GROWTH TIGHTNESS FOR WORD HYPERBOLIC GROUPS

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Abstract. We show that non-elementary word hyperbolic groups are growth tight. This means that, given such a group G and a finite set A of its generators, for any infinite normal subgroup N of G, the exponential growth rate of G/N with respect to the natural image of A is strictly less than the exponential growth rate of G with respect to A.

1. Introduction

Let G be a finitely generated group and A a finite set of generators for G. By $|x|$ we denote the *geodesic length* of an element $x \in G$ in the generators A, i.e. the length of a shortest word in the alphabet $A^{\pm 1}$ representing x. Let $B(n)$ denote the ball $\{g \in G \mid |g| \leq n\}$ of radius n in G.

The *exponential growth rate* of the pair (G, A) is the limit

$$\lambda(G, A) = \lim_{n \to \infty} \sqrt[n]{\#B(n)}$$

where $\#X$ denotes the number of elements of a finite set X. The existence of the limit follows from the submultiplicativity property of the function $\#B(n)$: $\#B(m+n) \leq \#B(n)\#B(m)$ for any $n, m \geq 0$, see for example [8, VI.C]. The *uniform exponential growth rate* $\lambda(G)$ of G is the infinum $\inf_A \lambda(G, A)$ over all finite generating sets A of G.

We say that a pair (G, A) is *growth tight* if $\lambda(G, A) > \lambda(G/N, \underline{A})$ for all infinite normal subgroups N of G, with $\underline{A}$ denoting the canonical image of A in G/N. (This is a modification of the definition in [5].) Observe that, for a torsion-free group G, "infinite" means "any nontrivial" in this definition and it coincides then with one given in [5]. On the other hand, if N is finite it is not hard to see that we always have $\lambda(G, A) = \lambda(G/N, \underline{A})$.

It is known that for a free group F_k of rank $k \geq 2$ and for a free basis X_k of F_k, the pair (F_k, X_k) is growth tight. The direct product $F_k \times F_k$ generated by $X = (X_k \times \{1\}) \cup (\{1\} \times X_k)$ is an example of a pair which is not growth tight [5].

For results and applications related to the exponential growth rate and the growth tightness property, see [5, 8, 9]. We mention here that the exponential growth rate

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of the pair (G, A) and its logarithm, sometimes called the entropy of the pair (G, A), give rise to bounds of the growth of the volume of balls in a Riemannian manifold M with fundamental group G as well as to bounds of the topological entropy of the geodesic flow on M, see [6, Section 5.B].

Recall that a group G is called *Hopfian* if every epimorphism $G \to G$ is an isomorphism, i.e. there is no a proper quotient G/N of G isomorphic to G. As an immediate consequence of the above definitions we have the following

Observation. *Let G be a finitely generated group. If there exists a finite generating set A of G such that $\lambda(G, A) = \lambda(G)$ and if (G, A) is growth tight, then G/N cannot be isomorphic to G for any infinite normal subgroup N of G.*

In this paper, we restrict our attention to the class of word hyperbolic groups G in the sense of M.Gromov [7]. Note that any word hyperbolic group is finitely generated and even finitely presented [7, 2.1A, 2.2A], [4, Proposition 4.17]. It is known also that there are only finitely many conjugacy classes of finite subgroups in a word hyperbolic group G, see for example [2, Ch.III.Γ, Theorem 3.2]. This implies in particular that G/N is not isomorphic to G for any nontrivial finite normal subgroup N of G. Indeed, if such an isomorphism $\phi : G \to G/N$ would exist then we could define an infinite strictly increasing sequence N_i $(i \geq 0)$ of finite subgroups of G by $N_0 = N$ and $N_{i+1} = \psi^{-1}\phi(N_i)$ where $\psi : G \to G/N$ is the canonical epimorphism.

Thus, the observation provides a possible way to prove the Hopf property for word hyperbolic groups. Note that torsion-free word hyperbolic groups are already known to be Hopfian [10]. For word hyperbolic groups with torsion, the question whether or not they are Hopfian is still open.

Our main result concerns the second condition in the observation above and gives an affirmative answer to the question about growth tightness of word hyperbolic groups, posed by R. Grigorchuk and P. de la Harpe [5]. Recall that a word hyperbolic group is called *elementary* if it is either finite or a finite extension of the infinite cyclic group.

Theorem 1. *Let G be a non-elementary word hyperbolic group and A any finite set of generators for G. Then, for any infinite normal subgroup N of G,*

$$\lambda(G, A) > \lambda(G/N, \underline{A}),$$

where $\underline{A}$ is the canonical image of A in G/N.

To prove the theorem we first observe that the growth tightness of G is related to the growth tightness of certain languages over $A^{\pm 1}$ and we obtain a stronger result (Theorem 2 below) which implies Theorem 1.

We fix an arbitrary linear ordering on the alphabet $A^{\pm 1}$. A word w in the alphabet $A^{\pm 1}$ is called *geodesic* if w has the minimal possible length among all the words representing the same element of G. If w is geodesic and is minimal in the induced lexical ordering, among all geodesic words representing the same element of G, it is called *shortlex geodesic*. By $\mathcal{L}$ we denote the set of all shortlex geodesic words in $A^{\pm 1}$. Clearly, every element $x \in G$ has a unique representative in $\mathcal{L}$ which we denote $\overline{x}$. It is obvious also that any subword of a shortlex geodesic word is itself shortlex geodesic.

Definition 1. Let $C > 0$. We say that two elements $x, y \in G$ are *C-close* if there are $u, v \in G$ such that $x = uyv$ and $|u|, |v| \le C$. We say that x *C-contains* y if $\overline{x}$ has a subword representing an element which is C-close to y. For a given $w \in G$, we denote

$$Z_{w,C} = \{x \in G \mid x \ C\text{-contains } w\}.$$

We define the exponential growth rate of the set $G \setminus Z_{w,C}$ by

$$\lambda_{w,C} = \limsup_{n\to\infty} \sqrt[n]{\#(B(n) \setminus Z_{w,C})} \tag{1}$$

Theorem 2. *Let G be a non-elementary word hyperbolic group and A a finite set of generators for G. Let $\lambda = \lambda(G, A)$ be the exponential growth rate of G with respect to A. Then there is a number $C = C(G, A)$ such that $\lambda_{w,C} < \lambda$ for any $w \in G$.*

Note that Theorem 2 is of somewhat technical nature because the definition of $\lambda_{w,C}$ uses the language of shortlex geodesic words in G. But it easily implies a stronger result which does not make use of any specific language. To formulate this result, we use, in an informal sense, "the strongest" form of the relation "x C-contains y" which leads to the "smallest possible" appropriate set $Z_{w,C}$ and, consequently, to the "largest possible" appropriate parameter $\lambda_{w,C}$.

We say that x *strongly C-contains* y if any geodesic word representing x contains a subword which is C-close to y. Let

$$\overline{Z}_{w,C} = \{x \in G \mid x \text{ strongly } C\text{-contains } w\}$$

and let $\overline{\lambda}_{w,C}$ be the exponential growth rate of the set $G \setminus \overline{Z}_{w,C}$, defined as in (1).

Theorem 3. *Let G be a non-elementary word hyperbolic group and A a finite set of generators for G. Let $\lambda = \lambda(G, A)$ be the exponential growth rate of G with respect to A. Then there is a number $C = C(G, A)$ such that $\overline{\lambda}_{w,C} < \lambda$ for any $w \in G$.*

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