EXAMPLES OF RANDOM GROUPS

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ABSTRACT. We present Gromov's construction of a random group with no coarse embedding into a Hilbert space.

> "Si nous n'étions pas ignorants, il n'y aurait pas de probabilité, il n'y aurait de place que pour la certitude;"[1]
>
> *Henri Poincaré*,
> La Science et l'hypothèse, Chapitre XI (1902).

1. Introduction

In the late 1950's, working on the uniform classification of metric spaces, Smirnov asked whether every separable metric space is uniformly homeomorphic to a subset of a Hilbert space [G]. This was settled negatively by Enflo, who proved that the Banach space of null sequences c_0 does not embed uniformly homeomorphically into any Hilbert space [E].

Initiating a new theory, Gromov introduced the concept of a *coarse embedding* (also termed as a *uniform embedding*) of metric spaces and asked whether every separable metric space coarsely embeds into a Hilbert space [$\mathrm{Gr}_{\mathrm{AI}}$, p. 218].

Definition 1.1 (Coarse embedding)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is said to be a *coarse embedding* if for $x_n, y_n \in X, n \in \mathbb{N}$,

$$d_X(x_n, y_n) \to \infty \quad \text{if and only if} \quad d_Y(f(x_n), f(y_n)) \to \infty.$$

Gromov's question was answered negatively in [Dr. *et al.*], where the authors adapt Enflo's original construction and build an infinite family of finite graphs of growing degrees admitting no coarse embedding into a Hilbert space.

In *"Spaces and questions"* [$\mathrm{Gr}_{\mathrm{SQ}}$], Misha Gromov observed that the above non-embedding phenomenon is due to the expanding properties of the graphs. He predicted and then, in *"Random walk in random groups"* [$\mathrm{Gr}_{\mathrm{RWRG}}$], constructed a finitely generated and even finitely presented *group* that, equipped with a word length metric, coarsely contains an infinite expander. This group has rather surprising properties: it admits no coarse embedding into a Hilbert space (or

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[1] "If we were not ignorant there would be no probability, there could only be certainty;"–*Henri Poincaré*, La Science et l'hypothèse, Chapter XI (1902).

into any ℓ^p with $1 \leqslant p < \infty$) and any finite-dimensional linear representation of this group has finite image. We call this group the *monster* although in the words of its inventor it is a "quite simple two-dimensional creature."

Our objective is to explain Gromov's construction. Let $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ be an infinite family of finite graphs. We shall always consider Θ as the disjoint union of Θ_n's endowed with a metric which coincides with the standard edge-path metric on each Θ_n and such that $\mathrm{dist}(\Theta_n, \Theta_{n'}) \to \infty$ as $n + n' \to \infty$.

A *labelling* of Θ (see Section 2 for details) is a simplicial map

$$m\colon \Theta \to W,$$

where W is a bouquet of oriented loops labelled by letters from a finite alphabet S. A choice of a base point in each of the connected components of Θ yields an induced map on the fundamental groups

$$m_*\colon \pi_1(\Theta) \to \pi_1(W)$$

and a label preserving simplicial map

$$f_m\colon \Theta \to \mathrm{Ca}(G(m))$$

to the Cayley graph of the group $G(m)$ which is, by definition, the quotient of the free group $F(S) := \pi_1(W)$ by the normal subgroup generated by the images of m_* of the fundamental groups of connected components of Θ.

The goal is to prove

Theorem 1.2 (cf. Theorem7.7)**.** *There exist an expander graph $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ and a labelling m of Θ such that the map $f_m\colon \Theta \to \mathrm{Ca}(G(m))$ is a coarse embedding.*

The proof goes in three steps.

Step I – Geometry. We start by observing that if the graph is finite and the labelling m satisfies a certain small cancellation condition, the quotient $G(m)$ is a non-elementary hyperbolic group and the graph embeds isometrically into its Cayley graph. This is explained in Section 2, where we extend the classical small cancellation theory to such *graphical quotients* of the free group.

In order to embed an infinite family of finite graphs, or, more precisely, the metric space (Θ, dist), we shall iterate the previous construction by taking successive graphical quotients of non-elementary hyperbolic groups (starting with the free group as above or with a given non-elementary hyperbolic group G_0). The inductive step of such an iteration is done in Section 3, where we appeal to small cancellation theory over a hyperbolic group and to hyperbolic geometry of orbispaces to prove the very small cancellation theorem (Theorem 3.10). It follows that, under the hypothesis of that theorem at the n-th inductive step, the Cayley graph of the $(n+1)$-th graphical quotient group G_{n+1} coarsely contains Θ_n. Then, by iteration, (Θ, dist) is coarsely embeddable into the Cayley graph of the direct limit of groups G_n, which is now a *lacunary hyperbolic* group, see Section 4.

Step II – Probability and harmonic analysis. It remains to show the existence of a labelling required at each inductive step and of a subset $I \subseteq \mathbb{N}$ such that Θ_I admits the iteration. We obtain the former in Section 5. We endow the space of all possible labellings of Θ_n (and hence, those of Θ) with the Kolmogorov probability measure and prove (see Proposition 5.9) that simple combinatorial conditions on a given $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ imply that, with asymptotic probability 1 as $n \to \infty$, a random labelling of Θ_n by generators of a non-elementary hyperbolic group satisfies the hypotheses of the very small cancellation theorem (Theorem 3.10). Then, in Section 6 (see Theorem 6.3), we extract from Θ a sub-family Θ_I for some recursive set $I \subseteq \mathbb{N}$ that admits the iteration and, therefore, a coarse embedding into the Cayley graph of the resulting lacunary hyperbolic group.

Step III – Arithmetics. To complete the proof, we have to present an infinite expander $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ satisfying the above mentioned combinatorial conditions. Indeed, in the last section, we show that results of Section 6 apply to certain regular infinite expanders. Namely, one can take the Selberg expanding family which is known to have arbitrarily large girth. It follows that the constructed lacunary hyperbolic group $G_I(\omega)$ defined by a random labelling ω of a sub-family $\Theta_I = (\Theta_n)_{n\in I\subseteq\mathbb{N}}$ of such a family admits no coarse embedding into a Hilbert space.

Our presentation emphasizes the asymptotic viewpoint and does not aim to provide explicit estimates on the parameters involved in the construction. Here are some of them:

- The number of generators k of a hyperbolic group G, the hyperbolicity constant δ, the spectral radius κ_{reg} of the simple random walk on G, and the Kazhdan constant κ (whenever G has Kazhdan's property (T));
- Parameters α, β, γ of a local quasi-isometric embedding induced by a random labelling of a given graph;
- The parameter λ of the geometric small cancellation condition;
- An infinite family $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ of graphs of girth ρ_n and of thinness b.

The main technical result (cf. Theorem 6.3) is the following.

Let $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ *be a family of finite graphs with* $\lim_{n\to\infty} \operatorname{girth} \Theta_n = \infty$ *and a uniformly bounded ratio* $\frac{\operatorname{diam} \Theta_n}{\operatorname{girth} \Theta_n}$. *Let* (G, S) *be a torsion-free non-elementary hyperbolic group and* κ_{reg} *the corresponding spectral radius. Assume that* Θ *is* b*-thin with a small enough* $b > 0$ *(namely,* $b + \ln \kappa_{\mathrm{reg}} < 0$*). Then, for any given* $p \in (0, 1)$*, there exists an infinite* $I \subseteq \mathbb{N}$ *such that, with probability at least* p*, the quotient group* $G_I(\omega)$ *defined by a random labelling* ω *of* $\Theta_I = (\Theta_n)_{n\in I\subseteq\mathbb{N}}$ *is an infinite finitely generated group and* $(\Theta_I, \operatorname{dist})$ *is coarsely embedded into its Cayley graph.*

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Disclaimer. The technical arguments inevitably require a good understanding of fundamentals such as hyperbolic groups, geometry of orbispaces, and asymptotic invariants of infinite groups.

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2. Small cancellation theory over free group

2.1. **Graph labelling.** Let S be a finite set of k elements and $F(S) \cong F_k$ be the free group on S of rank k. The group $F(S)$ is the fundamental group of a bouquet W of k oriented loops labelled by letters from S: $F(S) = \pi_1(W)$.

Definition 2.1 (Graph labelling)**.** Let Θ be a graph (i.e. a 1-dimensional simplicial complex) with no vertex of degree 1. A *labelling* of Θ by S is a map m that to each edge of the graph assigns an orientation and a letter from S. In other words, m is a simplicial map[2]

$$m\colon \Theta \to W.$$

A labelling is *reduced* if Θ contains no vertex v such that two edges adjacent to v have the same label and orientation at v (starting or ending simultaneously at v).

A choice of a base point in each of the connected components of Θ yields an induced map on the fundamental groups

$$m_*\colon \pi_1(\Theta) \to \pi_1(W).$$

Definition 2.2 (Group defined by a labelled graph)**.** Let m be a labelling of a graph Θ by S. The group $G(m)$ is defined to be the quotient of the free group $F(S)$ by the normal subgroup generated by the images of m_* of the fundamental groups of the connected components of Θ.

We view such a group $G(m)$ geometrically as follows. Let $\dot\Theta$ be the cone over Θ, that is the disjoint union of the topological cones over the connected components of Θ such that $\Theta = \partial\dot\Theta$. We build the CW-complex $\Pi_m = W \cup_m \dot\Theta$ through obvious identifications according to the labelling $m\colon \Theta \to W$. The van Kampen theorem implies

$$\pi_1(\Pi_m) = G(m).$$

[2]We identify m with its geometric realization which is a continuous map from the geometric realization of Θ to that of W.

For example, if Θ is a disjoint union of circle graphs of length $n_1, \ldots, n_p$, a labelling m of Θ is given by a choice of words $R_1, \ldots, R_p$ of length n_i in the alphabet $S \cup S^{-1}$, the group $G(m)$ is defined by the presentation $\langle S \mid R_1, \ldots, R_p \rangle$, and Π_m is the van Kampen polyhedron associated to this presentation [LS, III.2.3]. Extending the terminology, we say that Π_m is the *van Kampen polyhedron* of $G(m)$.

Let $\mathrm{Ca}(G(m))$ denote the Cayley graph of $G(m)$ with respect to the generating set S. The map m_* induces a label preserving simplicial map

$$f_m \colon \Theta \to \mathrm{Ca}(G(m)).$$

In general, such a map does not preserve the graph Θ in any natural way. Thus, the problem is to find an infinite expander Θ and an appropriate labelling m such that f_m is a coarse embedding.

2.2. **Small cancellation theory.**

Definition 2.3 (Piece)**.** A *piece* in a labelled graph $m(\Theta)$ is a simple path $I \subset \Theta$ such that there exists another simple path $J \subset \Theta$ and a label preserving isometry from I to J.

The *small cancellation conditions for* Θ are defined as follows (for $p \in \mathbb{N}$ and $\lambda \in]0, 1[$):

$C(p)$, no cycle in Θ is a disjoint union of fewer than p pieces;

$C'(\lambda)$, the length of a cycle in Θ is strictly greater than λ^{-1} times the length of the longest piece in this cycle;

$C''(\lambda)$, the girth of Θ is strictly greater than λ^{-1} times the length of the longest piece in Θ.

Obviously, $C''(\lambda) \Rightarrow C'(\lambda) \Rightarrow C([\lambda^{-1}]+1)$. Observe that if Θ is a disjoint union of circle graphs this recovers the usual small cancellation conditions. Moreover, as pointed out by Gromov [$\mathrm{Gr}_{\mathrm{RWRG}}$] and previously by Rips and Segev [RS], the classical small cancellation theory (works of Tartakovskii, Greendlinger, Lyndon, Schupp, etc., see details and references in [LS]) extends to this context: the $C(p)$ condition for $p \geqslant 6$ implies the asphericity of the van Kampen polyhedron and the $C(7)$ condition guarantees the hyperbolicity of the group $G(m)$. For a detailed study of the $C'(1/6)$ condition, see [Oll].

The idea to produce new groups using small cancellation conditions on general graphs instead of circle graphs has appeared for the first time in a work of Rips and Segev related to the Kaplansky problem on zero divisors in group rings [RS].

The next result will not be used in our further arguments. We omit the proof which can be obtained by standard arguments using van Kampen diagrams.

Theorem 2.4 (Small cancellation theorem)**.** (cf. [$\mathrm{Gr}_{\mathrm{RWRG}}$, Sec. 2]) *Let Θ be a finite connected graph and m be a reduced $C(7)$ labelling of Θ. Then the group $G(m)$ is hyperbolic and the map $f_m \colon \Theta \to \mathrm{Ca}(G(m))$ is an isometric embedding. Moreover, the van Kampen polyhedron Π_m is aspherical and the rational Euler characteristic satisfies $\chi(G(m)) = \chi(F_k) + b_1(\Theta)$.*

3. Small cancellation theory over a hyperbolic group.

Let $\delta \geqslant 0$. A geodesic metric space is said to be *δ-hyperbolic* if every geodesic triangle in this space is *δ-thin* [$\mathrm{Gr}_{\mathrm{HG}}$, 6.3], [CDP, Ch.1, §3]. It is *hyperbolic* if it is δ-hyperbolic for some $\delta \geqslant 0$. From now on, X denotes a proper δ-hyperbolic space and ∂X its boundary.

3.1. **Quasiconvexity and quasigeodesics.**

Definition 3.1 (Cylinder)**.** The *cylinder* $C(Y)$ of a subset $Y \subseteq X \cup \partial X$ is the set of points lying at a distance not greater than 100δ to a geodesic with endpoints in $Y \cup \partial Y$.

Definition 3.2 (Quasiconvex subset)**.** A subset $Y \subseteq X$ is *D-quasiconvex* if any geodesic $[x, y]$ with endpoints in Y belongs to the D-neighbourhood of Y. A subset is *quasiconvex* if it is D-quasiconvex for some $D \geqslant 0$.

Every geodesic quadrilateral in X with vertices in $X \cup \partial X$ is 8δ-quasiconvex, cf. [CDP, Ch.2]. Therefore, every cylinder is 10δ-quasiconvex.

Definition 3.3 (Projection)**.** Let Y be a subset of X. A map $P : X \to Y$ is a *projection* if for all $y \in Y$ and $x \in X$ we have $|x - P(x)| \leqslant |x - y| + \delta$.

If δ is strictly positive, or if Y is closed, then such a projection does exist. The next result is well-known, cf. the proof of [BH, III. Γ.3.11].

Proposition 3.4. *Let Y_1 and Y_2 be two D-quasiconvex subsets of X. The diameter of the projection of Y_1 onto Y_2 is at most* $2D + 100\delta + \operatorname{diam}(C(Y_1) \cap C(Y_2))$.□

Definition 3.5 (Quasigeodesics)**.** Let $\alpha, \beta, \gamma \in \mathbb{R}^+, \alpha \leqslant 1$. A rectifiable path $c \colon I \subset \mathbb{R} \to X$ is an $(\alpha, \beta; \gamma)$*-local quasigeodesic* if for all $x, y \in I$,

$$|x - y| \leqslant \gamma \implies \alpha|x - y| - \beta \leqslant |c(x) - c(y)| \leqslant \frac{1}{\alpha}|x - y| + \beta.$$

If $\gamma = \infty$, we say that c is an (α, β)-global quasigeodesic or, simply, a *quasigeodesic.*

Definition 3.6 (Quasi-isometric embedding)**.** Let Y be a geodesic metric space. A map $c : Y \to X$ is an $(\alpha, \beta; \gamma)$*-local quasi-isometric embedding* if the restriction of c to each geodesic path in Y of length at most γ satisfies the preceding inequalities.

If $\gamma = \infty$, we say that c is an (α, β)-quasi-isometric embedding or a *quasi-isometric embedding.*

Theorem 3.7 (Stability of local quasigeodesics)**.** ([$\mathrm{Gr}_{\mathrm{HG}}$, 7.2.B], [CDP, Ch.3]) *Let $\alpha, \beta \in \mathbb{R}^+$. There exist constants $\gamma = \gamma_\delta(\alpha, \beta), D = D_\delta(\alpha, \beta) \in \mathbb{R}^+$ such that every $(\alpha, \beta; \gamma)$-local quasigeodesic in X is a D-quasiconvex $(\frac{\alpha}{2}, \beta)$-global quasigeodesic.*

In addition, $\gamma_\delta(\alpha, \beta) = \eta(\alpha, \delta) + 8\beta$ *and* $D_\delta(\alpha, \beta) = \varsigma(\alpha, \delta) + 8\beta$ *for some* η *and* ς *independent of* β*, and the function* D *satisfies* $\lim_{\delta \to 0, \beta \to 0} D_\delta(\alpha, \beta) = 0$.

More generally, if $c : Y \to X$ *is an* $(\alpha, \beta; \gamma)$*-local quasi-isometric embedding, then* c *is an* $(\frac{\alpha}{2}, \beta)$*-quasi-isometric embedding and* $c(Y)$ *is* D*-quasiconvex.* □

Remark 3.8 (Homogeneity of γ and D). By rescaling, the functions γ and D are determined by their values at $\delta = 1$. For instance, one can choose $D_x(\alpha, x\beta) = xD_1(\alpha, \beta)$.

3.2. **Hyperbolic groups: the very small cancellation theorem.** A group G is *hyperbolic* if there exists a discrete cocompact action of G by isometries on a proper hyperbolic space. An equivalent definition is that G is hyperbolic if its Cayley graph with respect to a finite generating set is a δ-hyperbolic metric space for some $\delta \geqslant 0$. In this case, we say that G is δ-hyperbolic.

Let G be a *non-elementary* (that is, with no cyclic subgroups of finite index) torsion-free δ-hyperbolic group generated by a finite set S. Let X be the universal cover of the van Kampen polyhedron associated to some presentation of (G, S). Each 2-cell of X is a regular Euclidean polygon with edges of length 1 (we assume that the presentation of G has no relations of length 1 and 2). Then X is a proper simply connected 2δ-hyperbolic space and the group G acts freely discretely and cocompactly on it. We have $Ca(G) \subset X$ and for any $g, h \in G$,

$$(*) \qquad |g - h|_X \leqslant |g - h|_{Ca(G)} \leqslant 2|g - h|_X.$$

Let Θ be a finite connected graph and m be a labelling of Θ by S. After choosing a base point, the labelling m induces a homomorphism $m_*\colon \pi_1(\Theta) \to G$ and a simplicial $\pi_1(\Theta)$-equivariant map c from the universal covering tree T (viewed with the induced labelling) of Θ to the Cayley graph $\mathrm{Ca}(G) \subset X$, sending the base point of T to the identity vertex of $\mathrm{Ca}(G)$. We denote by $Y \subset X$ the image of T under c and by H the subgroup $m_*(\pi_1(\Theta))$ in G. Observe that H acts freely on Y.

The following invariants of the actions of G on X and of H on Y will allow us to geometrize the small cancellation condition.

Definition 3.9. Let $g \in G$ and x_0 be a point in X.

(1) The *stable length* of g, denoted by $[g]_X$, is the limit

$$\lim_{n \to \infty} \frac{|g^n x_0 - x_0|}{n}$$

It does not depend on the choice of x_0.

(2) The *injectivity radius* of the action of H on $Y \subset X$ is

$$\rho = \min_{g \in H \setminus \{e\}} [g]_X$$

(3) The *length of the largest piece* is

$$\Delta = \max_{g \in G \setminus H} \mathrm{diam}\,(\text{projection of } gY \text{ onto } Y)$$

(4) The *geometric cancellation parameter* of the action of H on Y is

$$\lambda = \frac{\Delta}{\rho}$$

The next result is crucial for the construction. It specifies the conditions that guarantee the hyperbolicity of the quotient $\overline{G} = G/\langle\langle H\rangle\rangle$, where $\langle\langle H\rangle\rangle$ is the normal subgroup of G generated by H.

The main part of this result has been proved in [Gr$_{\mathrm{MCH}}$], see also [DG], except for claim (iii) which will be needed later on and claim (iv) which is given for completeness. We keep the above notation.

Theorem 3.10 (The very small cancellation theorem)**.** *Given $\alpha > 0$ and $r_0 > 4 \cdot 10^5$, there exist two constants λ_0 and δ_0 such that the following holds.*

Let $\beta \in \mathbb{R}$ and $D = D_\delta(\alpha, \beta)$, $\gamma = \gamma_\delta(\alpha, \beta)$ be the constants given by Theorem 3.7. Assume that $c\colon T \to Ca(G) \subset X$ is an $(\alpha, \beta; \gamma)$-local quasi-isometric embedding and the very small cancellation conditions are satisfied:

$$\frac{\delta}{\rho} \leqslant \delta_0, \frac{D}{\rho} \leqslant 10\delta_0, \text{ and } \lambda \leqslant \lambda_0.$$

Then

(i) *the quotient $\overline{G} = G/\langle\langle H\rangle\rangle$ is a non-elementary torsion-free hyperbolic group;*
(ii) *the induced map $B(e, \frac{r_0\,\rho}{1200 \sinh r_0}) \subset G \to \overline{G}$ is injective;*
(iii) *the map $f_m\colon \Theta \to \mathrm{Ca}(\overline{G})$ induced by the labelling satisfies*

$$(**) \qquad |f_m(x) - f_m(y)| \geqslant \frac{\rho}{\operatorname{diam}\Theta} \cdot \frac{r_0}{160 \cdot 2\pi \sinh r_0}\left(\frac{\alpha}{2}|x - y| - \beta\right)$$

(iv) *the hyperbolicity constant of $\mathrm{Ca}(\overline{G})$ can be estimated explicitly (in terms of $r_0, \rho, \operatorname{diam}\Theta$, and δ);*
(v) *the rational Euler characteristic satisfies $\chi(\overline{G}) = \chi(G) + b_1(H)$.*

We postpone the proof till Section 3.4. The usual $C'(\lambda)$ small cancellation theorem is a version of the previous statement for $\alpha = 1, \beta = \delta = 0$, and $\lambda \leqslant 1/6$.

3.3. **The cone off construction.** Let $Y \subset X$ be a subset connected by rectifiable paths. We consider the cylinder $C(Y)$ of Y as a length space, with the induced length metric. By definition, $C(Y)$ is invariant under the action of the group of isometries of X preserving Y.

Let $r_0 > 0$ be fixed and

$$\dot{C}(Y) = (C(Y) \times]0, r_0]) \cup \{o\}$$

be the topological cone[3] endowed with the length metric $ds^2 = dr^2 + \frac{\sinh^2 r}{\sinh^2 r_0} dx^2$.

[3]Our cone coincides with the ball of radius r_0 centered at o of the (-1)-cone over the scaled metric space $C(Y)/\sinh r_0$, see [BH, Ch.I.5] for introductory details on the cone construction.

Let us recall the definition of the curvature in the sense of Alexandroff, for more information on CAT(-1) spaces (and, more general, CAT(κ) spaces with κ a real number), see [BH, Part II].

Definition 3.11 (The CAT(-1) inequality)**.** A geodesic metric space X is a *CAT(−1) space* if all geodesic triangles $[x, y, z]$ in X satisfy the CAT(-1) *inequality*: for all $u, v \in [x, y, z]$, we have

$$|u - v| \leqslant |u^* - v^*|,$$

where u^*, v^* are images of u, v under the unique path-isometric map from $[x, y, z]$ to a hyperbolic triangle $[x^*, y^*, z^*] \subset \mathbb{H}^2_{-1}$ with the same sides as $[x, y, z]$.

For example, if Y is a tree, then $\dot{C}(Y)$ is a CAT(-1) space[4].

Definition 3.12 (The CAT($-1, \varepsilon$) inequality)**.** A geodesic metric space is a CAT($-1, \varepsilon$) *space* if all geodesic triangles in this space satisfy the CAT(-1) inequality up to a given constant $\varepsilon \geqslant 0$, that is, keeping the notation above:

$$|u - v| \leqslant |u^* - v^*| + \varepsilon.$$

All geodesic triangles in the hyperbolic plane $\mathbb{H}^2_{-1}$ are known to be δ-thin with $\delta = \ln(1 + \sqrt{2})$. It follows that every CAT($-1, \varepsilon$) space is δ-hyperbolic with $\delta = \ln(1 + \sqrt{2}) + \varepsilon$. Hence, such a space is 1-hyperbolic whenever ε is small enough, for example, if $\varepsilon = 10^{-10}$.

Observe that the CAT($-1, \varepsilon$) inequality involves only 5 points and geodesics connecting them. As the cylinder $C(Y)$ is 10δ-quasiconvex, it is $\varepsilon_0(\delta)$-hyperbolic with the hyperbolicity constant $\varepsilon_0(\delta) > 0$ such that $\lim_{\delta\to 0} \varepsilon_0(\delta) = 0$. This follows immediately from the fact that the inclusion map $C(Y) \hookrightarrow X$ is a quasi-isometry and Theorem 3.7. The lemma of approximation of a δ-hyperbolic space by a tree [CDP, Ch.8, Th.1] implies that a geodesic pentagon in $C(Y)$ can be approximated by a finite metric tree, up to $\varepsilon_1(\delta) > 0$ such that $\lim_{\delta\to 0} \varepsilon_1(\delta) = 0$. Since the cone over such a tree is a CAT(-1) space and the cone over the limit (of uniformly bounded metric spaces over a non-principal ultrafilter) is isometric to the limit of the cones, we immediately obtain (cf. [DG, Prop.3.2.8]):

Lemma 3.13. *There exists a function $\varepsilon(\delta)$ with $\lim_{\delta\to 0} \varepsilon(\delta) = 0$ such that the cone $\dot{C}(Y)$ satisfies the* CAT($-1, \varepsilon(\delta)$) *inequality in the $r_0/2$ neighbourhood of every point at distance at least $\frac{r_0}{2}$ from the apex o of the cone.* □

Let H be a subgroup of G acting discretely and cocompactly on X. Given an H-invariant quasiconvex subset $Y \subset X$, we consider the induced action of H on $\dot{C}(Y)$ and equip the corresponding quotient $\dot{C}(Y)/H$ with the natural length structure. Let $\widetilde{\rho} = \inf_{h\in H\setminus\{e\}, x\in X} |hx - x|$ denote the minimal displacement.

[4]By definitions or by Berestovskii's theorem [BH, Th. 3.14, p.188] which implies that the cone over a CAT(1) space is a CAT(-1) space.

Lemma 3.14. *There exists a function $\varepsilon(\delta)$ with $\lim_{\delta\to 0}\varepsilon(\delta) = 0$ such that if $\widetilde{\rho} \geqslant 2\pi \sinh r_0$, then the quotient $\dot{C}(Y)/H$ is a* $\mathrm{CAT}(-1, \varepsilon(\delta))$ *space.*

Proof. The quotient $\dot{C}(Y)/H$ of the cone is isometric to the cone over the quotient $C(Y)/H$, see [BH, Ex. I.5.22(3)]. The $\mathrm{CAT}(-1, \varepsilon)$ inequality involves only five points and we can apply the approximation lemma [CDP, Ch.8, Th.1] as above. Thus, it suffices to check that if $C(Y)$ is a tree, that is, $C(Y)/H$ is a graph, then $\dot{C}(Y)/H$ is a $\mathrm{CAT}(-1)$ space; but if $\widetilde{\rho} \geqslant 2\pi \sinh r_0$ (= the circumference of the hyperbolic circle of radius r_0) the graph $C(Y)/H$ is a $\mathrm{CAT}(1)$ space, cf. [BH, Ex. II.3.17]. Then, Berestovskii's theorem [BH, Th. 3.14, p.188] implies that the cone over such a graph $C(Y)/H$ is a $\mathrm{CAT}(-1)$ space, as required. □

3.4. **Proof of the very small cancellation theorem.** We keep the above notation and suppose that the hypotheses of the very small cancellation theorem are satisfied. We proceed as in $[\mathrm{Gr}]_{\mathrm{MCH}}$, see also [DG].

Normalization. First, we rescale the length structure on X: we divide the length metric on X by $\frac{\rho}{2\pi \sinh r_0}$ so that the injectivity radius ρ of the action of H on $Y \subset X$ remains bounded below by $2\pi \sinh r_0$ with respect to this normalized length structure.

Cone-off. Let $\dot{X} = X \cup_{g\in G/H} g\dot{C}(Y)$ be the the space obtained from X by attaching a copy of the cone $\dot{C}(Y)$ to each $gC(Y) \subset X$. We endow $\dot{X}$ with the natural length structure induced by the (rescaled) metric on X and the corresponding length structure on the cone $\dot{C}(Y)$.

Charts. Let U be the $\frac{r_0}{2}$–neighborhood of X in $\dot{X}$, that is,

$$U = X \cup_{g\in G/H} g\dot{C}_{r_0/2}(Y),$$

where $\dot{C}_{r_0/2}(Y)$ consists of all points of the cone $\dot{C}(Y) \subset \dot{X}$ at distance at least $r_0/2$ from the apex.

Let $V = \dot{C}(Y)/H$. The length structure on this space is naturally induced by the length structure on $C(Y) \subset X$ and the length structure on the cone $\dot{C}(Y)$. This space can also be viewed as a cone over $C(Y)/H$, see [BH, Ex.I.5.22(3)].

Local curvature. We use the following

Proposition 3.15. (cf.[$\mathrm{Gr}_{\mathrm{MCH}}$, 7.B],[DG, Theorem 5.2.5]) *Under the notation above, there exist constants δ_0' and λ_0 such that if $\delta < \delta_0', D < 10\delta_0', \lambda < \lambda_0$, the space U is a* $\mathrm{CAT}(-1, 10^{-10})$ *space in the $r_0/2$–neighbourhood of every point.*

On the other hand, by Lemma 3.14 (as, obviously, $\widetilde{\rho} \geqslant \rho$), there exists a constant δ_0'' such that if $\delta < \delta_0''$ the space V is a $\mathrm{CAT}(-1, 10^{-10})$ space, hence 1-hyperbolic.

Thus, U and V are 1-hyperbolic spaces at the $r_0/2$ neighbourhood of every point. Set $\delta_0 = \min\{\delta_0', \delta_0''\}$.

New van Kampen polyhedron. We have $G = \pi_1(X/G)$ and H is the image of $\pi_1(C(Y)/H)$ under the induced homomorphism $\pi_1(C(Y)/H) \to \pi_1(X/G)$. Let us consider $\Pi = X/G \cup \dot{C}(Y)/H$, obtained by gluing X/G and $\dot{C}(Y)/H$ along $C(Y)/H$ under the natural maps $C(Y)/H \hookrightarrow X/G$ and $C(Y)/H \hookrightarrow \dot{C}(Y)/H$. We equip this space with the normalized length structure on X/G and with the corresponding length structure on $\dot{C}(Y)/H$. By the van Kampen theorem, the fundamental group of $\Pi = X/G \cup \dot{C}(Y)/H$ is the group $\overline{G}$ we study.

Local-to-global. The space Π has an orbispace structure (defined for CW-complexes, see, for example, [BH, Ch. III, $\mathcal{G}$], [DG, 4.1.1]) given by two charts U and V defined above, with respect to natural maps $U \to U/G \subset \Pi$ and $V \xrightarrow{id} V \subset \Pi$.

Observe that $\dot{C}(Y)/H$ and $\dot{C}(Y)$ are isometric in the $2\pi \sinh \frac{r_0}{2}$ neighbourhood of every point at distance at least $r_0/2$ from the apex as the distance between any two points of $\dot{C}(Y)$ in the same H-orbit is at least $2\pi \sinh r_0$. Therefore, images of U and V in Π are compatible on their intersection. Thus, U and V do form an atlas on Π.

By construction, the ball of radius $r_0/4$ centered at any point of Π has a preimage either in U or in V. This means that the length structure on Π is $r_0/4$-*useful* [DG, 4.1.3]. Thus, Π is a compact $r_0/2$-local 1-hyperbolic orbispace and the following result can be applied.

Theorem 3.16 (Cartan-Gromov-Hadamard)**.** ([$\mathrm{Gr}_{\mathrm{MCH}}$],[DG, Theorem 4.3.1]) *Let $\delta > 0$ and $\sigma > 10^5\delta$. Let Π be a compact σ-local δ-hyperbolic orbispace. Suppose that for every $x \in \Pi$ there exists a σ-useful chart $(U, \varphi, \tilde{x})$. Then,*

(1) *Π is developable;*
(2) *The universal cover $\widetilde{\Pi}$ of Π at x is 200δ-hyperbolic;*
(3) *If $(U, \varphi, \tilde{x})$ is a σ-useful chart of a neighborhood of $x \in \Pi$ and x' is a preimage of x in the universal cover $\widetilde{\Pi}$, the developing map $(U, \tilde{x}) \to (\widetilde{\Pi}, x')$ is an isometric embedding of the ball $B(\tilde{x}, \sigma/10)$ centered at $\tilde{x} \in U$ of radius $\sigma/10$ onto its image.*

It follows that $\widetilde{\Pi}$ is 200-hyperbolic. Since $\overline{G}$ acts discretely and cocompactly on $\widetilde{\Pi}$, this shows the hyperbolicity of $\overline{G}$.

Local injectivity. By (3), the ball $B(\tilde{x}, \frac{r_0}{40}) \subset U$ maps isometrically onto its image in $\widetilde{\Pi}$. This means that if $h \in \langle\langle H \rangle\rangle$, then $|h - e|_{Ca(G)} \geqslant \frac{r_0}{40} \cdot \frac{\rho}{2\pi \sinh r_0}$ (e denotes the identity element). That is, the map $B(e, \frac{1}{4} \cdot \frac{r_0}{40} \cdot \frac{\rho}{2\pi \sinh r_0}) \subset G \hookrightarrow \overline{G}$ is injective. This shows (ii).

The quotient is non-elementary and torsion-free. By construction, $\overline{G}$ acts freely on $\widetilde{\Pi}$. Passing from X to the appropriate Rips complex [CDP, Ch.5], we can assume that $\widetilde{\Pi}$ is contractible. It follows from Smith theory (see, for example, Th. III.7.11 in [Br]) that $\overline{G}$ is torsion-free: If the group has torsion then it contains a cyclic p-group, but a finite p-group acting on a finite dimensional contractible CW-complex has a fixed point. If $\overline{G}$ is elementary, then it would be isomorphic to $\mathbb{Z}$, contradicting to (ii).

Let us now obtain estimate $(**)$ in (iii) and that of the hyperbolicity constant.

Quasi-isometry. Let $x, y \in \Theta$. Since c is a local quasi-isometric embedding, using (ii), we see that if $|f_m(x) - f_m(y)|_{Ca(\overline{G})} \leqslant \frac{1}{4} \cdot \frac{r_0}{40} \cdot \frac{\rho}{2\pi \sinh r_0}$, then

$$|f_m(x) - f_m(y)|_{Ca(\overline{G})} = |m(x) - m(y)|_{Ca(G)} \geqslant \frac{\alpha}{2}|x - y| - \beta.$$

On the other hand, if $|f_m(x) - f_m(y)|_{Ca(\overline{G})} > \frac{1}{4} \cdot \frac{r_0}{40} \cdot \frac{\rho}{2\pi \sinh r_0}$, then

$$|f_m(x) - f_m(y)|_{Ca(\overline{G})} > \frac{1}{4} \cdot \frac{r_0}{40} \cdot \frac{\rho}{\operatorname{diam} \Theta \cdot 2\pi \sinh r_0}|x - y|.$$

Summarizing,

$$|f_m(x) - f_m(y)|_{Ca(\overline{G})} \geqslant \frac{\rho}{\operatorname{diam} \Theta} \cdot \frac{r_0}{160 \cdot 2\pi \sinh r_0}\left(\frac{\alpha}{2}|x - y| - \beta\right)$$

as required in $(**)$.

Hyperbolicity constant. By construction, $Ca(\overline{G})$ is quasi-isometric to $\widetilde{\Pi}$. Since $\widetilde{\Pi}$ is 200-hyperbolic, in order to estimate the hyperbolicity constant of $Ca(\overline{G})$ it suffices to estimate the parameters of such a quasi-isometry.

Let $x, y \in \Theta$. First, we get a lower bound on the distance between their images $m(x)$ and $m(y)$ in the cone $\dot{C}(Y)/H$ through that in its base $C(Y)/H$ as follows. By definition of the length structure on the cone,

$$\begin{aligned} |m(x) - m(y)|_{\dot{C}(Y)/H} &= 2 \operatorname{arg\,sinh}\left(\sinh r_0 \cdot \sin\left\{\frac{2|m(x) - m(y)|_{C(Y)/H}}{\rho} \cdot \frac{\pi}{2}\right\}\right), \\ &\qquad \text{whenever } |m(x) - m(y)|_{C(Y)/H} \leqslant \frac{\rho}{2}; \\ |m(x) - m(y)|_{\dot{C}(Y)/H} &= 2r_0, \text{ otherwise.} \end{aligned}$$

In particular, $|m(x) - m(y)|_{\dot{C}(Y)/H} = 2r_0$ at the maximal value of $|m(x) - m(y)|_{C(Y)/H}$ which is at most $\operatorname{diam} \Theta \cdot \frac{2\pi \sinh r_0}{\rho} + 200$ (V is 1-hyperbolic, we refer to the 100-neighborhood in the definition of the cylinder $C(Y)$). As the right side function above is concave,

$$|m(x) - m(y)|_{\dot{C}(Y)/H} \geqslant \frac{2r_0}{\operatorname{diam} \Theta \cdot \frac{2\pi \sinh r_0}{\rho} + 200}|m(x) - m(y)|_{C(Y)/H}.$$

That is, denoting $\varpi = \frac{2r_0}{\operatorname{diam} \Theta \cdot \frac{2\pi \sinh r_0}{\rho} + 200}$, we have

$$(***) \qquad |m(x) - m(y)|_{\dot{C}(Y)/H} \geqslant \varpi |m(x) - m(y)|_{C(Y)/H}.$$

Next we use the following fact. Its proof is an exercise.

Lemma 3.17. *Let Z, Z_1, Z_2 be proper geodesic metric spaces such that Z is the union of Z_1 and Z_2 along their boundaries, that is, $Z = Z_1 \cup Z_2$ and $Z_1 \cap Z_2 = \partial Z_1 = \partial Z_2$. Let $p : [0,1] \to Z_2$ be a path with the endpoints $p(0), p(1) \in \partial Z_2$ having no other intersection points with ∂Z_2. Suppose that for any such p there exists a path p_1 in Z_1 with the same endpoints such that their lengths satisfy*

$$\ell_{Z_1}(p_1) \leqslant K \ell_{Z_2}(p)$$

for some constant $K > 0$. Then

$$|x - y|_{Z_1} \leqslant K|x - y|_Z.$$

We apply this result to $Z := \widetilde{\Pi}$, $Z_1 := X/\langle\langle H\rangle\rangle$, $Z_2 := \cup_{g\in G\setminus H}\dot{C}(Y)/H$. The hypotheses are satisfied as, by $(***)$, we have $\ell_{X/\langle\langle H\rangle\rangle}(p_1) \leqslant \varpi^{-1}\ell_{\cup_{g\in G\setminus H}\dot{C}(Y)/H}(p)$. Using the notation above, we obtain

Claim 1. $|g - h|_{X/\langle\langle H\rangle\rangle} \leqslant \varpi^{-1}|g - h|_{\widetilde{\Pi}}$.

On the other hand, the lemma applies to $Z := X/\langle\langle H\rangle\rangle$, $Z_1 := \left(\frac{2\pi \sinh r_0}{\rho}\right) Ca(\overline{G})$, $Z_2 := \overline{Z \setminus Z_1}$ as it follows from $(*)$ that $\ell_{Ca(\overline{G})}(p_1) \leqslant 2\ell_{Z_2}(p)$. This gives

Claim 2. $\frac{2\pi \sinh r_0}{\rho}|g - h|_{Ca(\overline{G})} \leqslant 2|g - h|_{X/\langle\langle H\rangle\rangle}$.

Now we are able to conclude. We have $\frac{\rho}{2\pi \sinh r_0}|g - h|_{\widetilde{\Pi}} \leqslant |g - h|_{Ca(\overline{G})}$ as the (scaled) graph $Ca(\overline{G})$ is embedded into $\widetilde{\Pi}$. On the other hand, by the above claims,

$$|g - h|_{\widetilde{\Pi}} \geqslant \varpi|g - h|_{X/\langle\langle H\rangle\rangle} \geqslant \frac{1}{2} \cdot \varpi \cdot \frac{2\pi \sinh r_0}{\rho}|g - h|_{Ca(\overline{G})}.$$

In other words, $\left(\frac{2\pi \sinh r_0}{\rho}\right) Ca(\overline{G}) \hookrightarrow \widetilde{\Pi}$ is a $\left(\frac{2}{\varpi}, 0\right)$-quasi-isometric embedding. According to Theorem 3.7, if $M = 200D_1\left(\frac{2}{\varpi}, 0\right)$ then every $\left(\frac{2}{\varpi}, 0\right)$-quasigeodesic triangle in $\widetilde{\Pi}$ is $(4M + 200)$-thin (we use 200-hyperbolicity of $\widetilde{\Pi}$ and the fact that a path in the M-neighborhood of a geodesic with the same endpoints contains this geodesic in its $2M$-neighborhood [CDP, Ch. 3, Lemme 1.11]). Therefore, every geodesic triangle in $\left(\frac{2\pi \sinh r_0}{\rho}\right) Ca(\overline{G})$ is $\frac{2}{\varpi}(4M + 200)$-thin. Thus, $\overline{G}$ is $\frac{\rho}{2\pi \sinh r_0} \cdot \frac{2}{\varpi}(4M + 200)$-hyperbolic.

Euler characteristic. As above, we can assume that $\widetilde{\Pi}$ is contractible and therefore is a model for $E\overline{G}$, the universal covering of the classifying space. Then the Euler characteristic formula follows immediately from the inclusion-exclusion principle. □

4. Renormalization and lacunary hyperbolic groups

The following notion, suggested by Gromov [$\mathrm{Gr}_{\mathrm{RWRG}}$], aims to capture the asymptotic geometry of the iterated small cancellation groups. It goes back to

Rips [R], Bowditch [B], Thomas and Velickovic [TV], and has been made precise and systematically studied by Ol'shanskii, Osin, and Sapir [OOS].

Definition 4.1 (Lacunary hyperbolic group)**.** A finitely generated group is said to be *lacunary hyperbolic* if there exists a sequence $(R_i)_{i\in\mathbb{N}}$ of radii such that $\lim_{i\to\infty}\frac{R_{i+1}}{R_i}=\infty$ and the balls $B(e,R_i)$ of its Cayley graph are δ_i-hyperbolic with $\delta_i=o(R_i)$.

Let $(\Theta_n;m_n)_{n\in\mathbb{N}}$ be a family of finite graphs and m_n a labelling of Θ_n by generators of a given non-elementary torsion-free hyperbolic group G_0; For $n\geqslant 0$, denote by G_{n+1} the quotient of G_n by the normal subgroup generated by $H_n=(m_n)_*\left(\pi_1(\Theta_n)\right)$. The group H_n acts freely on $Y_n\subset X_n=\mathrm{Ca}(G_n)$, the image of the universal covering tree T_n of Θ_n under the natural map $c_n\colon T_n\to Ca(G_n)$ induced by the labelling. We denote by ρ_n,Δ_n, and λ_n the corresponding geometric invariants of this action.

The next result provides a way to construct a lacunary hyperbolic group starting from $(\Theta_n;m_n)_{n\in\mathbb{N}}$.

Proposition 4.2. *Let $r_0>4\cdot 10^5$ and δ_0,λ_0 be constants provided by the very small cancellation theorem. Assume that G_n is δ_n-hyperbolic for $n\geqslant 0$.*

Suppose that there exist a constant α and a sequence β_n with $\lim_{n\to\infty}\beta_n=\infty$ such that for every $n\geqslant 0$, we have the following.

(i) *The sequence of graphs is lacunary: $\lim_{n\to\infty}\frac{\delta_n}{\rho_n}=0$ and $\beta_n\operatorname{diam}\Theta_n<\frac{r_0\rho_{n+1}}{1200\sinh r_0}$.*
(ii) *$\gamma_{\delta_n}(\alpha,\beta_n)\leqslant\frac{\rho_n}{2}$ and $\frac{D_{\delta_n}(\alpha,\beta_n)}{\rho_n}\leqslant\delta_0$.*
(iii) *$\lambda_n=\frac{\Delta_n}{\rho_n}\leqslant\lambda_0$.*
(iv) *$c_n:T_n\to\mathrm{Ca}(G_n)$ is an $(\alpha,\beta_n;\frac{\rho_n}{2})$-local quasi-isometric embedding.*

Then the direct limit G_* of groups G_n is a lacunary hyperbolic group. Moreover, the induced maps $f_{m_n}:\Theta_n\to\mathrm{Ca}(G_*)$ satisfy

$$|f_{m_n}(x)-f_{m_n}(y)|\geqslant\frac{\rho_n}{\operatorname{diam}\Theta_n}\cdot\frac{r_0}{160\cdot 2\pi\sinh r_0}\left(\frac{\alpha}{2}|x-y|-\beta_n\right)$$

Proof. Set $R_n=\frac{r_0\,\rho_n}{1200\sinh r_0}$ for $n\geqslant 0$. We have $\rho_n\leqslant\operatorname{girth}\Theta_n\leqslant 2\operatorname{diam}\Theta_n$. It follows from assumption (i) that $\lim_{n\to\infty}\frac{R_{n+1}}{R_n}=\lim_{n\to\infty}\frac{\rho_{n+1}}{\rho_n}=\infty$ and $\delta_n=o(R_n)$.

The above conditions (i)–(iv) imply the hypotheses of the very small cancellation theorem for G_n. Therefore, the map $B(e,\frac{r_0\,\rho_n}{1200\sinh r_0})\subset G_n\to G_{n+1}$ is injective for every $n\geqslant 0$. Since $R_n<R_{n+1}$ and $\operatorname{diam}\Theta_n<R_{n+1}$, the balls $B(e,R_n)$ in the Cayley graph of G_* are δ_n-hyperbolic and natural maps $\Theta_n\to\mathrm{Ca}(G_{n+1})$ induce the required embeddings $f_{m_n}\colon\Theta_n\hookrightarrow\mathrm{Ca}(G_*)$. □

In the next section, given a family $(\Theta_n)_{n\in\mathbb{N}}$ of graphs and a non-elementary hyperbolic group G_0, we shall see how to construct appropriate labellings of Θ_n by generators of G_0. The main idea is to give a sufficient condition on the asymptotic geometry of the sequence $(\Theta_n)_{n\in\mathbb{N}}$ so that a *random* labelling satisfies the hypothesis of Proposition 4.2 with a positive probability, arbitrary close to 1 as $n \to \infty$.

5. Random labellings and groups

5.1. **Random group associated to a graph.** Let G be a group generated by a set S of k generators and Θ be a finite graph. To each labelling m of Θ by generators from S, we associate the group $G(m)$ which is the quotient of G by the normal subgroup generated by the image under m_* of the fundamental group of Θ (cf. Definition 2.2, where G is the free group on S). The set of all possible labellings of Θ is naturally equipped with the normalized counting measure which assigns equal probabilities to all labellings: there are $(2k)^E$ of them, where E denotes the number of edges in Θ. Now, let $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ be a family of finite graphs. We consider the space Ω of all possible labellings of Θ as the product of the above probability spaces endowed with the probability product measure so that a labelling m of Θ is now considered as an alea $\omega \in \Omega$. Such an element $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ defines a sequence $(G_n(\omega))_{n\in\mathbb{N}}$ of groups, where $G_n(\omega) := G(\omega_n)$ is the group associated to a random labeling ω_n of the finite graph Θ_n.

The objective is to analyze the typical behavior of $G_n(\omega)$ as $n \to \infty$. The main question is the following: given appropriate a group G and a family Θ, does the group $G_n(\omega)$ have interesting properties for a randomly chosen element $\omega \in \Omega$ as $n \to \infty$? In this section we show (see Proposition 5.9 below) that if the initial group G is non-elementary torsion-free hyperbolic and the family $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ is *thin* enough (that is, parameters b and ξ_0 in Definition 5.3 are small enough) then, with asymptotic probability 1 as $n \to \infty$, a random labelling of Θ_n by generators of G satisfies the hypotheses of the very small cancellation theorem (Theorem 3.10).

5.2. **Asymptotic characteristics of graphs and groups.** Denote by $B(e, r)$ the ball of radius r at the identity $e \in G$ with respect to the word length metric associated to S and by $\#B(e, r)$ the cardinality of this ball.

Definition 5.1 (Growth entropy). The *entropy of the pair* (G, S) is the limit

$$\operatorname{ent}(G, S) = \lim_{r\to\infty} \frac{\ln \#B(e, r)}{r}.$$

The limit does exist by the submultiplicativity of the growth function $\#B(e, r)$. It is the logarithm of the *exponential growth rate* of the pair (G, S). Since $\#B(e, r)$ is majorized by $2k(2k-1)^{r-1}$ (which is the growth function of the free group of rank k with respect to a free generating set), the above limit satisfies

$$0 \leqslant \operatorname{ent}(G, S) \leqslant \ln(2k-1).$$

The group G is of exponential growth if and only if $\operatorname{ent}(G,S)>0$. In particular, $\operatorname{ent}(G,S)>0$ for all non-elementary hyperbolic groups G (however, it is an open question whether the entropy is *uniformly* bounded from zero for all hyperbolic groups, say, on two generators).

Let μ be the standard symmetric probability measure on G, that is, μ is equidistributed over S: the associated random walk is the simple random walk on G with the transition probabilities given by numbers

$$\mu(g^{-1}h)=\begin{cases}\frac{1}{2k} & \text{if } g^{-1}h\in S\cup S^{-1}\\ 0 & \text{otherwise}\end{cases}$$

The distribution after r steps is given by the convolution power $\mu^{*r}=\mu*\mu^{*(r-1)}$.

For $g\in G$, $\mu^{*r}(g)$ is the probability that a word of length r in the generators of G represents the element g. More generally, for a subset $B\subset G$, $\mu^{*r}(B)$ is the probability that a word of length r represents an element from B.

Definition 5.2 (Spectral radius)**.** The *spectral radius* of the simple random walk on G is the number $\kappa_{\text{reg}}=\kappa_{\text{reg}}(\mathrm{G},\mathrm{S})$ such that

$$\ln\kappa_{\text{reg}}=\limsup_{\mathrm{r}\to\infty}\frac{\ln\mu^{*\mathrm{r}}(\mathrm{e})}{\mathrm{r}}.$$

By a classical result of Kesten [Kes],

$$\ln\frac{\sqrt{2k-1}}{k}\leqslant\ln\kappa_{\text{reg}}\leqslant 0$$

with the right-hand equality if and only if G is amenable. Thus, the number $\ln\kappa_{\text{reg}}(\mathrm{G},\mathrm{S})$ is strictly negative whenever G is a non-elementary hyperbolic group.

Observe that[5]

$$\ln\kappa_{\text{reg}}(\mathrm{G},\mathrm{S})\geqslant-\frac{\operatorname{ent}(\mathrm{G},\mathrm{S})}{2}.$$

From now on let $\Theta=(\Theta_n)_{n\in\mathbb{N}}$ be a family of graphs of girth $\rho_n=\operatorname{girth}\Theta_n$ with $\lim_{n\to\infty}\rho_n=\infty$. Assume that vertices of Θ are of uniformly bounded degree. Given $t>0$, denote by $b_n(t)$ the number of distinct simple paths of length t in Θ_n.

Definition 5.3 (Thin family of graphs)**.** Let $b>0$ and $\xi_0\in]0,1/2[$. The family Θ is said to be b-*thin* if there exists a constant $C>0$ such that for all $1/2>\xi\geqslant\xi_0$ we have

$$b_n(\xi\rho_n)\leqslant C\exp\xi\rho_n b.$$

For instance, a union over $n\in\mathbb{N}$ of a constant number of disjoint circle graphs on n vertices is b-thin for all $b>0$.

[5]By Cauchy-Schwartz inequality, $1=\left(\sum_{g\in B(e,r)}\mu^{*r}(g)\right)^2\leqslant\#B(e,r)\sum_{g\in B(e,r)}\mu^{*r}(g)^2=\#B(e,r)\mu^{*2r}(e)$, whence the required inequality.

The notion of b-thinness allows a quantitative control on the number of new relators we add on each inductive step of the construction (that we alluded to in the introduction). If such a number at the n-th step was too large comparing with girth Θ_n, the resulting quotient group would collapse (and the inductive procedure would not success) like in Gromov's density model of random groups whenever the density parameter is strictly larger than $1/2$ [$\mathrm{Gr}_{\mathrm{AI}}$, Ch. 9].

In our arguments below we omit explicit estimates on the parameter ξ_0. An acute reader can choose ξ_0 much smaller than λ_0 provided by the very small cancellation theorem, see Subsection 5.5 and a constant $a > 0$ satisfying $\operatorname{card} \Theta_n \geqslant \exp\{a \operatorname{girth} \Theta_n\}$, given by the specific family we consider, see Theorem 7.4 (or just set $\xi_0 = 10^{-10}$).

5.3. **The entropy language.** Let $(\Omega_n)_{n\geqslant 0}$ be a sequence of probability spaces and A_n be a measurable subset of Ω_n for $n \geqslant 0$.

Definition 5.4 (Entropy of the sequence of events)**.** The *entropy of the sequence* $A = (A_n)_{n\geqslant 0}$ is the number

$$\operatorname{ent}(A) = \limsup_{n\to\infty} \frac{\ln \Pr(A_n)}{n}.$$

Assume now that for every $n \geqslant 0$ the space Ω_n is a given finitely generated group G endowed with a symmetric probability measure μ as above.

Example 1. For $\varepsilon \geqslant 0$, consider the event "a word of length n in the generators of G represents an element of length $< \varepsilon n$". The entropy of this sequence is

$$\operatorname{ent}(\varepsilon) = \limsup_{n\to\infty} \frac{\ln \mu^{*n}(B(e, \varepsilon n))}{n}.$$

In particular, $\operatorname{ent}(0) = \ln \kappa_{\mathrm{reg}}$.

Example 2. For $\varepsilon \geqslant 0$, consider the event "a word of length n in the generators of G represents an element at distance $< \varepsilon n$ from a given element $g \in G$". Then the entropy of this sequence is at most

$$\overline{\operatorname{ent}}(\varepsilon) = \limsup_{n\to\infty} \frac{\ln \sup_{g\in G} \mu^{*n}(B(g, \varepsilon n))}{n}.$$

Example 3. For $\varepsilon \geqslant 0$, consider the event "two independently chosen words w_1 and w_2 of length $\max\{|w_1|, |w_2|\} = n$ satisfy $|w_1 - gw_2| + |g| < \varepsilon n$ for some $g \in G$ such that $w_1 \neq g^{-1}w_2 g$". Then the entropy of this sequence is at most

$$\operatorname{ent}^+(\varepsilon) = \limsup_{n\to\infty} \frac{\ln \Pr\left(\sup_{g\in G, w_1\neq g^{-1}w_2 g, \max\{|w_1|,|w_2|\}=n}(|w_1 - gw_2| + |g|) < \varepsilon n\right)}{n}.$$

Lemma 5.5. *We have the following general inequalities.*

(i) $\operatorname{ent}(\varepsilon) \leqslant \overline{\operatorname{ent}}(\varepsilon) \leqslant \operatorname{ent}(0) + \varepsilon \operatorname{ent}(G, S) \leqslant \ln \kappa_{\mathrm{reg}} + \varepsilon \ln(2\mathrm{k} - 1)$;
(ii) $\operatorname{ent}^+(\varepsilon) \leqslant \operatorname{ent}(0) + 2\varepsilon \operatorname{ent}(G, S)$.

Proof. (i) First and third inequalities are obvious from the definitions. Now for an element $g \in G$, we see that

$$\mu^{*n}(B(g, \varepsilon n)) = \sum_{h \in B(g,\varepsilon n)} \mu^{*n}(h) \leqslant \#B(e, \varepsilon n)\sqrt{\mu^{*2n}(e)} \tag{1}$$

as for any element $h \in G$ we have $(\mu^{*n}(h))^2 \leqslant \mu^{*2n}(e)$. (Indeed, if w and w' are two words representing h then $w(w')^{-1}$ represents the identity e.) This immediately implies the second inequality.

(ii) For a fixed w_1 of length at most n, the probability that a word w_2 of length at most n satisfies $w_2 = h_1 w_1 h_2$ with $|h_1| + |h_2| < \varepsilon n$ for some $h_1, h_2 \in G$ is at most

$$\mu^{*n}(h_1 w_1 h_2) \sum_{0 \leqslant t \leqslant \varepsilon n} \#B(e, t)\#B(e, \varepsilon n - t) \leqslant$$
$$\varepsilon n \#B^2(e, \varepsilon n)\sqrt{\mu^{*2n}(e)}.$$

Thus, $\mathrm{ent}^+(\varepsilon) \leqslant \mathrm{ent}(0) + 2\varepsilon\,\mathrm{ent}(G, S)$. $\square$

The above condition $|h_1| + |h_2| < \varepsilon n$ means that words w_1 and w_2 have a *large* "common" part. Such an intuition comes from the specific case when w_1 and w_2 would label paths in a tree, see Figure 1.

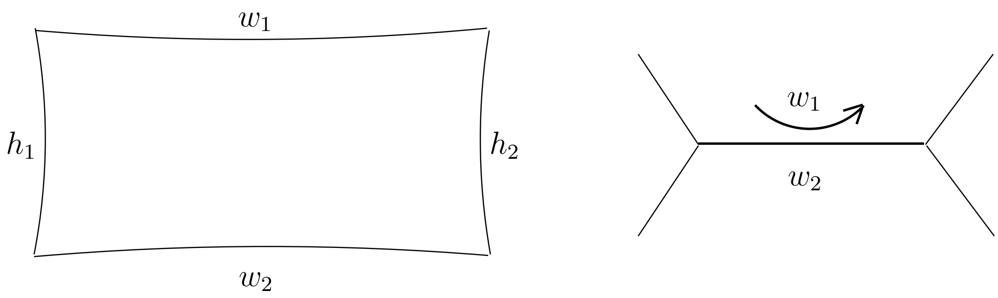

FIGURE 1

Definition 5.6 (Rare events)**.** A sequence $A = (A_n)_{n \geqslant 0}$ of events is said to be *rare* if $\mathrm{ent}(A) \neq 0$.

In other words, a sequence $A = (A_n)_{n \geqslant 0}$ of events is rare (or the complement of A is *very probable*) if the probability of A_n is at most $\exp \beta n$ whenever $\beta \in]\,\mathrm{ent}(A), 0[$ for n large enough.

Example 4.

(i) If G is a non-amenable group, then $\ln \kappa_{\mathrm{reg}} \neq 0$. Therefore, if A_n is the event "a word of length n in the generators of G represents the identity element", then $\mathrm{ent}(A) = \mathrm{ent}(0) = \ln \kappa_{\mathrm{reg}} \neq 0$. That is, A is rare.

(ii) By the previous lemma, the sequence of events of Example 1 is rare whenever $\varepsilon < \frac{-\operatorname{ent}(0)}{\ln(2k-1)} \leqslant \frac{-\operatorname{ent}(0)}{\operatorname{ent}(G,S)}$. In other words, for n and r such that $\frac{r}{n} < -\frac{\operatorname{ent}(0)}{\operatorname{ent}(G,S)}$, it is rare that a word of length $n \gg r$ in the generators of G represents a group element of length $< r$.

Let $(a_n)_{n\geqslant 0}$ be a non-decreasing sequence of positive real numbers and a its entropy, that is, $a = \limsup_{n\to\infty} \frac{\ln a_n}{n}$. Then for every $\epsilon > 0$ there exists a constant $M > 0$ such that for every $n \geqslant 0$ we have

$$a_n \leqslant M \exp{(a+\epsilon)n}.$$

In particular, inequalities (1), applied to the ball $B(e,r)$, show that for every $\epsilon > 0$ there exists $M > 0$ such that for every $r \geqslant 0$ and an integer $n > 0$ we have

$$\mu^{*n}(B(e,r)) \leqslant M \exp\{(\operatorname{ent}(0)+\epsilon)n + (\operatorname{ent}(G,S)+\epsilon)r\}. \tag{2}$$

5.4. **Quasigeodesic labellings.** Let (G,S) be a non-elementary torsion-free hyperbolic group, $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ be b-thin (with respect to a given constant $\xi_0 \in]0,1/2[$, see Definition 5.3), and ω_n denote a random labellings of Θ_n.

Lemma 5.7 (Random labelling is quasigeodesic)**.** *Let $\varepsilon > 0$ and $l(n)$ be a function with $\lim_{n\to\infty} l(n) = \infty$. Suppose that $b + \ln\kappa_{\text{reg}} < 0$ and $\varepsilon < \frac{|b+\ln\kappa_{\text{reg}}|}{\ln(2k-1)}$. Then the probability that there exists a simple path w of length $\|w\|$ between $\xi_0\rho_n$ and $\frac{\rho_n}{2}$ in Θ_n such that $|\omega_n(w)|_G < \varepsilon\|w\| - l(\rho_n)$ is at most*

$$B\exp\{-l(\rho_n)\operatorname{ent}(G,S)\}$$

for $n \gg 1$ and some constant $B > 0$.

In particular, if $b + \ln\kappa_{\text{reg}} < 0$ and $\alpha = \frac{|b+\ln\kappa_{\text{reg}}|}{2\ln(2k-1)}$, then, with asymptotic probability 1 as $n \to \infty$, a random labelling ω_n of Θ_n satisfies, for every simple path w of length $\|w\| \leqslant \frac{\rho_n}{2}$ in Θ_n,

$$|\omega_n(w)|_G \geqslant \alpha\|w\| - l(\rho_n) - \alpha\xi_0\rho_n.$$

Proof. The probability that there exists a simple path w of length $\|w\|$ in Θ_n such that $\xi_0\rho_n \leqslant \|w\| \leqslant \frac{\rho_n}{2}$ and $|\omega_n(w)|_G < \varepsilon\|w\| - l(\rho_n)$ is at most

$$\sum_{\xi_0\rho_n \leqslant \|w\| \leqslant \frac{\rho_n}{2}} b_n(\|w\|)\mu^{*\|w\|}\left(B(e,\varepsilon\|w\| - l(\rho_n))\right) \leqslant$$

$$\sum_{\xi_0\rho_n \leqslant \|w\| \leqslant \frac{\rho_n}{2}} b_n(\|w\|)\#B(e,\varepsilon\|w\| - l(\rho_n))\sqrt{\mu^{*2\|w\|}(e)} \leqslant$$

$$\sum_{\|w\| \leqslant \frac{\rho_n}{2}} M\exp\{(b' + \operatorname{ent}(0) + \varepsilon\operatorname{ent}(G,S))\|w\|\}\exp\{-l(\rho_n)\operatorname{ent}(G,S)\}$$

for $M > 0$ and $b' > b$ such that $b' + \ln \kappa_{\mathrm{reg}} < 0$ and $b' + \ln \kappa_{\mathrm{reg}} + \varepsilon \ln(2\mathrm{k} - 1) < 0$ (cf. (2)). It follows that the series

$$\sum_{\|w\| \leqslant \frac{\rho_n}{2}} M \exp\{(b' + \operatorname{ent}(0) + \varepsilon \operatorname{ent}(G,S))\|w\|\}$$

converges. Therefore, the required probability is at most

$$\frac{M \exp\{-l(\rho_n) \operatorname{ent}(G,S)\}}{1 - \exp\{(b' + \operatorname{ent}(0) + \varepsilon \operatorname{ent}(G,S))\}}, \tag{3}$$

whence the result for n large enough so that

$$l(\rho_n) > \frac{\ln\left(\frac{M}{1-\exp\{(b' + \operatorname{ent}(0) + \lambda \operatorname{ent}(G,S))\}}\right)}{\operatorname{ent}(G,S)}.$$

□

5.5. **Small cancellation condition.** Recall that T_n denotes the universal covering tree of Θ_n with respect to a fixed base point. A labelling of $\Theta = (\Theta_n)_{n \in \mathbb{N}}$ by generators of G induces naturally a map $c\colon T_n \to \mathrm{Ca}(G)$. By Lemma 5.7, if $b + \ln \kappa_{\mathrm{reg}} < 0$ and

$$\alpha = -\frac{b + \ln \kappa_{\mathrm{reg}}}{2 \ln(2k - 1)},$$

then, with asymptotic probability 1 as $n \to \infty$, a random labelling ω_n of Θ induces a local quasi-isometric embedding. More precisely, let us define a sequence β_n by

$$\beta_n := l(\rho_n) + \alpha \xi_0 \rho_n \leqslant 2\alpha \xi_0 \rho_n \ll \rho_n, \tag{4}$$

where $l(n)$ is a function with $\lim_{n \to \infty} l(n) = \infty$. Then, with asymptotic probability 1 as $n \to \infty$, we have

$$|c(u) - c(v)| \geqslant \alpha |u - v| - \beta_n$$

for all $u, v \in T_n$ at distance $\leqslant \frac{\rho_n}{2}$ and the map c given by a random labelling ω_n.

Assume now, in addition, that $l(n) = o(n)$. Then, for large enough integer $n > 0$ and small enough $\xi_0 > 0$ we have $\gamma_\delta(\alpha, \beta_n) \leqslant \frac{\rho_n}{2}$. Indeed, it follows, for example, from an explicit formula for $\gamma_\delta(\alpha, \beta_n)$ obtained in [CDP, Ch.3] (see also Theorem 3.7) whenever $\lim_{n \to \infty} \rho_n = \infty$. Therefore, by Theorem 3.7, applied to an $(\alpha, \beta_n; \frac{\rho_n}{2})$-local quasigeodesic in $\mathrm{Ca}(G)$ for n large enough, the Hausdorff distance from any geodesic path $[c(u), c(v)]$ to the path $c([u, v])$, and vise versa, is at most $D = D_\delta(\alpha, \beta_n)$. In fact, in view of the first inequality of (4) and an explicit estimate for $D_\delta(\alpha, \beta_n)$ [CDP, Ch.3], we have $D = 8\alpha \xi_0 \rho_n + \varsigma(\alpha, \delta)$ for large enough integer n and a function $\varsigma(\alpha, \delta)$. Finally, we consider n large enough so that

$$8D + 700\delta \leqslant 200\alpha \xi_0 \rho_n. \tag{5}$$

Fix $\lambda > 0$. Let us evaluate the length of the largest piece in the context of small cancellation theory with the geometric cancellation parameter λ, see Section 3. By Definition 3.9 and Proposition 3.4,

$$\Delta(\Theta_n) \leqslant \max_{g \in G \setminus (\omega_n)_*(\pi_1(\Theta_n))} \operatorname{diam}\left(C(gc(T_n)) \cap C(c(T_n))\right) + 2D + 100\delta,$$

where $C(c(T_n))$ denotes the cylinder of the image $c(T_n)$ in $\mathrm{Ca}(G)$, see Definition 3.1. As the image is D-quasiconvex, the cylinder $C(c(T_n))$ is at distance of at most $D + 100\delta$ from $c(T_n)$. Then

$$\Delta(\Theta_n) \leqslant \max_{g \in G \setminus (\omega_n)_*(\pi_1(\Theta_n))} \operatorname{diam}\left(c(T_n)^{D+100\delta} \cap gc(T_n)^{D+100\delta}\right) + 2D + 100\delta,$$

where $c(T_n)^{D+100\delta}$ denotes the $(D + 100\delta)$-neighbourhood of $c(T_n)$ in $\mathrm{Ca}(G)$.

Assume that for an element $g \in G$ we have

$$\operatorname{diam}\left(c(T_n)^{D+100\delta} \cap gc(T_n)^{D+100\delta}\right) + 2D + 100\delta > \lambda\rho_n.$$

Then there exists two simple paths $[u_1, v_1]$ and $[u_2, v_2]$ in Θ_n and an element $h \in G$ such that, after passing to the lifts of u_i, v_i in T_n, $|c(u_i) - c(v_i)| \geqslant \lambda\rho_n - 4D - 300\delta$ for $i = 1, 2$, $|h| = |c(u_1) - c(u_2)| \leqslant 2D + 200\delta$, $|c(v_1) - c(v_2)| \leqslant 2D + 200\delta$, and the distance from every point of the path $c([u_1, v_1])$ to a point of $c([u_2, v_2])$ is at most $4D + 300\delta$, see Figure 2.

Up to a possible extension of paths $c([u_1, v_1])$ and $c([u_2, v_2])$ at their endpoints $c(v_1)$ and $c(v_2)$ by simple paths of length at most $2D + 150\delta$ and up to cutting the paths into two parts, we find two *disjoint* simple paths in Θ_n labelled by two (independent) words w_1 and w_2 of length $|w_i| \leqslant \lambda\rho_n$ for $i = 1, 2$, and an element $h \in G$ such that

$$|w_i| \geqslant \frac{\lambda\rho_n}{2}, \quad |c(w_1) - hc(w_2)| + |h| < 8D + 700\delta.$$

That is, by choosing (once again) a small enough ξ_0 (so that, $\xi_0 < \lambda/200$), we obtain, for large enough n given by (5), $\frac{\lambda\rho_n}{2} \leqslant |w_i| \leqslant \lambda\rho_n$ and

$$|c(w_1) - hc(w_2)| + |h| < 8D + 700\delta \leqslant 200\alpha\xi_0\rho_n < \alpha\lambda\rho_n.$$

By Lemma 5.5 (ii), the event "two independent randomly chosen words of length $\leqslant \lambda\rho_n$ satisfy this inequality" is a *rare* event of entropy at most $\lambda(\operatorname{ent}(0) + 2\alpha \operatorname{ent}(G, S))$.

The number of pairs of distinct simple paths in Θ_n of length in the interval $[\frac{\lambda\rho_n}{2}, \lambda\rho_n]$ is of order $\exp 2b\lambda\rho_n$. Thus, the entropy of the event $\Delta(\Theta_n) > \lambda\rho_n$ is at most $2\lambda(\operatorname{ent}(0) + \alpha \operatorname{ent}(G, S) + b)$ which, for $\alpha = \frac{|b + \operatorname{ent}(0)|}{2\ln(2k-1)}$, gives

Lemma 5.8 (Random labelling satisfies the small cancellation condition)**.** *Let $\lambda > 0$ and $b > 0$. If $b + \operatorname{ent}(0) < 0$, then, with asymptotic probability 1 as $n \to \infty$, a random labelling of Θ_n satisfies $\Delta(\Theta_n) \leqslant \lambda\rho_n$.* □

Summarizing, if Θ is thin enough, Lemmas 5.7 and 5.8 provide a labelling that satisfies the hypotheses of the very small cancellation theorem, see Theorem 3.10.

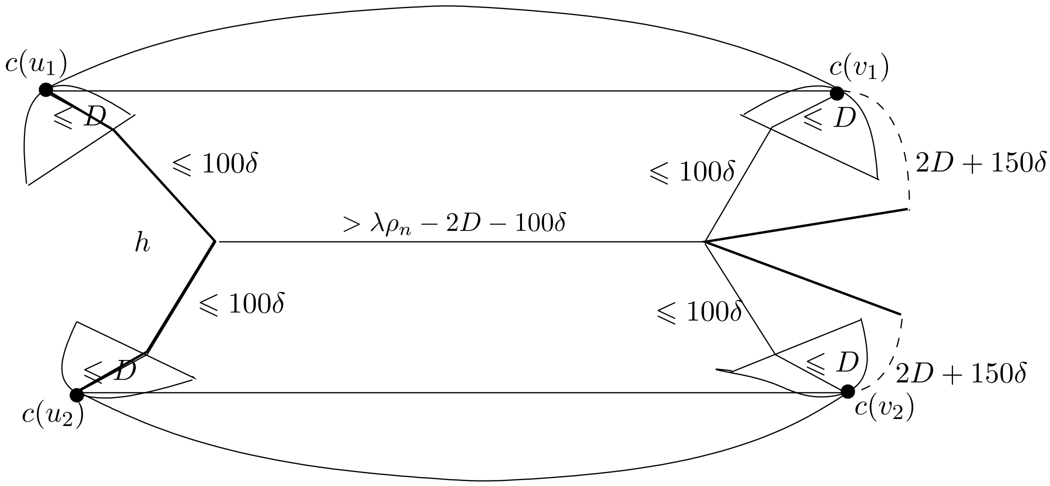

FIGURE 2

Proposition 5.9. *Let (G, S) be a non-elementary torsion-free hyperbolic group generated by a set S of k generators, $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ be a b-thin family with $\lim_{n\to\infty} \operatorname{girth} \Theta_n = \infty$, and $l(n)$ a function such that $\lim_{n\to\infty} l(n) = \infty$ and $l(n) = o(n)$. Let $\kappa > 0$ be a constant satisfying $\kappa_{\mathrm{reg}}(\mathrm{G}, \mathrm{S}) < \kappa$.*

If $b + \ln \kappa < 0$ and $\alpha = \frac{|\,b+\ln\kappa\,|}{2\ln(2k-1)}$, then, for every fixed $\lambda > 0$, with asymptotic probability 1 as $n \to \infty$, a random labelling of Θ_n induces an $(\alpha, \beta_n; \frac{\operatorname{girth}\Theta_n}{2})$-local quasi-isometric embedding $T_n \hookrightarrow Ca(G)$ and satisfies $\Delta(\Theta_n) \leqslant \lambda \operatorname{girth} \Theta_n$. □

6. RANDOM HYPERBOLIC AND LACUNARY HYPERBOLIC GROUPS

We are ready now to analyze typical properties of groups associated to random labellings ω of Θ. First, we obtain such properties for groups $G_n(\omega)$ as $n \to \infty$, and then for their direct limits for an appropriately chosen recursive subset $I = \{n_i\} \subseteq \mathbb{N}$.

6.1. **Random hyperbolic groups.**

Theorem 6.1. *Let (G, S) be a non-elementary torsion-free hyperbolic group generated by a set S of k generators, $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ be a b-thin family of graphs with $\lim_{n\to\infty} \operatorname{girth} \Theta_n = \infty$, and $l(n)$ a non-decreasing function with $\lim_{n\to\infty} l(n) = \infty$ and $l(n) = o(n)$. Let $\kappa > 0$ be a constant satisfying $\kappa_{\mathrm{reg}}(\mathrm{G}, \mathrm{S}) < \kappa$.*

If $b + \ln \kappa < 0$ and $\alpha = \frac{|\,b+\ln\kappa\,|}{2\ln(2k-1)}$, then the following properties hold with asymptotic probability 1 as $n \to \infty$ for a random labelling ω of Θ.

(i) *The group $G_n(\omega)$ is non-elementary torsion-free hyperbolic.*
(ii) *For a fixed $R > 0$, the restriction of the canonical projection $G \twoheadrightarrow G_n(\omega)$ onto the ball $B(e, R) \subset G$ is injective.*

(iii) *The map* $f_{\omega_n} : \Theta_n \to \mathrm{Ca}(G_{n+1}(\omega))$ *induced by the labelling satisfies*

$$|f_{\omega_n}(x) - f_{\omega_n}(y)| \geqslant \frac{\operatorname{girth}\Theta_n}{\operatorname{diam}\Theta_n} \cdot \frac{\alpha r_0}{320 \cdot 2\pi \sinh r_0} \left(\frac{\alpha}{2}|x-y| - \beta_n\right)$$

Proof. It follows from Proposition 5.9, Definition 3.9 of the stable length and Theorem 3.7 that, with asymptotic probability 1 as $n \to \infty$ for a random labelling ω of Θ, we have $\operatorname{girth}\Theta_n \geqslant \rho_n \geqslant \frac{\alpha}{2}\operatorname{girth}\Theta_n$, where ρ_n denotes the injectivity radius of the action of $H_n = (\omega_n)_*(\pi_1(\Theta_n))$. It suffices to apply the very small cancellation theorem (Theorem 3.10) and the fact that $l(n)$ is a non-decreasing function. ☐

A detailed proof of the next result has been given by Silberman [S], see also the Bourbaki seminar by Ghys [Gh].

Theorem 6.2. *Suppose that* Θ *is an* expander *(see Definition 7.3). Under the hypothesis of Theorem 6.1, with asymptotic probability* 1 *as* $n \to \infty$, *the group* $G_n(\omega)$ *satisfies Kazhdan's property (T) (see Definition 7.1).* ☐

Theorem 6.2, together with Proposition 7.2 below, give a uniform upper bound on the spectral radii of the simple random walk on all random quotients of (G, S) and hence allow us to *iterate* the construction. An alternative way is to start with a non-elementary torsion-free hyperbolic group which is already Kazhdan, for instance, with a co-compact lattice in $Sp(n,1), n \geqslant 2$.

6.2. **Random lacunary hyperbolic groups.** Let $I \subseteq \mathbb{N}$ be an infinite sequence of integers. We denote by $G_I(\omega)$ the quotient of G by the normal subgroup generated by the images under $\omega_* = ((\omega_n)_*)_{n\in I}$ of the fundamental groups of all $\Theta_n, n \in I$.

Theorem 6.3. *Let* (G, S) *be a non-elementary torsion-free hyperbolic group generated by a set* S *of* k *generators,* $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ *be a* b*-thin family of graphs with* $\lim_{n\to\infty}\operatorname{girth}\Theta_n = \infty$, *and* $l(n)$ *a non-decreasing function with* $\lim_{n\to\infty} l(n) = \infty$ *and* $l(n) = o(n)$. *Let* $\kappa > 0$ *be a constant satisfying* $\kappa_{\mathrm{reg}}(\mathrm{G},\mathrm{S}) < \kappa$.

If $b + \ln\kappa < 0$, *and* $\alpha = \frac{|b+\ln\kappa|}{2\ln(2k-1)}$, *then, for any given* $p \in (0,1)$, *there exists an infinite sequence of integers* $I = \{n_i\} \subset \mathbb{N}$ *such that with probability at least* p, *the group* $G_I(\omega)$ *is infinitely presented, of spectral radius at most* κ, *and the image of the graph* Θ_{n_i} *in the Cayley graph of* $G_I(\omega)$ *satisfies*

$$|f_{\omega_{n_i}}(x) - f_{\omega_{n_i}}(y)| \geqslant \frac{\operatorname{girth}\Theta_{n_i}}{\operatorname{diam}\Theta_{n_i}} \cdot \frac{\alpha r_0}{320 \cdot 2\pi \sinh r_0} \left(\frac{\alpha}{2}|x-y| - \beta_{n_i}\right),$$

where $\beta_{n_i} = l(\operatorname{girth}\Theta_{n_i}) + \alpha\xi_0 \operatorname{girth}\Theta_{n_i}$.

Proof. We proceed as in the proof of Theorem 6.1 but iterate the argument and apply Proposition 4.2 instead of the very small cancellation theorem. This is possible as for large enough integer $n > 0$ and small enough $\xi_0 > 0$ we have $\gamma_{\delta_n}(\alpha, \beta_n) \leqslant \frac{\rho_n}{2}$ and $\frac{D_{\delta_n}(\alpha,\beta_n)}{\rho_n} \leqslant \delta_0$. Indeed, it suffices to have $\lim_{n\to\infty}\frac{\rho_n}{\delta_n} = \infty$, use

$\lim_{n\to\infty}\rho_n = \infty$ and, for example, explicit formulae for $\gamma_{\delta_n}(\alpha,\beta_n)$ and $D_{\delta_n}(\alpha,\beta_n)$ given in [CDP, Ch.3], see Theorem 3.7. □

7. Expanders, property (T) and construction of the monster

7.1. **Property (T) and co-growth.** Let G be a group generated by a finite set S of cardinality k. The *Markov operator* is the element

$$M = \frac{1}{2k}\sum_{s\in S}\left(s+s^{-1}\right)$$

of the group ring $\mathbb{Q}G$.

If $\pi\colon G\to\mathcal{U}(\mathcal{H})$ is a unitary representation of G and $x\in\mathcal{H}$, then

$$Mx = \frac{1}{2k}\sum_{s\in S}\left(\pi(s)x+\pi(s^{-1})x\right).$$

Abusing notation, we denote by M both the Markov operator and its image under π. The operator M is symmetric with spectrum *spec*(M) in $[-1,1]$.

Definition 7.1 (Kazhdan's property (T))**.** A group G has *Kazhdan's property (T)* (or G is Kazhdan) if there exists $\kappa=\kappa(G,S)\in(0,1)$ such that for any unitary representation π of G

$$spec(M)\subseteq[-1,\kappa]\cup\{1\}.$$

A beautiful account of the theory of Kazhdan groups can be found in a recent monograph [BHV]. The following result is well-known.

Proposition 7.2. *Let G be a group with Kazhdan's property (T), S a set of group generators, and κ is the Kazhdan constant of the pair (G,S). Then every* infinite *quotient $(\overline{G},\overline{S})$ of (G,S) satisfies $\kappa_{\mathrm{reg}}(\overline{\mathrm{G}},\overline{\mathrm{S}})\leqslant\kappa$.*

Proof. Let us consider the natural action of G on the space $L^2(\overline{G})$. Since $\overline{G}$ is infinite the constant functions are not in $L^2(\overline{G})$. We have $\langle M^n x,x\rangle\leqslant\kappa^n\langle x,x\rangle$. Applying this to the Dirac function $f=\delta_e$ at the identity, we obtain $Mf=\frac{1}{2k}\sum_{a\in S^{\pm1}}f(a)$, then

$$M^n f = \frac{1}{(2k)^n}\sum_{\text{words } w \text{ of length } n} f(w),$$

and

$$M^n\delta_e = \frac{\text{number of words of length } n \text{ representing the identity } e}{\text{number of all words of length } n} = \mu^{*n}(e),$$

that is, $\mu^{*n}(e)^{1/n}\leqslant\kappa$, whence $\kappa_{\mathrm{reg}}(\overline{\mathrm{G}},\overline{\mathrm{S}})\leqslant\kappa$. □

7.2. **Expanders: definition and examples.** Let (V, E) be a finite connected graph. The combinatorial Laplace operator $\Delta \colon \ell^2(V) \to \ell^2(V)$ is given by the quadratic form

$$\langle f, \Delta f \rangle := \sum_{(u,v)\in E} |f(u) - f(v)|^2.$$

It is a positive semidefinite operator. Its smallest eigenvalue is zero and the corresponding eigenfunction is the constant function. Let λ_1 denote the smallest non-zero eigenvalue.

Definition 7.3 (Expander)**.** A family $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ of finite connected graphs is an *expander* if the following properties hold:

(i) vertices of Θ have uniformly bounded degree;
(ii) the number of vertices in Θ_n tends to infinity as $n \to \infty$;
(iii) $\lambda_1(\Theta_n) \geqslant \lambda_\infty > 0$ uniformly over $n \in \mathbb{N}$ for some constant λ_∞.

We consider Θ endowed with a metric d on the disjoint union of Θ_n's which coincides with the standard edge-path metric on each Θ_n and such that $d(x_n, x_{n'}) > n + n'$ whenever $x_n \in \Theta_n, x_{n'} \in \Theta_{n'}$, $n \neq n'$.

In order to construct the monster we shall apply the theorem of random lacunary hyperbolic groups to a *specific* family $\Theta = (\Theta_n)_{n\in\mathbb{N}}$, that is, a family with $\lim_{n\to\infty} \operatorname{girth}\Theta_n = \infty$, a uniformly bounded ratio $\frac{\operatorname{diam}\Theta_n}{\operatorname{girth}\Theta_n}$, and a small enough thinness $b(\Theta)$. Unfortunately, an easy available expander, that is, a random graph, has small girth. Therefore, we choose an appropriate family among several explicit expanders. For instance, we take graphs $\Theta_q = X^{p,q}$ which are the Cayley graphs of the projective general linear group $PGL_2(q)$ over the field of q elements for a particular set $S_{p,q}$ of $(p+1)$ generators, where p and q are distinct primes congruent to 1 modulo 4 with the Legendre symbol $\left(\frac{p}{q}\right) = -1$.

Theorem 7.4. ([M, LPS], cf. [DSV, V]) *Let p be fixed.*

(i) *The graph Θ_q is $p+1$ regular on $N = q(q^2-1)$ vertices.*
(ii) $\operatorname{girth}\Theta_q \geqslant 4\log_p q - \log_p 4$.
(iii) *The family $(\Theta_q)_{q\,\mathrm{prime}}$ is a family of Ramanujan graphs.*

In particular, $\lim_{q\to\infty} \operatorname{girth}\Theta_q = \infty$ and the family $(\Theta_q)_{q\,\mathrm{prime}}$ is an expander. By a standard result, known as the Expander mixing lemma, the graph Θ_q is of diameter $O(\log N)$. Thus, the ratio $\frac{\operatorname{diam}\Theta_q}{\operatorname{girth}\Theta_q}$ is uniformly bounded over primes q.

Lemma 7.5. *The family $(\Theta_q)_{q\ prime}$ is b-thin for some (finite) constant $b > 0$.*

Proof. Given $\xi \geqslant \xi_0$, let us estimate the number of distinct simple paths of length $\xi\rho_q$ in Θ_q, where $\rho_q = \operatorname{girth}\Theta_q$. Since graphs are $p+1$ regular the number of such paths at a fixed vertex is of order $p^{\xi\rho_q}$. On the other hand, the number of vertices is of order $p^{\operatorname{diam}\Theta_q} \leqslant p^{C\rho_q} \leqslant p^{\frac{C\xi\rho_q}{\xi_0}}$ with a constant $C > 0$ as the ratio

$\frac{\operatorname{diam}\Theta_q}{\operatorname{girth}\Theta_q}$ is uniformly bounded over primes q. Therefore, the number of paths we consider is at most $B\exp\xi\,\rho_q b$, where $b=\left(1+\frac{C}{\xi_0}\right)\ln p$ and $B>0$. $\square$

Given a non-elementary torsion-free hyperbolic group G and $b>0$ from the preceding lemma, we could have $b+\ln\kappa_{\mathrm{reg}}>0$. That is, the theorem of random lacunary hyperbolic groups cannot be applied. The next result provides a way to surmount this obstacle (the proof is immediate from the definitions).

Lemma 7.6. *Let $\Theta=(\Theta_n)_{n\in\mathbb{N}}$ be a family of graphs. If $\Theta^{(j)}$ denotes the graph obtained by subdivision of each edge of Θ into j new edges, then*

$$\lambda_1(\Theta_n^{(j)})\geqslant\frac{\lambda_1(\Theta_n)}{j^2}\quad\textit{and}\quad b(\Theta^{(j)})\leqslant\frac{b(\Theta)}{j},$$
$$\operatorname{girth}\Theta_n^{(j)}=j\ \operatorname{girth}\Theta_n\quad\textit{and}\quad\operatorname{diam}\Theta_n^{(j)}=j\ \operatorname{diam}\Theta_n.$$

$\square$

7.3. **The monster.** We are now ready to apply the theorem of random lacunary hyperbolic groups, see Theorem 6.3, to a non-elementary torsion-free hyperbolic group G generated by a set S of k generators and to a specific expander Θ from the preceding section: a constant $\kappa>0$ is given by Theorem 6.2 (or by Kazhdan's constant of (G,S) whenever G satisfies Kazhdan's property T) and $\alpha=\frac{|\,b+\ln\kappa\,|}{2\ln(2k-1)}$.

Theorem 7.7. *Let $0<p<1$. There exists an infinite sequence of integers $I=\{n_i\}\subset\mathbb{N}$ such that, with probability at least p, the group $G_I(\omega)$ is infinite and the image of the graph Θ_{n_i} in the Cayley graph of $G_I(\omega)$ satisfies*

$$|f_{\omega_{n_i}}(x)-f_{\omega_{n_i}}(y)|\geqslant\frac{\operatorname{girth}\Theta_{n_i}}{\operatorname{diam}\Theta_{n_i}}\cdot\frac{\alpha r_0}{320\cdot 2\pi\sinh r_0}\left(\frac{\alpha}{2}|x-y|-\beta_{n_i}\right),$$

where $\beta_{n_i}=l(\operatorname{girth}\Theta_{n_i})+\alpha\xi_0\operatorname{girth}\Theta_{n_i}$. $\square$

This group is called the *Gromov monster*: it contains a coarse image of an expander.

It is worth noticing that one can easily merge this construction with that of Tarski monster by Ol'shanskii (see, for example, [Olsh, Ch. 9, § 28.1]) and obtain a Tarski monster coarsely containing an expander. On the other hand, the construction can be made recursive, in a non-probabilistic way, providing a recursively presented group with the above properties. Indeed, the construction of the expander above is recursive, all labellings can be enumerated, and the geometric small cancellation condition can be checked in an algorithmic way. Using the Higman embedding theorem [LS, Th. IV.7.3], the resulting lacunary hyperbolic group can be embedded into a *finitely presented* group.

7.4. **No coarse embedding into a Hilbert space.** The following fundamental observation is due to Enflo [E], see also [Gh].

Theorem 7.8. *Let $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ be an expander. Suppose that there is an 1-Lipschitz map from Θ to a Hilbert space $\mathcal{H}$. Then there exist two sequences of points x_n, y_n in Θ_n at a distance at least $\ln(\operatorname{card}\Theta_n)$ with images at a uniformly bounded distance in $\mathcal{H}$.*

Corollary 7.9. *Let X be a graph, $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ be an expander with $\operatorname{card}\Theta_n \geqslant \exp\{a \operatorname{girth}\Theta_n\}$ for some fixed $a > 0$, and l a non-decreasing function with $\lim_{n\to\infty} l(n) = \infty$ and $l(n) = o(n)$. There exists a constant $\xi_0 > 0$ satisfying the following. Suppose that there exists a simplicial map $f : \Theta \to X$ and a constant $A > 0$ such that for all $x, y \in \Theta_n$,*

$$|f(x) - f(y)| \geqslant A\left(\frac{\alpha}{2}|x-y| - l(\operatorname{girth}\Theta_n) - \alpha\xi_0 \operatorname{girth}\Theta_n\right).$$

Then there is no coarse embedding of X into a Hilbert space.

Proof. Otherwise, we could have $\frac{\alpha}{2}\ln(\operatorname{card}\Theta_n) \leqslant C + l(\operatorname{girth}\Theta_n) + \alpha\xi_0 \operatorname{girth}\Theta_n$ for some constant $C > 0$, contradicting the hypothesis whenever $\xi_0 < \frac{a}{2}$. ☐

Corollary 7.10. *Gromov's monster admits no coarse embedding into a Hilbert space.* ☐

More generally, an expander cannot be coarsely embedded into any uniformly convex Banach space with unconditional basis [Oz] (neither into any ℓ^p with $1 \leqslant p < \infty$, see [Roe, Prop. 11.30]). It follows that Gromov's monster admits no such an embedding (cf. 7.6 below).

7.5. **Variant of Kapovich.** Using an argument of Misha Kapovich [K], one can construct a finitely generated group with no coarse embedding into a Hilbert space and, in addition, with the fixed point property on all buildings and all symmetric spaces. In order to do this, in the random group construction above, one starts with the hyperbolic group G_0 being a co-compact lattice of $\mathrm{Sp}(n,1)$. The super-rigidity of this group implies that every infinite quotient of G_0, hence the monster, satisfies this fixed point property. More results in this vein can be found in [NS].

7.6. **Around the Baum-Connes conjecture.** Following a remarkable result of Yu [Yu] on the coarse Baum-Connes conjecture, Higson [Hig] established the injectivity of the Baum-Connes homomorphism with coefficients (hence the Novikov conjecture on the homotopy invariance of higher signatures) for every discrete group endowed with an amenable action on a Hausdorff compact space (or, in other words, for every C^*-exact[6] group). This result was generalized to all groups with coarse embeddings into a Hilbert space [STY].

[6]A group G is C^*-*exact* if the spatial tensor product by the reduced C^*-algebra $C^*_r(G)$ of the group preserves short exact sequences of C^*-algebras. A finitely generated C^*-exact group is known to have a coarse embedding into a Hilbert space [A]. The converse is a well-known open problem.

Gromov's random group construction is actually the only known producing a finitely generated (and even finitely presented) group with no coarse embedding into a Hilbert space. Moreover, the following result is true.

Theorem 7.11. [HLS] *Let G be a finitely generated group. If the Cayley graph of G contains a coarsely embedded expander, then G does not satisfy the Baum-Connes conjecture with coefficients.*

Corollary 7.12. *Gromov's monster does not satisfy the Baum-Connes conjecture with coefficients.*

Note that the Novikov conjecture does hold for this group as it holds for every direct limit of hyperbolic groups, see [STY] and [Ro, Prop. 2.4]. Except for this monster, there is no other known finitely generated groups which are not C^*-exact.

Kasparov and Yu proved the Novikov conjecture for every finitely generated group having a coarse embedding into a uniformly convex Banach space [KY]. On the other hand, V. Lafforgue constructed an expander with no such an embedding [Laf]. Unfortunately, his expander is of small girth so Gromov's construction cannot be immediately applied to produce a new monster group.

The large girth condition plays also a crucial role in a recent work of Willet and Yu [WY1, WY2], who obtained a refinement of Theorem 7.11.

7.7. **Further applications.** Gromov's monster appears to be a notable example in other deep results. For instance, it is strongly relevant to the study of asphericity problems. The Borel conjecture is known for direct limits of hyperbolic groups and hence for Gromov's monster [L, Remark 4.12]). The geometric method of Delzant and Gromov [DG] we applied to prove the very small cancellation theorem (Theorem 3.10) has been further developed by Coulon in a more general context of the small cancellation theory for rotating families of groups. He used it to build new aspherical polyhedra [C]. Finitely presented aspherical groups and closed aspherical manifolds with extreme properties have been recently produced by Sapir [Sap], and again Gromov's monster plays its exceptional part in the construction.

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