QUANTIFYING METRIC APPROXIMATIONS OF DISCRETE GROUPS

GOULNARA ARZHANTSEVA AND PIERRE-ALAIN CHERIX

ABSTRACT. We introduce and systematically study a profile function whose asymptotic behaviour quantifies the dimension or the size of a metric approximation of a finitely generated group G by a family of groups $\mathcal{F} = \{(G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)\}_{\alpha \in I}$, where each group G_α is equipped with a bi-invariant metric d_α and a dimension k_α, for strictly positive real numbers ε_α such that $\inf_\alpha \varepsilon_\alpha > 0$. Through the notion of a residually amenable profile that we introduce, our approach generalises classical isoperimetric (aka Følner) profiles of amenable groups and recently introduced functions quantifying residually finite groups. Our viewpoint is much more general and covers hyperlinear and sofic approximations as well as many other metric approximations such as weakly sofic, weakly hyperlinear, and linear sofic approximations.

1. INTRODUCTION

Approximation is ubiquitous in mathematics. In the theory of groups, it is particularly natural to approximate infinite groups by finite ones. A fundamental realisation of this idea has lead Malcev (1940's) and P. Hall (1955) to the notion of a residually finite group: a group where the algebraic structure on any finite fixed set of elements is exactly as if these elements were in a suitable finite quotient of the group.

Once a concept of approximation is coined, a crucial question is how to compare distinct approximations of the same object, and, in particular, how to quantify the way an object is approximated. For residually finite groups, there are two main ways of quantifying the approximation of an infinite group by finite ones. The first way is to compute how many subgroups of a given finite index the group possesses. This is a classical subject of research on the subgroup growth, initiated by M. Hall (1949), which allows to enumerate how the group can be approximated by a finite quotient of a prescribed cardinality. The second way of quantifying is to compute the minimal cardinality among all possible finite quotients that detect the algebraic structure of the fixed finite set of elements of the residually finite group. This viewpoint is more recent and it is about the so-called full residual finiteness growth, see below for the definition.

In this paper, we push this second idea of quantifying of approximations of infinite groups significantly beyond the class of residually finite groups and apply it to much more general *metric* approximations of infinite groups in contrast to classical *algebraic* approximations. Metric approximations are approximations by groups equipped with bi-invariant metrics (see the next section for precise definitions) and they are very natural to study. Intuitively, we require that the algebraic operation on a finite set of group elements of the approximated group is almost as if these elements were in the approximating group, where 'almost' refers to the fixed bi-invariant

metric. This simple idea has gained a major importance following Gromov's introduction of *sofic* groups (= groups metrically approximated by symmetric groups of finite degrees, endowed with the normalised Hamming distance) and his settlement, for sofic groups, of Gottschalk's surjunctivity conjecture (1973) in topological dynamics. Another renowned example of metric approximation is that by unitary groups of finite rank, endowed with the normalised Hilbert-Schmidt distance. This defines the class of *hyperlinear* groups, appeared in the context of Connes' embedding problem (1972) in operator algebra.

We encompass both sofic and hyperlinear groups as well as their generalisations such as linear sofic groups, weakly sofic groups, and weakly hyperlinear groups into a general framework of metric approximations by groups with, in addition to a prescribed bi-invariant metric, a dimension or a size, associated with each of the approximating groups. For instance, the dimension of a finite symmetric group is chosen to be its degree, of a unitary group – its rank, of a finite group – its cardinality, etc. Our general quantification function, called *metric profile*, is then defined to be, given a finite set of group elements in the approximated group, e.g. the ball of finite radius with respect to the word length metric, the minimal dimension among all possible metric approximations which 'almost' preserve the algebraic structure of this finite set. Viewed within sofic groups, our approach is orthogonal to the recently emerged theory of sofic entropy started in the seminal work of L. Bowen (such a theory is not yet available for an a priori wider class of hyperlinear groups). Restricted to residually finite groups, the contrast between Bowen's viewpoint and our approach is exactly the distinction between the subgroup growth of a group and the full residual finiteness growth, respectively.

Since metric approximations generalise classical algebraic approximations, the previously known functions, quantifying 'exact' approximations (versus 'almost' ones), occur to be upper bounds for our metric profile. For example, a knowledge about the full residual finiteness growth of a residually finite group gives an estimate on the sofic and on the hyperlinear profiles of such a group. If the approximating groups are amenable, then besides a chosen dimension, they carry an associated isoperimetric function, the famous Følner function. We make use of this classical function and of our metric profile philosophy to define the *residually amenable profile* for every residually amenable group (and more generally, for every group locally embeddable into amenable ones). This allows to extend a classical study of Følner functions of amenable groups to non-amenable groups metrically approximable by amenable ones.

The main aim of this paper is to provide a necessary theoretical base for a further more specific quantitative analysis of metric approximations of concrete discrete groups. We meticulously compare our metric profile with previously investigated quantifying functions alluded to above. Since the classes of groups we study are preserved under several group-theoretical operations such as taking subgroups, direct and free products, extensions by amenable groups, restricted wreath products, etc., we also provide the corresponding estimates on the suitable metric profiles. On the way, we collect some crucial examples and finally formulate a number of open problems.

2. $\mathcal{F}$-APPROXIMATIONS AND $\mathcal{F}$-PROFILE

Let I denote an *index set*. We let $\mathcal{F} = (G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)_{\alpha \in I}$, where G_α is a group with a bi-invariant distance d_α and identity element e_α, k_α is a natural number that can be thought as the *dimension* of G_α, and ε_α is a strictly positive real number such that $\inf_\alpha \varepsilon_\alpha > 0$.

Let G be a countable discrete group with identity e_G and a distinguished generating set $S \subseteq G$. We denote by $|g|_S$ the length of an element $g \in G$ with respect to the word length metric defined by S. We let $B_{G,S}(n)$ be the ball of center e_G and radius n with respect to the word length metric induced by S – that is, $B_{G,S}(n) = \{g \in G \mid |g|_S \leqslant n\}$.

Definition 1 (Approximation). Let $n \in \mathbb{N}$ and $\varepsilon_\alpha > 0$. An (n, ε_α)-*approximation* of (G, S) by a group G_α is a function $\pi \colon G \to G_\alpha$ satisfying the following:

(1) $d_\alpha(\pi(g)\pi(h), \pi(gh)) < 1/n$ for every $g, h \in B_{G,S}(n)$ with $gh \in B_{G,S}(n)$, and

(2) $d_\alpha(\pi(g), \pi(h)) > \varepsilon_\alpha - 1/n$ for every $g, h \in B_{G,S}(n)$ such that $g \neq h$.

Such an (n, ε_α)-approximation is said to be *of dimension k_α*.

An $\mathcal{F}$-*approximation* of (G, S) is a sequence $(\pi_n)_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, π_n is an (n, ε_α)-approximation of (G, S) by G_α for some $\alpha \in I$.

A finitely generated group G is $\mathcal{F}$-*approximable* if (G, S) admits an $\mathcal{F}$-approximation for some (or, equivalently, any) finite generating set $S \subseteq G$.

The above condition (1) is often called 'almost homomorphism on the ball' while condition (2) is termed as 'uniform injectivity'.

In the definition below we convene that the minimum of the empty set is $+\infty$.

Definition 2 (Profile and dimension). Let G be a finitely generated group with a finite generating set S. The $\mathcal{F}$-*profile* of G is the function $\mathcal{D}_{G,S}^{\mathcal{F}} \colon \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by

$$\mathcal{D}_{G,S}^{\mathcal{F}}(n) = \min\{k \in \mathbb{N} \mid \exists \text{ an } (n, \varepsilon_\alpha)\text{-approximation of } G \text{ by } G_\alpha \text{ of dimension } k_\alpha = k\}.$$

The $\mathcal{F}$-*dimension* of G is defined by

$$\dim_{G,S}^{\mathcal{F}} = \limsup_{n \to +\infty} \frac{1}{n} \log \mathcal{D}_{G,S}^{\mathcal{F}}(n).$$

Observe that G is $\mathcal{F}$-approximable if and only if the function $\mathcal{D}_{G,S}^{\mathcal{F}}$ is everywhere finite. We write simply $\mathcal{D}_{G,S}$ when the family is irrelevant.

Remark 3. One can consider families $\mathcal{F}$ as above where d_α is not necessarily a metric but just a bi-invariant *pseudometric*. One can always reduce to the case of bi-invariant metric (rather than pseudometric) by replacing (G_α, d_α) with $(G_\alpha/N_\alpha, \overline{d}_\alpha)$, where N_α is the normal subgroup $\{g \in G_\alpha : d_\alpha(g, e_\alpha) = 0\}$ and $\overline{d}_\alpha$ is the bi-invariant metric induced by d_α on the quotient.

We consider the quasi-order $\preccurlyeq$ for functions $\mathcal{D}_1, \mathcal{D}_2 \colon \mathbb{N} \to \mathbb{N}$ defined by $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$ if and only if there exists a constant $C \in \mathbb{N}$ such that $\mathcal{D}_1(n) \leqslant C \mathcal{D}_2(Cn)$ for every $n \in \mathbb{N}$. We also let $\simeq$ be the equivalence relation associated with the quasi-order $\preccurlyeq$. Thus $\mathcal{D}_1 \simeq \mathcal{D}_2$ iff $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$ and $\mathcal{D}_2 \preccurlyeq \mathcal{D}_1$.

If S, S' are two finite generating sets of G, then there exists $C \in \mathbb{N}$ such $S \subseteq B_{G,S'}(C)$ and $S' \subseteq B_{G,S}(C)$. Therefore $B_{G,S}(n) \subseteq B_{G,S'}(Cn)$ and $B_{G,S'}(n) \subseteq B_{G,S}(Cn)$ for every $n \in \mathbb{N}$. This easily implies that $\mathcal{D}_{G,S}(n) \leqslant \mathcal{D}_{G,S'}(Cn)$ and $\mathcal{D}_{G,S'}(n) \leqslant \mathcal{D}_{G,S}(Cn)$. In particular, the $\simeq$-equivalence class of $\mathcal{D}_{G,S}$ does not depend on the choice of the finite generating set S. We denote such an equivalence class by $\mathcal{D}_G$.

Lemma 4. *Let G be a group and $F \subseteq G$ be a finite symmetric subset containing the identity e_G of G. Let $\varepsilon > 0$ and H be a bi-invariant metric group with metric d_H and $\pi \colon G \to H$ be a function such that $d_H(\pi(g)\pi(h), \pi(gh)) < \varepsilon$ for every $g, h \in F$ such that $gh \in F$.*

For every $n \in \mathbb{N}$ and $g, g_1, \ldots, g_n \in G$ such that $g, g_k, g_1^{\varepsilon_1} \cdots g_k^{\varepsilon_k} \in F$ for every $\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}$ and $k \in \{1, 2, \ldots, n\}$ the following assertions hold:

(1) $d_H(\pi(e_G), e_H) < \varepsilon$;
(2) $d_H(\pi(g^{-1}), \pi(g)^{-1}) < 2\varepsilon$;
(3) $d_H(\pi(g_1 \cdots g_n), \pi(g_1) \cdots \pi(g_n)) < (n-1)\varepsilon$, *whenever $n > 1$*;
(4) $d_H(\pi(g_1^{\varepsilon_1}) \cdots \pi(g_n^{\varepsilon_n}), \pi(g_1)^{\varepsilon_1} \cdots \pi(g_n)^{\varepsilon_n}) < 2n\varepsilon$;
(5) $d_H(\pi(g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n}), \pi(g_1)^{\varepsilon_1} \cdots \pi(g_n)^{\varepsilon_n}) < (3n-1)\varepsilon$.

Proof. Since $e_G \in F$, using the hypothesis on π and the bi-invariance of d_H we have

$$d_H(\pi(e_G), e_H) = d_H(\pi(e_G)^2, \pi(e_G)) < \varepsilon.$$

This proves (1). Similarly, for (2) we have

$$d_H(\pi(g^{-1}), \pi(g)^{-1}) = d_H(\pi(g)\pi(g^{-1}), e_H) \leqslant d_H(\pi(e_G), e_H) + \varepsilon < 2\varepsilon.$$

One can easily prove (3) by induction on n using the hypothesis, and (4) using (2). Finally (5) follows from (3) and (4). $\qquad\square$

In the following we suppose that G is a finitely generated group with a presentation $\langle X \mid R \rangle$ given by generators X and relators R. We denote by S the symmetric generating set of G associated with X, i.e. $S = X \sqcup X^{-1}$. Without loss of generality, we assume that $R = R^{-1}$, i.e. R contains inverses of all of its elements. We denote by F_X the free group over the alphabet X. For a word w over the letters from $X \sqcup X^{-1}$ we let $|w|_{F_X}$ be its length. In this situation, in order to define a homomorphism from G to another group H, it is often convenient to define a homomorphism from F_X to H, in such a way that any word in R is mapped to the identity element $e_H \in H$. This defines a unique group homomorphism from G to H, and any group homomorphism arises in this way. In this spirit, we define natural variants of the $\mathcal{F}$-profile as follows (recall our convention that the minimum of the empty set is $+\infty$):

- $\mathcal{W}_{G,S}^{\mathcal{F}}(n)$ is the least $k \in \mathbb{N}$ such that for some α with $k_\alpha = k$ there exists a homomorphism $\varphi \colon F_X \to G_\alpha$ such that for any word $w \in F_X$ of length at most n one has that $d_\alpha(\varphi(w), e_\alpha) > \varepsilon_\alpha - 1/n$ if w does not represent the identity of G and $d_\alpha(\varphi(w), e_\alpha) < 1/n$ otherwise;

- $\mathcal{R}_{G,S}^{\mathcal{F}}(n)$ is the least $k \in \mathbb{N}$ such that for some α with $k_\alpha = k$ there exists a homomorphism $\varphi \colon F_X \to G_\alpha$ such that for any word $w \in F_X$ of length at most n that does not represent the identity of G one has that $d_\alpha(\varphi(w), e_\alpha) > \varepsilon_\alpha - 1/n$, and $d_\alpha(\varphi(r), e_\alpha) < 1/n$ for any relator $r \in R$ of length at most n.

In the following proposition, we establish precise quantitative relations among the notions of profile $\mathcal{D}^{\mathcal{F}}_{G,S}, \mathcal{W}^{\mathcal{F}}_{G,S}, \mathcal{R}^{\mathcal{F}}_{G,S}$ we have just introduced. To begin with, we recall some notions from combinatorial group theory.

If $w \in F_X$ is a word that represents the identity of $G = \langle X \mid R \rangle$, then the corresponding combinatorial *area* $\mathcal{A}_{G,S}(w)$ is defined to be the least ℓ such that w can be written as a product of ℓ conjugates of relators from R. The *Dehn function*, denoted by $Dehn_{G,S}(m)$, is defined to be the largest combinatorial area $\mathcal{A}_{G,S}(w)$, where w ranges among all the words of length at most m representing the identity of G. We let $N_{G,S}(m)$ be the minimum, among all representations $w = \left(\eta_1 r_1 \eta_1^{-1}\right) \cdots \left(\eta_\ell r_\ell \eta_\ell^{-1}\right)$ as product of $\ell \leqslant Dehn_{G,S}(m)$ conjugates of relators $r_1, \ldots, r_\ell \in R$, of the maximum of the length of $r_1, \ldots, r_\ell$.

Proposition 5. *Under the notation above, the following relations between the functions $\mathcal{D}^{\mathcal{F}}_{G,S}$, $\mathcal{W}^{\mathcal{F}}_{G,S}, \mathcal{R}^{\mathcal{F}}_{G,S}$ hold for all sufficiently large m's:*

(1) $\mathcal{W}^{\mathcal{F}}_{G,S}(m) \leqslant \mathcal{D}^{\mathcal{F}}_{G,S}((3m+1)m)$;
(2) $\mathcal{D}^{\mathcal{F}}_{G,S}(m) \leqslant \mathcal{W}^{\mathcal{F}}_{G,S}(3m)$;
(3) $\mathcal{W}^{\mathcal{F}}_{G,S}(m) \leqslant \mathcal{R}^{\mathcal{F}}_{G,S}(\max\{Dehn_G(m)m, N_{G,S}(m)\})$;
(4) $\mathcal{R}^{\mathcal{F}}_{G,S}(m) \leqslant \mathcal{W}^{\mathcal{F}}_{G,S}(m)$.

In particular, if one of the functions $\mathcal{D}^{\mathcal{F}}_{G,S}, \mathcal{W}^{\mathcal{F}}_{G,S}, \mathcal{R}^{\mathcal{F}}_{G,S}$ is everywhere finite, then all the others are everywhere finite.

Proof. We prove the nontrivial inequalities below.

(1): Fix $m \in \mathbb{N}$ and consider $n = (3m+1)m$. If $k = \mathcal{D}^{\mathcal{F}}_{G,S}(n)$, then for some α with $k_\alpha = k$ there exists an (n, ε_α)-approximation $\pi \colon G \to G_\alpha$. Let $\varphi \colon F_X \to G_\alpha$ be the homomorphism induced by the restriction of π to X. Suppose that $w = x_1 \cdots x_l$ is a word in F_X of length $l \leqslant m$. If w represents the identity $e_G \in G$, then by the triangle inequality and by (1) and (5) of Lemma 4 we have that

$$d_\alpha(\varphi(w), e_\alpha) \leqslant d_\alpha(\varphi(w), \pi(e_G)) + d_\alpha(\pi(e_G), e_\alpha) < (3m-1)/n + 1/n < 1/m.$$

Suppose now that w represents a nontrivial element g of G. Then we have again, by the triangle inequality together with (1) and (5) of Lemma 4,

$$d_\alpha(\varphi(w), e_\alpha) \geqslant d_\alpha(\pi(g), \pi(e_G)) - d_\alpha(\pi(e_G), e_\alpha) - d_\alpha(\pi(g), \varphi(w)) > \varepsilon_\alpha - 1/m.$$

(2): Suppose that $m \in \mathbb{N}$ and set $n = 3m$. If $k = \mathcal{W}^{\mathcal{F}}_{G,S}(n)$, then for some α with $k_\alpha = k$ there exists a homomorphism $\varphi \colon F_X \to G_\alpha$ such that for any word $w \in F_X$ of length at most n one has that $d_\alpha(\varphi(w), e_\alpha) > \varepsilon_\alpha - 1/n$ if w does not represent the identity of G and $d_\alpha(\varphi(w), e_\alpha) < 1/n$ otherwise. One can choose for any element $g \in B_{G,S}(m)$ a word $w_g \in F_X$ of minimal length that represents g in such a way that $w_{g^{-1}} = w_g^{-1}$ for $g \in G$. Define then $\pi(g) = \varphi(w_g)$ for every $g \in B_{G,S}(m)$ and arbitrarily for $g \notin B_{G,S}(m)$. We claim that π is an (m, ε_α)-approximation for G. Indeed, suppose that $g, h \in B_{G,S}(m)$ are such that $gh \in B_{G,S}(m)$. Observe that the word $w_g w_h w_{gh}^{-1} \in F_X$ has length at most n and represents the identity of G. Hence, by assumption, we have that

$$d_\alpha(\pi(g)\pi(h), \pi(gh)) = d_\alpha(\varphi(w_g w_h w_{gh}^{-1}), e_\alpha) < 1/n < 1/m.$$

Suppose now that g, h are distinct elements of $B_{G,S}(m)$. We have that $w_g w_h^{-1}$ is an element of F_X of length at most $2m < n$ that does not represent the identity of G. Hence

$$d_\alpha(\pi(g), \pi(h)) = d_\alpha(\varphi(w_g w_h^{-1}), e_\alpha) > \varepsilon_\alpha - 1/n > \varepsilon_\alpha - 1/m.$$

(3): Fix $m \in \mathbb{N}$. Set $n = \max\{Dehn_{G,S}(m) \, m, N_{G,S}(m)\}$. If $k = \mathcal{R}^{\mathcal{F}}_{G,S}(n)$, then for some α with $k_\alpha = k$ there exists a homomorphism $\varphi \colon F_X \to G_\alpha$ such that for any word $w \in F_X$ of length at most n that does not represent the identity of G one has that $d_\alpha(\varphi(w), e_\alpha) > \varepsilon_\alpha - 1/n$, and $d_\alpha(\varphi(r), e_\alpha) < 1/n$ for any relator $r \in R$ of length at most n. Suppose now that w is a word in G of length at most m that represents the identity of G. Then we can write w as the product $(\eta_1 r_1 \eta_1^{-1}) \cdots (\eta_\ell r_\ell \eta_\ell^{-1})$ where $\ell \leqslant Dehn_{G,S}(m)$, $r_i \in R$ have length at most $N_{G,S}(m) \leqslant n$ and $\eta_i \in F_X$. Thus, we have, for each $i \leqslant \ell$,

$$d_\alpha(\varphi(\eta_i r_i \eta_i^{-1}), e_\alpha) = d_\alpha(\varphi(r_i), e_\alpha) < 1/n$$

and, hence, $d_\alpha(\varphi(w), e_\alpha) < Dehn_{G,S}(m)/n \leqslant 1/m$. $\square$

Proposition 6. *Let $\mathcal{F} = (G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)_{\alpha \in I}$ be a family as above such that*

  (i) for every $k \in \mathbb{N}$, $\{\alpha \in I : k_\alpha \leqslant k\}$ is finite, and

  (ii) for every $\alpha \in I$ the bi-invariant metric group (G_α, d_α) is compact.

  If G is a finitely generated group, then the following assertions are equivalent:

  (1) the $\mathcal{F}$-profile $\mathcal{D}^{\mathcal{F}}_{G,S}$ of G is bounded (equivalently, each of the profiles $\mathcal{W}^{\mathcal{F}}_{G,S}, \mathcal{R}^{\mathcal{F}}_{G,S}$ is bounded);

  (2) there exists an injective group homomorphism $\iota \colon G \hookrightarrow G_\alpha$ for some $\alpha \in I$ such that $d_\alpha(\iota(g), \iota(h)) > \varepsilon_\alpha$ for every $g, h \in G$ such that $g \neq h$.

Proof. The implication (2)$\Rightarrow$(1) is obvious, so we focus on the implication (1)$\Rightarrow$(2). Fix a finite generating set S for G. Suppose that the $\mathcal{F}$-profile of G is bounded. Then there exists $k \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ there exist $\alpha_n \in I$ such that $k_{\alpha_n} \leqslant k$ and an $(n, \varepsilon_{\alpha_n})$-approximation $\varphi_n \colon G \to G_{\alpha_n}$. Since by assumption $\{\alpha \in I : k_\alpha \leqslant k\}$ is finite, without loss of generality we can assume that $\alpha_n = \alpha$ does not depend on n, and thus $\varphi_n \colon G \to G_\alpha$ for every $n \in \mathbb{N}$. Fix a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ and define $\varphi \colon G \to G_\alpha$ by $\varphi(g) := \lim_{n \to \mathcal{U}} \varphi_n(g)$. Observe that this is well defined since, by assumption, (G_α, d_α) is compact. Since, for every $n \in \mathbb{N}$, φ_n is an (n, ε_α)-approximation, it follows that φ is an injective group homomorphism as required. $\square$

Remark 7. It is clear from the preceding proof that assumption (ii) can be weakened. For example, Proposition 6 remains true under condition (ii') that (G_α, d_α) is a proper metric space (i.e. a metric space where every closed ball is compact) and for each $g \in G$ and some $l > 0$ we have $\{n \in \mathbb{N} \mid d_\alpha(\varphi_n(g), e_\alpha) \leqslant l\} \in \mathcal{U}$, where α is the index appearing in the proof. Indeed, such (ii') ensures that $\lim_{n \to \mathcal{U}} \varphi_n(g)$ exists and is unique.

2.1. $\mathcal{F}$-approximations and metric ultraproducts.

Adopting the notation from the beginning of the section, we let $\mathcal{F}$ be the family $(G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)_{\alpha \in I}$, where k_α is a nonzero natural number, ε_α is a strictly positive real number such that $\inf_\alpha \varepsilon_\alpha > 0$, and G_α is a bi-invariant metric group with distance d_α and identity element e_α. Fix a non-principal ultrafilter $\mathcal{U}$ over the index set I. The *metric ultraproduct* $\prod_{\mathcal{U}} (G_\alpha, d_\alpha)$ of the family of bi-invariant metric groups $(G_\alpha, d_\alpha)_{\alpha \in I}$ can be defined as in [38, Section 4]. This is the quotient of the

direct product $\prod_{\alpha \in I} G_\alpha$ by the normal subgroup consisting of those elements (g_α) such that $\lim_{\alpha \to \mathcal{U}} d_\alpha (g_\alpha, e_\alpha) = 0$. The metric ultraproduct $\prod_{\mathcal{U}} (G_\alpha, d_\alpha)$ is endowed with a canonical bi-invariant metric $d_{\mathcal{U}}$, obtained as the quotient of the bi-invariant pseudometric on $\prod_{\alpha \in I} G_\alpha$ defined by $d_{\mathcal{U}} ((g_\alpha), (h_\alpha)) = \lim_{\alpha \to \mathcal{U}} d_\alpha (g_\alpha, h_\alpha)$. Such a construction is in fact a particular instance, in the case of bi-invariant metric groups, of the notion of ultraproduct in the logic for metric structures; see [7, Section 5] and [16, Section 2.6].

In the case when the family $(\varepsilon_\alpha)_{\alpha \in I}$ is constantly equal to a given strictly positive real number ε, one can reformulate the notion of $\mathcal{F}$-approximability in terms of embeddings into a metric ultraproduct $\prod_{\mathcal{U}} (G_\alpha, d_\alpha)$. Precisely, a countable group G is $\mathcal{F}$-approximable if and only if there exist an ultrafilter $\mathcal{U}$ over I and a group homomorphism $\iota \colon G \to \prod_{\mathcal{U}} (G_\alpha, d_\alpha)$ such that $d_{\mathcal{U}} (\iota (g), \iota (h)) \geqslant \varepsilon$ for every distinct $g, h \in G$; see also [46, Proposition 1.8].

Example 8 (Varieties and non-varieties). It follows from the preceding paragraph that if $\mathcal{C}$ is a class of groups that is closed under taking arbitrary direct products, subgroups, and quotients (equivalently, by the Birkhoff theorem, if $\mathcal{C}$ is a *variety of groups*, i.e. a class of groups defined by a given set of *identities* or, using an alternative terminology, *laws*), and that contains G_α for every $\alpha \in I$, then $\mathcal{C}$ contains every $\mathcal{F}$-approximable group.

For instance, for any positive integer ℓ, groups approximable by solvable groups of derived length at most ℓ are solvable of derived length at most ℓ. In contrast, there exists a non-solvable group which is approximable by solvable groups (with no uniform bound on the derived length). These are not using any consideration about a metric[1].

A more 'metric' example is given be the family $\mathcal{F}_{cc}^{fin} = (H_\alpha, d_\alpha^{cc}, |H_\alpha|, 1)_{\alpha \in I}$, see Section 3.4, where H_α, $\alpha \in I$ are all finite groups. It is clear that there exists an $\mathcal{F}_{cc}^{fin}$-approximable group which does not belong to $\mathcal{F}_{cc}^{fin}$ but, moreover, there exists a finitely presented group, the famous Higman group, which is not $\mathcal{F}_{cc}^{fin}$-approximable [46], cf. Question 49.

Example 9 (Constant dimension). Let G be a finitely generated $\mathcal{F}$-approximable group. Assume that $k_\alpha = k$ for all $\alpha \in I$ and a fixed $k > 0$. By Definition 2, the $\mathcal{F}$-profile $\mathcal{D}_{G,S}^{\mathcal{F}}$ and, by Proposition 5, profiles $\mathcal{W}_{G,S}^{\mathcal{F}}, \mathcal{R}_{G,S}^{\mathcal{F}}$ are constant. In particular, (1) of Proposition 6 holds. However, (i) of Proposition 6 is not fulfilled. It is natural to ask whether or not conclusion (2) of Proposition 6 remains true in this case of constant dimension of approximating groups.

For instance, if G_α are finite for all $\alpha \in I$ and $k_\alpha = |G_\alpha| = k$ for some $k > 0$, is the fixed cardinality of these finite groups[2], then Definition 1(2) implies that G is finite and injects in some G_α (cf. Section 3.4, for a general setting when sizes of finite groups vary). Thus, Proposition 6(2) does hold in this case.

In the same vein, let $G_\alpha = GL(k, \mathbb{K}_\alpha)$ be the group of $k \times k$ invertible matrices for a fixed $k > 0$, with coefficients in a field $\mathbb{K}_\alpha$, and equipped with the *trivial $\{0,1\}$-valued metric* $d_{\{0,1\}}$, for $\alpha \in I$, defined by $d_{\{0,1\}}(g, h) = 1$ if $g \neq h$ and 0 otherwise. Then G is linear, i.e. G is a subgroup of $GL(n, \mathbb{K})$ for some positive integer n and a field $\mathbb{K}$ (which is not required to coincide with $\mathbb{K}_\alpha$ for some $\alpha \in I$). Indeed, n can be chosen to be k and $\mathbb{K}$ to be the *algebraic ultraproduct* of fields $\mathbb{K}_\alpha$. This result is due to Malcev [36] and it can be also formulated using

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[1]In fact, the metric is the trivial $\{0,1\}$-valued metric $d_{\{0,1\}}$, see Example 9.

[2]There are finitely many finite groups of given cardinality, that is, at least one group among $G_\alpha, \alpha \in I$ appears infinitely many times in the metric ultraproduct $\prod_{\mathcal{U}} (G_\alpha, d_\alpha)$.

the language of the first-order classical logic: if the universal first-order theory of G contains that of a linear group $GL(k, \mathbb{K}_\alpha)$ (this is the case, for instance, when G and $GL(k, \mathbb{K}_\alpha)$ have the same elementary theories), then G is linear.

Note that we can relax the assumption on finite generation of G as by Malcev's local theorem a group has a faithful linear representation of degree n over a field of characteristic $p \geqslant 0$ if and only if each of its finitely generated subgroups has such a representation over a field of characteristic p [36].

Since we deal with *metric* ultraproducts (associated to arbitrary bi-invariant distances d_α) it is natural to ask whether or not Malcev's result still holds in the continuous logic setting, see Question 50.

However, it is clear that for an arbitrary group G_α one cannot expect that an embedding of G into an ultrapower of G_α induces an embedding $G \hookrightarrow G_\alpha$. Here is a concrete example in the case of algebraic ultrapower: if $G_\alpha = F_X$ is a free non-abelian group, then, by a result of Remeslennikov [41], a finitely generated subgroup G of an algebraic ultrapower of F_X is a fully residually free group. Take such a non-free group G, i.e. any non-free Sela's limit group. See also Question 53.

3. Classes of $\mathcal{F}$-approximable groups

The notion of $\mathcal{F}$-approximable group for various choices of $\mathcal{F}$ captures many classes of groups that are currently a subject of intensive study. We mention only main examples below, for more details about these wide classes of groups and a broad range of applications see [39, 38, 18, 1, 3, 16]. The reader is also invited to analyze the $\mathcal{F}$-profile and $\mathcal{F}$-dimension, given her/his favorite family $\mathcal{F}$. Moreover, note that our quantifying of metric approximations extends immediately to quantifying of constraint metric approximations [4], using suitable *constraint* metric profiles.

Observe that the classes of groups below are so that all residually finite and all amenable groups belong to these classes.

3.1. **Sofic groups.** This class of groups has been first introduced by Gromov in [30] in the context of symbolic dynamics; see also a work of Weiss [50].

Let $\mathcal{F}^{sof}$ be the collection of permutation groups $\mathrm{Sym}(n), n \in \mathbb{N}$, endowed with the normalised *Hamming distance*: for permutations $\sigma, \tau \in \mathrm{Sym}(n)$, we define

$$d_{\mathrm{Ham}}(\sigma, \tau) = \frac{1}{n} \left| \left\{ i \mid \sigma(i) \neq \tau(i) \right\} \right|.$$

Definition 10 (Sofic group via permutations; sofic profile and dimension)**.** A group G is said to be *sofic* if it is approximable by $\mathcal{F}^{sof} = (\mathrm{Sym}(n), d_{\mathrm{Ham}}, n, 1)_{n \in \mathbb{N}}$, in the sense of Definition 1.

We call *sofic profile* $\mathcal{D}_G^{sof}$ and *sofic dimension* $\dim_G^{sof}$ of a sofic group G the $\mathcal{F}^{sof}$-profile and $\mathcal{F}^{sof}$-dimension (respectively) for such a choice of the approximating family $\mathcal{F} = \mathcal{F}^{sof}$.

We stipulate that the dimension k_n of $\mathrm{Sym}(n)$ is chosen to be equal to n and $\varepsilon_n = 1$ for every $n \in \mathbb{N}$. Equivalently, ε_n can be chosen to be constantly equal to a fixed strictly positive real number $\varepsilon \leqslant 1$.

More precisely, $\varepsilon_n = 1$ and $\varepsilon_n = \varepsilon < 1$ in Definition 10 yield the same class of sofic groups but the corresponding sofic profile functions are not necessarily $\simeq$-equivalent. Indeed, one can

enlarge a fixed ε using the so-called "amplification trick" (see the discussions in [39, §3] and [3, Section 5]) but this enlarges the dimension of permutation matrices from n to n^l, whenever l amplification steps are required.

In contrast, the $\simeq$-equivalence class of $\mathcal{D}_G^{sof}$ and, hence, the value of $\dim_G^{sof}$, do not change if one defines the sofic profile only considering an $\mathcal{F}^{sof}$-approximation, i.e. a sofic approximation, defined by maps $g \mapsto \sigma_g$ such that σ_{e_G} is the identity permutation and, for $g \neq e_G$, $\sigma_{g^{-1}} = \sigma_g^{-1}$ has no fixed points; see [16, Exercise 2.1.10].

It follows from Proposition 6 that a finitely generated group has bounded sofic profile if and only if it is finite.

Gromov's original definition of soficity of a group G uses approximations of its Cayley graph by finite labeled graphs.

Definition 11 (Sofic group via graphs)**.** A group G with a finite symmetric generating set S is called *sofic*, if for each $\delta > 0$ and each $n \in \mathbb{N}$ there is a finite directed graph Γ edge-labeled by S, and a subset $\Gamma_0 \subseteq \Gamma$ with the properties, that:

(i) For each point $v \in \Gamma_0$ there is a map $\psi_v \colon B_{G,S}(n) \to \Gamma$ which is a label-preserving isomorphism between the ball $B_{G,S}(n)$ in the Cayley graph of G with respect to S and the n-ball in Γ around v, and

(ii) $|\Gamma_0| \geqslant (1 - \delta)|\Gamma|$.

Such a graph Γ is called an $[n, \delta]$-*approximation* of the Cayley graph $Cay(G, S)$.

We can now give another, in addition to the above-defined $\mathcal{D}_{G,S}^{sof}$ (and, hence, $\mathcal{W}_{G,S}^{sof}$ and $\mathcal{R}_{G,S}^{sof}$), natural definition of a profile of a sofic group:

- $\mathcal{G}_{G,S}^{sof}(n)$ is the least cardinality (= number of vertices) of the graph Γ in an $[n, 1/n]$-approximation of the Cayley graph $Cay(G, S)$.

This new profile is $\simeq$-equivalent to our initial definition of the sofic profile.

Proposition 12. $\mathcal{G}_{G,S}^{sof}(n) \simeq \mathcal{D}_{G,S}^{sof}(n)$.

Proof. The above two definitions of soficity, via permutations and via graphs, are equivalent, see, for instance, the proof of the equivalence of Definition 4.2 and Definition 4.3 in [23]. Analysing the details of this proof, we see that $\mathcal{G}_{G,S}^{sof}(2n) \geqslant \mathcal{D}_{G,S}^{sof}(n)$ and $\mathcal{G}_{G,S}^{sof}(n) \leqslant \mathcal{D}_{G,S}^{sof}(2n+2)$, whence the $\simeq$-equivalence of the two functions. $\square$

Example 13 (Sofic profiles of free abelian groups)**.** We begin with the group of integers: $G = \mathbb{Z}$ and $S = \{+1, -1\}$. Clearly, $\mathcal{D}_{\mathbb{Z},S}^{sof}(n) \leqslant 2n + 1$ by considering the sofic approximation coming from the action υ of $\mathbb{Z}$ on $\mathbb{Z}/(2n+1)\mathbb{Z}$ defined by

$$\upsilon(i) \;:\; \begin{array}{ccc} \mathbb{Z}/(2n+1)\mathbb{Z} & \to & \mathbb{Z}/(2n+1)\mathbb{Z} \\ \bar{j} & \mapsto & \upsilon(i)(\bar{j}) = \overline{i+j}, \end{array}$$

where $i, j \in \mathbb{Z}$ and $j \mapsto \bar{j}$ is the canonical epimorphism $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/(2n+1)\mathbb{Z}$. This shows that $\mathcal{D}_{\mathbb{Z}}^{sof}(n) \preccurlyeq n$.

Let us check that $\mathcal{D}_{\mathbb{Z}}^{sof}(n) \simeq n$. Suppose that $k < 2n + 1$ and assume that $\varphi \colon \mathbb{Z} \to \mathrm{Sym}(k)$ is a $(2n, 1)$-approximation of $\mathbb{Z}$. Then, since d_{Ham} has values in $\{0, 2/k, \ldots, (k-1)/k, 1\}$, we have that for every $i, j \in [-n, n]$, $\varphi(i + j) = \varphi(i)\varphi(j)$ and $\varphi(i)$ is a nontrivial element of $\mathrm{Sym}(k)$

whenever $i \neq 0$. In particular, $\varphi(i) = \varphi(1)^i$ and $\varphi(1)$ is an element of order $> n$. We stipulate that $\varepsilon = 1$, then for every $i \in [-n, n]$ such that $i \neq 1$ we have

$$d_{\text{Ham}}(\varphi(1), \varphi(1)^i) > 1 - \frac{1}{2n}.$$

This implies $d_{\text{Ham}}(\varphi(1), \varphi(1)^i) = 1$. Therefore, $\varphi(1)$ is a cycle of length $> n$. Hence, $k > n$. This shows that $n \preccurlyeq \mathcal{D}_{\mathbb{Z}}^{sof}(n)$. It follows from this and Proposition 34 that any virtually-$\mathbb{Z}$ group G has $\mathcal{D}_G^{sof}(n) \simeq n$. It also follows from this and the estimate on the sofic profile of the direct product that $\mathcal{D}_{\mathbb{Z}^d}^{sof}(n) \preccurlyeq n^d$ for every $d \in \mathbb{N}$, see Section 5.2. Moreover, $\mathcal{D}_{\mathbb{Z}^d}^{sof}(n) \simeq n^d$ and $\mathcal{D}_G^{sof}(n) \simeq n^d$ for any virtually-$\mathbb{Z}^d$ group G. This follows from the same argument as above combined with a result on the stability of the commutator relator words with respect to the Hamming distance [2, Corollary 6.5], see Corollary 32 and Example 33.

3.2. **Hyperlinear groups.** This class of groups appeared in relation to the concept of hyperlinearity in operator algebras. The definition below is due to a result of Rădulescu [42]; see also [16, Proposition 2.2.9] and [34, Section 4.2].

Let $\mathcal{F}^{hyp}$ be the collection of finite-rank unitary groups $U(n), n \in \mathbb{N}$, endowed with the normalised *Hilbert-Schmidt distance*: for unitary matrices $u = (u_{ij}), v = (v_{ij}) \in U(n)$, we define

$$d_{\text{HS}}(u, v) = \|u - v\|_2 = \sqrt{\frac{1}{n} \sum_{i,j=1}^n |u_{ij} - v_{ij}|^2} = \sqrt{\frac{1}{n} \text{tr}((u - v)^*(u - v))}.$$

Definition 14 (Hyperlinear group; hyperlinear profile and dimension)**.** A group G is said to be *hyperlinear* if it is approximable by $\mathcal{F}^{hyp} = (U(n), d_{\text{HS}}, n, \sqrt{2})_{n \in \mathbb{N}}$, in the sense of Definition 1.

We call *hyperlinear profile* $\mathcal{D}_G^{hyp}$ and *hyperlinear dimension* $\dim_G^{hyp}$ of a hyperlinear group G the $\mathcal{F}^{hyp}$-profile and $\mathcal{F}^{hyp}$-dimension (respectively) for such a choice of the approximating family $\mathcal{F} = \mathcal{F}^{hyp}$.

Again we convene that the dimension k_n of $U(n)$ is equal to n and $\varepsilon_n = \sqrt{2}$ for every $n \in \mathbb{N}$. Equivalently, ε_n can be chosen to be constantly equal to a fixed strictly positive real number $\varepsilon \leqslant \sqrt{2}$. That is, we obtain the same class of hyperlinear groups.

The flexibility in the choice of ε_n that we observe in the definitions of both sofic and hyperlinear groups does not hold a priori for arbitrary $\mathcal{F}$-approximations. Indeed, it strongly depends on specific properties of the metric we use. In fact, both the Hamming and the Hilbert-Schmidt metrics behave well under the above mentioned "amplification trick", whence this freedom in the choice of ε_n's in the definitions of sofic and hyperlinear groups, respectively.

It follows from Proposition 6 that a finitely generated group has bounded hyperlinear profile if and only if it embeds into $U(n)$ for some $n \in \mathbb{N}$.

Given two permutations $\sigma, \tau \in \text{Sym}(n)$, let $u_\sigma, v_\tau \in U(n)$ denote the corresponding permutation matrices. Then,

$$d_{\text{Ham}}(\sigma, \tau) = \frac{1}{2}(d_{\text{HS}}(u_\sigma, v_\tau))^2.$$

It follows that sofic groups are hyperlinear. Noting $B_{G,S}(n) \subseteq B_{G,S}(2n^2)$, we immediately obtain that, for a sofic group G, we have:

$$\mathcal{D}_{G,S}^{hyp}(n) \leqslant \mathcal{D}_{G,S}^{sof}(2n^2).$$

The converse is not yet known: whether or not all hyperlinear groups are sofic is a well-known open problem. Observe that the Hamming distance is an ℓ^1-type metric and the Hilbert-Schmidt distance is the Euclidean, hence, an ℓ^2-type metric. Therefore, the above square root distortion of the distance under the canonical map $\mathrm{Sym}(n) \hookrightarrow U(n)$ sending permutations to the permutation matrices: $\sigma \mapsto u_\sigma, \tau \mapsto v_\tau$, can a priori not be improved into an isometric embedding. However, we deal with approximations, whence the following

Conjecture 15. *If G is sofic, then $\mathcal{D}_G^{sof}(n) \preccurlyeq \mathcal{D}_G^{hyp}(n)$.*

The conjecture holds for sofic $\mathcal{F}^{hyp}$-stable groups (see Definition 30), e.g. for virtually abelian groups and the Heisenberg groups, see Corollary 32(2) and Example 33. See also Question 51 and Question 52.

Here is a useful modification of the hyperlinear profile. The finite-rank unitary groups $U(n), n \in \mathbb{N}$, can be endowed with the normalised *projective Hilbert-Schmidt pseudo-distance*: for unitary matrices $u, v \in U(n)$, we define

$$d_{\overline{\mathrm{HS}}}(u, v) = \inf_{\lambda \in \mathbb{T}} \sqrt{\frac{1}{n}\mathrm{tr}\left((u - \lambda v)^*(u - \lambda v)\right)},$$

where $\mathbb{T}$ denotes the set of complex numbers of modulus 1.

Definition 16 (Projective hyperlinear profile and dimension)**.** Let G be a finitely generated group with a finite generating set S. The *projective hyperlinear profile* of G is the function $\mathcal{D}_{G,S}^{\overline{hyp}}\colon \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by setting $\mathcal{D}_{G,S}^{\overline{hyp}}(n)$ to be the least k such that there exists a function $\sigma\colon G \to U(k)$ such that

    (1) $d_{\mathrm{HS}}\left(\sigma(g)\sigma(h), \sigma(gh)\right) < 1/n$ for every $g, h \in B_{G,S}(n)$ with $gh \in B_{G,S}(n)$, and
    (2) $d_{\overline{\mathrm{HS}}}\left(\sigma(g), \sigma(h)\right) > \sqrt{2} - 1/n$ for every $g, h \in B_{G,S}(n)$ such that $g \neq h$.

The *projective hyperlinear dimension* of G is defined by

$$\dim_{G,S}^{\overline{hyp}} = \limsup_{n\to+\infty} \frac{1}{n}\log \mathcal{D}_{G,S}^{\overline{hyp}}(n).$$

Two distinct (pseudo)distances on $U(k)$ are used in the two conditions above. One might consider the existence of a map $\sigma\colon G \to U(k)$ from Definition 16 as an alternative definition of a hyperlinear group. Indeed, a result of Rădulescu [42] is that this is actually equivalent to Definition 14. Our next result shows a precise relationship between these two approaches on the quantifying level.

For the proof of the following proposition, observe that $d_{\mathrm{HS}}\left(u, e_{U(n)}\right)^2 = 2 - \frac{2}{n}\mathrm{Re}\left(\mathrm{tr}(u)\right)$, and $d_{\overline{\mathrm{HS}}}\left(u, e_{U(n)}\right)^2 = 2 - \frac{2}{n}\left|\mathrm{tr}(u)\right|$ for $u \in U(n)$. Observe furthermore that $d_{\overline{\mathrm{HS}}}(u, v) \leqslant d_{\mathrm{HS}}(u, v)$ for $u, v \in U(n)$. Our next result is based on the "amplification trick"; see [42], [16, Proposition 2.2.9], [34, Section 4.2].

Proposition 17. *Let G be a group with a finite generating set S. Then*

$$\mathcal{D}_{G,S}^{hyp}(n) \leqslant \mathcal{D}_{G,S}^{\overline{hyp}}(n)$$

and, for $n \geqslant \left(\frac{4\sqrt{2}}{5} - 1\right)^{-1}$,

$$\mathcal{D}_{G,S}^{\overline{hyp}}(n) \leqslant \left(2\mathcal{D}_{G,S}^{hyp}(80n)\right)^\ell,$$

where $\ell = \left\lceil \log\left(\frac{1}{\frac{\sqrt{2}}{20n} - \frac{1}{200n^2}} \right) \log\left(\frac{5}{4}\right)^{-1} \right\rceil$.

Proof. The first inequality is obvious. Let us check the second inequality. For $u \in U(k)$, we set $\tau(u) = \frac{1}{k}\mathrm{tr}(u)$. Suppose that $n \geqslant \left(\frac{4\sqrt{2}}{5} - 1\right)^{-1}$. Suppose that $\sigma\colon G \to U(k)$ is a $(80n, \sqrt{2})$-approximation for (G, S). Then for $g \in B_{G,S}(n)$ such that $g \neq e_G$, using the triangle inequality, the uniform injectivity condition, and Lemma 4 (1), we have that $d_{\mathrm{HS}}(\sigma(g), e_{U(k)}) \geqslant d_{\mathrm{HS}}(\sigma(g), \sigma(e_G)) - d_{\mathrm{HS}}(\sigma(e_G), e_{U(k)}) > \sqrt{2} - \frac{1}{80n} - \frac{1}{80n} = \sqrt{2} - \frac{1}{40n}$. Therefore,

$$2 - 2\mathrm{Re}(\tau(\sigma(g))) = d_{\mathrm{HS}}(\sigma(g), e_{U(k)})^2 > 2 - \frac{1}{10n}.$$

Hence $\mathrm{Re}(\tau(\sigma(g))) < \frac{1}{20n}$. Consider the map $\widetilde{\sigma}\colon G \to U(2k)$ defined by

$$\widetilde{\sigma}(g) = \begin{bmatrix} \sigma(g) & 0 \\ 0 & 1 \end{bmatrix}.$$

Then one has that, since $n \geqslant \left(\frac{4\sqrt{2}}{5} - 1\right)^{-1}$,

$$|\tau(\widetilde{\sigma}(g))| \leqslant \frac{|1 + \tau(\sigma(g))|}{2} \leqslant \frac{\sqrt{2} + \frac{1}{20n}}{2} \leqslant \frac{1 + \frac{1}{20n}}{\sqrt{2}} \leqslant \frac{4}{5}.$$

Set $\delta := \frac{\sqrt{2}}{20n} - \frac{1}{200n^2}$. Fix $\ell \in \mathbb{N}$ such that

$$\ell \log\left(\frac{5}{4}\right) \geqslant \log\left(\frac{1}{\delta}\right).$$

Consider the map $\widetilde{\sigma}^{\otimes\ell}\colon G \to U\left((2k)^\ell\right)$ defined by $\widetilde{\sigma}^{\otimes\ell}(g) = \widetilde{\sigma}(g) \otimes \widetilde{\sigma}(g) \otimes \cdots \otimes \widetilde{\sigma}(g)$. Then we have that, for $g \in B_{G,S}(n)$ such that $g \neq e_G$, since the trace is multiplicative under tensor products,

$$\left|\tau(\widetilde{\sigma}^{\otimes\ell}(g))\right| = |\tau(\widetilde{\sigma}(g))|^\ell \leqslant \left(\frac{4}{5}\right)^\ell \leqslant \delta.$$

Therefore, we have that

$$d_{\overline{\mathrm{HS}}}(\widetilde{\sigma}^{\otimes\ell}(g), e_{U((2k)^\ell)})^2 = 2 - 2\left|\tau\left(\widetilde{\sigma}^{\otimes\ell}(g)\right)\right| \geqslant 2 - 2\delta.$$

Hence,

$$d_{\overline{\mathrm{HS}}}(\widetilde{\sigma}^{\otimes\ell}(g), e_{U((2k)^\ell)}) \geqslant \sqrt{2 - 2\delta} > \sqrt{2} - 1/10n.$$

Observe now, using the bi-invariance of d_{HS}, the mixed-product property of the tensor product, and the definition of $\widetilde{\sigma}$, that for $g, h \in B_{G,S}(n)$ with $gh \in B_{G,S}(n)$ one has that

$$d_{\mathrm{HS}}(\widetilde{\sigma}^{\otimes\ell}(g)\,\widetilde{\sigma}^{\otimes\ell}(h), \widetilde{\sigma}^{\otimes\ell}(gh)) \leqslant d_{\mathrm{HS}}(\sigma(g)\,\sigma(h), \sigma(gh)) \leqslant 1/80n < 1/n.$$

Finally, if $g, h \in B_{G,S}(n)$ are such that $g \neq h$, then, using the bi-invariance of $d_{\overline{\mathrm{HS}}}$, the triangle inequality, and the almost homomorphism condition on d_{HS}, and hence on $d_{\overline{\mathrm{HS}}}$, (cf. Lemma 4 (5)) we have that

$$\begin{aligned} d_{\overline{\mathrm{HS}}}(\widetilde{\sigma}^{\otimes\ell}(h), \widetilde{\sigma}^{\otimes\ell}(g)) &\geqslant d_{\overline{\mathrm{HS}}}(\widetilde{\sigma}^{\otimes\ell}(g^{-1}h), e_{U((2k)^\ell)}) - \frac{5}{80n} \\ &\geqslant \sqrt{2} - 1/10n - 1/16n \geqslant \sqrt{2} - 1/n. \end{aligned}$$

This concludes the proof. $\square$

3.3. **Linear sofic groups.** This class of groups has been introduced and systematically studied by Arzhantseva-Păunescu [3]. The next definition is due to [3, Proposition 4.4].

Let $\mathcal{F}^{lin}$ be the collection of groups $GL(n, \mathbb{K})$ of $n \times n$ invertible matrices with coefficients in a given field $\mathbb{K}$, endowed with the normalised *rank distance*: for invertible matrices $u, v \in GL(n, \mathbb{K})$, we define

$$d_{\mathrm{rank}}(u, v) = \frac{1}{n} \mathrm{rank}(u - v).$$

Definition 18 (Linear sofic group; linear sofic profile and dimension)**.** A group G is said to be *linear sofic over a field* $\mathbb{K}$ if it is approximable by $\mathcal{F}^{lin} = (GL(n, \mathbb{K}), d_{\mathrm{rank}}, n, 1/4)_{n \in \mathbb{N}}$, in the sense of Definition 1.

We call *linear sofic profile* $\mathcal{D}_G^{lin}$ and *linear sofic dimension* $\dim_G^{lin}$ over a field $\mathbb{K}$ of a linear sofic group G over a field $\mathbb{K}$ the $\mathcal{F}^{lin}$-profile and $\mathcal{F}^{lin}$-dimension (respectively) for such a choice of the approximating family $\mathcal{F} = \mathcal{F}^{lin}$.

Thus, the dimension k_n of $GL(n, \mathbb{K})$ is declared to be n and ε_n is constantly equal to $1/4$. Definition 18 is an equivalent characterisation of linear soficity proved in [3, Proposition 5.13] (see also [3, Pages 2289–2290] for the case of an arbitrary field $\mathbb{K}$). The value of $1/4$ comes from the so-called *rank amplification*, a construction introduced in [3]. It is not known whether $1/4$ can be replaced by a larger value (fixed or arbitrarily chosen between $1/4$ and 1), see Question 55.

It follows from Proposition 6 that a finitely generated group has bounded linear sofic profile over a field $\mathbb{K}$ if and only if it is linear over $\mathbb{K}$, namely, if and only if it embeds into $GL(n, \mathbb{K})$ for some $n \in \mathbb{N}$.

As above, we represent permutations by permutation matrices: $\mathrm{Sym}(n) \ni \sigma \mapsto u_\sigma \in GL(n, \mathbb{K})$, for a fixed arbitrary field $\mathbb{K}$. Observe that [3, Proposition 4.5]:

$$d_{\mathrm{rank}}(u_\sigma, e_{GL(n, \mathbb{K})}) \leqslant d_{\mathrm{Hamm}}(\sigma, e_{\mathrm{Sym}(n)}) \leqslant 2 d_{\mathrm{rank}}(u_\sigma, e_{GL(n, \mathbb{K})}).$$

Therefore, sofic groups are linear sofic over any given field $\mathbb{K}$ and, for a sofic group G, we have:

$$\mathcal{D}_{G,S}^{lin}(n) \leqslant \mathcal{D}_{G,S}^{sof}(n).$$

Here is a useful modification of the linear sofic profile, which is inspired by the projective variant of the hyperlinear profile. The groups $GL(n, \mathbb{K}), n \in \mathbb{N}$, can be endowed with the normalised *Jordan* or *projective rank pseudodistance*: for matrices $u, v \in GL(n, \mathbb{K})$, following [34], we define

$$d_{\overline{\mathrm{rank}}}(u, v) = \frac{1}{n} \min_{\lambda \in \mathbb{F}^\times} \mathrm{rank}(u - \lambda v),$$

where $\mathbb{F}$ is the algebraic closure of $\mathbb{K}$ and the rank is computed in $\mathbb{F}^n$.

Definition 19 (Projective linear sofic profile and dimension)**.** Let G be a finitely generated group with a finite generating set S. The *projective linear sofic profile* of G is the function $\mathcal{D}_{G,S}^{\overline{lin}} \colon \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by setting $\mathcal{D}_{G,S}^{\overline{lin}}(n)$ to be the least k such that there exists a function $\sigma \colon G \to GL(k, \mathbb{K})$ such that

(1) $d_{\mathrm{rank}}(\sigma(g)\sigma(h), \sigma(gh)) < 1/n$ for every $g, h \in B_{G,S}(n)$ with $gh \in B_{G,S}(n)$, and
(2) $d_{\overline{\mathrm{rank}}}(\sigma(g), \sigma(h)) > \frac{1}{8} - 1/n$ for every $g, h \in B_{G,S}(n)$ such that $g \neq h$.

The *projective linear sofic dimension* of G is defined by

$$\dim_{G,S}^{\overline{lin}} = \limsup_{n \to +\infty} \frac{1}{n} \log \mathcal{D}_{G,S}^{\overline{lin}}(n).$$

As in the projective hyperlinear case, two distinct (pseudo)distances on $GL(k, \mathbb{K})$ are used in the two conditions above and one might consider the existence of a map $\sigma : G \to GL(k, \mathbb{K})$ from Definition 19 as an alternative definition of a linear sofic group. Indeed, a result of Arzhantseva-Păunescu [3, Theorem 5.10] shows that this is actually equivalent to Definition 18. Moreover, our next result is that these two approaches are equivalent on the quantifying level as well.

Proposition 20. *Let G be a group with a finite generating set S. Then $\mathcal{D}_{G,S}^{lin}(n) \leqslant \mathcal{D}_{G,S}^{\overline{lin}}(n)$ and $\mathcal{D}_{G,S}^{\overline{lin}}(n) \leqslant 2\mathcal{D}_{G,S}^{lin}(2n)$.*

Proof. The first inequality is obvious. The second inequality is proved in [34, Proposition 4.8], using [3, Theorem 5.10]. $\square$

We obtain further useful variations of $\mathcal{D}_{G,S}^{lin}$ when restricting to other meaningful classes of matrices, still equipped with the normalised rank distance[3]. For example, we use $\mathcal{D}_{G,S}^u$ when approximate by unitary matrices, $\mathcal{D}_{G,S}^{sa}(n)$ by self-adjoint matrices and $\mathcal{D}_{G,S}^{nor}(n)$ by normal matrices with respect to d_{rank}. See also Question 63.

3.4. Weakly sofic groups.
This class of groups has been introduced by Glebsky-Rivera [28].

Let $\mathcal{F}^{fin}$ be the collection of all finite groups $H_\alpha, \alpha \in I$, each of which is endowed with a normalised bi-invariant distance d_α. For example, such a distance can be induced by the bi-invariant distances on the ambient groups via Cayley's embeddings $H_\alpha \hookrightarrow \mathrm{Sym}(|H_\alpha|) \hookrightarrow U(|H_\alpha|)$, where $|H_\alpha|$ denotes the cardinality of H_α.

Definition 21 (Weakly sofic group; weakly sofic profile and dimension). A group G is said to be *weakly sofic* if it is approximable by $\mathcal{F}^{fin} = (H_\alpha, d_\alpha, |H_\alpha|, 1)_{\alpha \in I}$, in the sense of Definition 1.

We call *weakly sofic profile* $\mathcal{D}_G^{fin}$ and *weakly sofic dimension* $\dim_G^{fin}$ of a weakly sofic group G the $\mathcal{F}^{fin}$-profile and $\mathcal{F}^{fin}$-dimension (respectively) for such a choice of the approximating family $\mathcal{F} = \mathcal{F}^{fin}$.

Here, the dimension k_α is the cardinality $|H_\alpha|$ of the finite group H_α and $\varepsilon_\alpha = 1$ for every $\alpha \in I$. It follows from Proposition 6 that a finitely generated group has bounded weakly sofic profile if and only if it is finite.

Sofic groups are clearly weakly sofic and, for a sofic group G, we have ('!' denotes the factorial):

$$\mathcal{D}_{G,S}^{fin}(n) \leqslant \mathcal{D}_{G,S}^{sof}(n)!$$

Linear sofic groups are weakly sofic [3, Theorem 8.2]. However, for a linear sofic group G, the exact relationship between $\mathcal{D}_{G,S}^{fin}(n)$ and $\mathcal{D}_{G,S}^{lin}(n)$ yet remains to establish. See Questions 56, 57, and 58.

An interesting subclass of weakly sofic groups has been introduced by Thom [46]. Namely, if $\mathcal{F}_{cc}^{fin} = (H_\alpha, d_\alpha^{cc}, |H_\alpha|, 1)_{\alpha \in I}$ is a family of finite groups where each bi-invariant distance

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[3]Naturally, one can also vary the distance by taking, for example, the normalised operator norm, the Frobenius norm, the p-Schatten norm with $1 \leqslant p \leqslant \infty$, etc.

d_α^{cc} is in addition *commutator-contractive*, then the famous Higman group H_4 is not $\mathcal{F}_{cc}^{fin}$-approximable [46], hence, the corresponding metric profile of the Higman group diverges (i.e. $\mathcal{D}_{H_4,S}^{\mathcal{F}_{cc}^{fin}}(n) = +\infty$ for a large enough $n \in \mathbb{N}$).

A great freedom in the choice of bi-invariant distances d_α on finite groups $H_\alpha, \alpha \in I$ suggests that sofic groups might be a proper subset of weakly sofic groups. This is unknown. It is intriguing that the Hamming distance on symmetric groups $\mathrm{Sym}(n), n \in \mathbb{N}$ plays a distinguished role: the soficity can be defined with no reference to any distance [27], or accurately speaking, with a reference to the trivial $\{0,1\}$-valued distance $d_{\{0,1\}}$ only, see Example 9. See also Question 67.

3.5. Weakly hyperlinear groups.
This class of groups has been introduced by Gismatullin [25].

Let $\mathcal{F}^{ct}$ be the collection of all compact groups $H_\alpha, \alpha \in I$, each of which is endowed with a normalised bi-invariant distance d_α. Examples of such distances are the trivial $\{0,1\}$-valued distance $d_{\{0,1\}}$ and the (normalised) *Alexandroff-Urysohn distance*[4] d_{AU} or, in specific compact groups, the *conjugacy distance* d_{conj} induced by the Haar measure on centerless compact groups[5] and the well-known bi-invariant Riemannian distances on compact Lie groups.

Definition 22 (Weakly hyperlinear group; weakly hyperlinear profile and dimension). A group G is called *weakly hyperlinear* if it is approximable by $\mathcal{F}^{ct} = (H_\alpha, d_\alpha, \dim H_\alpha, 1)_{\alpha \in I}$, in the sense of Definition 1.

We call *weakly hyperlinear profile* $\mathcal{D}_G^{ct}$ and *weakly hyperlinear dimension* $\dim_G^{ct}$ of a weakly hyperlinear group G the $\mathcal{F}^{ct}$-profile and $\mathcal{F}^{ct}$-dimension (respectively) for such a choice of the approximating family $\mathcal{F} = \mathcal{F}^{ct}$.

Here, the dimension $k_\alpha = \dim H_\alpha$ can be the Lebesgue covering dimension or cohomological dimension of the compact group H_α, and $\varepsilon_\alpha = 1$ for every $\alpha \in I$. Hyperlinear groups are clearly weakly hyperlinear and, hence, for a hyperlinear group G, we have:

$$\mathcal{D}_{G,S}^{ct}(n) \leqslant \mathcal{D}_{G,S}^{hyp}(n),$$

whenever $\dim U(n)$ of the unitary group $U(n) \in \mathcal{F}^{ct}$ is chosen to be at most n.

3.6. LE-$\mathcal{F}$ groups.
The introduction of these classes of groups goes back to Malcev [36], see also a work of Vershik-Gordon [48].

Let us consider an arbitrary family $\mathcal{F}_{\{0,1\}}$ of discrete groups, where a discrete group is endowed with the trivial $\{0,1\}$-valued metric $d_{\{0,1\}}$ (induced by the length function such that the length of each non-trivial element is 1). In this case, the choice of the parameters ε_α is of course irrelevant. We call an $\mathcal{F}_{\{0,1\}}$-approximable group simply an *LE-$\mathcal{F}$-group*. For example, the famous class of groups that are *locally embeddable into finite groups* (LEF) coincides with the class of $\mathcal{F}_{\{0,1\}}^{fin}$-approximable groups, where $\mathcal{F}_{\{0,1\}}^{fin} = (H_\alpha, d_{\{0,1\}}, |H_\alpha|, 1)_{\alpha \in I}$ is the collection of all finite discrete groups each of which is endowed with the trivial metric $d_{\{0,1\}}$. When $\mathcal{F}_{\{0,1\}}^a$ is

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[4]In a topological group H, if $\{U_n\}_{n=0}^\infty$ is so that for all $h \in H, n \geqslant 0$, one has $U_n = U_n^{-1} = hU_nh^{-1} \subseteq H$ and $U_0 = H, e_H \in U_n, U_{n+1}^3 \subseteq U_n$, then $d_{AU}(g,h) = \inf\{l(s_1) + \ldots + l(s_k) \mid s_i \in H, gh^{-1} = s_1 \cdots s_k\}$, where $l(s_i) = \inf\{2^{-n} \mid s_i \in U_n\}$, is a bi-invariant distance on H. For a compact H, this distance is continuous, (H, d_{AU}) is of finite diameter, and d_{AU} is compatible with the topology if and only if H is first-countable.

[5]$d_{\mathrm{conj}}(g,h) = \frac{\log \mu(C(gh^{-1}))}{\log \mu(H)}$, where $C(gh^{-1})$ denotes the conjugacy class of element gh^{-1} for $g, h \in H$, and μ is the Haar measure on a compact group H with $\mu(H) \neq 1$ and $\mu(C(gh^{-1})) \neq 0$.

the family of all amenable groups endowed with the trivial metric $d_{\{0,1\}}$, one obtains the notion of an *initially subamenable* group or, in other terminology, of a group *locally embeddable into amenable groups* (LEA). Accordingly, we have the concepts of the *LE-$\mathcal{F}$ profile* (also called the LE-$\mathcal{F}$ growth in this case, see Definition 27 below) and dimensions. In particular, we have the LE-$\mathcal{F}^{fin}$ and LE-$\mathcal{F}^a$ profiles whenever the dimensions of groups from $\mathcal{F}^{fin}_{\{0,1\}}$ and $\mathcal{F}^a_{\{0,1\}}$ are chosen. For instance, for a finite group one can again take its cardinality and for an amenable group its asymptotic dimension or its cohomological dimension. Alternatively, generalising our approach further, one can use a non-constant 'dimension' function of approximating groups. For instance, for approximating amenable groups one can use their Følner functions; we formalise this in more detail below by introducing the concept of the LEA profile, see Section 4.3.

4. Relations to other famous quantifying functions

Many examples of quantifying functions associated with a given finitely generated group have been investigated in the literature. Since any group is trivially approximated by itself equipped with the trivial metric $d_{\{0,1\}}$, we can consider such quantifying functions as very specific instances of our much more general approach.

Residually finite groups and amenable groups (and, hence, residually amenable groups) are basic examples of groups which are approximable by families we have mentioned in the preceding subsection. Therefore, functions quantifying the residual finiteness and amenability can be used to produce interesting upper bounds to our metric $\mathcal{F}$-profiles.

Lower bounds are more difficult to provide as our general setting of $\mathcal{F}$-approximable groups encompasses many different classes of groups with a priori very distinct quantifying features.

4.1. **Growth of balls and the metric profile.** A famous quantifying function associated to every finitely generated group is the *growth* function

$$\beta_{G,S}(n) = |B_{G,S}(n)|,$$

the cardinality of the ball at the identity of G with respect to the word length distance induced by the generating set S. This function gives a lower bound for an arbitrary $\mathcal{F}$-profile. Indeed, the uniform injectivity condition, condition (2) of Definition 1, ensures that the ball $B_{G,S}(n)$ is injected into the corresponding group G_α. It remains to estimate from below the minimal dimension k_α such that G_α can have this fixed ball injected. Usually, such an estimate of k_α is immediate (although, in general, it depends on the dimension one has chosen for each G_α). For example, for a sofic group G, we have:

$$\beta_{G,S}(n) \leqslant \mathcal{D}^{fin}_{G,S}(n) \leqslant \mathcal{D}^{sof}_{G,S}(n)!$$

4.2. **Følner function.** This renowned function has been introduced by Vershik [47]. Let G be a group with finite symmetric generating set S. The *Følner function* of G with respect to S, denoted by $\mathrm{Føl}_{G,S}(n)$, is defined to be the smallest size $|A|$ of a finite subset $A \subseteq G$ with the property that

$$\sum_{g \in B_{G,S}(n)} |gA \triangle A| \leqslant |A|/n,$$

with $\mathrm{Føl}_{G,S}(n) = \infty$ whenever there is no such a subset $A \subseteq G$ with respect to S; see [37]. Such a subset A is called $\frac{1}{n}$-*Følner set* corresponding to the ball $B_{G,S}(n)$. It is clear that the

$\simeq$-equivalence class Føl_G of this Følner function does not depend on the chosen finite generating set S.

Remark 23. There is a great flexibility in the choice of the definition of a Følner function as it is in the choice of the definition of a Følner set. For instance, instead of the symmetric difference above one can take $|gA \setminus A|$ or instead of left translation by $g \in B_{G,S}(n)$ in $gA \triangle A$ one can use right translation $Ag \triangle A$, etc. We leave to the reader to check that all these natural variations lead to $\simeq$-equivalent Følner functions and they do not depend on the choice of the finite generating set of G.

It follows from the proof that amenable groups are sofic—see for instance [39, Example 4.4]—that, for an amenable group G, we have:

$$\mathcal{D}_{G,S}^{sof}(n) \leqslant \text{Føl}_{G,S}(2n).$$

This gives

$$\mathcal{D}_{G,S}^{lin}(n) \leqslant \text{Føl}_{G,S}(2n) \text{ and } \mathcal{D}_{G,S}^{hyp}(n) \leqslant \text{Føl}_{G,S}(4n^2).$$

Basic examples of amenable groups include groups of subexponential growth and elementary amenable groups. Among the latter are virtually nilpotent groups. It is not hard to estimate the Følner functions of such groups[6] and, hence, to obtain the upper bounds for their metric profiles using the preceding inequalities. On the lower bound see Question 59.

Example 24 (Groups of subexponential growth)**.** Suppose that G has subexponential growth and S is a finite generating set of G. Define a_n to be the size of the ball $B_{G,S}(n)$ for $n \in \mathbb{N}$. Since G has subexponential growth we have that $\lim_{n \to \infty} \sqrt[n]{a_n} = 1$ and hence $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$ [18, Lemma 6.11.1]. Given $n \in \mathbb{N}$, let $m \in \mathbb{N}$ be so that $a_{k+1}/a_k \leqslant \left(1 + \frac{1}{2na_n}\right)^{\frac{1}{n}}$ for every $k \geqslant m$, then we have that, for every $g \in B_{G,S}(n)$,

$$\frac{|gB_{G,S}(m) \setminus B_{G,S}(m)|}{|B_{G,S}(m)|} \leqslant \frac{a_{n+m} - a_m}{a_m} \leqslant \frac{1}{2na_n}$$

and hence

$$\frac{1}{|B_{G,S}(m)|} \sum_{g \in B_{G,S}(n)} |gB_{G,S}(m) \triangle B_{G,S}(m)| \leqslant \frac{1}{n}.$$

This shows that, for a group G of subexponential growth, we have:

$$\mathcal{D}_G^{sof}(n) \preccurlyeq \text{Føl}_G(n) \leqslant \min \left\{ m : a_{k+1}/a_k \leqslant \left(1 + \frac{1}{2na_n}\right)^{\frac{1}{n}} \text{ for } k \geqslant m \right\}.$$

Example 25 (Virtually nilpotent groups)**.** Suppose that G is a virtually nilpotent group with finite generating set S. By Wolf's result [51] (cf. Gromov's polynomial growth theorem [29]), G has polynomial growth. Let d be the order of polynomial growth of G. Then, using the notation of Example 24, for some constant $c = c_S > 0$ one has, for all but finitely many $n \in \mathbb{N}$, $n^d/c \leqslant a_n \leqslant cn^d$. Therefore, we have that $a_{m+1}/a_m \leqslant c^2(1 + 1/m)^d$ and $\left(1 + (2na_n)^{-1}\right)^{\frac{1}{n}} \geqslant \left(1 + n^{-(d+1)}/2c\right)^{\frac{1}{n}}$. Hence, we get from Example 24 that

$$\text{Føl}_{G,S}(n) \leqslant \left\lceil \left(\left(c^{-\frac{2}{d}}\left(1 + \frac{1}{2cn^{d+1}}\right)^{\frac{1}{dn}} - 1\right)^{-1}\right)\right\rceil.$$

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[6]We give direct estimates. A more careful study can be done for specific groups and classes of groups.

This gives, for example, in case $c = 1$ that

$$\mathrm{F\o l}_{G,S}(n) \leqslant 2dn^{d+2} + dn.$$

Indeed, observe that, for any number $x > 0$,

$$\frac{1}{n} \log(1+x) > \frac{x}{n(1+x)}$$

and hence,

$$(1+x)^{\frac{1}{n}} > 1 + \frac{x}{n(1+x)}.$$

Therefore,

$$\left(1 + \frac{1}{2n^{d+1}}\right)^{\frac{1}{dn}} - 1 > \frac{1}{dn(2n^{d+1}+1)}$$

and

$$\left(\left(1 + \frac{1}{2n^{d+1}}\right)^{\frac{1}{dn}} - 1\right)^{-1} < dn(2n^{d+1}+1).$$

It follows that, for such a virtually nilpotent group G, we have:

$$\mathcal{D}_G^{sof}(n) \preccurlyeq \mathrm{F\o l}_G(n) < 2dn^{d+2} + dn.$$

The two examples above use the fact that in these groups subsequences of balls form a collection of Følner sets. Since the growth of balls is prescribed one can erroneously expect to have this prescribed (e.g. subexponential or polynomial) behaviour of the corresponding Følner functions. However, our quantitative statements on Følner functions require more information than just the knowledge of the asymptotic type of the growth function or of the classical isoperimetric profile of an amenable group. By Definition 1, given $n \in \mathbb{N}$, one has to determine an exact dependence between the radius N of the $\frac{1}{n}$-Følner set $B_{G,S}(N)$ corresponding to the ball $B_{G,S}(n)$ and n. This dependence is exponential in our direct estimates above.

4.3. **LE-$\mathcal{F}$ growth and LEA profile.** Suppose that G is a residually finite group with a finite generating set S. Quantifying functions associated with residually finite groups have been investigated, for example, in [9, 12, 13, 11, 14]. The most popular such a function is the *residual finiteness growth* (also called, the *depth function*), denoted by $F_{G,S}(n)$: it is defined to be the least integer $k > 0$ such that for any nontrivial element $g \in B_{G,S}(n)$ there exists a normal subgroup of G of index at most k that does not contain g.

Our approach is closer in the spirit to another, less known, function quantifying residual finiteness, the *full residual finiteness growth* (or the *full depth function*), denoted by $\Phi_{G,S}(n)$: it is defined to be the least k such that there exists a normal subgroup $N \trianglelefteq G$ of index at most k that meets $B_{G,S}(n)$ only at the identity. It is clear that the $\simeq$-equivalence class of $\Phi_{G,S}(n)$ does not depend from the generating set S, and can be denoted by Φ_G.

By Cayley's theorem, a finite quotient G/N embeds into the symmetric group acting on the quotient itself: $G/N \hookrightarrow \mathrm{Sym}(|G/N|)$. Therefore, for a residually finite group G, we have:

$$\mathcal{D}_{G,S}^{sof}(n) \leqslant \Phi_{G,S}(n), \quad \mathcal{D}_{G,S}^{hyp}(n) \leqslant \Phi_{G,S}(n), \quad \mathcal{D}_{G,S}^{lin}(n) \leqslant \Phi_{G,S}(n) \text{ and}$$

$$\beta_{G,S}(n) \leqslant \mathcal{D}_{G,S}^{fin}(n) \leqslant \Phi_{G,S}(n).$$

Example 26 (Linear / Nilpotent / Virtually abelian groups). Finitely generated linear groups have at most exponential function $\Phi_{G,S}(n)$, see [10] (where the full residual finiteness growth is termed the *residual girth* and the *normal systolic growth*). It follows that all the main metric profiles, $\mathcal{D}_{G,S}^{hyp}(n)$, $\mathcal{D}_{G,S}^{sof}(n)$, $\mathcal{D}_{G,S}^{lin}(n)$ and $\mathcal{D}_{G,S}^{fin}(n)$, are at most exponential. If, in addition, such a group is not virtually nilpotent then it has at least exponential growth. We conclude that $\mathcal{D}_{G,S}^{fin}(n)$ is exponential whenever G is a finitely generated linear group that is not virtually nilpotent. See also Question 61.

If G is a finitely generated nilpotent group, then $\beta_{G,S}(n) \simeq \Phi_{G,S}(n)$ if and only if G is virtually abelian [14]. Hence, $\mathcal{D}_{G,S}^{fin}(n) \simeq \beta_{G,S}(n) \simeq \Phi_{G,S}(n)$ if and only if G is virtually abelian. Classical examples of nilpotent groups which are not virtually abelian include discrete Heisenberg groups $H_{2n+1} = H_{2l+1}(\mathbb{Z})$ of dimension $2l + 1$ with $l \geqslant 1$. By the Bass-Guivarc'h formula [32, 5] for growth in terms of the derived series of a finitely generated nilpotent group, $\beta_{H_{2l+1},S}(n) \simeq n^{2l+2}$. The upper and lower central series of H_{2l+1} coincide, then by [14, Theorem 1] we have $\Phi_{H_{2l+1},S}(n) \simeq n^{2(2l+1)}$. That is, $n^{2l+2} \preccurlyeq \mathcal{D}_{H_{2l+1},S}^{fin}(n) \preccurlyeq n^{2(2l+1)}$.

If H_{2l+1} is $\mathcal{F}^{fin}$-stable, then by Corollary 32(2), we have $\mathcal{D}_{H_{2l+1},S}^{fin}(n) \simeq n^{2(2l+1)}$. See Conjecture 64 and more generally Conjecture 65. See also Example 33.

The above way of quantifying and the above estimates extend immediately from the class of residually finite groups to a more general setting of LE-$\mathcal{F}$ groups, discussed in Section 3.6: one can introduce the corresponding quantifying function $\Phi_{G,S}^{\mathcal{F}}(n)$ as follows.

Definition 27 (LE-$\mathcal{F}$ growth). Let G by a finitely generated group with a finite generating set S. The *LE-$\mathcal{F}$ growth* of G is the function $\Phi_{G,S}^{\mathcal{F}}: \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by

$$\Phi_{G,S}^{\mathcal{F}}(n) = \min\{k \in \mathbb{N} \mid \exists \text{ a group } G_\alpha \in \mathcal{F} \text{ of size } k_\alpha = k \text{ with } B_{G,S}(n) \hookrightarrow G_\alpha\},$$

where $B_{G,S}(n) \hookrightarrow G_\alpha$ is a *monomorphism on the ball*, that is, it preserves the algebraic operation of G on the elements from $B_{G,S}(n)$ (i.e., it is a homomorphism on those elements) and it is an injection.

Observe that this LE-$\mathcal{F}$ growth is nothing else that our $\mathcal{F}_{\{0,1\}}$-profile and, for an LE-$\mathcal{F}$ group G, we have:

$$\mathcal{D}_{G,S}^{\mathcal{F}}(n) \leqslant \Phi_{G,S}^{\mathcal{F}}(n) = \mathcal{D}_{G,S}^{\mathcal{F}_{\{0,1\}}}(n),$$

whenever $B_{G,S}(n) \hookrightarrow G_\alpha \in \mathcal{F}_{\{0,1\}}$ satisfies the uniform injectivity condition with respect to d_α. This applies to an arbitrary fixed family $\mathcal{F} = \{(G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)\}_{\alpha \in I}$.

When $\mathcal{F} = \mathcal{F}_{\{0,1\}}^{fin}$, that is, for an LEF group G, the inequalities above specify to

$$\beta_{G,S}(n) \leqslant \mathcal{D}_{G,S}^{fin}(n) \leqslant \Phi_{G,S}^{\mathcal{F}_{\{0,1\}}^{fin}}(n) = \mathcal{D}_{G,S}^{\mathcal{F}_{\{0,1\}}^{fin}}(n).$$

It would be interesting to find an example of an LEF group with the second inequality being strict, cf. Question 66.

When $\mathcal{F} = \mathcal{F}_{\{0,1\}}^{a}$ is a family of amenable groups, every amenable group $G_\alpha \in \mathcal{F}_{\{0,1\}}^{a}$ generated by a finite generating set S_α has the associated Følner function $\text{Føl}_{G_\alpha,S_\alpha}(n)$. Therefore, for an $\mathcal{F}_{\{0,1\}}^{a}$-approximable group we extend our quantifying viewpoint further.

Definition 28. (LEA-$\mathcal{F}^a$ profile) Let G by a finitely generated group with a finite generating set S. The *LEA-$\mathcal{F}^a$ profile* of G is the function $LEA_{G,S}^{\mathcal{F}^a}\colon \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by

$$LEA_{G,S}^{\mathcal{F}^a}(n) = \min \left\{ \mathrm{F\o l}_{G_\alpha, S_\alpha}(n) \mid \exists \text{ a group } G_\alpha \in \mathcal{F}_{\{0,1\}}^a \text{ with } B_{G,S}(n) \hookrightarrow G_\alpha \right\},$$

where S_α is a finite generating set of G_α fixed in advance and the minimum is taken over all such monomorphisms on the ball.

This allows to study Følner like profile functions for non-amenable groups which possess 'exact' approximations by amenable groups. If $\mathcal{F}_{\{0,1\}}^a$ consists of *all* amenable groups, we denote the corresponding LEA profile by $LEA_{G,S}(n)$. Also, if $\mathcal{F}_{\{0,1\}}^a$ consists of amenable quotients of G, this gives the *residually amenable profile*, denoted by $RA_{G,S}(n)$. Without loss of generality, we assume no any relationship between the generating sets S and S_α. Also, the assumption on the finiteness of $S_\alpha, \alpha \in I$, is not necessary. If the approximating groups G_α are not finitely generated, given a group G_α with $B_{G,S}(n) \hookrightarrow G_\alpha$, we can consider a subgroup of G_α generated by finitely many images under this injection of elements from $B_{G,S}(n)$ and take the Følner function of this finitely generated subgroup of G_α. Properties of usual Følner functions immediately extend to our LEA profile and RA profile. In particular, the $\simeq$-equivalence class of $LEA_{G,S}$, respectively of $RA_{G,S}$, does not depend on the choice of the finite generating set S.

For a group G, and a family $\mathcal{F} \supseteq \mathcal{F}^a \supseteq \{$ amenable quotients of $G\}$, we have:

$$\mathcal{D}_{G,S}^{\mathcal{F}}(n) \leqslant LEA_{G,S}(n) \leqslant RA_{G,S}(n) \leqslant \mathrm{F\o l}_{G,S}(n),$$

whenever $B_{G,S}(n) \hookrightarrow G_\alpha \in \mathcal{F}_{\{0,1\}}^a$ satisfies the uniform injectivity condition with respect to d_α. Here, $\mathrm{F\o l}_{G,S}(n) = \infty$ whenever G is non-amenable and the dimensions of $G_\alpha \in \mathcal{F}^a$ approximating $B_{G,S}(n)$ satisfy $k_\alpha \leqslant \mathrm{F\o l}_{G_\alpha, S_\alpha}(n)$. See also Question 60.

Subsequent to the present work and partially on the suggestion of the first author, the functions LE-$\mathcal{F}^{fin}$, $LEA_{G,S}(n)$ and $RA_{G,S}(n)$ have been also studied in [8] and [17].

4.4. **Metric profiles of stable metric approximations.** As above, G denotes a group generated by a finite set $S = X \sqcup X^{-1}$, with, say, $X = \{x_1, \ldots, x_m\}$, subject to a finite set of relators $R \subseteq F_m = F_X$ and $\mathcal{F} = (G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)_{\alpha \in I}$ is an approximating family of G.

If $r \in F_m$ and $g_1, \ldots, g_m$ are elements in a group H, we denote by $r(g_1, \ldots, g_m) \in H$ the image of r under the unique group homomorphism $F_m \to H$ such that $x_i \mapsto g_i$.

Definition 29 (Solution and almost solution). Elements $g_1^\alpha, \ldots, g_m^\alpha \in G_\alpha$ are a *solution* of R in G_α if

$$r(g_1^\alpha, \ldots, g_m^\alpha) = e_\alpha, \forall r \in R,$$

where e_α denotes the identity of G_α.

Elements $g_1^\alpha, \ldots, g_m^\alpha \in G_\alpha$ are a *δ-solution* of R in G_α, for some $\delta > 0$, if

$$d_\alpha \left(r(g_1^\alpha, \ldots, g_m^\alpha), e_\alpha \right) < \delta, \forall r \in R.$$

The following notion is due to Arzhantseva-Păunescu [2], see also [4] for a more general setting.

Definition 30 ($\mathcal{F}$-stable groups). The set R is *$\mathcal{F}$-stable* if $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \alpha \in I \ \forall g_1, \ldots, g_m \in G_\alpha$ a δ-solution of R, there exist $\tilde{g}_1, \ldots, \tilde{g}_m \in G_\alpha$ a solution of R such that $d_\alpha(g_i, \tilde{g}_i) < \varepsilon$.

The group $G = F_m / \langle\!\langle R \rangle\!\rangle$ is called *$\mathcal{F}$-stable* if its set of relator words R is $\mathcal{F}$-stable.

The definition of $\mathcal{F}$-stability does not depend on the particular choice of finite presentation of the group: Tietze transformations preserve stability as the metrics d_α are bi-invariant, see [2, Section 3]. The following theorem is due to Arzhantseva-Păunescu, cf. [2, Definition 4.1, Theorem 4.2 and lines after it, and Theorem 4.3], see again [4] for a more general variant.

Theorem 31. *Assume that $(\varepsilon_\alpha)_{\alpha \in I}$ is constantly equal to a real number $\varepsilon > 0$. Then the following holds.*

(1) *The set R is $\mathcal{F}$-stable if and only if any group homomorphism $\iota \colon G \to \prod_{\mathcal{U}} (G_\alpha, d_\alpha)$ lifts to $\prod_{\alpha \in I} G_\alpha$.*

(2) *Assume that all subgroups of every $G_\alpha \in \mathcal{F}$ belong to $\mathcal{F}$. If G is $\mathcal{F}$-approximable and $\mathcal{F}$-stable, then G is LE-$\mathcal{F}$, equivalently, fully residually-$\mathcal{F}$.*

A homomorphism ι is not necessarily injective and the lifting property means that there exists $g_\alpha^i \in G_\alpha, i = 1, \ldots, m$ such that $\{g_\alpha^1, \ldots, g_\alpha^m\}$ is a solution of R for any $\alpha \in I$ and $\iota(x_i) = \prod_{\mathcal{U}} g_\alpha^i$, see [2, Definition 4.1]. A group G is *fully residually-$\mathcal{F}$* if for every finite subset F of G there exists a normal subgroup $N \trianglelefteq G$ such that F injects into G/N and $G/N \cong G_\alpha$ for some $\alpha \in I$. The equivalence between the LE-$\mathcal{F}$ property and the fully residually $\mathcal{F}$ property is because, in this section, G is assumed to be finitely presented.

Corollary 32. *Under the hypothesis above, we have the following inequalities.*

(1) *If the set R is $\mathcal{F}$-stable, then*

$$\mathcal{D}_{G,S}^{\mathcal{F}_{\{0,1\}}}(n) \leqslant \mathcal{D}_{G,S}^{\mathcal{F}}(n).$$

(2) *If G is $\mathcal{F}$-approximable and $\mathcal{F}$-stable, then*

$$\Phi_{G,S}^{\mathcal{F}}(n) \simeq \mathcal{D}_{G,S}^{\mathcal{F}_{\{0,1\}}}(n) \simeq \mathcal{D}_{G,S}^{\mathcal{F}}(n),$$

whenever $B_{G,S}(n) \hookrightarrow G_\alpha \in \mathcal{F}_{\{0,1\}}$ satisfies the uniform injectivity condition with respect to d_α.

Proof. (1): By convention, the upper bound is $+\infty$, whenever G does not admit any $\mathcal{F}$-approximation. Otherwise, by definition of $\mathcal{F}$-stability, an almost homomorphism on the ball lifts to a homomorphism on the ball. It follows that (as uniform injectivity lifts to injectivity), given an (n, ε)-approximation by $(G_\alpha, d_\alpha) \in \mathcal{F}$ of dimension k_α, there exists an $(n, 1)$-approximation by $(G_\alpha, d_{\{0,1\}}) \in \mathcal{F}_{\{0,1\}}$ of the same dimension k_α.

(2): This is immediate from (1) and our observation in Section 4.3. □

In particular, for such an $\mathcal{F}^{sof}$-stable sofic group G one has

$$\Phi_{G,S}(n) \simeq \mathcal{D}_{G,S}^{sof}(n).$$

Similarly, for an $\mathcal{F}^{hyp}$-stable hyperlinear, respectively $\mathcal{F}^{lin}$-stable linear sofic, and yet, respectively $\mathcal{F}^{fin}$-stable weakly sofic group G one has $\Phi_{G,S}(n) \simeq \mathcal{D}_{G,S}^{hyp}(n)$, respectively $\Phi_{G,S}(n) \simeq \mathcal{D}_{G,S}^{lin}(n)$, and yet, respectively $\Phi_{G,S}(n) \simeq \mathcal{D}_{G,S}^{fin}(n)$. Here we imply that, under the hypothesis of Corollary 32(2) and by definitions of the growths, the corresponding LE-$\mathcal{F}$ growths coincide, up to the $\simeq$-equivalence, e.g. $\Phi_{G,S}^{\mathcal{F}^{sof}}(n) \simeq \Phi_{G,S}(n)$, etc.

The assumption on finite presentation of G can be relaxed, cf. [4, Remark 2.10]. Moreover, the above statements can be generalised to a wider context of *constraint* metric approximations [4].

Corollary 32 allows us to explicit various metric profiles for groups known to be stable.

Example 33 ($\mathcal{F}$-profiles of $\mathcal{F}$-stable groups).

(1) Let G be a finitely generated virtually abelian group and $\operatorname{rank} G$ denote the rank of any finite index free-abelian subgroup of G. Then

$$\mathcal{D}_{G,S}^{sof}(n) \simeq \mathcal{D}_{G,S}^{hyp}(n) \simeq \mathcal{D}_{G,S}^{fin}(n) \simeq n^{\operatorname{rank} G}.$$

Indeed, we use [2] for $\mathcal{F}^{sof}$-stability, [26] for $\mathcal{F}^{hyp}$-stability, Example 26 for $\mathcal{D}_{G,S}^{fin}(n)$, [14] for $\Phi_{G,S}(n) \simeq n^{\operatorname{rank} G}$ (or simply argue as in Example 13) and Corollary 32(2) together with Proposition 34(ii) for conclusion. See Conjecture 62 on $\mathcal{D}_{G,S}^{lin}(n)$.

(2) Let G be a finitely generated virtually nilpotent group so that a finite-index nilpotent subgroup $H \leqslant G$ satisfies $[Z(H) : \gamma_c(H)] < \infty$, where $Z(H)$ and $\gamma_c(H)$ denote the center and the last nontrivial term of the lower central series of H (i.e. c is the nilpotency class of H). Let $\dim H$ denote the number of infinite cyclic factors in a composition series of H with cyclic factors. Then

$$\mathcal{D}_{G,S}^{sof}(n) \simeq n^{\dim H}.$$

Indeed, we apply Proposition 34(ii) to restrict to a nilpotent group H as above, then [6] for $\mathcal{F}^{sof}$-stability, [14] for $\Phi_{H,S}(n) \simeq n^{\dim H}$ and Corollary 32(2) for conclusion. In particular, for the $(2l + 1)$-dimensional Heisenberg group we have $\mathcal{D}_{H_{2l+1},S}^{sof}(n) \simeq n^{2(2l+1)}$.

The assumption $[Z(H) : \gamma_c(H)] < \infty$ is required by [14, Theorem 1]. For an arbitrary virtually nilpotent group, analogous conclusions hold using [14, Theorem 2] which provides a polynomial upper bound on $\Phi_{G,S}(n)$, and hence, by Corollary 32(2), on the sofic profile $\mathcal{D}_{G,S}^{sof}(n)$. This improves the upper bound from Example 25.

Furthermore, all virtually {polycyclic-by-finite} groups which are not virtually nilpotent have exponential $\mathcal{D}_{G,S}^{sof}(n)$. For we use Proposition 34(ii) to restrict to a polycyclic-by-finite group H, then [6] for $\mathcal{F}^{sof}$-stability, [10] for $\Phi_{H,S}(n) \simeq 2^n$ (note that every polycyclic-by-finite group is linear [43, Section 5.C]) and Corollary 32(2) for conclusion.

On the hyperlinear profile, we have $\mathcal{D}_{H_{2l+1},S}^{hyp}(n) \simeq n^{2(2l+1)}$ for the $(2l+1)$-dimensional Heisenberg group, since it is $\mathcal{F}^{hyp}$-stable [33] (the argument extends from 3-dimensional to $(2l + 1)$ dimensional Heisenberg group) and $\Phi_{H_{2l+1},S}(n) \simeq n^{2(2l+1)}$ [14, Theorem 1].

See Conjecture 64 on other metric profiles of the Heisenberg groups and Conjecture 65 on arbitrary finitely generated virtually nilpotent groups.

4.5. **Other metric profiles.** Subsequent to our work other metric profiles have been introduced, with alternative (not equivalent!) to Definition 2 formulations, and restricted to certain metric approximations. Notably, a sofic profile was introduced in [19] and, by analogy, a hyperlinear profile in [44]. In both cases, the formulation is 'transversal' to our line of thought. Indeed, one can parametrise an $\mathcal{F}$-approximation by two parameters m and n (instead of one parameter n in Definition 1): m being the radius of the ball to be approximated and $1/n$ being the constant involved in the definitions of almost homomorphism and uniform injectivity on that ball. This can seem to give a greater flexibility but in fact provides an equivalent definition of an

$\mathcal{F}$-approximation. Even so, this yields distinct approaches to quantifying when one prescribes different constraints on the corresponding quantifying function $\mathcal{D}_{G,S}^{\mathcal{F}}(m,n)$ of two variables. Our Definition 1, and hence Definition 2, take $m = n$ so that we consider the values of the quantifying function on the diagonal of m-n plane (viewing, for example, m on the horizontal and n on the vertical axes). In contrast, both [19] and [44] choose to fix m and consider such a function on the vertical n-line, when n varies. A careful reader is invited to pay attention to such differences and variations in the existing terminology. See also the notion of sofic dimension [21] which mirrors an 'orthogonal' concept of subgroup growth we alluded to in the introduction.

5. Metric profile and group-theoretical constructions

In this section, we observe how the $\mathcal{F}$-profile behaves with respect to various group-theoretical constructions such as taking subgroups, direct and free products, and restricted wreath products.

5.1. **Subgroups.** It follows from Definition 1 that every subgroup of an $\mathcal{F}$-approximable group is $\mathcal{F}$-approximable and from definition of an LE-$\mathcal{F}$ group in Section 3.6 that a subgroup of an LE-$\mathcal{F}$ group is an LE-$\mathcal{F}$ group. Quantifying these statements we obtain the following easy but instructive result.

Proposition 34. *Let G be a finitely generated $\mathcal{F}$-approximable group and H be a finitely generated subgroup of G. Then the following holds.*

(i) $\mathcal{D}_H^{\mathcal{F}} \preccurlyeq \mathcal{D}_G^{\mathcal{F}}$.
(ii) If H has finite index in G, then $\mathcal{D}_G^{sof} \simeq \mathcal{D}_H^{sof}, \mathcal{D}_G^{hyp} \simeq \mathcal{D}_H^{hyp}$, and $\mathcal{D}_G^{lin} \simeq \mathcal{D}_H^{lin}$.
(iii) If G is an LE-$\mathcal{F}$ group, then $\Phi_H^{\mathcal{F}}(n) \preccurlyeq \Phi_G^{\mathcal{F}}(n)$.
(iv) If G is an LEA group, then $LEA_H(n) \preccurlyeq LEA_G(n)$.

Proof. The first assertion is clear and it holds for an arbitrary $\mathcal{F}$-profile. The statements about LE-$\mathcal{F}$ and LEA profiles are by definitions.[7]

We now prove the second assertion. By (i) above, it suffices to show that $\mathcal{D}_G^{sof} \preccurlyeq \mathcal{D}_H^{sof}$. We proceed as in the definition of an induced linear representation and define an induced $\mathcal{F}$-approximation of G starting from a given $\mathcal{F}$-approximation of H. Let ℓ be the index of H in G, and $T = \{g_1, \ldots, g_\ell\}$ be a choice of representatives for left cosets of H in G. Given g, we denote by $\overline{g} \in T$ the representative of gH. We assume for simplicity that the representative of H is the identity element $e_G \in T$. If S is a generating set of G, then, by Schreier's lemma, $S_H = \{(\overline{st})^{-1}st \mid s \in S, t \in T\} \subseteq H$ generates H. Moreover, $B_{G,S}(n) \subseteq T \cdot B_{H,S_H}(n)$. We fix these finite generating sets for convenience: the $\simeq$-equivalence class of the corresponding $\mathcal{F}$-profile does not depend on the choice of a finite generating set.

Observe that for every $g \in G$ and $i \leqslant \ell$ there exist unique $\alpha_g(i) \leqslant \ell$ and $h_{g,i} \in H$ such that $gg_i = g_{\alpha_g(i)}h_{g,i}$. From the uniqueness of such a representation one can deduce that $\alpha_g \circ \alpha_k = \alpha_{gk}$ and $h_{gk,i} = h_{g,\alpha_k(i)}h_{k,i}$. Furthermore, the map $i \mapsto \alpha_g(i)$ is a permutation of $\{1, \ldots, \ell\}$. Suppose now that $\varphi \colon H \to \mathrm{Sym}(m)$ is an $(n, 1)$-approximation of H. We define a map $\psi \colon G \to \mathrm{Sym}(\ell m)$ by identifying $\mathrm{Sym}(\ell m)$ with the group of permutations of $\{1, \ldots, \ell\} \times \{1, \ldots, m\}$ and defining $\psi(g)$ to be the map $(i, j) \mapsto (\alpha_g(i), \varphi(h_{g,i})(j))$. It is straightforward to check that for every

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[7]The $\preccurlyeq$ inequalities in (i), (iii), and (iv) can be replaced by usual $\leqslant$ inequalities, whenever we consider the metric profiles with respect to fixed finite generating sets $S \subseteq G$ and $S_H \subseteq H$ such that $S_H \subseteq S$.

$g \in G$ the map $\psi(g)$ is a permutation and that ψ is an $(n, 1)$-approximation of G of dimension ℓm. For the latter, we use a general fact that given two permutations $\sigma \in \mathrm{Sym}(m)$ and $\tau \in \mathrm{Sym}(q)$, the normalised Hamming distance of the direct sum $\sigma \oplus \tau \in \mathrm{Sym}(m + q)$ satisfies

$$d_{\mathrm{Ham}}(\sigma \oplus \tau, e_{\mathrm{Sym}(m+q)}) = \frac{md_{\mathrm{Ham}}(\sigma, e_{\mathrm{Sym}(m)}) + qd_{\mathrm{Ham}}(\tau, e_{\mathrm{Sym}(q)})}{m + q}.$$

This shows the required $\mathcal{D}_G^{sof} \preccurlyeq \mathcal{D}_H^{sof}$. The proofs for the hyperlinear and linear sofic profiles are analogous, using corresponding properties of the normalised Hilbert-Schmidt and rank distances, cf. [3, Proposition 2.2]. Namely, we use a general fact that given two unitary matrices $u \in U(m)$ and $v \in U(q)$, the normalised Hilbert-Schmidt distance of the block-diagonal matrix $u \oplus v \in U(m + q)$ satisfies

$$d_{\mathrm{HS}}^2(u \oplus v, e_{U(m+q)}) = \frac{md_{\mathrm{HS}}^2(u, e_{U(m)}) + qd_{\mathrm{HS}}^2(v, e_{U(q)})}{m + q}.$$

Similarly, given $u \in GL(m, \mathbb{K})$ and $v \in GL(q, \mathbb{K})$, the normalised rank distance of the block-diagonal matrix $u \oplus v \in GL(m + q, \mathbb{K})$ satisfies

$$d_{\mathrm{rank}}(u \oplus v, e_{GL(m+q,\mathbb{K})}) = \frac{md_{\mathrm{rank}}(u, e_{GL(m,\mathbb{K})}) + qd_{\mathrm{rank}}(v, e_{GL(q,\mathbb{K})})}{m + q}.$$

$\square$

Remark 35. Proposition 34(ii) extends to other metric $\mathcal{F}$-profiles. The proof proceeds as above constructing $\mathcal{F}$-approximations of G induced (as induced representations) by a given $\mathcal{F}$-approximation of H. For this to work, assumptions on the family $\mathcal{F}$ and the distances d_α are required. For example, one assumes that $G_\alpha \oplus G_\beta$ is defined and is isomorphic to some G_γ and the distances $d_\alpha, d_\beta, d_\gamma$ satisfy that $f(d_\gamma)$ is a convex combination of $f(d_\alpha)$ and $f(d_\beta)$ for some function f. We call such functions f *diagonally block-convex*, which include so-called *diagonally block-additive* functions where the convexity is given by the sum: $f(d_\gamma) = f(d_\alpha) + f(d_\beta)$. In our proof above, $f(x) = x$ for the normalised Hamming and rank distances and $f(x) = x^2$ for the Hilbert-Schmidt distance. Obviously, we can take $f(x) = x^p, 1 \leqslant p \leqslant \infty$ which yields the convex equality as above for the p-Schatten norm. Therefore, we have the result as in Proposition 34(ii) for the metric approximations by matrices endowed with the normalised p-Schatten norm, for $1 \leqslant p \leqslant \infty$, and hence, for the normalised trace and operator norms (the rank of a matrix can be viewed as its p-Schatten norm for $p = 0$). Similarly, $\mathcal{D}_G^{fin} \simeq \mathcal{D}_H^{fin}$, whenever the distances d_α on the finite approximating groups admit some diagonally block-convex function as above.

5.2. **Direct and free products.** It follows from Proposition 34 that for every $n \in \mathbb{N}$ we have

$$\max\{\mathcal{D}_G^{\mathcal{F}}(n), \mathcal{D}_H^{\mathcal{F}}(n)\} \leqslant \mathcal{D}_{G \times H}^{\mathcal{F}}(n), \ \ \max\{\Phi_G^{\mathcal{F}}(n), \Phi_H^{\mathcal{F}}(n)\} \leqslant \Phi_{G \times H}^{\mathcal{F}}(n) \ \text{ and}$$

$$\max\{LEA_G(n), LEA_H(n)\} \leqslant LEA_{G \times H}(n).$$

On the other hand, taking a family $\mathcal{F} \in \left\{\mathcal{F}^{sof}, \mathcal{F}^{hyp}, \mathcal{F}^{fin}, \mathcal{F}_{cc}^{fin}, \mathcal{F}^{lin}, \mathcal{F}_{\{0,1\}}^{fin}, \mathcal{F}_{\{0,1\}}^a\right\}$, the known proofs that if G and H are $\mathcal{F}$-approximable, then $G \times H$ is again $\mathcal{F}$-approximable (see, for example, [24, Theorem 1] for the proof for sofic groups) give

$$\mathcal{D}_{G \times H}^{\mathcal{F}}(n) \preccurlyeq \mathcal{D}_G^{\mathcal{F}}(n)\mathcal{D}_H^{\mathcal{F}}(n).$$

Analogously, for LE-$\mathcal{F}$ groups G and H, we have, under the assumption that $\mathcal{F}$ is closed under taking direct product of two groups,

$$\Phi_{G \times H}^{\mathcal{F}}(n) \preccurlyeq \Phi_G^{\mathcal{F}}(n) \Phi_H^{\mathcal{F}}(n).$$

It is also not hard to check that, for LEA groups G and H, for the Følner type profile, we have:

$$LEA_{G \times H}(n) \preccurlyeq LEA_G(n) LEA_H(n).$$

The only point in the above estimates is to choose a suitable finite generating set of the direct product $G \times H$ starting from finite generating sets S_G and S_H. A natural choice is satisfactory: one can take $S_{G \times H} = S_G \times \{e_H\} \sqcup \{e_G\} \times S_H$.

The case of free products is less understood. Some of the classes of $\mathcal{F}$-approximable groups are not known to be closed under taking free products, e.g. it is not yet proved for weak soficity and for weak hyperlinearity. As usual, we focus on the quantifying aspects.

Let G and H be sofic groups, and F_r denote a free group of rank $r \geqslant 2$. It follows from Proposition 34(iii) that $\Phi_{F_r}(n) \leqslant \Phi_{F_2}(n)$. This and Elek-Szabó's proof of soficity of the free product $G * H$ [24, Theorem 1] give the estimate

$$\mathcal{D}_{G*H}^{sof}(n) \preccurlyeq \mathcal{D}_G^{sof}(n) \mathcal{D}_H^{sof}(n) \Phi_{F_2}(n).$$

Analogously, a free product of linear sofic groups is linear sofic [45, Theorem 5.6] and one might extract an upper bound on $\mathcal{D}_{G*H}^{lin}(n)$ whenever G and H are linear sofic, see Conjecture 68.

A free product of hyperlinear groups is hyperlinear [40, 49] or [15] but a meaningful upper bound on $\mathcal{D}_{G*H}^{hyp}(n)$, whenever G and H are hyperlinear, seems unknown, see Question 69.

For residually amenable G and H, an upper bound on $RA_{G*H}(n)$ is obtained in [8, Theorem 6.2.7 and lines before Example 6.2.9]. See Question 70 and Question 71.

5.3. **Extensions by amenable groups.** Let G be a group, A be a finite set of size n, and G^A be the group of all functions from A to G. Let $\text{Sym}(A)$ be the group of permutations on A, which is clearly isomorphic to the group $\text{Sym}(n)$ of permutations on $\{1, 2, \ldots, n\}$.

The *permutation wreath product* is the semidirect product $\text{Sym}(A) \ltimes G^A$ with respect to the action of $\text{Sym}(A)$ on G^A that permutes the coordinates. An element of $\text{Sym}(A) \ltimes G^A$ is represented by a pair (σ, b) where $\sigma \in \text{Sym}(A)$ and $b \in G^A$ is a function $b \colon A \to G$. When G is a bi-invariant metric group, one can endow $\text{Sym}(A) \ltimes G^A$ with a canonical bi-invariant metric, which was defined in [35, Section 5] as follows. If $\sigma_0, \sigma_1 \in \text{Sym}(A)$ and $b_0, b_1 \in G^A$, then

$$d_{\text{Sym}(A) \ltimes G^A}((\sigma_0, b_0), (\sigma_1, b_1)) = \frac{1}{n} \sum \{d_G(b_0(a), b_1(a)) : a \in A, \sigma_0(a) = \sigma_1(a)\} +$$
$$d_{\text{Sym}(A)}(\sigma_0, \sigma_1),$$

where $d_{\text{Sym}(A)}$ denotes the normalised Hamming distance on $\text{Sym}(A)$.

We now introduce a useful notion of controlled Følner function, which is implied by the controlled Følner sequence introduced in [20, Section 3.2].

Definition 36 (Controlled Følner function)**.** Let G be a finitely generated amenable group with finite symmetric generating set S. The *controlled Følner function* of G with respect to S, denoted by $\text{Føl}_{G,S}^{con}(n)$, is defined to be the smallest $k \geqslant n$ such that there exists a subset A of

$B_{G,S}(k)$ of size at most k with the property that

$$\sum_{g \in B_{G,S}(n)} |gA \triangle A| \leqslant |A|/n.$$

As usual, choosing a different finite generating set would yield a function with the same asymptotic type of growth. We let Føl_G^{con} be the corresponding $\simeq$-equivalence class of functions.

Let now H be a subgroup of G generated by a finite set R. We now recall the notion of *distortion* for the subgroup H in G as defined in [31, Chapter 3].

Definition 37 (Distortion function). The *distortion function* of H in G with respect to R and S, denoted by $\Delta_{H \leqslant G, R, S}(n)$, is defined to be the smallest $k \geqslant n$ such that $H \cap B_{G,S}(n)$ is contained in $B_{H,R}(k)$.

Since different choices of generating sets yield functions with the same asymptotic type of growth, the $\simeq$-equivalence class of $\Delta_{H \leqslant G}$ of the function defined above is well-defined.

Let $\mathcal{F}$ be an approximating family $\mathcal{F} = \{(G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)\}_{\alpha \in I}$ such that for every $\alpha \in I$, ε_α is equal to a fixed strictly positive constant, which up to normalisation can be assumed to be equal to 1. We furthermore assume that for every $\alpha \in I$ and a finite set A there exists $\beta \in I$ such that $k_\beta \leqslant |A| k_\alpha$, and the permutation wreath product $\text{Sym}(A) \ltimes G_\alpha^A$ is isometrically isomorphic (when endowed with the canonical bi-invariant metric $d_{\text{Sym}(A) \ltimes G_\alpha^A}$ described above) to G_β. For example, this applies when $I = \mathbb{N}$ and, for every $n \in \mathbb{N}$, G_n is the permutation group $\text{Sym}(n)$ endowed with the normalised Hamming distance. The following result can be seen as a quantified version of [35, Theorem 5.1].

Theorem 38. *Let G be a finitely generated group and $\mathcal{F}$ be a family as above, N be a normal subgroup of G such that the quotient G/N is amenable. Then we have:*

$$\mathcal{D}_G^{\mathcal{F}}(n) \preccurlyeq \text{Føl}_{G/N}^{con}(n) \, \mathcal{D}_N^{\mathcal{F}}\left(\Delta_{N \leqslant G}(\text{Føl}_{G/N}^{con}(n))\right).$$

Proof. For $g \in G$ we denote by $\overline{g}$ the image of g under the quotient map $G \twoheadrightarrow G/N$. Similarly, if A is a subset of G, then we let $\overline{A}$ be the collection $\{\overline{g} : g \in A\}$. Fix a finite symmetric subset T of G such that $\overline{T}$ is a generating set of G/N. Fix also a right inverse $\sigma \colon G/N \to G$ for the quotient map $G \twoheadrightarrow G/N$ such that $\sigma(B_{G/N, \overline{T}}(n)) \subseteq B_{G,T}(n)$ for every $n \in \mathbb{N}$. In particular, this implies that $\sigma(\overline{e}_G) = e_G$ and $\overline{\sigma(\overline{t})} = \overline{t}$ for $t \in T$. Let now R be a finite symmetric generating subset of N with the property that, for every $t \in T$, $t^{-1}\sigma(\overline{t}) \in R$. Finally, let S be the finite symmetric generating set $R \cup T$ for G.

Let n be a natural number. Observe that $\overline{B_{G,S}(n)} \subseteq B_{G/N, \overline{T}}(n)$ for every $n \in \mathbb{N}$. Set $k := \text{Føl}_{G/N, \overline{T}}^{con}(10n) \geqslant 10n$. By definition of the controlled Følner function of G/N, one has that there exists a finite subset B of G such that $B \subseteq B_{G/N, \overline{T}}(k)$, $|B| \leqslant k$, and $\frac{1}{|B|}\sum_{g \in B_{G,S}(10n)} |\overline{g}B \triangle B| \leqslant (10n)^{-1}$. By the choice of σ, $A = \sigma(B)$ is a subset of $B_{G,T}(k)$ such that $B = \overline{A}$. Consider now a function $\phi \colon G \to \text{Sym}(A)$, $g \mapsto \phi_g$, such that $\phi_g(a) = \sigma(\overline{ag})$ if $\overline{ag} \in \overline{A}$ (and it is defined arbitrarily otherwise). We have that

$$A \cdot B_{G,S}(10n) \cdot A^{-1} \subseteq B_{G,S}(2k + 10n) \subseteq B_{G,S}(20k)$$

and hence

$$N \cap A \cdot B_{G,S}(10n) \cdot A^{-1} \subseteq B_{N,R}(\Delta_{N \leqslant G}(20k)).$$

By definition of $\mathcal{D}_{N,R}^{\mathcal{F}}$, we deduce that there exist $\alpha \in I$ such that $k_\alpha \leqslant \mathcal{D}_{N,R}^{\mathcal{F}}(\Delta_{N \leqslant G}(20k))$, and a $(20k, 1)$-approximation $\psi \colon N \to G_\alpha$. By hypothesis on $\mathcal{F}$, there exists $\beta \in I$ such that

$$k_\beta \leqslant |A| \, k_\alpha \leqslant k \mathcal{D}_{N,R}^{\mathcal{F}}(\Delta_{N \leqslant G}(20k))$$

and G_β is isometrically isomorphic to $\mathrm{Sym}(A) \ltimes G_\alpha^A$. Define now the function $\Upsilon \colon G \to \mathrm{Sym}(A) \ltimes G_\alpha^A$ by setting $\Upsilon(g) = (\psi_g, b_g)$, where $b_g \in G_\alpha^A$ is defined by $a \mapsto \psi\left(\sigma\left(ag^{-1}\right)ga\right)$. Then the argument at the end of the proof [35, Theorem 5.1] shows that Υ is an $(n, 1)$-approximation for G. This concludes our proof. $\qquad\square$

Corollary 39. *Suppose that G is a finitely generated group, and N is a finitely generated normal subgroup of G such that the quotient G/N is amenable. Then we have*

$$\mathcal{D}_G^{\mathrm{sof}}(n) \preccurlyeq \mathrm{F\o l}_{G/N}^{\mathrm{con}}(n) \, \mathcal{D}_N^{\mathrm{sof}}(\Delta_{N \leqslant G}(\mathrm{F\o l}_{G/N}^{\mathrm{con}}(n))).$$

5.4. **Restricted wreath products.** Let G and H be two groups. The *regular restricted wreath product $G \wr H$* is the semidirect product $B \rtimes H$, where $B = \bigoplus_H G$ is the group of finitely-supported functions from H to G, and the action $\upsilon \colon H \curvearrowright B$ is the Bernoulli shift. An element of $G \wr H$ can be represented by a pair (b, h) where $h \in H$ and $b \in B, b \colon H \to G$. It is clear that if G and H are finite, then $G \wr H$ is finite and $|G \wr H| = |H| \, |G|^{|H|}$.

Suppose now that G is a bi-invariant metric group and F is a finite group. Then the wreath product $G \wr F = B \rtimes F$ is endowed with a canonical bi-invariant metric, defined in [35, Section 3] as follows. For $x_0, x_1 \in F$ and $b_0, b_1 \in B$, we set

$$d_{G \wr F}((b_0, x_0), (b_1, x_1)) = \begin{cases} \max_{x \in F} d_G(b_0(x), b_1(x)) & \text{if } x_0 = x_1, \\ 1 & \text{otherwise.} \end{cases}$$

It is proved in [35, Lemma 3.2] that if G is endowed with a commutator-contractive invariant length function, then for any finite group F the bi-invariant metric $d_{G \wr F}$ on $G \wr F$ described above is commutator-contractive.

5.4.1. *The metric profile of wreath products by residually finite groups.* Let $\mathcal{F}$ be an approximating family $\mathcal{F} = \{(G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)\}_{\alpha \in I}$. We assume that $\varepsilon_\alpha = 1$ for every $\alpha \in I$, and furthermore that for any $\alpha \in I$ and any finite group F there exists $\beta \in I$ such that $k_\beta \leqslant |F| \, k_\alpha^{|F|}$ and G_β is isometrically isomorphic to the wreath product $G_\alpha \wr F$ endowed with the bi-invariant metric $d_{G \wr F}$ described above. These assumptions are fulfilled for example for $\mathcal{F} \in \{\mathcal{F}^{fin}, \mathcal{F}_{cc}^{fin}, \mathcal{F}_{\{0,1\}}^{fin}\}$.

Recall that $\Phi_{G,S}(n)$ denotes the full residual finiteness growth function of a residually finite group G, see Section 4.3.

Theorem 40. *Let $\mathcal{F}$ be a family as above. Let G be a group with finite generating set R, and H be a group with finite generating set T. Consider the finite generating set S of $G \wr H = \bigoplus_H G \rtimes H$ consisting of the pairs (b, h), where $h \in T$ and $b \in \bigoplus_H G$ has support contained in $\{e_H\}$ and range contained in R. Then for every $n \in \mathbb{N}$ we have:*

$$\mathcal{D}_{G \wr H, S}^{\mathcal{F}}(n) \leqslant \Phi_{H,T}(4n) \cdot \mathcal{D}_{G,R}^{\mathcal{F}}(\Phi_{H,T}(4n))^{\Phi_{H,T}(4n)}$$

Proof. Clearly, we can assume that G is $\mathcal{F}$-approximable and H is residually finite, otherwise there is nothing to prove. Observe that if $n \in \mathbb{N}$ then any element of $B_{G \wr H, S}(n)$ is of the form (b, h), where $h \in B_{H,T}(n)$ and $b \in \bigoplus_H G$ has support contained in $B_{H,T}(n)$ and range contained in $B_{G,R}(n)$. Set $m = \Phi_{H,T}(4n)$. Then by definition of $\Phi_{H,T}$ there exists a normal subgroup N

of H of index at most m such that $N \cap B_{H,T}(4n) = \{e_H\}$. Also by definition of $\mathcal{D}_{G,R}^{\mathcal{F}}$ there exists $\alpha \in I$ and a $(m, 1)$-approximation $\phi \colon G \to G_\alpha$ such that $k_\alpha \leqslant \mathcal{D}_G^{\mathcal{F}}(m)$. By hypothesis on $\mathcal{F}$, there exists $\beta \in I$ such that $k_\beta \leqslant m \cdot k_\alpha^m \leqslant m \cdot \mathcal{D}_G^{\mathcal{F}}(m)^m$ and G_β is isometrically isomorphic to $G_\alpha \wr H/N$. Define now the function $\psi \colon G \wr H \to G_\alpha \wr H/N$ by setting $\psi(g, h) = (\widehat{g}, hN)$ where $\widehat{g} \colon H/N \to G_\alpha$ is defined by

$$\widehat{g}(kN) = \begin{cases} e_\alpha & \text{if } B_{H,T}(n) \cap kN = \varnothing, \text{ and} \\ \varphi(g_{k'}) & \text{if } B_{H,T}(n) \cap kN = \{k'\}. \end{cases}$$

Then the proof of [35, Theorem 3.1] shows that ψ is a $(n, 1)$-approximation of $G \wr H$. $\qquad\square$

5.4.2. *The sofic profile of wreath products of sofic groups by sofic groups.* In this section, we suppose that G and H are groups with finite generating sets R and T, respectively. We let S be the set of elements of $G \wr H = \bigoplus_H G \rtimes H$ of the form (b, h) where $h \in T$ and $b \in \bigoplus_H G$ is such that the support of b is contained in $\{e_H\}$ and the range is contained in R.

Let K denote a group with a bi-invariant distance d. If F is a finite subset of G and $c, \varepsilon > 0$, then a function $\phi \colon G \to K$ is (F, ε)-*multiplicative* if $d(\phi(xy), \phi(x)\phi(y)) < \varepsilon$ for $x, y \in F$, and (F, c)-*injective* if $d(\phi(x), \phi(y)) \geqslant c$ for $x, y \in F$ distinct, cf. terminology of Definition 1. Proceeding as in the proof of [34, Lemma 2.8] gives the following.

Lemma 41. *Suppose that $\Psi \colon G \wr H \to K$ is a function. Fix $n \in \mathbb{N}$ and let G_n be the set of elements of $\bigoplus_H G$ whose support is contained in $B_{H,T}(n)$ and whose range is contained in $B_{G,R}(n)$. Suppose that Ψ satisfies the following (the identities $e_{\bigoplus_H G}$ and e_H are denoted by 1):*

- $d(\Psi(xy, 1), \Psi(x, 1)\Psi(y, 1)) < \varepsilon_1$ *whenever* $x, y \in G_n$,
- $d(\Psi(1, x)\Psi(1, y), \Psi(1, xy)) < \varepsilon_0$ *whenever* $x, y \in B_{H,T}(n)$,
- $\Psi(x, 1)\Psi(1, y) = \Psi(x, y)$ *whenever* $x \in G_n$ *and* $y \in B_{H,T}(n)$, *and*
- $\Psi(1, y)\Psi(x, 1) = \Psi(v_y(x), y)$.

Then $d(\Psi(zw), \Psi(z)\Psi(w)) < \varepsilon_0 + \varepsilon_1$ for any $z, w \in B_{G\wr H, S}(n)$.

Suppose that K is a bi-invariant metric group. Let d' be any bi-invariant metric on $\bigoplus_B K$ that restricts to the original metric on K on each copy of K, and d' is also the corresponding metric on $\bigoplus_B K \wr_B \mathrm{Sym}(B)$. Consider also the maximum metric $d_{\max}$ on $\bigoplus_B K$ and the corresponding metric on $\bigoplus_B K \wr_B \mathrm{Sym}(B)$. The proof of [34, Proposition 3.3] gives the following.

Lemma 42. *Fix $\varepsilon > 0$. Suppose that $\sigma \colon H \to \mathrm{Sym}(B)$ is a $(B_{H,T}(4n), \varepsilon)$-multiplicative and $(B_{H,T}(4n), 1 - \varepsilon)$-injective function. Suppose also that $\theta \colon G \to K$ is a $(B_{G,R}(4n), \varepsilon)$-multiplicative and $(B_{G,R}(4n), 1 - \varepsilon)$-injective function. Then there exists a function*

$$\Psi \colon G \wr H \to \bigoplus_B K \wr_B \mathrm{Sym}(B)$$

that is $(B_{G\wr H, S}(n), 48\,|B_{H,T}(4n)|^2\,\varepsilon)$-multiplicative with respect to the metric d' and $(B_{G\wr H, S}(n), 1 - 48\,|B_{H,T}(n)|^2\,\varepsilon)$-injective with respect to the metric $d_{\max}$.

More precisely, [34, Proposition 3.3] states that under the assumptions of Lemma 42, which are more restrictive than the hypotheses of [34, Proposition 3.3], for any finite subset $F \subseteq G \wr H$, there exists a set $E \subseteq H$, such that the previously defined function Ψ is (F, ε')-multiplicative, for ε' satisfying $\varepsilon < \frac{\varepsilon'}{48|E|^2}$. Since the assumptions of Lemma 42 are more restrictive, the set E can

be chosen to be $B_{H,T}(4n)$ and with $\varepsilon' = 48|E|^2\varepsilon$. Moreover, applying a stronger version of [34, Proposition 3.3], formulated in [34, Remark 3.5], gives that Ψ is $(B_{G\wr H,S}(n), 1-48\left|B_{H,T}(n)\right|^2\varepsilon)$-injective with respect to the metric $d_{\max}$.

In a similar manner, a quantitative analysis of the proof of [34, Theorem 4.1] then shows the following.

Lemma 43. *Fix* $\varepsilon > 0$. *Suppose that* $\sigma\colon H \to \mathrm{Sym}\,(B)$ *is a* $(B_{H,T}(4n), \varepsilon)$-*multiplicative and* $(B_{H,T}(4n), 1-\varepsilon)$-*injective function. Suppose that* $\theta\colon G \to \mathrm{Sym}\,(A)$ *is a* $(B_{G,R}(4n), \varepsilon)$-*multiplicative and* $(B_{G,R}(4n), 1-\varepsilon)$-*injective function. Let*

$$\Psi\colon G\wr H \to \bigoplus_B \mathrm{Sym}\,(A) \wr_B \mathrm{Sym}\,(B)$$

be obtained from σ *and* θ *as in Lemma 42. Define*

$$\Theta\colon \bigoplus_B \mathrm{Sym}\,(A) \wr_B \mathrm{Sym}\,(B) \to \mathrm{Sym}\,(A^B \times B)\ \ by$$

$$\Theta\,(\pi, \tau)\colon\ ((a_\beta), b) \mapsto ((\pi_{b,\beta}\,(a_\beta)), \tau\,(b))\,.$$

Then the composition $\Theta\circ\Psi\colon G\wr H \to \mathrm{Sym}\,(B \times A^B)$ *is* $(B_{G\wr H,S}(n), 48\left|B_{H,T}(4n)\right|^2\varepsilon)$-*multiplicative and* $(B_{G\wr H,S}(n), 1-48\left|B_{H,T}(n)\right|^2\varepsilon)$-*injective when* $\mathrm{Sym}\,(B \times A^B)$ *is endowed with the normalised Hamming distance.*

We extract from Lemma 43 the following upper bound on the sofic profile of the wreath product of sofic groups. Recall that the sofic profile $\mathcal{D}^{sof}_{G,R}(n)$ of a group G with finite generating set R is the $\mathcal{F}$-profile $\mathcal{D}^{\mathcal{F}}_{G,R}(n)$ where $\mathcal{F}$ is the family $\mathcal{F}^{sof} = (\mathrm{Sym}\,(A), d_{\mathrm{Ham}}, |A|, 1)_{n\in\mathbb{N}}$ where A is a finite set and $\mathrm{Sym}\,(A)$ is endowed with the normalised Hamming distance.

Recall that $\beta_{G,S}(n) = |B_{G,S}(n)|$ is the growth function of G with respect to a finite generating set S. We let β_G denote the $\simeq$-equivalence class of $\beta_{G,S}$, which is independent of the choice of the generating set S.

Theorem 44. *Let* G *be a group with finite generating set* R, *and* H *be a group with finite generating set* T. *Consider the finite generating set* S *of* $G \wr H = \bigoplus_H G \rtimes H$ *consisting of the pairs* (b, h) *where* $h \in H$ *and* $b \in \bigoplus_H G$ *has support contained in* $\{e_H\}$ *and range contained in* R. *Then for every* $n \in \mathbb{N}$ *one has that*

$$\mathcal{D}^{sof}_{G\wr H,S}(n) \leqslant \mathcal{D}^{sof}_{H,T}(48\beta_{H,T}(n)^2\,n) \cdot \mathcal{D}^{sof}_{G,R}(48\beta_{H,T}(n)^2\,n)^{\mathcal{D}^{sof}_{H,T}(48\beta_{H,T}(n)^2 n)}.$$

5.4.3. *The hyperlinear profile of the wreath product of a hyperlinear group by a sofic group.* We adopt the preceding notations: G and H are groups with finite generating sets R and T, respectively. We let S be the finite generating set for $G \wr H$ of elements (b, h) for $h \in B_{H,T}(n)$ and $b \in \bigoplus_H G$ with support contained in $\{e_H\}$ and range contained in $B_{G,R}(n)$.

Suppose that B is a finite set. Then we denote by $\mathcal{H}_B$ the finite-dimensional Hilbert space with basis $\{|b\rangle : b \in B\}$. If $\mathcal{H}$ is a finite-dimensional Hilbert space, then we define $\mathcal{H}^{\otimes B}$ to be the tensor product of a family of $|B|$ copies of $\mathcal{H}$ indexed by B. We denote by $U(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$ equipped with the *projective* normalised Hilbert-Schmidt pseudometric $d_{\overline{\mathrm{HS}}}$ as defined in Section 3.2. We analyze the proof of [34, Section 4.2] and obtain the following.

Lemma 45. *Fix $\varepsilon > 0$. Suppose that $\sigma\colon H \to \mathrm{Sym}\,(B)$ is a $(B_{H,T}\,(4n)\,,\varepsilon)$-multiplicative and $(B_{H,T}\,(4n)\,,1-\varepsilon)$-injective function. Suppose that $\theta\colon G \to U\,(\mathcal{H})$ is a $(B_{G,R}\,(4n)\,,\varepsilon)$-multiplicative and $(B_{G,R}\,(4n)\,,1-\varepsilon)$-injective function for some finite-dimensional Hilbert space $\mathcal{H}$. Let $\Psi\colon G \wr H \to \bigoplus_B U\,(\mathcal{H}) \wr \mathrm{Sym}\,(B)$ be the function obtained in Lemma 42. Define the function $\Theta\colon \bigoplus_B U\,(\mathcal{H}) \wr_B \mathrm{Sym}\,(B) \to U\left(\mathcal{H}^{\otimes B} \otimes \mathcal{H}_B\right)$ by*

$$\Theta\,(\pi,\tau)\colon \bigotimes_\gamma \xi_\gamma \otimes |b\rangle \mapsto \bigotimes_\gamma \pi_{b,\gamma}\,(\xi_\gamma) \otimes |\sigma\,(b)\rangle\,.$$

Then $\Theta \circ \Psi$ is $\left(B_{G\wr H,S}\,(n)\,,48\,|B_{H,T}\,(4n)|^2\,\varepsilon^{\frac{1}{2}}\right)$-multiplicative function and $\left(B_{G\wr H,S}\,(n)\,,1-48\,|B_{H,T}\,(n)|^2\,\varepsilon^{\frac{1}{2}}\right)$-injective function when $U\left(\mathcal{H}^{\otimes B} \otimes \mathcal{H}_B\right)$ is endowed with the normalised Hilbert-Schmidt distance.

We deduce the following.

Theorem 46. *Let G be a group with finite generating set R, and H be a group with finite generating set T. Consider the finite generating set S of $G \wr H = \bigoplus_H G \rtimes H$ consisting of the pairs (b,h) where $h \in H$ and $b \in \bigoplus_H G$ has support contained in $\{e_H\}$ and range contained in R. Then for every $n \in \mathbb{N}$ one has that*

$$\mathcal{D}_{G\wr H,S}^{\overline{hyp}}\,(n) \leqslant \mathcal{D}_{H,T}^{sof}(2500\beta_{H,T}\,(n)^4\,n^2) \cdot \mathcal{D}_{G,R}^{\overline{hyp}}(2500\beta_{H,T}\,(n)^4\,n^2)^{\mathcal{D}_{H,T}^{sof}(2500\beta_{H,T}(n)^4 n^2)}\,.$$

5.4.4. *The linear sofic profile of the wreath product of a linear sofic group by a sofic group.* Let $\mathbb{K}$ be a field. If B is a finite set, then we let $\mathbb{K}^B$ to be the $\mathbb{K}$-vector space obtained as the direct sum of $|B|$ copies of $\mathbb{K}$ indexed by B with basis $\{|b\rangle : b \in B\}$. If V is a finite-dimensional $\mathbb{K}$-vector space, then we let $V^{\otimes B}$ be the tensor product of a family of $|B|$ copies of V indexed by B. We denote by $GL\,(V,\mathbb{K})$ the group of invertible operators on V equipped with the *projective normalised rank pseudometric* $d_{\overline{\mathrm{rank}}}$ as defined in Section 3.3. We analyse the proof of [34, Proposition 4.12] and obtain the following.

Lemma 47. *Fix $\varepsilon > 0$. Suppose that $\sigma\colon H \to \mathrm{Sym}\,(B)$ is a $(B_{H,T}\,(4n)\,,\varepsilon)$-multiplicative and $(B_{H,T}\,(4n)\,,1-\varepsilon)$-injective function. Suppose also that $\theta\colon G \to GL\,(V,\mathbb{K})$ is a $(B_{G,R}\,(4n)\,,\varepsilon)$-multiplicative and $(B_{G,R}\,(4n)\,,1-\varepsilon)$-injective function. Let $\Psi\colon G \wr H \to \bigoplus_B \mathrm{GL}\,(V,\mathbb{K}) \wr_B \mathrm{Sym}\,(B)$ be the obtained as in Lemma 42. Let also*

$$\Theta\colon \bigoplus_B \mathrm{GL}\,(V,\mathbb{K}) \wr_B \mathrm{Sym}\,(B) \to \mathrm{GL}\left(V^{\otimes B} \otimes \mathbb{K}^B, \mathbb{K}\right)$$

be defined by

$$\Theta\,(\pi,\tau)\colon \bigotimes_\gamma a_\gamma \otimes |b\rangle \mapsto \bigotimes_\gamma \pi_{b,\gamma}\,(a_\gamma) \otimes |\tau\,(b)\rangle\,.$$

Then $\Theta \circ \Psi$ is $\left(B_{G\wr H,S}\,(n)\,,48\,|B_{H,T}\,(4n)|^2\,\varepsilon\right)$-multiplicative function and $\left(B_{G\wr H,S}\,(n)\,,1-48\,|B_{H,T}\,(4n)|^2\,\varepsilon\right)$-injective function.

The following is an immediate consequence of the preceding lemma.

Theorem 48. *Let G be a group with finite generating set R, and H be a group with finite generating set T. Consider the finite generating set S of $G \wr H = \bigoplus_H G \rtimes H$ consisting of the*

pairs (b, h) where $h \in H$ and $b \in \bigoplus_H G$ has support contained in $\{e_H\}$ and range contained in R. Then for every $n \in \mathbb{N}$ one has that

$$\mathcal{D}^{\overline{lin}}_{G \wr H, S}(n) \leqslant \mathcal{D}^{sof}_{H,T}(48n\beta_{H,T}(n)^2) \cdot \mathcal{D}^{\overline{lin}}_{G,R}(48n\mathcal{D}^{sof}_{H,T}(48n\beta_{H,T}(n)^2))^{\mathcal{D}^{sof}_{H,T}(48n\beta_{H,T}(n)^2)}.$$

6. FURTHER REMARKS AND OPEN QUESTIONS

The following question incites, in particular, a thorough study of possible definitions of bi-invariant metrics on solvable groups; see also our observation in Example 8.

Question 49. Does there exist an infinite group which is not approximable by solvable groups (with no uniform bound on the derived length)?

We denote by $Th^{\forall}_c(G)$ the universal theory of G in the continuous logic setting [7]. An affirmative answer to the next question would generalise Malcev's result, see Example 9.

Question 50. Let G be a group such that $Th^{\forall}_c(GL(k, \mathbb{K}_\alpha)) \subseteq Th^{\forall}_c(G)$ for some $k > 0$ and a field $\mathbb{K}_\alpha$. Is G linear?

Answers to the next two questions will clarify the status of Conjecture 15.

Question 51. Does there exist a sofic group G which is not $\mathcal{F}^{hyp}$-stable but satisfies $\mathcal{D}^{sof}_G(n) \preccurlyeq \mathcal{D}^{hyp}_G(n)$?

Question 52. Does there exist a sofic group G which is not $\mathcal{F}^{sof}$-stable but $\mathcal{F}^{hyp}$-stable?

Let $\bar{U} = U(R)$ be the unitary group of the hyperfinite factor R of type II_1 equipped with the ultraweak topology. A group G is hyperlinear if and only if G embeds into a metric ultrapower of $\bar{U}$ [38, Corollary 4.3]. In [1, Proposition 7], the first author observed that all Gromov hyperbolic groups G are residually finite (respectively, LE-$\mathcal{F}^{fin}$, LEA, etc.) if and only if every G embeds into $\bar{U}$.

Question 53. Let G be a non-elementary Gromov hyperbolic group such that $G \hookrightarrow \prod_{\mathcal{U}} (\bar{U}, d)$. Does it imply the existence of an embedding $G \hookrightarrow \bar{U}$?

A positive answer will establish the following conjecture.

Conjecture 54. [1, Conjecture 2.8] *All Gromov hyperbolic groups are residually finite $\iff$ all Gromov hyperbolic groups are sofic.*

An answer to the next question will give a better understanding of the rank metric on linear groups, and hence, of linear sofic groups and their profile functions.

Question 55. [3] Does the class of linear sofic groups, i.e. $\mathcal{F}^{lin} = (GL(n, \mathbb{K}), d_{\text{rank}}, n, 1/4)_{n \in \mathbb{N}}$-approximable groups, coincide with the class of $(GL(n, \mathbb{K}), d_{\text{rank}}, n, \varepsilon_n)_{n \in \mathbb{N}}$-approximable groups, where ε_n is constantly equal to a fixed or to an arbitrarily chosen number between $1/4$ and 1?

The next three questions are about examples of sofic and linear sofic groups with extreme profile functions with respect to the ambient class of weakly sofic groups.

Question 56. Does there exist a sofic group G such that $\mathcal{D}^{fin}_{G,S}(n) \simeq n^n$?

Question 57. Does there exist an infinite linear sofic group G such that $\mathcal{D}_{G,S}^{fin}(n) \simeq \mathcal{D}_{G,S}^{lin}(n)$?

Question 58. What is a relationship between $\mathcal{D}_{G,S}^{fin}(n)$ and $\mathcal{D}_{G,S}^{lin}(n)$ for a linear sofic group G?

If affirmative, the answer to the next question will generalise the famous Coulhon-Saloff-Coste isoperimetric inequality which is, using our notation: $\beta_{G,S}(n) \preccurlyeq \text{Føl}_{G,S}(n)$.

Question 59. Is $\beta_{G,S}(n) \preccurlyeq \mathcal{D}_{G,S}^{sof}(n)$ for a sofic group G?

To answer this rather challenging question one can first focus on subclasses of sofic groups such as classical matrix groups and (elementary) amenable groups. Here is a variant for groups locally embeddable into amenable groups.

Question 60. Is $\beta_{G,S}(n) \preccurlyeq LEA_{G,S}(n)$ for an LEA group G?

Recall that a finitely generated linear group that is not virtually nilpotent has exponential weakly sofic profile $\mathcal{D}_{G,S}^{fin}(n)$, see Example 26.

Question 61. Does there exist a finitely generated linear group of exponential growth with polynomial/subexponential sofic profile $\mathcal{D}_{G,S}^{sof}(n)$?

Corollary 32 and known stability results on virtually abelian groups, see Example 33, yield the following.

Conjecture 62. *Let G be a finitely generated virtually abelian group. Then $\mathcal{D}_{G,S}^{lin}(n) \simeq n^{\text{rank } G}$.*

A direct approach to establish Conjecture 62 is to proceed as in Example 13, using the discreteness of values of the rank distance d_{rank}. Alternatively, Conjecture 62 would be established by proving the $\mathcal{F}^{lin}$-stability of virtually abelian groups. This remains unknown, see [22] for a partial result. In this vein, it is interesting to investigate the (non)-stability of commutator relator word with respect to the rank distance within various classes of matrices. Unitary, self-adjoint and normal matrices are natural classes to consider, see Section 3.3 for notation.

Question 63. Let G be a finitely generated virtually abelian group. Is it true that $\mathcal{D}_{G,S}^{u}(n) \simeq \mathcal{D}_{G,S}^{sa}(n) \simeq \mathcal{D}_{G,S}^{nor}(n) \simeq n^{\text{rank } G}$?

We have shown that sofic and hyperlinear profiles of the Heisenberg groups H_{2l+1} coincide with the full residual finiteness growth, see Example 33. We expect this to hold for the weakly sofic and linear sofic profiles as well, cf. Example 26.

Conjecture 64. *We have $\mathcal{D}_{H_{2l+1},S}^{fin}(n) \simeq \mathcal{D}_{H_{2l+1},S}^{lin}(n) \simeq n^{2(2l+1)}$.*

Here is an ambitious generalisation of Example 13 and Example 33.

Conjecture 65. *Let G be a finitely generated virtually nilpotent group. Then*

$$\mathcal{D}_{G,S}^{sof}(n) \simeq \mathcal{D}_{G,S}^{hyp}(n) \simeq \mathcal{D}_{G,S}^{lin}(n) \simeq \mathcal{D}_{G,S}^{fin}(n) \simeq \Phi_{G,S}(n).$$

To approach the weakly sofic part of the preceding conjecture from an ambient class of groups, we consider arbitrary residually finite groups.

Question 66. Does there exist a finitely generated residually finite group with $\mathcal{D}_{G,S}^{fin}(n) \not\asymp \Phi_{G,S}(n)$?

The next question explores a possibility to describe weakly sofic groups in purely algebraic terms (see [27] for such a result on sofic groups).

Question 67. Is there a characterisation of weakly sofic groups with no reference to any nontrivial bi-invariant distances d_α on finite approximating groups H_α?

In Section 5.2, we give an upper bound on the sofic profile of a free product of sofic groups. Other profiles of free products remain unexplored. Recall that F_2 denotes a free group of rank 2.

Conjecture 68. *Let G and H be linear sofic groups. Then*
$$\mathcal{D}_{G*H}^{lin}(n) \preccurlyeq \left(\mathcal{D}_G^{lin}(n) + \mathcal{D}_H^{lin}(n) \right) \Phi_{F_2}(n).$$

Question 69. Let G and H be hyperlinear groups. Find an upper bound on $\mathcal{D}_{G*H}^{hyp}(n)$ in terms of $\mathcal{D}_G^{hyp}(n)$ and $\mathcal{D}_H^{hyp}(n)$. Is it true that
$$\mathcal{D}_{G*H}^{hyp}(n) \preccurlyeq \left(\mathcal{D}_G^{hyp}(n) + \mathcal{D}_H^{hyp}(n) \right) \Phi_{F_2}(n)?$$

Question 70. Let G and H be LE-$\mathcal{F}$ groups. Estimate $\Phi_{G*H}^{\mathcal{F}}(n)$ in terms of $\Phi_G^{\mathcal{F}}(n)$ and $\Phi_H^{\mathcal{F}}(n)$.

Question 71. Let G and H be LEA groups. Estimate $LEA_{G*H}(n)$ in terms of $LEA_G(n)$ and $LEA_H(n)$.

Question 72. Let $\mathcal{F} \in \{ \mathcal{F}^{sof}, \mathcal{F}^{hyp}, \mathcal{F}^{lin}, \mathcal{F}^{fin}, \mathcal{F}^{ct}, \mathcal{F}_{\{0,1\}}^{fin} \}$ and $m \geqslant 3$. What is $\mathcal{D}_{SL_m(\mathbb{Z}),S}^{\mathcal{F}}(n)$?

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Universität Wien, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria.
Email address: `goulnara.arzhantseva@univie.ac.at`

Université de Genève, Section de Mathématiques, Uni Dufour, 24 rue du Général Dufour, Case postale 64, 1211 Genève 4, Switzerland.
Email address: `pierre-alain.cherix@unige.ch`