent, so this action is not growth tight.

Not only does this provide an example of a non-growth tight action on a hyperbolic space, but since augmented spaces with different horoball parameters are still equivariantly quasi-isometric to each other, we have:

Observation 7.9. Growth tightness is not invariant among equivariantly quasi-isometric G–spaces.

Dal'bo, Peigné, Picaud, and Sambusetti [18, Theorem 1.4] show that this is essentially the only way that growth tightness can fail for cusp uniform actions. Their proof is for geometrically finite Kleinian groups, but our Theorem 5.3 generalizes this result.

Definition 7.10. Let $G \curvearrowright \mathcal{X}$ be a cusp uniform action on a hyperbolic space. Let $\underline{P}$ be the collection of maximal parabolic subgroups of G. Then G satisfies the *Parabolic Gap Condition* if $\delta_P < \delta_G$ for all $P \in \underline{P}$.

Theorem 7.11. *Let G be a finitely generated, non-elementary group. Let $G \curvearrowright \mathcal{X}$ be a cusp uniform action on a hyperbolic space. Suppose that G satisfies the Parabolic Gap Condition. Then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

Proof. Let Q be the diameter of the quotient of the truncated space. The Q–complementary growth exponent is the maximum of the parabolic growth exponents, which, by the Parabolic Gap Condition, is strictly less than the growth exponent of G. Apply Theorem 5.3. ☐

Corollary 7.12. *Let G be a finitely generated group hyperbolic relative to a collection $\underline{P}$ of virtually nilpotent subgroups. Then there exists a hyperbolic G–space $\mathcal{X}$ such that $G \curvearrowright \mathcal{X}$ is cusp uniform and growth tight.*

Proof. Construct $\mathcal{X}$ as an augmented space by taking a Cayley Graph for G and attaching combinatorial horoballs to the cosets of the peripheral subgroups. Since the parabolic groups are virtually nilpotent, they have polynomial growth in any word metric [25]. It follows that the growth exponent of each parabolic group with respect to the horoball metric is bounded by a multiple of the horoball parameter. By choosing the horoball parameters small enough, we can ensure G satisfies the Parabolic Gap Condition. ☐

8. Rank 1 Actions on CAT(0) Spaces

Let $\mathcal{X}$ be a CAT(0) G–space. See, for example, [12] for background on CAT(0) spaces. Recall that our definition of 'G–space' includes the hypothesis that $\mathcal{X}$ is proper, so an element is strongly contracting if and only if it acts as a rank 1 isometry:

Theorem 8.1 ([8, Theorem 5.4])**.** *Let h be a hyperbolic isometry of a proper CAT(0) space $\mathcal{X}$ with axis $\mathcal{A}$. Closest point projection to $\mathcal{A}$ is strongly contracting if and only if $\mathcal{A}$ does not bound an isometrically embedded half-flat in $\mathcal{X}$.*

Theorem 8.2. *Let $h \in G$ act as a hyperbolic isometry on $\mathcal{X}$. The following are equivalent:*

1. *h is a $\mathcal{PS}$–contracting element for some equivariant path system $\mathcal{PS}$ on $\mathcal{X}$.*
2. *$E(h)$ exists and is hyperbolically embedded.*
3. *h acts as a rank 1 isometry of $\mathcal{X}$.*
4. *h is strongly contracting.*

Proof. (1) $\implies$ (2) by Theorem 1.13.

If $E(h)$ is hyperbolically embedded, a theorem of Sisto [49] says that h is Morse. If an axis of h bounds a half-flat in $\mathcal{X}$, then h is not Morse, so h is a rank 1 isometry for $G \curvearrowright \mathcal{X}$. Thus, (2) $\implies$ (3)

(3) $\implies$ (4) $\implies$ (1) by Theorem 8.1 and Proposition 6.6. ☐

Theorem 8.2 and Theorem 5.4 show:

Theorem 8.3. *If G is a non-elementary, finitely generated group and $\mathcal{X}$ is a quasi-convex, CAT(0) G–space such that G contains an element that acts as a rank 1 isometry on $\mathcal{X}$, then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

Theorem 8.4. *Let G be a non-elementary, finitely generated group that splits non-trivially as a graph of groups and is not an ascending HNN-extension. Suppose that the corresponding action of G on the Bass-Serre tree of the splitting has two edges whose stabilizers have finite intersection. Suppose further that there is a corresponding graph of spaces that is non-positively curved, so that its universal cover $\mathcal{X}$ admits a CAT(0) metric. Then $G \curvearrowright \mathcal{X}$ is a growth tight action.*

See Scott and Wall [44] for the graph of spaces construction.

Proof. By Corollary 1.17, G contains an infinite order element h such that $E(h)$ is hyperbolically embedded. The result follows by Theorem 8.2 and Theorem 8.3. □

The 'flip-graph-manifolds' of Kapovich and Leeb [30] are examples of non-relatively hyperbolic groups that admit growth tight actions by Theorem 8.4.

9. Mapping Class Groups

Let $\mathcal{S} = \mathcal{S}_{g,p}$ be a connected and oriented surface of genus g with p punctures. We require $\mathcal{S}$ to have negative Euler characteristic.

Given two orientation-preserving homeomorphisms $\phi, \psi \colon \mathcal{S} \to \mathcal{S}$, we will consider ϕ and ψ to be equivalent if $\phi \circ \psi^{-1}$ is isotopic to the identity map on $\mathcal{S}$. Each equivalence class is called a *mapping class* of $\mathcal{S}$, and the set $\mathrm{Mod}(\mathcal{S})$ of all equivalence classes naturally forms a group called the *mapping class group* of $\mathcal{S}$.

A mapping class $f \in \mathrm{Mod}(\mathcal{S})$ is called *reducible* is there exists an f–invariant curve system on $\mathcal{S}$ and *irreducible* otherwise. By the Nielsen-Thurston classification of elements of $\mathrm{Mod}(\mathcal{S})$, a mapping class is irreducible and infinite order if and only it is pseudo-Anosov [51].

Let $\mathcal{X}$ be the Teichmüller space of marked hyperbolic structures on $\mathcal{S}$, equipped with the Teichmüller metric. (See [29] and [37] for more information.)

Theorem 9.1 ([35])**.** *Pseudo-Anosov mapping classes are strongly contracting for* $\mathrm{Mod}(\mathcal{S}) \curvearrowright \mathcal{X}$.

The action of $\mathrm{Mod}(\mathcal{S})$ on Teichmüller space is not quasi-convex. For each $\epsilon > 0$ there is a decomposition of $\mathcal{X}$ into a 'thick part' $\mathcal{X}^{\geqslant\epsilon}$ and a 'thin part' $\mathcal{X}^{<\epsilon}$ according to whether the hyperbolic structure on $\mathcal{S}$ corresponding to the point $x \in \mathcal{X}$ has any closed curves of length $< \epsilon$. This decomposition is $\mathrm{Mod}(\mathcal{S})$–invariant, and $\mathrm{Mod}(\mathcal{S}) \curvearrowright \mathcal{X}^{\geqslant\epsilon}$ is cocompact (see [36] and [21]).

Theorem 9.2. *The action of the mapping class group* $\mathrm{Mod}(\mathcal{S})$ *of* $\mathcal{S} = \mathcal{S}_{g,p}$ *on its Teichmüller space* $\mathcal{X}$ *with the Teichmüller metric is a growth tight action.*

Proof. Let $\zeta = 6g - 6 + 2p \geqslant 2$. The growth exponent of $\mathrm{Mod}(\mathcal{S})$ with respect to its action on $\mathcal{X}$ is ζ [4]. (We remark that the result of [4] is stated for closed surfaces, but their proof works in general. For our interest, it is enough that the growth exponent of $\mathrm{Mod}(\mathcal{S})$ is bounded below by ζ. This can be obtained from [28] and [20].)

Choose an r_0 and a maximal r_0–separated set in moduli space $\mathrm{Mod}(\mathcal{S})\backslash\mathcal{X}$, and let $\mathcal{A}$ be its full lift to $\mathcal{X}$. Given r_0 as above and $\delta = \frac{1}{2}$, let ϵ be sufficiently small as in [20, Theorem 1.7]. Let Q be the smallest number such that the ϵ–thick part of $\mathcal{X}$ is contained in $\overline{\mathcal{N}_Q(\mathrm{Mod}(\mathcal{S}).o)}$. Choose a finite subset $\{a_1, \ldots, a_n\} \subset \mathcal{A}$ such that:

$$\overline{\mathcal{B}_Q(o)} \setminus \mathcal{N}_Q(\mathrm{Mod}(\mathcal{S}).o) \subset \bigcup_{i=1}^{n} \mathcal{B}_{r_0}(a_i)$$

Suppose that $g \in \mathrm{Mod}(\mathcal{S})$ is such that there exists a geodesic $[x, y]$ between $\overline{\mathcal{B}_Q(o)}$ and $\overline{\mathcal{B}_Q(g.o)}$ whose interior stays in $\mathcal{X} \setminus \overline{\mathcal{N}_Q(\mathrm{Mod}(\mathcal{S}).o)}$. Then there are indices i and j such that $x \in \mathcal{B}_{r_0}(a_i)$ and $y \in \mathcal{B}_{r_0}(g.a_j)$. This means that every element contributing to $Comp_{Q,r}^{\mathrm{Mod}(\mathcal{S})}$ of Definition 5.2 also contributes to some $N_1(\mathcal{Q}_{1,\epsilon}, a_i, a_j, r)$ of [20, Theorem 1.7]. The conclusion of [20, Theorem 1.7] is that $N_1(\mathcal{Q}_{1,\epsilon}, a_i, a_j, r) \leq G(a_i)G(a_j)\exp(r \cdot (\zeta - \frac{1}{2}))$ for all sufficiently large r, where G is a particular function on $\mathcal{X}$. There are finitely many such sets, and the function G is bounded on $\{a_1, \ldots, a_n\}$, so there is a constant C such that $Comp_{Q,r}^{\mathrm{Mod}(\mathcal{S})} \leq C \cdot \exp(r \cdot (\zeta - \frac{1}{2}))$ for all sufficiently large r. Thus, the Q–complementary growth exponent is at most $\zeta - \frac{1}{2} < \zeta$. The theorem now follows from Theorem 9.1 and Theorem 5.3. □

Remark. When the genus of $\mathcal{S}$ is at least 3 then there does not exist a cocompact, CAT(0) $\mathrm{Mod}(\mathcal{S})$–space [11], and $\mathrm{Mod}(\mathcal{S})$ has trivial Floyd boundary (so is not relatively hyperbolic) [1], so Theorem 9.2 does not follow from the results of the previous sections.

10. Snowflake Groups

Let $G := BB(1, r) = \langle a, b, s, t \mid aba^{-1}b^{-1} = 1,\ s^{-1}as = a^r b,\ t^{-1}at = a^r b^{-1} \rangle$ be a Brady-Bridson snowflake group with $r \geqslant 3$. Let $L := 2r$. These groups have an interesting mixture of positive and negative curvature properties. G splits as an amalgam of $\mathbb{Z}^2 = \langle a, b \rangle$ by two cyclic groups $\langle a^r b \rangle$ and $\langle a^r b^{-1} \rangle$, and the action of G on the Bass-Serre tree $\mathcal{T}$ of this splitting satisfies Corollary 1.17, so G has hyperbolically embedded subgroups. However, we can not automatically conclude that such a hyperbolically embedded subgroup gives rise to a strongly contracting element, as there does not exist a cocompact, CAT(0) G–space. If such a space existed, then the Dehn function of G would be at most quadratic, but Brady and Bridson have shown [10] that the Dehn function of $BB(1, r)$ is $n^{2\log_2 L} > n^2$.

We will fix a G–space $\mathcal{X}$ and demonstrate two different elements of G that act hyperbolically on $\mathcal{T}$ such that the pointwise stabilizer of any length 3 segment of their axes is finite. One of these elements will be strongly contracting for the action on $\mathcal{X}$, and the other will not. Hence:

Theorem 10.1. *G admits a cocompact growth tight action.*

Remark. G has a trivial Floyd boundary, thus is non-relatively hyperbolic, by a theorem of Anderson, Aramayona, and Shackleton [1], so growth tightness cannot be achieved using Yang's theorem.

10.1. **The Model Space $\mathcal{X}$.** Let $\mathcal{X}$ be the Cayley graph for G with respect to the generating set $\{a,\ a^r b,\ a^r b^{-1}, s, t\}$, where the edges corresponding to $a^r b$ and $a^r b^{-1}$ have been rescaled to have length $L := 2r$.

It is also useful to consider G as the fundamental group of the topological space obtained from a torus by gluing on two annuli. Choose a basepoint for the torus and for each boundary component of the annuli. For one annulus, the s–annulus, glue the two boundary curves to the curves a and $a^r b$ in the torus, gluing basepoints to the basepoint of the torus. For the other annulus, the t–annulus, glue the two boundary curves to the curves a and a^rb^{-1} of the torus. The resulting space is a graph of spaces [44] associated to the given graph of groups decomposition of G. The fundamental group of this space is G, which acts freely by deck transformations on the universal cover $\mathcal{X}'$. Choose the basepoint o of $\mathcal{X}'$ to be a lift of the basepoint of the torus. The correspondence between a vertex $g \in \mathcal{X}$ and the point $g.o \in \mathcal{X}'$ inspires the following terminology: A *plane* is a coset $g\langle a, b\rangle \in G/\langle a, b\rangle$, which corresponds to a lift of the torus at the point $g.o \in \mathcal{X}'$. An *s-wall* is the set of outgoing s–edges incident to a coset $g\langle a^rb\rangle \in G/\langle a^rb\rangle$. This corresponds to a lift of the s–annulus at the point $g.o \in \mathcal{X}'$. A *t-wall* is the set of outgoing t–edges incident to a coset $g\langle a^rb^{-1}\rangle \in G/\langle a^rb^{-1}\rangle$. This corresponds to a lift of the t–annulus at the point $g.o \in \mathcal{X}'$. Each wall separates $\mathcal{X}$ (and $\mathcal{X}'$) into two complementary components. Notice that the origins of consecutive edges in an s–wall are connected by a single a^rb–edge of length $2r$, while the termini of those edges are connected by a single a–edges of length 1. We say that crossing an s–wall in the positive direction scales distance by a factor of $\frac{1}{L}$. The same is true for the t–walls.

10.2. **Geodesics Between Points in a Plane.** We will define a family of $\mathcal{X}$–geodesics joining 1 to every point of $\langle a, b\rangle$. This is similar to the argument of [10].

For a point of the form $(a^rb)^m$ there is a geodesic of the form: $[1, s^{-1}]+s^{-1}[1, a^m]+s^{-1}a^m[s^{-1}, 1]$. To see this, first suppose that γ is any geodesic joining 1 to $(a^rb)^m$. Now, $s^{-1}a^ms = (a^rb)^m$, and $s^{-1}a^ms$ has length $2 + m$, which is already shorter than any path from 1 to $(a^rb)^m$ that stays in the plane $\langle a, b\rangle$. Thus, any γ must cross some walls. Since $\langle a, b\rangle$ is abelian we may assume that γ can be written as a concatenation of geodesics:

$$\begin{aligned}[1, s^{-1}] + s^{-1}[1, a^n] + s^{-1}a^n[s^{-1}, 1] + s^{-1}a^ns[1, t] + s^{-1}a^nst[1, a^p]\\ + s^{-1}a^nst^{-1}a^p[t^{-1}, 1] + s^{-1}a^nst^{-1}a^pt[1, a^q]\end{aligned}$$

This is a path from 1 to $s^{-1}a^nst^{-1}a^pta^q = (a^rb)^n(a^rb^{-1})^pa^q = a^{r(n+p)+q}b^{n-p} = a^{rm}b^m$, so $p = n - m$ and $q = -Lp$. If $p = q = 0$ we are done. Otherwise, let $\gamma_s = [1, s^{-1}] + s^{-1}[1, a^n] + s^{-1}a^n[s^{-1}, 1]$, $\gamma_t = s^{-1}a^ns[1, t^{-1}] + s^{-1}a^n\mathcal{S}t^{-1}[1, a^p] + s^{-1}a^nst^{-1}a^p[t^{-1}, 1]$, and $\gamma' = s^{-1}a^nst^{-1}a^pt[1, a^q]$, so that $\gamma = \gamma_s + \gamma_t + \gamma'$. There is a symmetry that exchanges γ_t with a geodesic $\gamma_t' = s^{-1}a^ns[1, s^{-1}]+s^{-1}a^nss^{-1}[1, a^{-p}]+s^{-1}a^nss^{-1}a^{-p}[s^{-1}, 1]$, but this means that $\gamma_s+\gamma_t'$ is a path from 1 to $(a^rb)^m$ of length $|\gamma_s| + |\gamma_t'| = |\gamma_s| + |\gamma_t| < |\gamma_s| + |\gamma_t| + |\gamma'| = d(1,(a^rb)^m)$, which is a contradiction.

For $0 \leqslant m \leqslant \frac{L}{2}+3$, the edge path a^m from 1 to a^m is a geodesic of length m. For $\frac{L}{2}+3 \leqslant m \leqslant L$ the edge path $s^{-1}ast^{-1}ata^{L-m}$ is a geodesic from 1 to a^m of length $6 + L - m$.

Using the fact that $\langle a, b\rangle$ is abelian, for every point a^xb^y there is a geodesic from 1 to a^xb^y of the form:

$$[1,(a^rb)^m] + (a^rb)^m[1,(a^rb^{-1})^n] + (a^rb)^m(a^rb^{-1})^n[1, a^p]$$

Moreover, $|p| < L$, since otherwise $[1, a^p]$ is of a similar form, and by rearranging geodesic subsegments we get a path from 1 to a^xb^y with backtracking across an s–wall and a t–wall, contradicting the fact that we started with a minimal length path. In particular, there is a geodesic from 1 to $(a^rb)^m(a^rb^{-1})^n$ of the form:

$$[1,(a^rb)^m] + (a^rb)^m[1,(a^rb^{-1})^n]$$

We can now find geodesics from 1 to $a^x b^y$ by induction. For example, Figure 3 shows two geodesics, each of length $5 \cdot 2^k - 4$ between 1 and a^{L^k}. (These form a geodesic loop that bears witness to the Dehn function.)

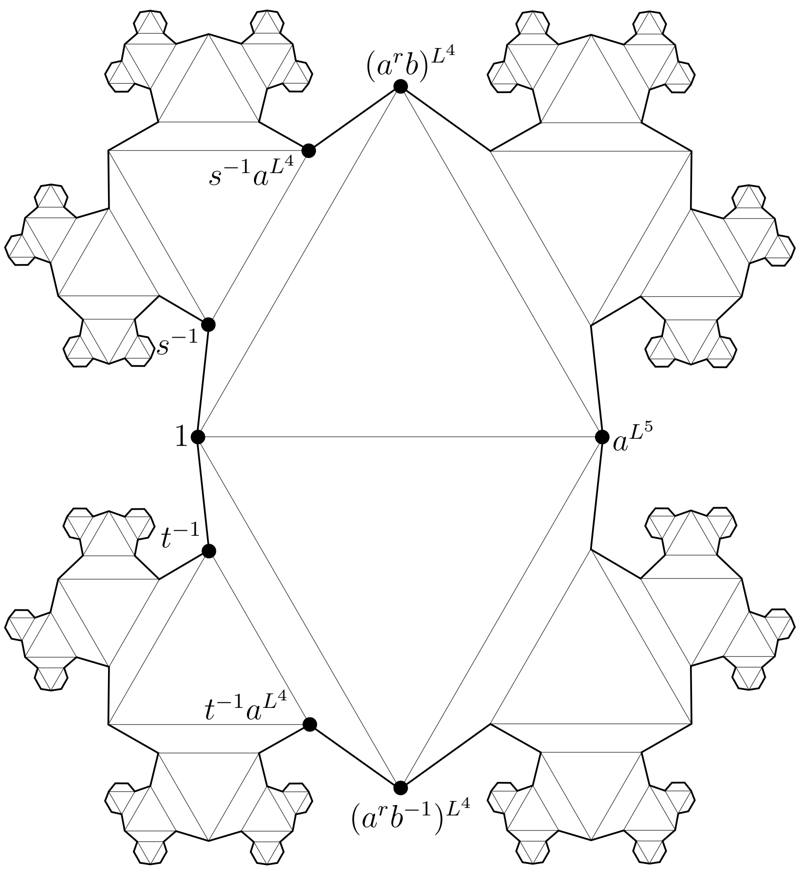

FIGURE 3. Snowflake - The boundary is a geodesic loop of length $2(5 \cdot 2^5 - 4)$

10.3. **Projections to Geodesics in $\mathcal{X}$.** In this section we consider two different geodesics:

$$\alpha(2n) = (s^{-1}t)^n$$
$$\beta(n) = s^{-n}$$

These are geodesics since for each of these paths, every edge crosses a distinct wall. Let $\mathcal{T}$ be the Bass-Serre tree of G, and let $o \in \mathcal{T}$ be the vertex fixed by the subgroup $\langle a, b\rangle$. The orbit map $g \mapsto g.o$ sends each of α and β isometrically to a geodesic in $\mathcal{T}$. We will use π_α to denote closest point projection to α, both in $\mathcal{X}$ and in $\mathcal{T}$, and similarly for β.

Both of these geodesics have the property that for any vertices at distance at least three in the corresponding geodesic of the Bass-Serre tree, the pointwise stabilizers of the pair of vertices is trivial. We might hope, in analogy to Theorem 8.4, that these would be strongly contracting geodesics. As in Theorem 8.4, $\langle s^{-1}t\rangle$ and $\langle s\rangle$ are hyperbolically embedded subgroups in G, but, of the two, we will see only $s^{-1}t$ is strongly contracting.

10.3.1. α. We claim that closest point projection $\pi_\alpha \colon \mathcal{X} \to \alpha$ is coarsely well defined and strongly contracting. First, consider π_α on $\langle a, b\rangle$. The geodesic α enters $\langle a, b\rangle$ through the incoming t–wall V at 1, and exits through the outgoing s^{-1}–wall W at 1.

Lemma 10.2. *For every $v \in V$ and every $w \in W$ there exists a geodesic from v to w that includes the vertex 1.*

Proof. The lemma follows from the discussion of geodesics in Section 10.2. □

Lemma 10.3. *The orbit map $\mathcal{X} \to \mathcal{T} : g \mapsto g.o$ coarsely commutes with closest point projection to α. In particular, closest point projection to α in $\mathcal{X}$ is coarsely well defined.*

Proof. Suppose $z \in \mathcal{X}$ is some vertex that is separated from 1 by V, and suppose there is an $n \geqslant 0$ such that $\alpha(n) \in \pi_\alpha(z)$. Let σ be a geodesic from z to $\alpha(n)$. Write $\sigma = \sigma_1 + \sigma_2 + \sigma_3$, where σ_2 is the subsegment of σ from the first time σ crosses V until the first time σ reaches W. By Lemma 10.2, we can replace σ_2 by a geodesic segment $\sigma_2' + \sigma_2''$ where the concatenation point is 1. This means that z is connected to $1 = \alpha(0)$ by a path $\sigma_1 + \sigma_2'$. By hypothesis, the length of this path is at least the length of σ, so σ_2'' and σ_3 are trivial and $n = 0$. It follows immediately that the orbit map $\mathcal{X} \to \mathcal{T}$ commutes with π_α up to an error of 4. (In fact, a little more work will show the error is at most 2.) □

Lemma 10.4 (Bounded Geodesic Image Property for π_α)**.** *For any geodesic σ in $\mathcal{X}$, if the diameter of $\pi_\alpha(\sigma.o)$ is at least 5, then $\sigma \cap \alpha \neq \emptyset$.*

Proof. Suppose $\alpha([-1, 3]).o \subset \pi_\alpha(\sigma.o)$. Then σ crosses the walls V, W, $s^{-1}tV$ and $s^{-1}tW$. Write σ as a concatenation of geodesic subsegments $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5$, where σ_1 is all of σ prior to the first V crossing, σ_2 is the part of σ between the first V crossing and the last W crossing, σ_3 is the part between the last W crossing and the first $s^{-1}tV$ crossing, which included edges labeled s^{-1} and t, σ_4 is the part from the first $s^{-1}tV$ crossing until the last $s^{-1}tW$ crossing, and σ_5 is the remainder of σ. We can apply Section 10.2 to replace σ_2 by a geodesic $\sigma_2' + \sigma_2''$ with the same endpoints and concatenated at 1. Similarly, we can replace σ_4 by a geodesic $\sigma_4' + \sigma_4''$ with the same endpoints and concatenated at $s^{-1}t$. But then we can replace the subsegment $\sigma_2 + \sigma_3 + \sigma_4$ of σ by the path $\sigma_2'' + [1, s^{-1}t] + \sigma_4''$ with the same endpoints. This path is strictly shorter unless σ_2'' and σ_4'' are trivial. This means that $[1, s^{-1}t] \subset \sigma \cap \alpha$. □

By Proposition 6.6, this means:

Corollary 10.5. *The element $s^{-1}t$ is strongly contracting for $G \curvearrowright \mathcal{X}$.*

Together with Theorem 5.4, this proves Theorem 10.1.

10.3.2. β. Using the arguments in Section 10.2, we see that $\pi_\beta(a^{L^j}) = \beta(j)$ for all $j \geqslant 0$.

In this case, the orbit map does not coarsely commute with closest point projection, as $\pi_\beta(a^{L^k}).o = \beta(k).o$, while $\pi_\beta(a^{L^k}.o) = \beta(0).o$. For $0 < j < k$ there is a geodesic $\sigma_{j,k}$ from a^{L^j} to a^{L^k} such that $d(\sigma_{j,k}, \beta) = d(a^{L^j}, \beta)$. It follows that π_β is not strongly contracting, since it does not enjoy the Bounded Geodesic Image Property. In fact, there are points of $\mathcal{X}$ for which π_β is not coarsely well defined.

There is another natural projection to consider. Define $\tau_\beta(x) = \beta(t)$ where $\pi_\beta(x.o) = \beta(t).o$. So $\tau_\beta(x)$ is just the preimage in β of closest point projection

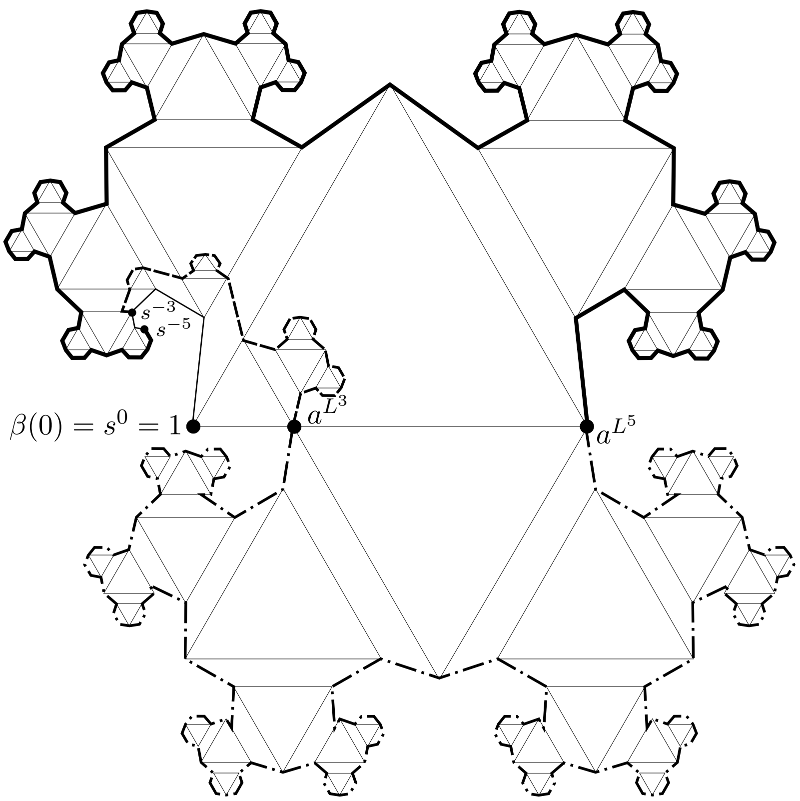

FIGURE 4. Geodesics $[a^{L^3}, \pi_\beta(a^{L^3})]$ (dashed), $[a^{L^5}, \pi_\beta(a^{L^5})]$ (solid), and $\sigma_{3,5} = [a^{L^3}, a^{L^5}]$ (dash-dot)

to $\beta.o$ in $\mathcal{T}$. π_β is a $\mathcal{PS}$–contraction where $\mathcal{PS}$ is the path system that consisting of:

- $\mathcal{PS}' = \{\text{geodesics } \gamma \in \mathcal{X} \mid \operatorname{diam}(\tau_\beta(\gamma)) \leqslant 1\}$, and
- for each pair of points $x, y \in \mathcal{X}$ with $d(\tau_\beta(x), \tau_\beta(y)) > 1$, every path of the form: $[x, \tau_\beta(x)] \in \mathcal{PS}'$, followed by the subsegment of β from $\tau_\beta(x)$ to $\tau_\beta(y)$, followed by $[\tau_\beta(y), y] \in \mathcal{PS}'$.

Thus, the element s^{-1} is a $\mathcal{PS}$–contracting element for $G \curvearrowright \mathcal{X}$, but not a strongly contracting element.

Acknowledgements

The first two authors supported by the European Research Council (ERC) grant of Goulnara ARZHANTSEVA, grant agreement #259527. The third author supported by NSF grant DMS-1311834.

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