

ON APPROXIMATION PROPERTIES OF SEMIDIRECT PRODUCTS OF GROUPS

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ABSTRACT. Let \mathcal{R} be a class of groups closed under taking (split) extensions with finite kernel and fully residually \mathcal{R} -groups. We prove that \mathcal{R} contains all (split) { finitely generated residually finite }-by- \mathcal{R} groups. It follows that a split extension with a finitely generated residually finite kernel and a surjective quotient is surjective. This remained unknown even for direct products of a surjective group with the integers \mathbb{Z} .

Sur les propriétés d'approximation des produits semi-directs des groupes

RRÉSUMÉ: Soit \mathcal{R} une classe de groupes fermée par rapport aux extensions (scindées) avec un noyau fini et par rapport aux groupes multi-résiduellement \mathcal{R} . Nous montrons que \mathcal{R} contient toutes les extensions (scindées) de type { finiment engendré résiduellement fini }-par- \mathcal{R} . Nous obtenons en corollaire qu'une extension scindée avec un noyau finiment engendré résiduellement fini et un quotient surjonctif est surjonctive. Cela restait inconnu, même pour les produits directs d'un groupe surjonctif avec les entiers \mathbb{Z} .

The concept of approximation is one of the most fundamental in science. In nowadays geometric group theory it refers to algebraic approximations such as residual finiteness, to approximations in the space of marked groups such as local embeddability into a given class of groups, e.g. [VG97, CG05], or to metric approximations such as soficity and hyperlinearity [PES08, CL15].

A current intensive study of sofic and hyperlinear groups, motivated by a variety of deep results on these wide – and still mysterious: is there a non sofic/hyperlinear group? – classes of groups, has given rise to a number of questions on group-theoretical properties both of these recently discovered groups and of their fundamental predecessors. One of such a major question is whether or not a given class of groups is preserved under taking (split or, in other words, semidirect) extensions.

Every semidirect extension of a residually finite group with a finitely generated residually finite kernel is residually finite by an elegant result of Mal'cev [MAL56]. A more general is a class of groups locally embeddable into finite ones (briefly, LEF-groups). There exist examples of semidirect extensions of LEF-groups which are not LEF [VG97].

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With regard to metric approximations, the only known and positive results on extensions are on (not necessarily split) extensions by an amenable group [PES08, CL15].

Our main result is the following general theorem which can be then applied to many groups, including all those just mentioned, and, in particular, to surjunctive groups (see the definition below), which came to light after Gromov's spectacular proof of the surjunctivity of all sofic groups [GRO99].

Theorem 1. *Assume that \mathcal{R} is a class of groups such that*

- *fully residually \mathcal{R} -groups belong to \mathcal{R} and*
- *(split) extension of \mathcal{R} -groups with finite kernel belong to \mathcal{R} .*

Let G be a (split) extension with a finitely generated residually finite kernel K and a quotient group Q in \mathcal{R} :

$$1 \rightarrow K \hookrightarrow G \twoheadrightarrow Q \rightarrow 1.$$

Then G belongs to \mathcal{R} .

Proof. Take a finite subset $S \subseteq G$. Define $F := S^{-1}S$ and choose a finite index subgroup $K_0 \trianglelefteq K$ such that $F \cap K_0 = \{e\}$. Since K_0 is of finite index and K is finitely generated one can find a finite index subgroup $K_1 \trianglelefteq K_0$ which is characteristic, that is, $\text{Aut}(K)$ -invariant. Then, for $N := K/K_1$ (note that, by construction, K_1 is normal in G , and hence, in K), the group $G_1 := G/K_1$ has the following properties:

- (1) S injects into G_1 ;
- (2) G_1 is a (split) extension of Q with finite kernel N .

Denote the quotient map by $\pi: G \twoheadrightarrow G_1$. Let $s_1, s_2 \in S$. If $\pi(s_1) = \pi(s_2)$, then $\pi(s_1^{-1}s_2) = e$. However, $s_1^{-1}s_2 \in S^{-1}S \cap \ker(\pi) = F \cap K_1 \subseteq F \cap K_0 = \{e\}$. Thus, $s_1 = s_2$ and we obtain (1). For (2), we observe that the quotient G_1/N is isomorphic to $G/K \cong Q$ and N is finite by construction. In addition, a splitting homomorphism $G/K \rightarrow G$ induces a splitting homomorphism $G_1/N \cong (G/K_1)/(K/K_1) \rightarrow G/K_1$.

By (2) and the hypothesis on \mathcal{R} , G_1 belongs to \mathcal{R} . As a consequence, since S is an arbitrary finite subset of G , the group G is fully residually \mathcal{R} . Thus, G belongs to \mathcal{R} . \square

Lemma 2. *Assume that \mathcal{R} is a class of groups such that*

- *direct products of a finite and of an \mathcal{R} -group belong to \mathcal{R} and*
- *all finite index subgroups of \mathcal{R} -groups belong to \mathcal{R} .*

Then any semidirect extension with finite kernel of an \mathcal{R} -group belongs to \mathcal{R} .

Proof. Let G be a split extension with finite kernel of an \mathcal{R} -group. Then G admits a retraction $\varphi: G \rightarrow H$ on its finite index subgroup which belongs to \mathcal{R} . Consider the homomorphism $G \hookrightarrow \text{Sym}(G/H) \times H: g \mapsto (fH \mapsto gfH, \varphi(g))$. It is injective and the image has finite index in $\text{Sym}(G/H) \times H$ which belongs to \mathcal{R} . Thus, G belongs to \mathcal{R} . \square

Let us apply our theorem to groups mentioned above. We begin with surjunctive groups.

The concept of surjunctivity was introduced by Gottschalk [GOT73] in topological dynamics in 1973. Applied to actions of a discrete group G , it can be viewed as an analogue of co-Hopf property of groups and of Artinian modules.

Definition 3. A continuous dynamical system (X, G) is called *surjunctive* if every continuous injective map $f: X \rightarrow X$ commuting with the action of G is surjective.

Definition 4. Given a finite alphabet Σ the associated *Bernoulli shift* is the space Σ^G of Σ -valued functions on G .

Definition 5. A group G is called *surjunctive* if the Bernoulli shift Σ^G is surjunctive for any finite alphabet Σ .

All sofic groups are surjunctive [GRO99, WEI00] and the class of sofic groups is the largest known class of surjunctive groups. It is still unknown whether every group is surjunctive and whether there exists a non-sofic surjunctive group. In particular, it is not clear whether there exists a group G such that $\{0, 1\}^G$ is surjunctive but $\{0, 1, 2\}^G$ is not.

A very few affirmative results are known regarding the class of surjunctive groups:

- Finite groups are surjunctive [CSC10, Proposition 3.1.3].
- Every subgroup of a surjunctive group is surjunctive [CSC10, Proposition 3.2.1].
- Locally surjunctive groups are surjunctive [CSC10, Proposition 3.2.2]. In particular, $S_\infty = \bigcup_{n \rightarrow \infty} S_n$, the group of finitely supported permutations of \mathbf{N} , is surjunctive.
- Fully residually surjunctive groups are surjunctive [CSC10, Lemma 3.3.4]. In particular, Abelian groups and free groups are surjunctive.

Lemma 6. *Virtually surjunctive groups are surjunctive. In particular, semidirect extension with finite kernel of a surjunctive group is surjunctive.*

Proof. Assume that $H \leq G$ is of finite index and H is surjunctive. Then we have $\Sigma^G = \left(\Sigma^{G/H}\right)^H$, as H -spaces, and every G -invariant map is H -invariant. This yields the claim. \square

It is unknown whether a product of two surjunctive groups is surjunctive [WEI00]. The question is open both for direct and free products. A particular case of free products with group of integers \mathbf{Z} is open as well.

The concept of surjunctivity extends to a more broad context where the space Σ^G of Σ -valued functions on G is considered for Σ being an object in a *surjunctive category* \mathcal{C} . This naturally leads to the notion of *\mathcal{C} -surjunctive group* [CSC13]. For instance, the above definition of surjunctive groups corresponds to \mathcal{C} being the category of finite

sets and if \mathcal{C} is the category of finite-dimensional vector spaces over an arbitrary field one obtains the notion of linear surjunctivity [CSC10, Chapter 8.14].

Without going into technical details (precise definitions are given in [CSC13]) a surjunctive category is a category where every injective \mathcal{C} -endomorphism (self-map) is surjective. Let us give examples of surjunctive categories (see [CSC13, Example 7.3] and references therein):

- the category of finite sets,
- the category of finite-dimensional vector spaces over an arbitrary field,
- the category of left Artinian modules over an arbitrary ring, of finitely generated left modules over an arbitrary left-Artinian ring,
- the category of affine algebraic sets over an arbitrary uncountable algebraically closed field, or
- the category of compact topological manifolds.

Let us note that all the categories above are closed under taking finite direct products (cf. property (FP) in [CSC13, Section 3.1]).

Our Lemma 6 remains valid for the class of \mathcal{C} -surjunctive groups with \mathcal{C} any surjunctive category closed under taking finite direct products; in particular, for any \mathcal{C} from the list above.

We extend Malcev's result who considered the case of \mathcal{R} being the class of residually finite groups [MAL56].

Theorem 7. *Let \mathcal{R} denote one of the following classes of groups:*

- *residually amenable groups,*
- *groups locally embeddable into finite groups (that is, LEF-groups),*
- *initially subamenable groups (in other words, LEA-groups),*
- *sofic groups,*
- *surjunctive groups, or*
- *\mathcal{C} -surjunctive groups, where \mathcal{C} is a surjunctive category closed under taking finite direct products.*

Then any semidirect extension of a group from class \mathcal{R} with residually finite, finitely generated kernel belongs to \mathcal{R} .

Proof. Fully residually surjunctive groups are surjunctive by [CSC10, Lemma 3.3.4]. A straightforward generalization of the proof of [CSC10, Lemma 3.3.4] shows that fully residually \mathcal{C} -surjunctive groups are \mathcal{C} -surjunctive. Analogous statements for other classes are obvious.

To check if an arbitrary semidirect extension with finite kernel of an \mathcal{R} -group belongs to \mathcal{R} , we apply Lemma 2 in case of residually amenable, LEF, or LEA groups, Lemma 6 in case of surjunctive groups and also, as mentioned above, in case of \mathcal{C} -surjunctive groups, as well as [CSC10, Proposition 7.5.14] in case of sofic groups.

Thus, all the discussed classes satisfy the hypothesis of Lemma 2 and, therefore, that of the split case of Theorem 1. \square

Question 1. Let G be a non-split extension with a finite kernel and a surjunctive (resp. \mathcal{C} -surjunctive) quotient. Is G surjunctive (resp. \mathcal{C} -surjunctive)?

A positive answer to this question, combined with the non-split case of Theorem 1, will imply that all such extensions with residually finite, finitely generated kernel are surjunctive (resp. \mathcal{C} -surjunctive). An analogous question for residually amenable, LEF, or LEA groups has a negative answer (since there exist extensions with a finite kernel and a residually finite quotient which have Kazhdan's property (T), and therefore are not LEA, e.g. [DEL78, MIL79]). It remains open for sofic and hyperlinear groups, c.f. [ABFSG19, Question 4.1]. In a particular instance when the kernel is a finite cyclic group, a positive answer to Question 1 would imply the surjunctivity of Deligne's non-residually finite central extensions [DEL78], of its SL_n analogues [MIL79] and of its recent arithmetic analogues [HIL19].

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