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The SQ-universality and residual properties of relatively hyperbolic groups ☆

G. Arzhantseva [a], A. Minasyan [a,*], D. Osin [b]

[a] *Université de Genève, Section de Mathématiques, 2-4 rue du Lièvre, Case postale 64, 1211 Genève 4, Switzerland*
[b] *NAC 8133, Department of Mathematics, The City College of the City University of New York, Convent Ave. at 138th Street, New York, NY 10031, USA*

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Abstract

In this paper we study residual properties of relatively hyperbolic groups. In particular, we show that if a group G is non-elementary and hyperbolic relative to a collection of proper subgroups, then G is SQ-universal.

Keywords: Relatively hyperbolic groups; SQ-universality

1. Introduction

The notion of a group hyperbolic relative to a collection of subgroups was originally suggested by Gromov [9] and since then it has been elaborated from different points of view [3,5,6,20]. The class of relatively hyperbolic groups includes many examples. For instance, if M is a complete finite-volume manifold of pinched negative sectional curvature, then $\pi_1(M)$ is hyperbolic with respect to the cusp subgroups [3,6]. More generally, if G acts isometrically and properly discontinuously on a proper hyperbolic metric space X so that the induced action of G

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* Corresponding author.
E-mail addresses: goulnara.arjantseva@math.unige.ch (G. Arzhantseva), aminasyan@gmail.com (A. Minasyan), denis.osin@gmail.com (D. Osin).

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on ∂X is geometrically finite, then G is hyperbolic relative to the collection of maximal parabolic subgroups [3]. Groups acting on $CAT(0)$ spaces with isolated flats are hyperbolic relative to the collection of flat stabilizers [13]. Algebraic examples of relatively hyperbolic groups include free products and their small cancellation quotients [20], fully residually free groups (or Sela's limit groups) [4], and, more generally, groups acting freely on $\mathbb{R}^n$-trees [10].

The main goal of this paper is to study residual properties of relatively hyperbolic groups. Recall that a group G is called *SQ-universal* if every countable group can be embedded into a quotient of G [24]. It is straightforward to see that any SQ-universal group contains an infinitely generated free subgroup. Furthermore, since the set of all finitely generated groups is uncountable and every single quotient of G contains (at most) countably many finitely generated subgroups, every SQ-universal group has uncountably many non-isomorphic quotients. Thus the property of being SQ-universal may, in a very rough sense, be considered as an indication of "largeness" of a group.

The first non-trivial example of an SQ-universal group was provided by Higman, Neumann and Neumann [11], who proved that the free group of rank 2 is SQ-universal. Presently many other classes of groups are known to be SQ-universal: various HNN-extensions and amalgamated products [7,14,23], groups of deficiency 2 [2], most $C(3)\,\&\,T(6)$-groups [12], etc. The SQ-universality of non-elementary hyperbolic groups was proved by Olshanskii in [18]. On the other hand, for relatively hyperbolic groups, there are some partial results. Namely, in [8] Fine proved the SQ-universality of certain Kleinian groups. The case of fundamental groups of hyperbolic 3-manifolds was studied by Ratcliffe in [22].

In this paper we prove the SQ-universality of relatively hyperbolic groups in the most general settings. Let a group G be hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ (called *peripheral subgroups*). We say that G is *properly hyperbolic relative to* $\{H_\lambda\}_{\lambda\in\Lambda}$ (or G is a *PRH group* for brevity), if $H_\lambda \neq G$ for all $\lambda \in \Lambda$. Recall that a group is *elementary*, if it contains a cyclic subgroup of finite index. We observe that every non-elementary PRH group has a unique maximal finite normal subgroup denoted by $E_G(G)$ (see Lemmas 4.3 and 3.3 below).

Theorem 1.1. *Suppose that a group G is non-elementary and properly relatively hyperbolic with respect to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then for each finitely generated group R, there exists a quotient group Q of G and an embedding $R \hookrightarrow Q$ such that*:

(1) *Q is properly relatively hyperbolic with respect to the collection $\{\psi(H_\lambda)\}_{\lambda\in\Lambda} \cup \{R\}$ where $\psi : G \to Q$ denotes the natural epimorphism*;
(2) *For each $\lambda \in \Lambda$, we have $H_\lambda \cap \ker(\psi) = H_\lambda \cap E_G(G)$, that is, $\psi(H_\lambda)$ is naturally isomorphic to $H_\lambda/(H_\lambda \cap E_G(G))$.*

In general, we cannot require the epimorphism ψ to be injective on every H_λ. Indeed, it is easy to show that a finite normal subgroup of a relatively hyperbolic group must be contained in each infinite peripheral subgroup (see Lemma 4.4). Thus the image of $E_G(G)$ in Q will have to be inside R whenever R is infinite. If, in addition, the group R is torsion-free, the latter inclusion implies $E_G(G) \leqslant \ker(\psi)$. This would be the case if one took $G = F_2 \times \mathbb{Z}/(2\mathbb{Z})$ and $R = \mathbb{Z}$, where F_2 denotes the free group of rank 2 and G is properly hyperbolic relative to its subgroup $\mathbb{Z}/(2\mathbb{Z}) = E_G(G)$.

Since any countable group is embeddable into a finitely generated group, we obtain the following.

Corollary 1.2. *Any non-elementary PRH group is SQ-universal.*

Let us mention a particular case of Corollary 1.2. In [7] the authors asked whether every finitely generated group with infinite number of ends is SQ-universal. The celebrated Stallings theorem [25] states that a finitely generated group has infinite number of ends if and only if it splits as a non-trivial HNN-extension or amalgamated product over a finite subgroup. The case of amalgamated products was considered by Lossov who provided the positive answer in [14]. Corollary 1.2 allows us to answer the question in the general case. Indeed, every group with infinite number of ends is non-elementary and properly relatively hyperbolic, since the action of such a group on the corresponding Bass–Serre tree satisfies Bowditch's definition of relative hyperbolicity [3].

Corollary 1.3. *A finitely generated group with infinite number of ends is SQ-universal.*

The methods used in the proof of Theorem 1.1 can also be applied to obtain other results:

Theorem 1.4. *Any two finitely generated non-elementary PRH groups G_1, G_2 have a common non-elementary PRH quotient Q. Moreover, Q can be obtained from the free product $G_1 * G_2$ by adding finitely many relations.*

In [17] Olshanskii proved that any non-elementary hyperbolic group has a non-trivial finitely presented quotient without proper subgroups of finite index. This result was used by Lubotzky and Bass [1] to construct representation rigid linear groups of non-arithmetic type thus solving in negative the Platonov Conjecture. Theorem 1.4 yields a generalization of Olshanskii's result.

Definition 1.5. Given a class of groups $\mathcal{G}$, we say that a group R is *residually incompatible with* $\mathcal{G}$ if for any group $A \in \mathcal{G}$, any homomorphism $R \to A$ has a trivial image.

If G and R are finitely presented groups, G is properly relatively hyperbolic, and R is residually incompatible with a class of groups $\mathcal{G}$, we can apply Theorem 1.4 to $G_1 = G$ and $G_2 = R * R$. Obviously, the obtained common quotient of G_1 and G_2 is finitely presented and residually incompatible with $\mathcal{G}$.

Corollary 1.6. *Let $\mathcal{G}$ be a class of groups. Suppose that there exists a finitely presented group R that is residually incompatible with $\mathcal{G}$. Then every finitely presented non-elementary PRH group has a non-trivial finitely presented quotient group that is residually incompatible with $\mathcal{G}$.*

Recall that there are finitely presented groups having no non-trivial recursively presented quotients with decidable word problem [15]. Applying the previous corollary to the class $\mathcal{G}$ of all recursively presented groups with decidable word problem, we obtain the following result.

Corollary 1.7. *Every non-elementary finitely presented PRH group has an infinite finitely presented quotient group Q such that the word problem is undecidable in each non-trivial quotient of Q.*

In particular, Q has no proper subgroups of finite index. The reader can easily check that Corollary 1.6 can also be applied to the classes of all torsion (torsion-free, Noetherian, Artinian, amenable, etc.) groups.

2. Relatively hyperbolic groups

We recall the definition of relatively hyperbolic groups suggested in [20] (for equivalent definitions in the case of finitely generated groups see [3,5,6]). Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a fixed collection of subgroups of G (called *peripheral subgroups*), X a subset of G. We say that X is a *relative generating set of* G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$ if G is generated by X together with the union of all H_λ (for convenience, we always assume that $X = X^{-1}$). In this situation the group G can be considered as a quotient of the free product

$$F = \Bigl(\mathop{\ast}_{\lambda\in\Lambda} H_\lambda\Bigr) \ast F(X), \tag{1}$$

where $F(X)$ is the free group with the basis X. Suppose that $\mathcal{R}$ is a subset of F such that the kernel of the natural epimorphism $F \to G$ is a normal closure of $\mathcal{R}$ in the group F, then we say that G has *relative presentation*

$$\bigl\langle X,\ \{H_\lambda\}_{\lambda\in\Lambda} \bigm| R = 1,\ R \in \mathcal{R}\bigr\rangle. \tag{2}$$

If sets X and $\mathcal{R}$ are finite, the presentation (2) is said to be *relatively finite*.

Definition 2.1. We set

$$\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} \bigl(H_\lambda \setminus \{1\}\bigr). \tag{3}$$

A group G is *relatively hyperbolic with respect to a collection of subgroups* $\{H_\lambda\}_{\lambda\in\Lambda}$, if G admits a relatively finite presentation (2) with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$ satisfying a *linear relative isoperimetric inequality*. That is, there exists $C > 0$ satisfying the following condition. For every word w in the alphabet $X \cup \mathcal{H}$ representing the identity in the group G, there exists an expression

$$w =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i \tag{4}$$

with the equality in the group F, where $R_i \in \mathcal{R}$, $f_i \in F$, for $i = 1, \dots, k$, and $k \leqslant C\|w\|$, where $\|w\|$ is the length of the word w. This definition is independent of the choice of the (finite) generating set X and the (finite) set $\mathcal{R}$ in (2).

For a combinatorial path p in the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ of G with respect to $X \cup \mathcal{H}$, p_-, p_+, $l(p)$, and $\mathbf{Lab}(p)$ will denote the initial point, the ending point, the length (that is, the number of edges) and the label of p respectively. Further, if Ω is a subset of G and $g \in \langle \Omega\rangle \leqslant G$, then $|g|_\Omega$ will be used to denote the length of a shortest word in $\Omega^{\pm 1}$ representing g.

Let us recall some terminology introduced in [20]. Suppose q is a path in $\Gamma(G, X \cup \mathcal{H})$.

Definition 2.2. A subpath p of q is called an H_λ*-component* for some $\lambda \in \Lambda$ (or simply a *component*) of q, if the label of p is a word in the alphabet $H_\lambda \setminus \{1\}$ and p is not contained in a bigger subpath of q with this property.

Two components p_1, p_2 of a path q in $\Gamma(G, X \cup \mathcal{H})$ are called *connected* if they are H_λ-components for the same $\lambda \in \Lambda$ and there exists a path c in $\Gamma(G, X \cup \mathcal{H})$ connecting a vertex of p_1 to a vertex of p_2 such that $\mathbf{Lab}(c)$ entirely consists of letters from H_λ. In algebraic terms this means that all vertices of p_1 and p_2 belong to the same coset gH_λ for a certain $g \in G$. We can always assume c to have length at most 1, as every non-trivial element of H_λ is included in the set of generators. An H_λ-component p of a path q is called *isolated* if no distinct H_λ-component of q is connected to p. A path q is said to be *without backtracking* if all its components are isolated.

The next lemma is a simplification of Lemma 2.27 from [20].

Lemma 2.3. *Suppose that a group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Then there exists a finite subset $\Omega \subseteq G$ and a constant $K \geqslant 0$ such that the following condition holds. Let q be a cycle in $\Gamma(G, X \cup \mathcal{H})$, $p_1, \ldots, p_k$ a set of isolated H_λ-components of q for some $\lambda \in \Lambda$, $g_1, \ldots, g_k$ elements of G represented by labels $\mathbf{Lab}(p_1), \ldots, \mathbf{Lab}(p_k)$ respectively. Then $g_1, \ldots, g_k$ belong to the subgroup $\langle \Omega \rangle \leqslant G$ and the word lengths of g_i's with respect to Ω satisfy the inequality*

$$\sum_{i=1}^{k} |g_i|_\Omega \leqslant K l(q).$$

3. Suitable subgroups of relatively hyperbolic groups

Throughout this section let G be a group which is properly hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$, X a finite relative generating set of G, and $\Gamma(G, X \cup \mathcal{H})$ the Cayley graph of G with respect to the generating set $X \cup \mathcal{H}$, where $\mathcal{H}$ is given by (3). Recall that an element $g \in G$ is called *hyperbolic* if it is not conjugate to an element of some H_λ, $\lambda \in \Lambda$. The following description of elementary subgroups of G was obtained in [19].

Lemma 3.1. *Let g be a hyperbolic element of infinite order of G. Then the following conditions hold.*

(1) *The element g is contained in a unique maximal elementary subgroup $E_G(g)$ of G, where*

$$E_G(g) = \left\{ f \in G\colon\ f^{-1} g^n f = g^{\pm n} \text{ for some } n \in \mathbb{N} \right\}. \tag{5}$$

(2) *The group G is hyperbolic relative to the collection $\{H_\lambda\}_{\lambda \in \Lambda} \cup \{E_G(g)\}$.*

Given a subgroup $S \leqslant G$, we denote by S^0 the set of all hyperbolic elements of S of infinite order. Recall that two elements $f, g \in G^0$ are said to be *commensurable* (in G) if f^k is conjugated to g^l in G for some non-zero integers k and l.

Definition 3.2. A subgroup $S \leqslant G$ is called *suitable*, if there exist at least two non-commensurable elements $f_1, f_2 \in S^0$, such that $E_G(f_1) \cap E_G(f_2) = \{1\}$.

If $S^0 \neq \emptyset$, we define

$$E_G(S) = \bigcap_{g \in S^0} E_G(g).$$

Lemma 3.3. *If $S \leqslant G$ is a non-elementary subgroup and $S^0 \neq \emptyset$, then $E_G(S)$ is the maximal finite subgroup of G normalized by S.*

Proof. Indeed, if a finite subgroup $M \leqslant G$ is normalized by S, then $|S : C_S(M)| < \infty$ where $C_S(M) = \{g \in S\colon\ g^{-1}xg = x,\ \forall x \in M\}$. Formula (5) implies that $M \leqslant E_G(g)$ for every $g \in S^0$, hence $M \leqslant E_G(S)$.

On the other hand, if S is non-elementary and $S^0 \neq \emptyset$, there exist $h \in S^0$ and $a \in S^0 \setminus E_G(h)$. Then $a^{-1}ha \in S^0$ and the intersection $E_G(a^{-1}ha) \cap E_G(h)$ is finite. Indeed if $E_G(a^{-1}ha) \cap E_G(h)$ were infinite, we would have $(a^{-1}ha)^n = h^k$ for some $n, k \in \mathbb{Z} \setminus \{0\}$, which would contradict to $a \notin E_G(h)$. Hence $E_G(S) \leqslant E_G(a^{-1}ha) \cap E_G(h)$ is finite. Obviously, $E_G(S)$ is normalized by S in G. $\square$

The main result of this section is the following

Proposition 3.4. *Suppose that a group G is hyperbolic relative to a collection $\{H_\lambda\}_{\lambda \in \Lambda}$ and S is a subgroup of G. Then the following conditions are equivalent.*

(1) *S is suitable*;
(2) *$S^0 \neq \emptyset$ and $E_G(S) = \{1\}$.*

Our proof of Proposition 3.4 will make use of several auxiliary statements below.

Lemma 3.5. *(See Lemma 4.4, [19].) For any $\lambda \in \Lambda$ and any element $a \in G \setminus H_\lambda$, there exists a finite subset $\mathcal{F}_\lambda = \mathcal{F}_\lambda(a) \subseteq H_\lambda$ such that if $h \in H_\lambda \setminus \mathcal{F}_\lambda$, then ah is a hyperbolic element of infinite order.*

It can be seen from Lemma 3.1 that every hyperbolic element $g \in G$ of infinite order is contained inside the elementary subgroup

$$E_G^+(g) = \bigl\{f \in G\colon\ f^{-1}g^n f = g^n \text{ for some } n \in \mathbb{N}\bigr\} \leqslant E_G(g),$$

and $|E_G(g) : E_G^+(g)| \leqslant 2$.

Lemma 3.6. *Suppose $g_1, g_2 \in G^0$ are non-commensurable and $A = \langle g_1, g_2 \rangle \leqslant G$. Then there exists an element $h \in A^0$ such that*:

(1) *h is not commensurable with g_1 and g_2*;
(2) *$E_G(h) = E_G^+(h) \leqslant \langle h, E_G(g_1) \cap E_G(g_2)\rangle$. If, in addition, $E_G(g_j) = E_G^+(g_j)$, $j = 1, 2$, then $E_G(h) = E_G^+(h) = \langle h\rangle \times (E_G(g_1) \cap E_G(g_2))$.*

Proof. By Lemma 3.1, G is hyperbolic relative to the collection of peripheral subgroups $\mathfrak{C}_1 = \{H_\lambda\}_{\lambda \in \Lambda} \cup \{E_G(g_1)\} \cup \{E_G(g_2)\}$. The center $Z(E_G^+(g_j))$ has finite index in $E_G^+(g_j)$, hence (possibly, after replacing g_j with a power of itself) we can assume that $g_j \in Z(E_G^+(g_j))$, $j = 1, 2$. Using Lemma 3.5 we can find an integer $n_1 \in \mathbb{N}$ such that the element $g_3 = g_2 g_1^{n_1} \in A$ is hyperbolic relatively to $\mathfrak{C}_1$ and has infinite order. Applying Lemma 3.1 again, we achieve hyperbolicity of G relative to $\mathfrak{C}_2 = \mathfrak{C}_1 \cup \{E_G(g_3)\}$. Set $\mathcal{H}' = \bigsqcup_{H \in \mathfrak{C}_2} (H \setminus \{1\})$.

Let $\Omega \subset G$ be the finite subset and $K > 0$ the constant chosen according to Lemma 2.3 (where G is considered to be relatively hyperbolic with respect to $\mathfrak{C}_2$). Using Lemma 3.5 two more times, we can find numbers $m_1, m_2, m_3 \in \mathbb{N}$ such that

$$g_i^{m_i} \notin \{y \in \langle \Omega \rangle \colon |y|_\Omega \leqslant 21K\}, \quad i = 1, 2, 3, \tag{6}$$

and $h = g_1^{m_1} g_3^{m_3} g_2^{m_2} \in A$ is a hyperbolic element (with respect to $\mathfrak{C}_2$) and has infinite order. Indeed, first we choose m_1 to satisfy (6). By Lemma 3.5, there is m_3 satisfying (6), so that $g_1^{m_1} g_3^{m_3} \in A^0$. Similarly m_2 can be chosen sufficiently large to satisfy (6) and $g_1^{m_1} g_3^{m_3} g_2^{m_2} \in A^0$. In particular, h will be non-commensurable with g_j, $j = 1, 2$ (otherwise, there would exist $f \in G$ and $n \in \mathbb{N}$ such that $f^{-1} h^n f \in E(g_j)$, implying $h \in f E(g_j) f^{-1}$ by Lemma 3.1 and contradicting the hyperbolicity of h).

Consider a path q labeled by the word $(g_1^{m_1} g_3^{m_3} g_2^{m_2})^l$ in $\Gamma(G, X \cup \mathcal{H}')$ for some $l \in \mathbb{Z} \setminus \{0\}$, where each $g_i^{m_i}$ is treated as a single letter from $\mathcal{H}'$. After replacing q with q^{-1}, if necessary, we assume that $l \in \mathbb{N}$. Let $p_1, \ldots, p_{3l}$ be all components of q; by the construction of q, we have $l(p_j) = 1$ for each j. Suppose not all of these components are isolated. Then one can find indices $1 \leqslant s < t \leqslant 3l$ and $i \in \{1, 2, 3\}$ such that p_s and p_t are $E_G(g_i)$-components of q, $(p_t)_-$ and $(p_s)_+$ are connected by a path r with $\mathbf{Lab}(r) \in E_G(g_i)$, $l(r) \leqslant 1$, and $(t - s)$ is minimal with this property. To simplify the notation, assume that $i = 1$ (the other two cases are similar). Then $p_{s+1}, p_{s+4}, \ldots, p_{t-2}$ are isolated $E_G(g_3)$-components of the cycle $p_{s+1} p_{s+2} \ldots p_{t-1} r$, and there are exactly $(t - s)/3 \geqslant 1$ of them. Applying Lemma 2.3, we obtain $g_3^{m_3} \in \langle \Omega \rangle$ and

$$\frac{t - s}{3} |g_3^{m_3}|_\Omega \leqslant K(t - s).$$

Hence $|g_3^{m_3}|_\Omega \leqslant 3K$, contradicting (6). Therefore two distinct components of q cannot be connected with each other; that is, the path q is without backtracking.

To finish the proof of Lemma 3.6 we need an auxiliary statement below. Denote by $\mathcal{W}$ the set of all subwords of words $(g_1^{m_1} g_3^{m_3} g_2^{m_2})^l$, $l \in \mathbb{Z}$ (where $g_i^{\pm m_i}$ is treated as a single letter from $\mathcal{H}'$). Consider an arbitrary cycle $o = rqr'q'$ in $\Gamma(G, X \cup \mathcal{H}')$, where $\mathbf{Lab}(q), \mathbf{Lab}(q') \in \mathcal{W}$; and set $C = \max\{l(r), l(r')\}$. Let p be a component of q (or q'). We will say that p is *regular* if it is not an isolated component of o. As q and q' are without backtracking, this means that p is either connected to some component of q' (respectively q), or to a component of r, or r'.

Lemma 3.7. *In the above notations*

(a) *if $C \leqslant 1$ then every component of q or q' is regular*;
(b) *if $C \geqslant 2$ then each of q and q' can have at most $15C$ components which are not regular.*

Proof. Assume the contrary to (a). Then one can choose a cycle $o = rqr'q'$ with $l(r), l(r') \leqslant 1$, having at least one $E(g_i)$-isolated component on q or q' for some $i \in \{1, 2, 3\}$, and such that $l(q) + l(q')$ is minimal. Clearly the latter condition implies that each component of q or q' is an isolated component of o. Therefore q and q' together contain k distinct $E(g_i)$-components of o where $k \geqslant 1$ and $k \geqslant \lfloor l(q)/3 \rfloor + \lfloor l(q')/3 \rfloor$. Applying Lemma 2.3 we obtain $g_i^{m_i} \in \langle \Omega \rangle$ and $k|g_i^{m_i}|_\Omega \leqslant K(l(q) + l(q') + 2)$, therefore $|g_i^{m_i}|_\Omega \leqslant 11K$, contradicting the choice of m_i in (6).

Let us prove (b). Suppose that $C \geqslant 2$ and q contains more than $15C$ isolated components of o. We consider two cases:

Case 1. No component of q is connected to a component of q'. Then a component of q or q' can be regular only if it is connected to a component of r or r'. Since q and q' are without backtracking, two distinct components of q or q' cannot be connected to the same component of r (or r'). Hence q and q' together can contain at most $2C$ regular components. Thus there is an index $i \in \{1, 2, 3\}$ such that the cycle o has k isolated $E(g_i)$-components, where $k \geqslant \lfloor l(q)/3 \rfloor + \lfloor l(q')/3 \rfloor - 2C \geqslant \lfloor 5C \rfloor - 2C > 2C > 3$. By Lemma 2.3, $g_i^{m_i} \in \langle \Omega \rangle$ and $k|g_i^{m_i}|_\Omega \leqslant K(l(q) + l(q') + 2C)$, hence

$$\left|g_i^{m_i}\right|_\Omega \leqslant K \frac{3(\lfloor l(q)/3 \rfloor + 1) + 3(\lfloor l(q')/3 \rfloor + 1) + 2C}{\lfloor l(q)/3 \rfloor + \lfloor l(q')/3 \rfloor - 2C} \leqslant K\left(3 + \frac{6 + 8C}{2C}\right) \leqslant 9K,$$

contradicting the choice of m_i in (6).

Case 2. The path q has at least one component which is connected to a component of q'. Let $p_1, \ldots, p_{l(q)}$ denote the sequence of all components of q. By part (a), if p_s and p_t, $1 \leqslant s \leqslant t \leqslant l(q)$, are connected to components of q', then for any j, $s \leqslant j \leqslant t$, p_j is regular. We can take s (respectively t) to be minimal (respectively maximal) possible. Consequently $p_1, \ldots, p_{s-1}, p_{t+1}, \ldots, p_{l(q)}$ will contain the set of all isolated components of o that belong to q.

Without loss of generality we may assume that $s - 1 \geqslant 15C/2$. Since p_s is connected to some component p' of q', there exists a path v in $\Gamma(G, X \cup \mathcal{H}')$ satisfying $v_- = (p_s)_-$, $v_+ = p'_+$, $\mathbf{Lab}(v) \in \mathcal{H}'$, $l(v) = 1$. Let $\bar{q}$ (respectively $\bar{q}'$) denote the subpath of q (respectively q') from q_- to $(p_s)_-$ (respectively from p'_+ to q'_+). Consider a new cycle $\bar{o} = r\bar{q}v\bar{q}'$. Reasoning as before, we can find $i \in \{1, 2, 3\}$ such that $\bar{o}$ has k isolated $E(g_i)$-components, where $k \geqslant \lfloor l(\bar{q})/3 \rfloor + \lfloor l(\bar{q}')/3 \rfloor - C - 1 \geqslant \lfloor 15C/6 \rfloor - C - 1 > C - 1 \geqslant 1$. Using Lemma 2.3, we get $g_i^{m_i} \in \langle \Omega \rangle$ and $k|g_i^{m_i}|_\Omega \leqslant K(l(\bar{q}) + l(\bar{q}') + C + 1)$. The latter inequality implies $|g_i^{m_i}|_\Omega \leqslant 21K$, yielding a contradiction in the usual way and proving (b) for q. By symmetry this property holds for q' as well. $\square$

Continuing the proof of Lemma 3.6, consider an element $x \in E_G(h)$. According to Lemma 3.1, there exists $l \in \mathbb{N}$ such that

$$xh^l x^{-1} = h^{\epsilon l}, \tag{7}$$

where $\epsilon = \pm 1$. Set $C = |x|_{X \cup \mathcal{H}'}$. After raising both sides of (7) in an integer power, we can assume that l is sufficiently large to satisfy $l > 32C + 3$.

Consider a cycle $o = rqr'q'$ in $\Gamma(G, X \cup \mathcal{H}')$ satisfying $r_- = q'_+ = 1$, $r_+ = q_- = x$, $q_+ = r'_- = xh^l$, $r'_+ = q'_- = xh^l x^{-1}$, $\mathbf{Lab}(q) \equiv (g_1^{m_1} g_3^{m_3} g_2^{m_2})^l$, $\mathbf{Lab}(q') \equiv (g_1^{m_1} g_3^{m_3} g_2^{m_2})^{-\epsilon l}$, $l(q) = l(q') = 3l$, $l(r) = l(r') = C$.

Let $p_1, p_2, \ldots, p_{3l}$ and $p'_1, p'_2, \ldots, p'_{3l}$ be all components of q and q' respectively. Thus, $p_3, p_6, p_9, \ldots, p_{3l}$ are all $E_G(g_2)$-components of q. Since $l > 17C$ and q is without backtracking, by Lemma 3.7, there exist indices $1 \leqslant s, s' \leqslant 3l$ such that the $E_G(g_2)$-component p_s of q is connected to the $E_G(g_2)$-component $p'_{s'}$ of q'. Without loss of generality, assume that $s \leqslant 3l/2$

(the other situation is symmetric). There is a path u in $\Gamma(G, X \cup \mathcal{H}')$ with $u_- = (p'_{s'})_-$, $u_+ = (p_s)_+$, $\mathbf{Lab}(u) \in E_G(g_2)$ and $l(u) \leqslant 1$. We obtain a new cycle $o' = up_{s+1} \dots p_{3l} r' p'_1 \dots p'_{s'-1}$ in the Cayley graph $\Gamma(G, X \cup \mathcal{H}')$. Due to the choice of s and l, the same argument as before will demonstrate that there are $E_G(g_2)$-components $p_{\bar{s}}$, $p'_{\bar{s}'}$ of q, q' respectively, which are connected and $s < \bar{s} \leqslant 3l$, $1 \leqslant \bar{s}' < s'$ (in the case when $s > 3l/2$, the same inequalities can be achieved by simply renaming the indices correspondingly).

It is now clear that there exist $i \in \{1, 2, 3\}$ and connected $E_G(g_i)$-components p_t, $p'_{t'}$ of q, q' ($s < t \leqslant 3l$, $1 \leqslant t' < s'$) such that $t > s$ is minimal. Let v denote a path in $\Gamma(G, X \cup \mathcal{H}')$ with $v_- = (p_t)_-$, $v_+ = (p_{t'})_+$, $\mathbf{Lab}(v) \in E_G(g_i)$ and $l(v) \leqslant 1$. Consider a cycle o'' in $\Gamma(G, X \cup \mathcal{H}')$ defined by $o'' = up_{s+1} \dots p_{t-1} v p'_{t'+1} \dots p'_{s'-1}$. By part (a) of Lemma 3.7, p_{s+1} is a regular component of the path $p_{s+1} \dots p_{t-1}$ in o'' (provided that $t - 1 \geqslant s + 1$). Note that p_{s+1} cannot be connected to u or v because q is without backtracking, hence it must be connected to a component of the path $p'_{t'+1} \dots p'_{s'-1}$. By the choice of t, we have $t = s + 1$ and $i = 1$. Similarly $t' = s' - 1$. Thus $p_{s+1} = p_t$ and $p'_{s'-1} = p'_{t'}$ are connected $E_G(g_1)$-components of q and q'.

In particular, we have $\epsilon = 1$. Indeed, otherwise we would have $\mathbf{Lab}(p_{s'-1}) \equiv g_3^{m_3}$ but $g_3^{m_3} \notin E_G(g_1)$. Therefore $x \in E_G^+(h)$ for any $x \in E_G(h)$, consequently $E_G(h) = E_G^+(h)$.

Observe that $u_- = v_+$ and $u_+ = v_-$, hence $\mathbf{Lab}(u)$ and $\mathbf{Lab}(v)^{-1}$ represent the same element $z \in E_G(g_2) \cap E_G(g_1)$. By construction, $x = h^\alpha z h^\beta$ where $\alpha = (3l - s')/3 \in \mathbb{Z}$, and $\beta = -s/3 \in \mathbb{Z}$. Thus $x \in \langle h, E_G(g_1) \cap E_G(g_2) \rangle$ and the first part of the claim (2) is proved.

Assume now that $E_G(g_j) = E_G^+(g_j)$ for $j = 1, 2$. Then $h = g_1^{m_1}(g_2 g_1^{n_1})^{m_3} g_2^{m_2}$ belongs to the centralizer of the finite subgroup $E_G(g_1) \cap E_G(g_2)$ (because of the choice of g_1, g_2 above). Consequently $E_G(h) = \langle h \rangle \times (E_G(g_1) \cap E_G(g_2))$. □

Lemma 3.8. *Let S be a non-elementary subgroup of G with $S^0 \neq \emptyset$. Then*

(i) *there exist non-commensurable elements $h_1, h'_1 \in S^0$ with $E_G(h_1) \cap E_G(h'_1) = E_G(S)$;*
(ii) *S^0 contains an element h such that $E_G(h) = \langle h \rangle \times E_G(S)$.*

Proof. Choose an element $g_1 \in S^0$. By Lemma 3.1, G is hyperbolic relative to the collection $\mathfrak{C} = \{H_\lambda\}_{\lambda \in \Lambda} \cup \{E_G(g_1)\}$. Since the subgroup S is non-elementary, there is $a \in S \setminus E_G(g_1)$, and Lemma 3.5 provides us with an integer $n \in \mathbb{N}$ such that $g_2 = a g_1^n \in S$ is a hyperbolic element of infinite order (now, with respect to the family of peripheral subgroups $\mathfrak{C}$). In particular, g_1 and g_2 are non-commensurable and hyperbolic relative to $\{H_\lambda\}_{\lambda \in \Lambda}$.

Applying Lemma 3.6, we find $h_1 \in S^0$ (with respect to the collection of peripheral subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$) with $E_G(h_1) = E_G^+(h_1)$ such that h_1 is not commensurable with g_j, $j = 1, 2$. Hence, g_1 and g_2 stay hyperbolic after including $E_G(h_1)$ into the family of peripheral subgroups (see Lemma 3.1). This allows to construct (in the same manner) one more element $h_2 \in \langle g_1, g_2 \rangle \leqslant S$ which is hyperbolic relative to $(\{H_\lambda\}_{\lambda \in \Lambda} \cup E_G(h_1))$ and satisfies $E_G(h_2) = E_G^+(h_2)$. In particular, h_2 is not commensurable with h_1.

We claim now that there exists $x \in S$ such that $E_G(x^{-1} h_2 x) \cap E_G(h_1) = E_G(S)$. By definition, $E_G(S) \subseteq E_G(x^{-1} h_2 x) \cap E_G(h_1)$. To obtain the inverse inclusion, arguing by the contrary, suppose that for each $x \in S$ we have

$$\big(E_G\big(x^{-1} h_2 x\big) \cap E_G(h_1)\big) \setminus E_G(S) \neq \emptyset. \tag{8}$$

Note that if $g \in S^0$ with $E_G(g) = E_G^+(g)$, then the set of all elements of finite order in $E_G(g)$ form a finite subgroup $T(g) \leqslant E_G(g)$ (this is a well-known property of groups, all of whose conjugacy classes are finite). The elements h_1 and h_2 are not commensurable, therefore

$$E_G(x^{-1}h_2x) \cap E_G(h_1) = T(x^{-1}h_2x) \cap T(h_1) = x^{-1}T(h_2)x \cap T(h_1).$$

For each pair of elements $(b, a) \in D = T(h_2) \times (T(h_1) \setminus E_G(S))$ choose $x = x(b, a) \in S$ so that $x^{-1}bx = a$ if such x exists; otherwise set $x(b, a) = 1$.

The assumption (8) clearly implies that $S = \bigcup_{(b,a)\in D} x(b, a)C_S(a)$, where $C_S(a)$ denotes the centralizer of a in S. Since the set D is finite, a well-know theorem of B. Neumann [16] implies that there exists $a \in T(h_1) \setminus E_G(S)$ such that $|S : C_S(a)| < \infty$. Consequently, $a \in E_G(g)$ for every $g \in S^0$, that is, $a \in E_G(S)$, a contradiction.

Thus, $E_G(xh_2x^{-1}) \cap E_G(h_1) = E_G(S)$ for some $x \in S$. After setting $h_1' = x^{-1}h_2x \in S^0$, we see that elements h_1 and h_1' satisfy the claim (i). Since $E_G(h_1') = x^{-1}E_G(h_2)x$, we have $E_G(h_1') = E_G^+(h_1')$. To demonstrate (ii), it remains to apply Lemma 3.6 and obtain an element $h \in \langle h_1, h_1'\rangle \leqslant S$ which has the desired properties. $\square$

Proof of Proposition 3.4. The implication (1) ⇒ (2) is an immediate consequence of the definition. The inverse implication follows directly from the first claim of Lemma 3.8 (S is non-elementary as $S^0 \neq \emptyset$ and $E_G(S) = \{1\}$). $\square$

4. Proofs of the main results

The following simplification of Theorem 2.4 from [21] is the key ingredient of the proofs in the rest of the paper.

Theorem 4.1. *Let U be a group hyperbolic relative to a collection of subgroups $\{V_\lambda\}_{\lambda\in\Lambda}$, S a suitable subgroup of U, and T a finite subset of U. Then there exists an epimorphism $\eta : U \to W$ such that*:

(1) *The restriction of η to $\bigcup_{\lambda\in\Lambda} V_\lambda$ is injective, and the group W is properly relatively hyperbolic with respect to the collection $\{\eta(V_\lambda)\}_{\lambda\in\Lambda}$.*
(2) *For every $t \in T$, we have $\eta(t) \in \eta(S)$.*

Let us also mention two known results we will use. The first lemma is a particular case of Theorem 1.4 from [20] (if $g \in G$ and $H \leqslant G$, H^g denotes the conjugate $g^{-1}Hg \leqslant G$).

Lemma 4.2. *Suppose that a group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then*

(a) *For any $g \in G$ and any $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, the intersection $H_\lambda^g \cap H_\mu$ is finite.*
(b) *For any $\lambda \in \Lambda$ and any $g \notin H_\lambda$, the intersection $H_\lambda^g \cap H_\lambda$ is finite.*

The second result can easily be derived from Lemma 3.5.

Lemma 4.3. *(See Corollary 4.5, [19].) Let G be an infinite properly relatively hyperbolic group. Then G contains a hyperbolic element of infinite order.*

Lemma 4.4. *Let the group G be hyperbolic with respect to the collection of peripheral subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ and let $N \lhd G$ be a finite normal subgroup. Then*

(1) *If H_λ is infinite for some $\lambda \in \Lambda$, then $N \leqslant H_\lambda$;*
(2) *The quotient $\bar{G} = G/N$ is hyperbolic relative to the natural image of the collection $\{H_\lambda\}_{\lambda\in\Lambda}$.*

Proof. Let K_λ, $\lambda \in \Lambda$, be the kernel of the action of H_λ on N by conjugation. Since N is finite, K_λ has finite index in H_λ. On the other hand, $K_\lambda \leqslant H_\lambda \cap H_\lambda^g$ for every $g \in N$. If H_λ is infinite this implies $N \leqslant H_\lambda$ by Lemma 4.2.

To prove the second assertion, suppose that G has a relatively finite presentation (2) with respect to the free product F defined in (1). Denote by $\bar{X}$ and $\bar{H}_\lambda$ the natural images of X and H_λ in $\bar{G}$. In order to show that $\bar{G}$ is relatively hyperbolic, one has to consider it as a quotient of the free product $\bar{F} = (\ast_{\lambda\in\Lambda} \bar{H}_\lambda) * F(\bar{X})$. As G is a quotient of F, we can choose some finite preimage $M \subset F$ of N. For each element $f \in M$, fix a word in $X \cup \mathcal{H}$ which represents it in F and denote by $\mathcal{S}$ the (finite) set of all such words. By the universality of free products, there is a natural epimorphism $\varphi : F \to \bar{F}$ mapping X onto $\bar{X}$ and each H_λ onto $\bar{H}_\lambda$. Define the subsets $\bar{\mathcal{R}}$ and $\bar{\mathcal{S}}$ of words in $\bar{X} \cup \bar{\mathcal{H}}$ (where $\bar{\mathcal{H}} = \bigsqcup_{\lambda\in\Lambda}(\bar{H}_\lambda \setminus \{1\})$) by $\bar{\mathcal{R}} = \varphi(\mathcal{R})$ and $\bar{\mathcal{S}} = \varphi(\mathcal{S})$. Then the group $\bar{G}$ possesses the relatively finite presentation

$$\left\langle \bar{X},\ \{\bar{H}_\lambda\}_{\lambda\in\Lambda} \;\middle|\; \bar{R} = 1,\ \bar{R} \in \bar{\mathcal{R}};\ \bar{S} = 1,\ \bar{S} \in \bar{\mathcal{S}} \right\rangle. \tag{9}$$

Let $\psi : F \to G$ denote the natural epimorphism and $D = \max\{\|s\|:\ s \in \mathcal{S}\}$. Consider any non-empty word $\bar{w}$ in the alphabet $\bar{X} \cup \bar{\mathcal{H}}$ representing the identity in $\bar{G}$. Evidently we can choose a word w in $X \cup \mathcal{H}$ such that $\bar{w} =_{\bar{F}} \varphi(w)$ and $\|w\| = \|\bar{w}\|$. Since $\ker(\psi) \cdot M$ is the kernel of the induced homomorphism from F to $\bar{G}$, we have $w =_F vu$ where $u \in \mathcal{S}$ and v is a word in $X \cup \mathcal{H}$ satisfying $v =_G 1$ and $\|v\| \leqslant \|w\| + D$. Since G is relatively hyperbolic there is a constant $C \geqslant 0$ (independent of v) such that

$$v =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i,$$

where $R_i \in \mathcal{R}$, $f_i \in F$, and $k \leqslant C\|v\|$. Set $\bar{R}_i = \varphi(R) \in \bar{\mathcal{R}}$, $\bar{f}_i = \varphi(f_i) \in \bar{F}$, $i = 1, 2, \ldots, k$, and $\bar{R}_{k+1} = \varphi(u) \in \bar{\mathcal{S}}$, $\bar{f}_{k+1} = 1$. Then

$$\bar{w} =_{\bar{F}} \prod_{i=1}^{k+1} \bar{f}_i^{-1} \bar{R}_i^{\pm 1} \bar{f}_i,$$

where

$$k + 1 \leqslant C\|v\| + 1 \leqslant C\big(\|w\| + D\big) + 1 \leqslant C\|\bar{w}\| + CD + 1 \leqslant (C + CD + 1)\|\bar{w}\|.$$

Thus, the relative presentation (9) satisfies a linear isoperimetric inequality with the constant $(C + CD + 1)$. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Observe that the quotient of G by the finite normal subgroup $N = E_G(G)$ is obviously non-elementary. Hence the image of any finite H_λ is a proper subgroup of G/N. On the other hand, if H_λ is infinite, then $N \leqslant H_\lambda \lneqq G$ by Lemma 4.4, hence its image is also proper in G/N. Therefore G/N is properly relatively hyperbolic with respect to the collection of images of H_λ, $\lambda \in \Lambda$ (see Lemma 4.4). Lemma 3.3 implies $E_{G/N}(G/N) = \{1\}$. Thus, without loss of generality, we may assume that $E_G(G) = 1$.

It is straightforward to see that the free product $U = G * R$ is hyperbolic relative to the collection $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{R\}$ and $E_{G*R}(G) = E_G(G) = 1$. Note that G^0 is non-empty by Lemma 4.3. Hence G is a suitable subgroup of $G * R$ by Proposition 3.4. Let Y be a finite generating set of R. It remains to apply Theorem 4.1 to $U = G * R$, the obvious collection of peripheral subgroups, and the finite set Y. $\square$

To prove Theorem 1.4 we need one more auxiliary result which was proved in the full generality in [20] (see also [6]):

Lemma 4.5. *(See Theorem 2.40, [20].) Suppose that a group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{S_1, \dots, S_m\}$, where $S_1, \dots, S_m$ are hyperbolic in the ordinary (non-relative) sense. Then G is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$.*

Proof of Theorem 1.4. Let G_1, G_2 be finitely generated groups which are properly relatively hyperbolic with respect to collections of subgroups $\{H_{1\lambda}\}_{\lambda\in\Lambda}$ and $\{H_{2\mu}\}_{\mu\in M}$ respectively. Denote by X_i a finite generating set of the group G_i, $i = 1, 2$. As above we may assume that $E_{G_1}(G_1) = E_{G_2}(G_2) = \{1\}$. We set $G = G_1 * G_2$. Observe that $E_G(G_i) = E_{G_i}(G_i) = \{1\}$ and hence G_i is suitable in G for $i = 1, 2$ (by Lemma 4.3 and Proposition 3.4).

By the definition of suitable subgroups, there are two non-commensurable elements $g_1, g_2 \in G_2^0$ such that $E_G(g_1) \cap E_G(g_2) = \{1\}$. Further, by Lemma 3.1, the group G is hyperbolic relative to the collection $\mathfrak{P} = \{H_{1\lambda}\}_{\lambda\in\Lambda} \cup \{H_{2\mu}\}_{\mu\in M} \cup \{E_G(g_1), E_G(g_2)\}$. We now apply Theorem 4.1 to the group G with the collection of peripheral subgroups $\mathfrak{P}$, the suitable subgroup $G_1 \leqslant G$, and the subset $T = X_2$. The resulting group W is obviously a quotient of G_1.

Observe that W is hyperbolic relative to (the image of) the collection $\{H_{1\lambda}\}_{\lambda\in\Lambda} \cup \{H_{2\mu}\}_{\mu\in M}$ by Lemma 4.5. We would like to show that G_2 is a suitable subgroup of W with respect to this collection. To this end we note that $\eta(g_1)$ and $\eta(g_2)$ are elements of infinite order as η is injective on $E_G(g_1)$ and $E_G(g_2)$. Moreover, $\eta(g_1)$ and $\eta(g_2)$ are not commensurable in W. Indeed, otherwise, the intersection $(\eta(E_G(g_1)))^g \cap \eta(E_G(g_2))$ is infinite for some $g \in G$ that contradicts the first assertion of Lemma 4.2. Assume now that $g \in E_W(\eta(g_i))$ for some $i \in \{1, 2\}$. By the first assertion of Lemma 3.1, $(\eta(g_i^m))^g = \eta(g_i^{\pm m})$ for some $m \neq 0$. Therefore, $(\eta(E_G(g_i)))^g \cap \eta(E_G(g_i))$ contains $\eta(g_i^m)$ and, in particular, this intersection is infinite. By the second assertion of Lemma 4.2, this means that $g \in \eta(E_G(g_i))$. Thus, $E_W(\eta(g_i)) = \eta(E_G(g_i))$. Finally, using injectivity of η on $E_G(g_1) \cup E_G(g_2)$, we obtain

$$E_W\big(\eta(g_1)\big) \cap E_W\big(\eta(g_2)\big) = \eta\big(E_G(g_1)\big) \cap \eta\big(E_G(g_2)\big) = \eta\big(E_G(g_1) \cap E_G(g_2)\big) = \{1\}.$$

This means that the image of G_2 is a suitable subgroup of W.

Thus we may apply Theorem 4.1 again to the group W, the subgroup G_2 and the finite subset X_1. The resulting group Q is the desired common quotient of G_1 and G_2. The last property, which claims that Q can be obtained from $G_1 * G_2$ by adding only finitely many relations, follows because $G_1 * G_2$ and G are hyperbolic with respect to the same family of peripheral subgroups and any relatively hyperbolic group is relatively finitely presented. □

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