INTERPOLATION OF DERIVATIVES AND ULTRADIFFERENTIABLE REGULARITY

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ABSTRACT. Interpolation inequalities for C^m functions allow to bound derivatives of intermediate order 0 < j < m by bounds for the derivatives of order 0 and m. We review various interpolation inequalities for L^p -norms $(1 \le p \le \infty)$ in arbitrary finite dimensions. They allow us to study ultradifferentiable regularity by lacunary estimates in a comprehensive way, striving for minimal assumptions on the weights.

1. INTRODUCTION

Ultradifferentiable classes are classes of C^{∞} functions defined by prescribed growth behavior of the infinite sequence of derivatives. The most classical among them are the Denjoy–Carleman classes, where the growth of the derivatives is dominated by a weight sequence $M = (M_j)_{j\geq 0}$. The Gevrey classes (in particular, the real analytic class), which play an important role in the theory of differential equations, are special cases thereof.

In applications, sometimes the question arises whether the ultradifferentiable regularity can be concluded if only lacunary information on the growth of the derivatives is available. For instance, Bolley, Camus, and Métivier [2] studied the analyticity of analytic vectors and Liess [17] partially extended their work to Denjoy–Carleman classes. Knowing suitable bounds for the sequence $(P^j f)_{j\geq 0}$, where P is an elliptic linear partial differential operator, one would like to conclude similar bounds for all derivatives f. In [20], this approach was applied to isotropic functions and used to prove Chevalley-type results.

The basic problem is the following. Suppose that we know that a smooth function satisfies certain ultradifferentiable bounds for the derivatives of order k_j , where (k_j) is a strictly increasing sequence of integers, can we deduce that its derivatives of all orders satisfy the ultradifferentiable bounds? An affirmative answer clearly depends on conditions for the base sequence (k_j) , the weight $M = (M_j)$, and on suitable interpolation inequalities.

In the recent paper [1], Albano and Mughetti, motivated by L^2 methods for proving local regularity of solutions for (degenerate) elliptic equations (cf. [3, 4, 5]), gave sufficient conditions for Denjoy–Carleman classes of Roumieu type on a compact interval of \mathbb{R} , based on the Cartan–Gorny inequality (cf. Proposition 2.6).

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The base sequence (k_j) is required to be such that k_{j+1}/k_j is bounded. For the weight sequence, the authors assume a rather strong condition (which we recall and discuss in Remark 6.4). In fact, we prove a version under a weaker and more natural condition in Theorem 6.1 (see also Theorem 4.1), namely, boundedness of $m_{k_{j+1}}/m_{k_j}$, where $m_j := M_j^{1/j}$. Note that Liess showed (in [17]) that, under certain assumptions on M, boundedness of $m_{k_{j+1}}/m_{k_j}$ is also a necessary condition for the interpolation problem (see Remark 4.4). We discuss at the end of Section 6 necessity of the conditions and how the different conditions are related.

The main purpose of this paper is to treat the interpolation problem in a broad and comprehensive way, striving for minimal assumptions on the weights. To this end, we review various interpolation inequalities for L^p -norms, where $1 \le p \le \infty$, and work in arbitrary (finite) dimensions. Note that the growth of the involved constants is crucial in the ultradifferentiable setting. We consider ultradifferentiable classes of local and global type and allow very general weight systems (so that also Braun–Meise–Taylor classes are covered). All the ingredients seem to be wellknown, even though somewhat scattered in the literature, but we think a unified treatment can be useful.

Let us outline the structure of the paper. In Section 2, we recall several interpolation inequalities with short proofs for the convenience of the reader. Building on these inequalities, we obtain three technical propositions in Section 3 on which most of the subsequent ultradifferentiable regularity results by lacunary estimates are based. In Section 4, we present these results for a broad variety of Denjoy– Carleman classes (defined in terms of a single weight sequence) and, in Section 5, we extend them to more general ultradifferentiable classes (defined by families of weight sequences) including the Braun–Meise–Taylor classes. In the last Section 6, we compare our results with the approach of Albano and Mughetti [1] and discuss optimality of the conditions imposed on the base sequence (k_j) and on the weights. In Section A, we generalize to general weight systems a construction of [1] which shows that, in general, boundedness of k_{j+1}/k_j cannot be omitted.

2. INTERPOLATION INEQUALITIES

2.1. The global setting. The Landau–Kolmogorov inequality states that a C^m function $f : \mathbb{R} \to \mathbb{R}$ with finite $||f||_{L^{\infty}(\mathbb{R})}$ and $||f^{(m)}||_{L^{\infty}(\mathbb{R})}$ satisfies

$$\|f^{(j)}\|_{L^{\infty}(\mathbb{R})} \leq K_{m,j} \|f\|_{L^{\infty}(\mathbb{R})}^{1-j/m} \|f^{(m)}\|_{L^{\infty}(\mathbb{R})}^{j/m}, \quad j = 1, \dots, m-1.$$

Due to Kolmogorov [12], the optimal constants $K_{m,j}$ are given by

$$K_{m,j} = \frac{k_{m-j}}{k_m^{1-j/m}},$$

where $k_r := \frac{4}{\pi} \sum_{i=0}^{\infty} \left[\frac{(-1)^i}{2i+1}\right]^{r+1}$ are the Favard constants. Note that $1 \le k_r \le 2$ so that $K_{m,j} \le 2$. By a simple functional-analytic argument, Certain and Kurtz [8] inferred that, if $(E, \|\cdot\|)$ is a real Banach space and A is the generator of a strongly continuous group of isometries, then

$$||A^{j}x|| \le K_{m,j} ||x||^{1-j/m} ||A^{m}x||^{j/m}, \quad j = 1, \dots, m-1,$$

for x in the domain of A^m . In particular (see also [16, Section 4.4]), we have the following lemma.

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Lemma 2.1. Let $1 \leq p \leq \infty$ and $m \in \mathbb{N}_{\geq 2}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^m function. Then, for all $v \in \mathbb{S}^{n-1}$,

$$\|d_v^j f\|_{L^p(\mathbb{R}^n)} \le 2 \|f\|_{L^p(\mathbb{R}^n)}^{1-j/m} \|d_v^m f\|_{L^p(\mathbb{R}^n)}^{j/m}, \quad j = 1, \dots, m-1,$$

if the right-hand side is finite, where $d_v^j f(x) := \partial_t^j f(x+tv)|_{t=0}$.

Proof. We may assume v = (1, 0, ..., 0) by choosing a suitable orthonormal system of coordinates. Let us first consider $p = \infty$. We have

$$\sup_{x \in \mathbb{R}^n} |\partial_1^j f(x)| \le \sup_{\substack{x_i \in \mathbb{R} \\ 2 \le i \le n}} \left[2 \left(\sup_{x_1 \in \mathbb{R}} |f(x)| \right)^{1-j/m} \left(\sup_{x_1 \in \mathbb{R}} |\partial_1^m f(x)| \right)^{j/m} \right]$$
$$\le 2 \left(\sup_{x \in \mathbb{R}^n} |f(x)| \right)^{1-j/m} \left(\sup_{x \in \mathbb{R}^n} |\partial_1^m f(x)| \right)^{j/m}.$$

For $p < \infty$, we apply the above remarks to $A = \partial_1$ and $E = L^p(\mathbb{R}^n)$. It generates the group of translations $T(s)f(x_1, \ldots, x_n) = f(x_1+s, x_2, \ldots, x_n)$ which is strongly continuous, by the dominated convergence theorem.

Remark 2.2. For p = 2, the factor 2 can be omitted as follows from an application of the Fourier transform:

$$\int |\xi_1^j \widehat{f}(\xi)|^2 d\xi = \int (|\xi_1|^m |\widehat{f}(\xi)|)^{\frac{2j}{m}} |\widehat{f}(\xi)|^{2(1-\frac{j}{m})} d\xi \le \|\xi_1^m \widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}^{2j/m} \|\widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}^{2(1-j/m)} d\xi \le \|\xi_1^m \widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}^{2(1-j/m)} d\xi \le \|\xi_1^m \widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}^{2(1-j/m)$$

by Hölder's inequality.

2.2. The local setting. The next two lemmas follow from an easy adaptation of the proof of [2, Lemmas 2.3-2.5]. We sketch the argument for the convenience of the reader.

Lemma 2.3. There is a constant C > 0 such that the following holds. Let $1 \le p \le \infty$, a > 0, $m \in \mathbb{N}$, $0 \le j \le m$, and $f \in C^m([-2a, 2a])$. Then

$$\frac{a^{j}}{j!} \|f^{(j)}\|_{L^{p}([-a,a])} \leq C^{m} \Big(\frac{a^{m}}{m!} \|f^{(m)}\|_{L^{p}([-2a,2a])} + \|f\|_{L^{p}([-2a,2a])} \Big).$$

Proof. By rescaling, it suffices to assume a = 1. Fix m. Let φ be a C^{∞} function with support in (-2, 2) that equals 1 near [-1, 1] and satisfies $\|\varphi^{(k)}\|_{L^{\infty}(\mathbb{R})} \leq (C_0 m)^k$ for $0 \leq k \leq m$ for a universal constant C_0 (cf. [11, Theorem 1.3.5]). By Taylor's theorem, for $t \in [-2, 2]$ and $0 \leq j \leq m - 1$,

$$(\varphi f)^{(j)}(t) = \int_{-2}^{t} \frac{(t-s)^{m-j-1}}{(m-j-1)!} (\varphi f)^{(m)}(s) \, ds = \sum_{i=0}^{m} \binom{m}{i} A_i(t),$$

where

$$A_i(t) := \frac{1}{(m-j-1)!} \int_{-2}^t (t-s)^{m-j-1} \varphi^{(m-i)}(s) f^{(i)}(s) \, ds.$$

For i = m, we have

$$|A_m(t)| \le \frac{1}{(m-j-1)!} \int_{-2}^{2} |t-s|^{m-j-1} |\varphi(s)f^{(m)}(s)| \, ds,$$

and thus, by Young's inequality,

 $||A_m||_{L^p([-1,1])} \le ||A_m||_{L^p([-2,2])}$

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$$\leq \frac{1}{(m-j-1)!} \|\mathbf{1}_{[-2,2]} t^{m-j-1}\|_{L^{1}(\mathbb{R})} \|\varphi f^{(m)}\|_{L^{p}(\mathbb{R})}$$
$$\leq \frac{2^{m-j+1}}{(m-j)!} \|f^{(m)}\|_{L^{p}([-2,2])}.$$

For $0 \le i \le m-1$ and $t \in [-1, 1]$, integration by parts gives

$$A_{i}(t) = \frac{(-1)^{i}}{(m-j-1)!} \int_{-2}^{-1} \partial_{s}^{i} [(t-s)^{m-j-1} \varphi^{(m-i)}(s)] f(s) \, ds$$
$$= \sum_{\ell=0}^{k} {\binom{i}{\ell}} \frac{(-1)^{i+\ell}}{(m-j-1-\ell)!} \int_{-2}^{-1} (t-s)^{m-j-1-\ell} \varphi^{(m-\ell)}(s) f(s) \, ds,$$

where $k := \min\{i, m - j - 1\}$. Using Young's inequality as before, we get

$$\|A_i\|_{L^p([-1,1])} \le \sum_{\ell=0}^k \binom{i}{\ell} \frac{2^{m-j-\ell+1}}{(m-j-\ell)!} (C_0 m)^{m-\ell} \|f\|_{L^p([-2,2])}.$$

Consequently,

$$\sum_{i=0}^{m-1} \binom{m}{i} \|A_i\|_{L^p([-1,1])} \le \sum_{i=0}^{m-1} \sum_{\ell=0}^k \binom{m}{i} \binom{i}{\ell} \frac{2^{m-j-\ell+1}}{(m-j-\ell)!} (C_0 m)^{m-\ell} \|f\|_{L^p([-2,2])}$$

and the lemma follows if we prove that

$$\sum_{i=0}^{m-1} \sum_{\ell=0}^{k} \binom{m}{i} \binom{i}{\ell} \frac{2^{m-j-\ell+1}}{(m-j-\ell)!} (C_0 m)^{m-\ell} \le C^m j!$$

for a universal constant C. The left-hand side equals

$$\sum_{\ell=0}^{m-j-1} \sum_{i=\ell}^{m-1} \frac{(m-\ell)!}{(m-i)!(i-\ell)!} \frac{m!}{\ell!(m-\ell)!} \frac{2^{m-j-\ell+1}}{(m-j-\ell)!} (C_0 m)^{m-\ell}$$

which is bounded by

$$\sum_{\ell=0}^{m-j-1} 2^{m-\ell} 2^m \frac{2^{m-j-\ell+1}}{(m-j-\ell)!} C_0^{m-\ell} e^m \frac{m!}{\ell!}$$

$$\leq e^m 2^{2m} 2^{j+1} C_0^j j! \sum_{\ell=0}^{m-j} \frac{(m-j)!}{\ell! (m-j-\ell)!} (2^2 C_0)^{m-j-\ell}$$

which is of the required form.

Lemma 2.4. There exist constants $C_0, C > 0$, depending only on the dimension n, such that the following holds. Let $1 \leq p \leq \infty$. Let U, V be bounded open subsets of \mathbb{R}^n such that $\overline{U} \subseteq V$. Let $f \in C^m(\overline{V})$. For all $0 < a \leq C_0 \operatorname{dist}(U, \partial V)$, all $0 \leq j \leq m$, and all $v \in \mathbb{S}^{n-1}$,

$$\frac{a^{j}}{j!} \|d_{v}^{j}f\|_{L^{p}(U)} \leq C^{m} \Big(\frac{a^{m}}{m!} \|d_{v}^{m}f\|_{L^{p}(V)} + \|f\|_{L^{p}(V)}\Big).$$

Proof. We may assume that v = (1, 0, ..., 0). Let us first show the assertion for $U = (-a, a)^n$ and $V = (-2a, 2a)^n$. For $x' := (x_2, ..., x_n) \in (-a, a)^{n-1}$, Lemma 2.3 gives

$$\frac{a^{j}}{j!} \|\partial_{1}^{j} f(\cdot, x')\|_{L^{p}((-a,a))} \leq C^{m} \Big(\frac{a^{m}}{m!} \|\partial_{1}^{m} f(\cdot, x')\|_{L^{p}((-2a,2a))} + \|f(\cdot, x')\|_{L^{p}((-2a,2a))} \Big).$$

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For $p = \infty$, the statement follows, by taking the supremum over all $x' \in (-a, a)^{n-1}$. For $p < \infty$, take the *p*-th power and integrate over $x' \in (-a, a)^{n-1}$.

In general, there is a constant $C_0 > 0$ depending only on n such that, if $a \leq C_0 \operatorname{dist}(U, \partial V)$, then U can be covered by a family \mathcal{Q} of cubes $Q = x + [-a, a]^n$ such that $2Q = x + [-2a, 2a]^n \subseteq V$ and such that any two cubes in \mathcal{Q} have disjoint interior. Then, for each $Q \in \mathcal{Q}$,

$$\frac{a^{j}}{j!} \|\partial_{1}^{j}f\|_{L^{p}(Q)} \leq C^{m} \Big(\frac{a^{m}}{m!} \|\partial_{1}^{m}f\|_{L^{p}(2Q)} + \|f\|_{L^{p}(2Q)} \Big).$$

For $p = \infty$, take the maximum over all $Q \in \mathcal{Q}$, for $p < \infty$, take the *p*-th power and the sum over all $Q \in \mathcal{Q}$.

Now we combine Lemma 2.4 with Lemma 2.1 (following [17, p. 193]).

Corollary 2.5. Let $1 \leq p \leq \infty$. Let U, V be bounded open subsets of \mathbb{R}^n such that $\overline{U} \subseteq V$. Let $f \in C^{\infty}(\overline{V})$. Then, for each integer $m \geq 2$, $j = 1, \ldots, m-1$, and all $v \in \mathbb{S}^{n-1}$,

$$\|d_v^j f\|_{L^p(U)} \le C^m \|f\|_{L^p(V)}^{1-j/m} \Big(\|d_v^m f\|_{L^p(V)}^{j/m} + m^j \|f\|_{L^p(V)}^{j/m} \Big),$$

where C > 0 is a constant depending only on U, V, and n.

Proof. Let W be the open δ -neighborhood of \overline{U} , where $\delta := \frac{1}{2} \operatorname{dist}(U, \partial V)$. Let φ be a C^{∞} function with support in W that equals 1 near \overline{U} and satisfies $\|\partial^{\alpha}\varphi\|_{L^{\infty}(\mathbb{R}^{n})} \leq C_{0}^{|\alpha|+1}m^{|\alpha|}$ for $|\alpha| \leq m$, where C_{0} only depends on U, V, and n (cf. [11, Theorem 1.4.2]). Then φf has a C^{∞} extension by zero outside W to all of \mathbb{R}^{n} which we also denote by φf . By Lemma 2.1,

$$\|d_v^j f\|_{L^p(U)} \le \|d_v^j(\varphi f)\|_{L^p(\mathbb{R}^n)} \le 2 \, \|(\varphi f)\|_{L^p(W)}^{1-j/m} \|d_v^m(\varphi f)\|_{L^p(W)}^{j/m}$$

and, by Lemma 2.4 with $a = C_1 \delta$, where $C_1 = C_1(n)$,

$$\begin{split} \|d_{v}^{m}(\varphi f)\|_{L^{p}(W)} &\leq \sum_{i=0}^{m} \binom{m}{i} \|d_{v}^{m-i}\varphi\|_{L^{\infty}(\mathbb{R}^{n})} \|d_{v}^{i}f\|_{L^{p}(W)} \\ &\leq \sum_{i=0}^{m} \binom{m}{i} C_{0}(C_{0}m)^{m-i}C^{m} \frac{i!}{(C_{1}\delta)^{i}} \Big(\frac{(C_{1}\delta)^{m}}{m!} \|d_{v}^{m}f\|_{L^{p}(V)} + \|f\|_{L^{p}(V)} \Big) \\ &\leq \sum_{i=0}^{m} \binom{m}{i} \frac{C_{0}(eC_{0}C\max\{C_{1}\delta,1\})^{m}}{(C_{0}C_{1}\delta)^{i}} \Big(\|d_{v}^{m}f\|_{L^{p}(V)} + m! \|f\|_{L^{p}(V)} \Big) \\ &\leq C_{2}^{m} \Big(\|d_{v}^{m}f\|_{L^{p}(V)} + m^{m} \|f\|_{L^{p}(V)} \Big). \end{split}$$

This implies the assertion.

2.3. The Cartan–Gorny inequality. The following result is due to Gorny [10] and independently to Cartan [7].

Proposition 2.6. Let $I \subseteq \mathbb{R}$ be a compact interval and $f \in C^m(I)$. Then, for $j = 1, \ldots, m-1$,

$$\|f^{(j)}\|_{L^{\infty}(I)} \leq 4e^{2j} \left(\frac{m}{j}\right)^{j} \|f\|_{L^{\infty}(I)}^{1-j/m} \max\left\{\|f^{(m)}\|_{L^{\infty}(I)}, \frac{m!}{|I|^{m}} \|f\|_{L^{\infty}(I)}\right\}^{j/m}.$$

The factor $(\frac{m}{i})^j$ is bounded by e^m .

3. Ultradifferentiable regularity by lacunary estimates

In this section, we prove three technical propositions on which most of the subsequent results are based.

Let $0 =: k_0 < k_1 < k_2 < \cdots$ be a strictly increasing infinite sequence of positive integers. A sequence (k_j) with these properties is called a *base sequence*. If additionally k_{j+1}/k_j , for $j \ge 1$, is bounded we say that (k_j) is a *special base sequence*.

Proposition 3.1. Let (k_j) be a base sequence. Let (m_j) and (m'_j) be positive increasing sequences of reals satisfying $m_j \leq m'_j$ for all j. Assume that $m_{k_{j+1}}/m'_{k_j}$ is bounded. Let $1 \leq p \leq \infty$ and $f \in C^{\infty}(\mathbb{R}^n)$.

If there are $C, \rho > 0$ such that

(1)
$$\max_{|\alpha|=k_j} \|\partial^{\alpha} f\|_{L^p(\mathbb{R}^n)} \le C(\rho m_{k_j})^{k_j}, \quad j \ge 0,$$

then there exist $C', \rho' > 0$ such that

(2)
$$\max_{|\alpha|=\ell} \|\partial^{\alpha}f\|_{L^{p}(\mathbb{R}^{n})} \leq C'(\rho'm'_{\ell})^{\ell}, \quad \ell \geq 0.$$

We have $\rho' = O(\rho)$ as $\rho \to 0$.

Proof. By (1), there are constants $C_1, \rho_1 > 0$ such that

(3)
$$\|d_v^{k_j}f\|_{L^p(\mathbb{R}^n)} \le C_1(\rho_1 m_{k_j})^{k_j}, \quad j \ge 0, v \in \mathbb{S}^{n-1}.$$

Let $\ell \geq 0$ and let $j \geq 0$ be such that $k_j \leq \ell < k_{j+1}$. By Lemma 2.1 and the assumptions on (k_j) , (m_j) , and (m'_j) ,

$$(4) \|d_v^\ell f\|_{L^p(\mathbb{R}^n)} \leq 2 \|d_v^{k_j} f\|_{L^p(\mathbb{R}^n)}^{1-\frac{\ell-k_j}{k_{j+1}-k_j}} \|d_v^{k_{j+1}} f\|_{L^p(\mathbb{R}^n)}^{\frac{\ell-k_j}{k_{j+1}-k_j}} \leq 2C_1 \left(\rho_1 m_{k_j}\right)^{k_j \left(1-\frac{\ell-k_j}{k_{j+1}-k_j}\right)} \left(\rho_1 m_{k_{j+1}}\right)^{k_{j+1} \frac{\ell-k_j}{k_{j+1}-k_j}} \leq 2C_1 \left(\rho_1 m'_{k_j}\right)^{k_j \left(1-\frac{\ell-k_j}{k_{j+1}-k_j}\right)} \left(C_2 \rho_1 m'_{k_j}\right)^{k_{j+1} \frac{\ell-k_j}{k_{j+1}-k_j}} \leq 2C_1 \left(C_2 \rho_1 m'_\ell\right)^\ell.$$

By polarization [13, Lemma 7.13], for $|\alpha| = \ell$ and v_1, \ldots, v_ℓ the list of standard unit vectors

$$\underbrace{e_1,\ldots,e_1}_{\alpha_1},\underbrace{e_2,\ldots,e_2}_{\alpha_2},\ldots,\underbrace{e_n,\ldots,e_n}_{\alpha_n},$$

we have

(5)
$$\partial^{\alpha} f = \frac{1}{\ell!} \sum_{\epsilon_1, \dots, \epsilon_\ell = 0}^{1} (-1)^{\ell - \sum \epsilon_\ell} \Big(\sum \epsilon_i \Big)^{\ell} d_{\sum \epsilon_i v_i / \sum \epsilon_i}^{\ell} f.$$

Therefore,

$$\|\partial^{\alpha} f\|_{L^{p}(\mathbb{R}^{n})} \leq 2C_{1}(C_{2}\rho_{1}m_{\ell}')^{\ell} \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} k^{\ell} \leq 2C_{1}(2eC_{2}\rho_{1}m_{\ell}')^{\ell}.$$

Thus (2) is proved.

Proposition 3.2. Let (k_j) be a special base sequence. Let (m_j) and (m'_j) be positive increasing sequences of reals satisfying $m_j \leq m'_j$ for all j and such that j/m_j is

bounded. Assume that $m_{k_{j+1}}/m'_{k_j}$ is bounded. Let $1 \leq p \leq \infty$. Let U, V be bounded open subsets of \mathbb{R}^n such that $\overline{U} \subseteq V$. Let $f \in C^{\infty}(\overline{V})$.

If there are $C, \rho > 0$ such that

(6)
$$\max_{|\alpha|=k_j} \|\partial^{\alpha} f\|_{L^p(V)} \le C(\rho m_{k_j})^{k_j}, \quad j \ge 0,$$

then there exist $C', \rho' > 0$ such that

(7)
$$\max_{|\alpha|=\ell} \|\partial^{\alpha}f\|_{L^{p}(U)} \leq C'(\rho'm'_{\ell})^{\ell}, \quad \ell \geq 0$$

Under the additional assumption that $j/m_j \to 0$, we have that $\rho' = O(\rho)$ as $\rho \to 0$.

Proof. By (6), there are constants $C_1, \rho_1 > 0$ such that

$$\|d_v^{k_j}f\|_{L^p(V)} \le C_1(\rho_1 m_{k_j})^{k_j}, \quad j \ge 0, \ v \in \mathbb{S}^{n-1}$$

Let $\ell \geq 0$ and let $j \geq 0$ be such that $k_j \leq \ell < k_{j+1}$. By Corollary 2.5 and the assumptions on (k_j) , (m_j) , and (m'_j) ,

$$\begin{split} \|d_{v}^{\ell}f\|_{L^{p}(U)} &\leq C^{k_{j+1}-k_{j}} \left(\|d_{v}^{k_{j}}f\|_{L^{p}(V)}^{1-\frac{\ell-k_{j}}{k_{j+1}-k_{j}}} \|d_{v}^{k_{j+1}}f\|_{L^{p}(V)}^{\frac{\ell-k_{j}}{k_{j+1}-k_{j}}} + (k_{j+1}-k_{j})^{\ell-k_{j}} \|d_{v}^{k_{j}}f\|_{L^{p}(V)} \right) \\ &\leq C_{1}C_{2}^{k_{j}} \left((\rho_{1}m_{k_{j}})^{k_{j}\left(1-\frac{\ell-k_{j}}{k_{j+1}-k_{j}}\right)} (\rho_{1}m_{k_{j+1}})^{k_{j+1}\frac{\ell-k_{j}}{k_{j+1}-k_{j}}} + (C_{3}k_{j})^{\ell-k_{j}} (\rho_{1}m_{k_{j}})^{k_{j}} \right) \\ &\leq C_{1}C_{2}^{k_{j}} \left((\rho_{1}m_{k_{j}}')^{k_{j}\left(1-\frac{\ell-k_{j}}{k_{j+1}-k_{j}}\right)} (C_{4}\rho_{1}m_{k_{j}}')^{k_{j+1}\frac{\ell-k_{j}}{k_{j+1}-k_{j}}} + (C_{5}m_{k_{j}}')^{\ell-k_{j}} (\rho_{1}m_{k_{j}}')^{k_{j}} \right) \\ &\leq C_{1}C_{2}^{k_{j}} \left((C_{4}\rho_{1}m_{k_{j}}')^{\ell} + ((C_{5}+\rho_{1})m_{k_{j}}')^{\ell} \right) \\ &\leq C_{1}(C_{6}m_{\ell}')^{\ell}. \end{split}$$

If we assume that $j/m_j \to 0$ (actually $k_j/m_{k_j} \to 0$ is enough), then for each $\tau > 0$ there is j_{τ} such that $k_j/m_{k_j} \leq \tau$ for all $j \geq j_{\tau}$. Thus $C_5 = C_3 \rho_1$ provided that $j \geq j_{\rho_1}$ and so

$$\|d_v^{\ell}f\|_{L^p(U)} \le C_1(C_7\rho_1 m_{\ell}')^{\ell}, \quad \ell \ge k_{j_{\rho_1}}$$

For $\ell < k_{j_{\rho_1}}$,

$$\|d_v^\ell f\|_{L^p(U)} \le C_1 (C_6 m_\ell')^\ell \le C_1 \max\{\rho_1^{-\ell} : \ell < k_{j\rho_1}\} \cdot (C_6 \rho_1 m_\ell')^\ell.$$

Since $\rho_1 = O(\rho)$, we get

$$\|d_v^\ell f\|_{L^p(U)} \le C'(\rho' m_\ell')^\ell, \quad \ell \ge 0,$$

where $\rho' = O(\rho)$ and $C' = C'(\rho')$.

To end the proof, it suffices to apply $\|\cdot\|_{L^p(U)}$ to (5) and use the estimate for $\|d_v^\ell f\|_{L^p(U)}$.

Remark 3.3. The requirement $k_0 = 0$ for the base sequence is important in Proposition 3.1. Without this assumption there is no reason why a function satisfying (1) should also fulfill (2) for $\ell = 0$. In the local setting of Proposition 3.2, the assumption $k_0 = 0$ can be made without loss of generality, by adjusting the constant C.

Note that, in the proof of Proposition 3.2, the assumption that k_{j+1}/k_j is bounded is necessary to estimate the factor $C^{k_{j+1}-k_j}$. For the term $(k_{j+1}-k_j)^{\ell-k_j}$ it is enough to use that j/m_j and $m_{k_{j+1}}/m'_{k_j}$ are bounded. **Proposition 3.4.** Let (k_j) be a base sequence. Let (m_j) and (m'_j) be positive increasing sequences of reals satisfying $m_j \leq m'_j$ for all j. Assume that $m_{k_{j+1}}/m'_{k_j}$ is bounded. Let $f \in C^{\infty}(\mathbb{R}^n)$.

If there are $C, \sigma > 0$ such that

(8)
$$\max_{|\alpha|=k_j} \|x^{\alpha}f\|_{L^{\infty}(\mathbb{R}^n)} \le C(\sigma m_{k_j})^{k_j}, \quad j \ge 0,$$

then there exist $C', \sigma' > 0$ such that

(9)
$$\max_{|\alpha|=\ell} \|x^{\alpha}f\|_{L^{\infty}(\mathbb{R}^{n})} \leq C'(\sigma'm'_{\ell})^{\ell}, \quad \ell \geq 0.$$

We have $\sigma' = O(\sigma)$.

Proof. Recall that

$$|x| \le n^{1/2 - 1/p} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$
 if $p \ge 2$.

So, if $|\alpha| = k \ge 2$, then

$$|x^{\alpha}| \le |x|^{k} \le n^{(k-2)/2} \sum_{i=1}^{n} |x_{i}|^{k} \le n^{(k-2)/2} \sum_{|\alpha|=k} |x^{\alpha}| \le n^{(3k-2)/2} \max_{|\alpha|=k} |x^{\alpha}|.$$

If $|\alpha| = 1$, then

$$|x^{\alpha}| \le |x| \le \sum_{i=1}^{n} |x_i| \le n \max_{|\alpha|=1} |x^{\alpha}|.$$

The assumption (8) gives

$$|x|^{k_j}|f(x)| \le C(n^{3/2}\sigma m_{k_j})^{k_j}, \quad j \ge 0, \, x \in \mathbb{R}^n.$$

Then, for $k_j \leq \ell < k_{j+1}$,

$$|x|^{\ell}|f(x)| = (|x|^{k_j}|f(x)|)^{1-\frac{\ell-k_j}{k_{j+1}-k_j}} (|x|^{k_{j+1}}|f(x)|)^{\frac{\ell-k_j}{k_{j+1}-k_j}}.$$

So (8) implies (9), by a computation analogous to (4).

4. Denjoy-Carleman classes

Let $M = (M_k)_{k\geq 0}$ be a positive sequence of real numbers. We say that M is a weight sequence if $M_0 = 1 \leq M_1 \leq \mu_2 \leq \mu_3 \leq \cdots$, where $\mu_k := M_k/M_{k-1}$. Then also the sequence $m_k := M_k^{1/k}$ is increasing. Let U be an open subset of \mathbb{R}^n and $1 \leq p \leq \infty$. The L^p -based local Denjoy-

Let U be an open subset of \mathbb{R}^n and $1 \leq p \leq \infty$. The L^p -based local Denjoy– Carleman class of Roumieu type on U is the set $\mathcal{E}_{L^p}^{\{M\}}(U)$ of all $f \in C^{\infty}(U)$ such that for all open relatively compact subsets $\Omega \Subset U$ there exist constants $C, \rho > 0$ such that

(10)
$$\max_{|\alpha|=k} \|\partial^{\alpha} f\|_{L^{p}(\Omega)} \leq C(\rho m_{k})^{k}, \quad k \in \mathbb{N}.$$

Analogously, the L^p -based local Denjoy-Carleman class of Beurling type is the set $\mathcal{E}_{L^p}^{(M)}(U)$ of all $f \in C^{\infty}(U)$ that satisfy (10) for all Ω and all ρ with $C = C(\Omega, \rho)$. We use the placeholder [·] for either the Roumieu case {·} or the Beurling case (·).

Replacing (10) with various global requirements we obtain natural global Denjoy– Carleman classes,

$$\mathcal{B}_{L^p}^{\{M\}}(\mathbb{R}^n) := \{ f \in C^\infty(\mathbb{R}^n) : \exists C, \rho > 0 \ \forall \alpha : \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \le C(\rho m_{|\alpha|})^{|\alpha|} \},$$

$$\mathcal{B}_{L^p}^{(M)}(\mathbb{R}^n) := \{ f \in C^{\infty}(\mathbb{R}^n) : \forall \rho > 0 \ \exists C > 0 \ \forall \alpha : \|\partial^{\alpha} f\|_{L^p(\mathbb{R}^n)} \le C(\rho m_{|\alpha|})^{|\alpha|} \},\$$

Gelfand-Shilov classes $\mathcal{S}^{\{M\}}(\mathbb{R}^n)$ (resp. $\mathcal{S}^{(M)}(\mathbb{R}^n)$) consisting of all $f \in C^{\infty}(\mathbb{R}^n)$ such that there exist $\rho, \sigma > 0$ such that (resp. for all $\rho, \sigma > 0$)

$$\sup_{\alpha,\beta\in\mathbb{N}^n}\sup_{x\in\mathbb{R}^n}\frac{|x^{\alpha}\partial^{\beta}f(x)|}{(\rho m_{|\alpha|})^{|\alpha|}(\sigma m_{|\beta|})^{|\beta|}}<\infty,$$

and the classes $\mathcal{D}^{[M]}(\mathbb{R}^n) := \mathcal{B}_{L^{\infty}}^{[M]}(\mathbb{R}^n) \cap C_c^{\infty}(\mathbb{R}^n)$ of all $\mathcal{B}_{L^{\infty}}^{[M]}$ -functions with compact support. (Cleary, $\mathcal{D}^{[M]}(\mathbb{R}^n)$ is nontrivial only in the non-quasianalytic setting.) Note that various aspects of these global classes have recently been studied in [14, 15, 19]. We have continuous (with respect to the natural locally convex topologies) inclusions

and, for $1 \le p \le q \le \infty$,

(11)
$$\mathcal{B}_{L^p}^{[M]}(\mathbb{R}^n) \longrightarrow \mathcal{B}_{L^q}^{[M]}(\mathbb{R}^n) \qquad \qquad \mathcal{E}_{L^p}^{[M]}(U) == \mathcal{E}_{L^q}^{[M]}(U),$$

provided that M is derivation closed, i.e.,

(12)
$$\sup_{k\geq 0} \left(\frac{M_{k+1}}{M_k}\right)^{1/(k+1)} < \infty$$

This follows from the Sobolev inequality, since the latter condition guarantees stability under taking derivatives. For the local classes, we have equality if (12) holds, because $\|\cdot\|_{L^p(\Omega)} \leq |\Omega|^{1/p-1/q} \|\cdot\|_{L^q(\Omega)}$.

If (k_j) is a base sequence, we consider the sets $\mathcal{E}_{L^p,(k_j)}^{[M]}(U)$, $\mathcal{B}_{L^p,(k_j)}^{[M]}(\mathbb{R}^n)$, $\mathcal{D}_{(k_j)}^{[M]}(\mathbb{R}^n)$ of C^{∞} functions that satisfy the defining estimates for multiindices α with $|\alpha| = k_j$.

Theorem 4.1. Let $1 \le p \le \infty$. Let (k_j) be a base sequence and $M = (M_j)$ a weight sequence such that $m_{k_{j+1}}/m_{k_j}$ is bounded. We have:

(a) $\mathcal{B}_{L^{p},(k_{j})}^{[M]}(\mathbb{R}^{n}) = \mathcal{B}_{L^{p}}^{[M]}(\mathbb{R}^{n}).$ (b) $\mathcal{D}_{(k_{j})}^{[M]}(\mathbb{R}^{n}) = \mathcal{D}^{[M]}(\mathbb{R}^{n}).$

Let (k_i) be a special base sequence and $U \subseteq \mathbb{R}^n$ open. Then:

(c) $\mathcal{E}_{L^p,(k_j)}^{\{M\}}(U) = \mathcal{E}_{L^p}^{\{M\}}(U)$ provided that j/m_j is bounded. (d) $\mathcal{E}_{L^p,(k_j)}^{(M)}(U) = \mathcal{E}_{L^p}^{(M)}(U)$ provided that $j/m_j \to 0$.

Proof. (a) follows from Proposition 3.1 and (b) is a consequence of (a) by intersecting with $C_c^{\infty}(\mathbb{R}^n)$. (c) and (d) follow from Proposition 3.2.

In (c) and (d), the assumption that the base sequence (k_j) is special cannot be omitted; this follows from Remark 6.5 below.

That j/m_j is bounded (resp. tends to zero) is equivalent to the fact that $\mathcal{E}_{L^{\infty}}^{\{M\}}(U)$ (resp. $\mathcal{E}_{L^{\infty}}^{(M)}(U)$) contains $C^{\omega}(U)$. Note that $\mathcal{E}_{L^p}^{[M]}(U) \subseteq \mathcal{E}_{L^q}^{[M]}(U)$ if $1 \leq q \leq p \leq \infty$. Before we come to a similar result in the Gelfand–Shilov case, let us recall a result due to [9]. (In that paper only the Roumieu case is treated, but the Beurling case can be proved analogously, see [20, Lemma 4].) To this end we need some further conditions on the weight sequence. We say that a weight sequence $M = (M_j)$ has moderate growth if

(13)
$$\sup_{j,k\geq 1} \left(\frac{M_{j+k}}{M_j M_k}\right)^{1/(j+k)} < \infty$$

Note that (13) implies (12).

Proposition 4.2 ([9]). Let $M = (M_j)$ be a weight sequence with moderate growth. Assume that j/m_j is bounded in the Roumieu case and tends to zero in the Beurling case. Then the following are equivalent:

- (a) $f \in \mathcal{S}^{[M]}(\mathbb{R}^n)$.
- (b) There are constants $C, \rho, \sigma > 0$ (resp. for all $\rho, \sigma > 0$ there is C > 0) such that

$$\sup_{x} |x^{\alpha} f(x)| \le C(\rho m_{|\alpha|})^{|\alpha|} \quad and \quad \sup_{x} |\partial^{\beta} f(x)| \le C(\sigma m_{|\beta|})^{|\beta|}$$

for all $\alpha, \beta \in \mathbb{N}^n$.

(c) There are constants $C, \rho, \sigma > 0$ (resp. for all $\rho, \sigma > 0$ there is C > 0) such that

 $\sup_{x} |x^{\alpha} f(x)| \leq C(\rho m_{|\alpha|})^{|\alpha|} \quad and \quad \sup_{x} |x^{\beta} \widehat{f}(x)| \leq C(\sigma m_{|\beta|})^{|\beta|}$ for all $\alpha, \beta \in \mathbb{N}^{n}$.

In view of this proposition, we allow a second base sequence (ℓ_j) and define $\mathcal{S}_{(k_j),(\ell_j)}^{[M]}(\mathbb{R}^n)$ to be the set of $f \in C^{\infty}(\mathbb{R}^n)$ such that there exist $C, \rho, \sigma > 0$ (resp. for all $\rho, \sigma > 0$ there is C > 0) such that

$$\max_{|\alpha|=k_j} \sup_{x} |x^{\alpha} f(x)| \le C(\rho m_{|\alpha|})^{|\alpha|} \quad \text{and} \quad \max_{|\beta|=\ell_j} \sup_{x} |\partial^{\beta} f(x)| \le C(\sigma m_{\ell_j})^{\ell_j}$$

for all $j \ge 0$.

Theorem 4.3. Let (k_j) and (ℓ_j) be base sequences. Let $M = (M_j)$ be a weight sequence with moderate growth. Assume that $m_{k_{j+1}}/m_{k_j}$ and $m_{\ell_{j+1}}/m_{\ell_j}$ are bounded. Then:

(a) $\mathcal{S}^{\{M\}}_{(k_j),(\ell_j)}(\mathbb{R}^n) = \mathcal{S}^{\{M\}}(\mathbb{R}^n)$ provided that j/m_j is bounded. (b) $\mathcal{S}^{(M)}_{(k_j),(\ell_j)}(\mathbb{R}^n) = \mathcal{S}^{(M)}(\mathbb{R}^n)$ provided that $j/m_j \to 0$.

Proof. This follows from Proposition 3.1, Proposition 3.4, and Proposition 4.2. \Box

Remark 4.4. Liess [17] (extending an argument of [2]) proved that, under a number of assumptions on the sequence $M = (M_j)$, the equality $\mathcal{E}_{L^2,(k_j)}^{\{M\}}(U) = \mathcal{E}_{L^2}^{\{M\}}(U)$ implies that $m_{k_{j+1}}/m_{k_j}$ is bounded. His assumptions are the following:

- (a) $m_j \leq m_{j+1}$ for all j and j/m_j is bounded.
- (b) m_{j+1}/m_j is bounded.
- (c) There exists $g: \mathbb{N} \to (0, \infty)$ such that $m_{kj} \leq g(k)m_j$ for all k and j.
- (d) For every c > 0 there is c' > 0 such that $m_j \le c m_k$ implies $j \le c'k$.

Let us assume that M is a weight sequence of moderate growth such that j/m_j is bounded. Then it is easily seen that the conditions (a)–(c) are satisfied. On the other hand, for a weight sequence, (c) is equivalent to (13) (take k = 2 and use [18, Theorem 1]).

Observe that, assuming $m_j \leq m_{j+1}$, (d) is equivalent to the existence of an integer $n \geq 2$ such that $\liminf_{k\to\infty} m_{nk}/m_k =: a > 1$. Indeed, for all $\ell \geq 1$ and large k, we have $a^{\ell}m_k \leq m_{n^{\ell}k}$. If a > 1 and $m_j \leq c m_k$, then $c/a^{-\ell} < 1$, provided that ℓ is large enough, and thus $m_j < m_{n^{\ell}k}$. As m_j is increasing, we conclude that $j \leq n^{\ell}k$. It is clear that a = 1 violates (d).

5. Other ultradifferentiable classes

Classically, besides Denjoy–Carleman classes a lot of attention was devoted to ultradifferentiable classes, sometimes called Braun–Meise–Taylor classes due to the foundational article [6], defined in terms of a weight function; they have their origin in work of Beurling.

A weight function is a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ satisfying $\omega(0) = 0$, $\omega(2t) = O(\omega(t))$, $\log t = o(\omega(t))$ as $t \to \infty$ and such that $\varphi(t) := \omega(e^t)$ is convex. It is no restriction to assume $\omega(t) = 0$ for $0 \le t \le 1$ (cf. [21, Section 11.1]).

The local classes $\mathcal{E}_{L^p}^{\{\omega\}}(U)$ and $\mathcal{E}_{L^p}^{(\omega)}(U)$ are defined in analogy to Denjoy–Carleman classes, where now

(14)
$$\max_{|\alpha|=k} \|\partial^{\alpha} f\|_{L^{p}(\Omega)} \leq C \exp\left(\frac{1}{\rho}\varphi^{*}(\rho k)\right), \quad k \geq 0$$

with $\varphi^*(s) := \sup_{t>0} (st - \varphi(t))$, plays the role of (10). More precisely, we have

$$\mathcal{E}_{L^p}^{\{\omega\}}(U) := \bigcap_{\Omega \Subset U} \bigcup_{\rho > 0} \bigcup_{C > 0} \{ f \in C^{\infty}(U) : f \text{ satisfies (14)} \},\$$
$$\mathcal{E}_{L^p}^{(\omega)}(U) := \bigcap_{\Omega \Subset U} \bigcap_{\rho > 0} \bigcup_{C > 0} \{ f \in C^{\infty}(U) : f \text{ satisfies (14)} \}.$$

The global classes $\mathcal{B}_{L^p}^{[\omega]}(\mathbb{R}^n)$, $\mathcal{S}^{[\omega]}(\mathbb{R}^n)$, and $\mathcal{D}^{[\omega]}(\mathbb{R}^n)$ are defined in a straightforward way. Furthermore, we consider $\mathcal{E}_{L^p,(k_j)}^{[\omega]}(U)$, $\mathcal{B}_{L^p,(k_j)}^{[\omega]}(\mathbb{R}^n)$, $\mathcal{S}_{(k_j),(\ell_j)}^{[\omega]}(\mathbb{R}^n)$, and $\mathcal{D}_{(k_j)}^{[\omega]}(\mathbb{R}^n)$, all of them defined in the obvious manner. (Note that $\mathcal{D}^{[\omega]}(\mathbb{R}^n)$ is non-trivial if and only if $\int_1^\infty t^{-2}\omega(t) dt < \infty$; cf. [6].)

There is an overarching framework for ultradifferentiable classes introduced in [22] which goes beyond Denjoy–Carleman and Braun–Meise–Taylor classes and, on a technical level, reduces the proofs to handling certain families of weight sequences. See [21] for a comprehensive survey of the theory.

From now on, let \mathfrak{M} be a totally ordered family of weight sequences, i.e., if $M, M' \in \mathfrak{M}$, then either $M_j \leq M'_j$ or $M'_j \leq M_j$ for all j.

We define

$$\mathcal{E}_{L^{p}}^{\{\mathfrak{M}\}}(U) := \bigcap_{\Omega \Subset U} \bigcup_{M \in \mathfrak{M}} \bigcup_{\rho > 0} \bigcup_{C > 0} \{ f \in C^{\infty}(U) : f \text{ satisfies (10)} \},\$$
$$\mathcal{E}_{L^{p}}^{(\mathfrak{M})}(U) := \bigcap_{\Omega \Subset U} \bigcap_{M \in \mathfrak{M}} \bigcap_{\rho > 0} \bigcup_{C > 0} \{ f \in C^{\infty}(U) : f \text{ satisfies (10)} \},\$$

and in the evident analogous way also the global classes $\mathcal{B}_{L^p}^{[\mathfrak{M}]}(\mathbb{R}^n)$, $\mathcal{S}^{[\mathfrak{M}]}(\mathbb{R}^n)$, and $\mathcal{D}^{[\mathfrak{M}]}(\mathbb{R}^n)$ as well as $\mathcal{E}_{L^p,(k_j)}^{[\mathfrak{M}]}(U)$, $\mathcal{B}^{[\mathfrak{M}]}_{L^p,(k_j)}(\mathbb{R}^n)$, $\mathcal{S}^{[\mathfrak{M}]}_{(k_j),(\ell_j)}(\mathbb{R}^n)$, and $\mathcal{D}^{[\mathfrak{M}]}_{(k_j)}(\mathbb{R}^n)$.

Theorem 5.1. Let $1 \le p \le \infty$. Let ω be a weight function. For $\mathfrak{M} := \{M^{(\rho)}\}_{\rho>0}$, where $M_k^{(\rho)} := \exp(\frac{1}{\rho}\varphi^*(\rho k))$, we have the identities:

$$\begin{aligned} \mathcal{E}_{L^{p}}^{[\omega]}(U) &= \mathcal{E}_{L^{p}}^{[\mathfrak{M}]}(U), \quad \mathcal{E}_{L^{p},(k_{j})}^{[\omega]}(U) = \mathcal{E}_{L^{p},(k_{j})}^{[\mathfrak{M}]}(U), \\ \mathcal{B}_{L^{p}}^{[\omega]}(\mathbb{R}^{n}) &= \mathcal{B}_{L^{p}}^{[\mathfrak{M}]}(\mathbb{R}^{n}), \quad \mathcal{B}_{L^{p},(k_{j})}^{[\omega]}(\mathbb{R}^{n}) = \mathcal{B}_{L^{p},(k_{j})}^{[\mathfrak{M}]}(\mathbb{R}^{n}), \\ \mathcal{S}^{[\omega]}(\mathbb{R}^{n}) &= \mathcal{S}^{[\mathfrak{M}]}(\mathbb{R}^{n}), \quad \mathcal{S}_{(k_{j}),(\ell_{j})}^{[\omega]}(\mathbb{R}^{n}) = \mathcal{S}_{(k_{j}),(\ell_{j})}^{[\mathfrak{M}]}(\mathbb{R}^{n}), \\ \mathcal{D}^{[\omega]}(\mathbb{R}^{n}) &= \mathcal{D}^{[\mathfrak{M}]}(\mathbb{R}^{n}), \quad \mathcal{D}_{(k_{j})}^{[\omega]}(\mathbb{R}^{n}) = \mathcal{D}_{(k_{j})}^{[\mathfrak{M}]}(\mathbb{R}^{n}), \end{aligned}$$

where (k_i) and (ℓ_i) are base sequences.

Proof. This is shown by the proof of [22, Lemma 5.14] which is based on the following property of \mathfrak{M} , see [22, (5.10)],

(15)
$$\forall \sigma > 0 \; \exists H \ge 1 \; \forall \rho > 0 \; \exists C \ge 1 \; \forall k \in \mathbb{N} : \sigma^k M_k^{(\rho)} \le C M_k^{(H\rho)}.$$

In fact, for given $\rho, \sigma > 0$ there exist $C, H \ge 1$ such that

$$\sup_{j \ge 0} \frac{\max_{|\alpha| = k_j} \|\partial^{\alpha} f\|_{L^p(\Omega)}}{M_{k_j}^{(H\rho)}} \le C \sup_{j \ge 0} \frac{\max_{|\alpha| = k_j} \|\partial^{\alpha} f\|_{L^p(\Omega)}}{\sigma^{k_j} M_{k_j}^{(\rho)}}$$

and $M_{k_j}^{(H\rho)} = \exp(\frac{1}{H\rho}\varphi^*(H\rho k_j))$. In the Gelfand–Shilov case, we observe that additionally

$$\sup_{j\geq 0} \frac{\max_{|\alpha|=\ell_j} \sup_x |x^{\alpha} f(x)|}{M_{\ell_j}^{(H\rho)}} \leq C \sup_{j\geq 0} \frac{\max_{|\alpha|=\ell_j} \sup_x |x^{\alpha} f(x)|}{\sigma^{\ell_j} M_{\ell_j}^{(\rho)}}.$$

The case $\mathcal{S}^{[\omega]}(\mathbb{R}^n) = \mathcal{S}^{[\mathfrak{M}]}(\mathbb{R}^n)$ follows from

$$\sup_{\alpha,\beta} \frac{\sup_{x} |x^{\alpha}\partial^{\beta}f(x)|}{M_{|\alpha|}^{(H\rho)}M_{|\beta|}^{(H\rho)}} \le C \sup_{\alpha,\beta} \frac{\sup_{x} |x^{\alpha}\partial^{\beta}f(x)|}{\tau^{|\alpha|}\sigma^{|\beta|}M_{|\alpha|}^{(\rho)}M_{|\beta|}^{(\rho)}}.$$

taking constants C, H which work for both σ and τ .

The family \mathfrak{M} from Theorem 5.1 has an inherent moderate growth property (see [22, (5.6)]):

(16)
$$M_{j+k}^{(\rho)} \le M_j^{(2\rho)} M_k^{(2\rho)}, \quad j,k \ge 0.$$

In general, we say that \mathfrak{M} has [moderate growth] if, in the Roumieu case (i.e., $[\cdot] = \{\cdot\}$),

$$\forall M \in \mathfrak{M} \exists M' \in \mathfrak{M} \exists C > 0 \; \forall j, k \ge 0 : M_{j+k} \le C^{j+k} M'_j M'_k,$$

and, in the Beurling case (i.e., $[\cdot] = (\cdot)$),

$$\forall M \in \mathfrak{M} \exists M' \in \mathfrak{M} \exists C > 0 \; \forall j, k \ge 0 : M'_{j+k} \le C^{j+k} M_j M_k$$

Lemma 5.2. If (k_j) is a special base sequence and \mathfrak{M} has [moderate growth], then, in the Roumieu case (i.e., $[\cdot] = \{\cdot\}$),

(17)
$$\forall M \in \mathfrak{M} \exists M' \in \mathfrak{M} : m_{k_{i+1}}/m'_{k_i} \text{ is bounded},$$

and, in the Beurling case (i.e., $[\cdot] = (\cdot)$),

 $\forall M \in \mathfrak{M} \exists M' \in \mathfrak{M} : m'_{k_{i+1}}/m_{k_i} \text{ is bounded.}$ (18)

Proof. By assumption, there exists a positive integer a such that $k_{j+1} \leq ak_j$ for all $j \geq 1$. Moderate growth in the Roumieu case guarantees that, for given $M \in \mathfrak{M}$ there exist C > 0 and $M' \in \mathfrak{M}$ (depending on a) such that

$$M_{ak_j} \le C^{ak_j} (M'_{k_j})^a.$$

This implies $m_{k_{i+1}} \leq m_{ak_i} \leq Cm'_{k_i}$. The Beurling case is similar.

Theorem 5.3. Let $1 \leq p \leq \infty$. Let $(k_j), (\ell_j)$ be base sequences. Let \mathfrak{M} satisfy (17) in the Roumieu case and (18) in the Beurling case. We have:

- (a) $\mathcal{B}_{L^{p},(k_{j})}^{[\mathfrak{M}]}(\mathbb{R}^{n}) = \mathcal{B}_{L^{p}}^{[\mathfrak{M}]}(\mathbb{R}^{n}).$ (b) $\mathcal{D}_{(k_{j})}^{[\mathfrak{M}]}(\mathbb{R}^{n}) = \mathcal{D}^{[\mathfrak{M}]}(\mathbb{R}^{n}).$

Assume additionally that, for all $M \in \mathfrak{M}$, j/m_j is bounded, in the Roumieu case, and tends to zero, in the Beurling case. Then:

- (c) \$\mathcal{E}_{L^{p},(k_{j})}^{[\mathcal{M}]}(U) = \mathcal{E}_{L^{p}}^{[\mathcal{M}]}(U)\$ for all open U ⊆ ℝⁿ, provided that (k_j) is a special base sequence.
 (d) \$\mathcal{S}_{(k_{j}),(\ell_{j})}^{[\mathcal{M}]}(\mathcal{R}^{n}) = \mathcal{S}^{[\mathcal{M}]}(\mathcal{R}^{n}), provided that \$\mathcal{M}\$ has [moderate growth].

Proof. Since the sequences in \mathfrak{M} are totally ordered, we may assume that the sequences provided by (17) and (18) also satisfy $m_j \leq m'_j$ or $m'_j \leq m_j$ for all j, respectively. Then (a), (b), and (c) follow easily from Proposition 3.1 and Proposition 3.2. Finally, (d) follows from Proposition 3.1, Proposition 3.4, and the generalization [20, Lemma 4] of Proposition 4.2. \square

To deduce a version for Braun–Meise–Taylor classes, it suffices to note that for a weight function ω and $\mathfrak{M} = \{M^{(\rho)}\}_{\rho>0}$ the associated family from Theorem 5.1,

- $\omega(t) = O(t)$ as $t \to \infty$ if and only if $j/m_j^{(\rho)}$ is bounded for some $\rho > 0$,
- $\omega(t) = o(t)$ as $t \to \infty$ if and only if $j/m_i^{(\rho)} \to 0$ for all $\rho > 0$;

cf. [22, Lemma 5.7 and Corollary 5.15].

Theorem 5.4. Let $1 \le p \le \infty$. Let (k_i) , (ℓ_i) be special base sequences. Let ω be a weight function. We have:

(a) $\mathcal{B}_{L^{p},(k_{j})}^{[\omega]}(\mathbb{R}^{n}) = \mathcal{B}_{L^{p}}^{[\omega]}(\mathbb{R}^{n}).$ (b) $\mathcal{D}_{(k_{j})}^{[\omega]}(\mathbb{R}^{n}) = \mathcal{D}^{[\omega]}(\mathbb{R}^{n}).$

Assume additionally that $\omega(t) = O(t)$ as $t \to \infty$, in the Roumieu case, and $\omega(t) =$ o(t), in the Beurling case. Then:

(c) $\mathcal{E}_{L^{p},(k_{j})}^{[\omega]}(U) = \mathcal{E}_{L^{p}}^{[\omega]}(U)$ for all open $U \subseteq \mathbb{R}^{n}$. (d) $\mathcal{S}_{(k_{j}),(\ell_{j})}^{[\omega]}(\mathbb{R}^{n}) = \mathcal{S}^{[\omega]}(\mathbb{R}^{n})$.

Proof. This is a corollary of Theorem 5.1, Lemma 5.2, and Theorem 5.3.

Remark 5.5. The family $\mathfrak{M} = \{M^{(\rho)}\}_{\rho>0}$ associated with a weight function in Theorem 5.1 has the following property:

(a) Let $0 < \sigma \leq \rho$. If there is $C_1 > 0$ such that $m_i^{(\sigma)} \leq C_1 m_k^{(\rho)}$, then there is $C_2 = C_2(\sigma, \rho, C_1) > 0$ such that $j \leq C_2 k$.

So, in this case, validity of (17) or (18) entails that (k_i) is a special base sequence; the converse holds by (16) and Lemma 5.2.

To see (a), note first that for any positive integer N we have

$$m_k^{(N\rho)} = \exp(\frac{1}{N\rho k}\varphi^*(N\rho k)) = m_{Nk}^{(\rho)}$$

By (15), there exist $C = C(\rho) \ge 1$ and an integer $H \ge 2$ such that

$$4m_k^{(\rho)} \le C^{1/k} m_k^{(H\rho)} = C^{1/k} m_{Hk}^{(\rho)}, \quad k \ge 0.$$

For $k \geq k_0$, we have $C^{1/k} < 2$ and hence $2m_k^{(\rho)} < m_{Hk}^{(\rho)}$ and, by iteration,

$$2^{\ell} m_k^{(\rho)} < m_{H^{\ell} k}^{(\rho)}$$

If $m_i^{(\sigma)} \leq C_1 m_k^{(\rho)}$, then, choosing ℓ such that $C_1 2^{-\ell} < 1$, we find

$$m_j^{(\sigma)} < m_{H^\ell k}^{(\rho)}.$$

Then the assumption $j \ge \left\lceil \frac{\rho}{\sigma} \right\rceil H^{\ell} k$ leads to

$$m_{H^{\ell}k}^{(\rho)} \le m_{H^{\ell}k}^{\left\lceil \frac{\rho}{\sigma} \rceil \sigma\right)} = m_{\left\lceil \frac{\rho}{\sigma} \rceil H^{\ell}k}^{(\sigma)} < m_{H^{\ell}k}^{(\rho)},$$

a contradiction. Thus $j < C_2 k$, where $C_2 := \left\lceil \frac{\rho}{\sigma} \right\rceil H^{\ell}$. For $k < k_0$, there are only finitely many j satisfying $m_j^{(\sigma)} \leq C_1 m_{k_0}^{(\rho)}$ (since $\lim_{j\to\infty} m_j^{(\sigma)} = \infty$; see [6, Remark 1.3]). So adding this bound to C_2 gives the assertion for all k.

Remark 5.6. Under the [moderate growth] assumption, we have, for $1 \le p \le q \le \infty$, the continuous inclusions $\mathcal{B}_{L^p}^{[\mathfrak{M}]}(\mathbb{R}^n) \subseteq \mathcal{B}_{L^q}^{[\mathfrak{M}]}(\mathbb{R}^n)$ and $\mathcal{E}_{L^p}^{[\mathfrak{M}]}(U) = \mathcal{E}_{L^q}^{[\mathfrak{M}]}(U)$, in particular, $\mathcal{B}_{L^p}^{[\omega]}(\mathbb{R}^n) \subseteq \mathcal{B}_{L^q}^{[\omega]}(\mathbb{R}^n)$ and $\mathcal{E}_{L^p}^{[\omega]}(U) = \mathcal{E}_{L^q}^{[\omega]}(U)$.

6. Comparison with some recent results

In this section, we compare our result with the ones obtained by Albano and Mughetti [1] and we discuss optimality of the conditions imposed on the base sequence (k_j) and on the weights.

Let $M = (M_i)$ be a weight sequence. Let $I \subseteq \mathbb{R}$ be a compact interval. We define

$$\mathcal{B}_{L^{\infty}}^{\{M\}}(I) := \{ f \in C^{\infty}(I) : \exists C, \rho > 0 \; \forall j \ge 0 : \| f^{(j)} \|_{L^{\infty}(I)} \le C(\rho m_j)^j \}, \\ \mathcal{B}_{L^{\infty}}^{(M)}(I) := \{ f \in C^{\infty}(I) : \forall \rho > 0 \; \exists C > 0 \; \forall j \ge 0 : \| f^{(j)} \|_{L^{\infty}(I)} \le C(\rho m_j)^j \},$$

as well as $\mathcal{B}_{L^{\infty}(k_i)}^{[M]}(I)$ in the obvious way.

Theorem 6.1. Let (k_i) be a special base sequence. Let $M = (M_i)$ be a weight sequence such that $m_{k_{j+1}}/m_{k_j}$ is bounded. Then:

- (a) $\mathcal{B}_{L^{\infty},(k_j)}^{\{M\}}(I) = \mathcal{B}_{L^{\infty}}^{\{M\}}(I)$ provided that j/m_j is bounded. (b) $\mathcal{B}_{L^{\infty},(k_j)}^{(M)}(I) = \mathcal{B}_{L^{\infty}}^{(M)}(I)$ provided that $j/m_j \to 0$.

Proof. Assume that $f \in \mathcal{B}_{L^{\infty},(k_j)}^{\{M\}}(I)$. Let $\ell \geq 0$ and let $j \geq 0$ be such that $k_j \leq \ell < k_{j+1}$. By Proposition 2.6 and the properties of (k_j) and (m_j) ,

$$\|f^{(\ell)}\|_{L^{\infty}(I)} \leq 4e^{2(\ell-k_j)}e^{k_{j+1}-k_j} \|f^{(k_j)}\|_{L^{\infty}(I)}^{1-\frac{\ell-k_j}{k_{j+1}-k_j}}$$

$$\cdot \max\left\{ \|f^{(k_{j+1})}\|_{L^{\infty}(I)}^{\frac{\ell-k_{j}}{k_{j+1}-k_{j}}}, \left(\frac{k_{j+1}-k_{j}}{|I|}\right)^{\ell-k_{j}}\|f^{(k_{j})}\|_{L^{\infty}(I)}^{\frac{\ell-k_{j}}{k_{j+1}-k_{j}}}\right\}$$

$$\leq 4C C_1^{\ell} \left(\rho m_{k_j}\right)^{k_j \left(1 - \frac{\ell - k_j}{k_{j+1} - k_j}\right)} \left(\left(\rho m_{k_{j+1}}\right)^{\frac{k_{j+1} \left(\ell - k_j\right)}{k_{j+1} - k_j}} + \left(\frac{C_2 k_j}{|I|}\right)^{\ell - k_j} \left(\rho m_{k_j}\right)^{\frac{k_j \left(\ell - k_j\right)}{k_{j+1} - k_j}} \right)$$

$$\leq 4C C_1^{\ell} \left((C_3 \rho m_{k_j})^{\ell} + \left(\frac{C_4 m_{k_j}}{|I|}\right)^{\ell - k_j} \left(\rho m_{k_j}\right)^{k_j} \right)$$

$$\leq 8C \left(C_5 \max\{\rho, C_4 |I|^{-1}\}\right)^{\ell} m_{\ell}^{\ell}.$$

Hence $f \in \mathcal{B}_{L^{\infty}}^{\{M\}}(I)$.

Assume that $f \in \mathcal{B}_{L^{\infty},(k_j)}^{(M)}(I)$. Let $\sigma > 0$ be given. If we assume that $j/m_j \to 0$, then we may proceed as in the proof of Proposition 3.2 to see that choosing $\rho > 0$ sufficiently small we may find

$$\|f^{(\ell)}\|_{L^{\infty}(I)} \le D(\sigma m_{\ell})^{\ell}, \quad \ell \ge 0,$$

where $D = D(\sigma)$.

It is now an easy exercise to deduce the following extensions.

Theorem 6.2. Let (k_i) be a special base sequence. Let \mathfrak{M} have [moderate growth] and assume that, for all $M \in \mathfrak{M}$, j/m_j is bounded, in the Roumieu case, and tends to zero, in the Beurling case. Then $\mathcal{B}_{L^{\infty},(k_j)}^{[\mathfrak{M}]}(I) = \mathcal{B}_{L^{\infty}}^{[\mathfrak{M}]}(I)$.

Theorem 6.3. Let (k_i) be a special base sequence. Let ω be a weight function such that $\omega(t) = O(t)$ as $t \to \infty$, in the Roumieu case, and $\omega(t) = o(t)$, in the Beurling case. Then $\mathcal{B}_{L^{\infty},(k_i)}^{[\omega]}(I) = \mathcal{B}_{L^{\infty}}^{[\omega]}(I).$

Remark 6.4. In [1], instead of assuming $m_{k_{i+1}}/m_{k_i}$ bounded, the authors require that

(19)
$$\exists i_0 \ge 0 \ \forall i, j > i_0 \text{ and } i < j < ik : m_j \le m_i m_k.$$

If there is $\epsilon > 0$ such that $m_j > \epsilon$ for all j, then it is not hard to check that (19) implies

(20)
$$\exists C \ge 1 \ \forall i, j \ge 1 \text{ and } i < j < ik : m_j \le C m_i m_k.$$

But (20) (for k = 3 and j = 2i) implies that M has moderate growth (cf. [18, Theorem 1). And, if (k_i) is a special base sequence, then Lemma 5.2 implies that $m_{k_{j+1}}/m_{k_j}$ is bounded.

Remark 6.5. In Theorem 6.1, the assumption that (k_i) is a special base sequence cannot be omitted. In fact, by [1, Theorem 1.6], for each positive sequence $M = (M_j)$ such that $\limsup_{j \to \infty} m_j = \infty$ there is a base sequence (k_j) and $f \in \mathcal{B}_{L^{\infty},(k_j)}^{\{M\}}(I)$ but $f \notin \mathcal{B}_{L^{\infty}}^{\{M\}}(I)$. More precisely, a weight sequence $N = (N_j)$ and sequences of positive integers (k_j) and (ℓ_j) with the following properties are constructed:

- $\dots < \ell_j < k_j < \ell_{j+1} < k_{j+1} < \dots$, $N_{k_j} = M_{k_j}$, and
- $N_{\ell_i} = 2^{2^{\ell_j}} M_{\ell_i}.$

It is well-known that for each weight sequence N there exists $f \in \mathcal{B}_{L^{\infty}}^{\{N\}}(\mathbb{R})$ with $|f^{(j)}(0)| \geq N_j$ for all $j \geq 0$ (cf. [23, Theorem 1] or [21, Lemma 2.4]) from which the assertion follows easily.

Note that this result does not contradict Theorem 4.1: if also $M = (M_j)$ is a weight sequence, then the above conditions imply

$$m_{k_{j+1}} = n_{k_{j+1}} \ge n_{\ell_{j+1}} = 2^{2^{\ell_{j+1}}/\ell_{j+1}} m_{\ell_{j+1}} \ge 2^{2^{\ell_{j+1}}/\ell_{j+1}} m_{k_j}$$

so that $m_{k_{j+1}}/m_{k_j}$ is unbounded.

It is possible to adapt the proof of this result to the Beurling case as well as to the general setting of a weight structure \mathfrak{M} ; see Section A.

Optimality of the conditions. Let us summarize the situation in the case of one weight sequence $M = (M_j)$ satisfying that j/m_j is bounded. Let $I \subseteq \mathbb{R}$ be a compact interval. By Theorem 6.1,

(21)
$$\mathcal{B}_{L^{\infty},(k_{i})}^{\{M\}}(I) = \mathcal{B}_{L^{\infty}}^{\{M\}}(I)$$

provided that k_{j+1}/k_j and $m_{k_{j+1}}/m_{k_j}$ are bounded.

As seen in Remark 4.4 and Lemma 5.2, the dependencies can be summarized by the following diagram:



For instance, for all Gevrey sequences $M = (k^{sk})_k$, where $s \ge 1$, k_{j+1}/k_j is bounded if and only if $m_{k_{j+1}}/m_{k_j} = (k_{j+1}/k_j)^s$ is bounded.

Theorem 1.4 in [1] states that (21) holds if k_{j+1}/k_j is bounded and M satisfies (19). In view of Remark 6.4, (19) entails moderate growth so that the conditions of [1, Theorem 1.4] imply those of Theorem 6.1.

Regarding Remark 6.5, one may ask if (ℓ_j) and (N_j) with analogous properties can be found for *each* given sequence (k_j) such that k_{j+1}/k_j is unbounded. That would imply that boundedness of k_{j+1}/k_j is necessary for the validity of (21). The construction in [1] alluded to in Remark 6.5 and generalized in Section A does not provide this. On the other hand, in the situation of Remark 4.4, in particular, in the Gevrey case, the validity of $\mathcal{E}_{L^2,(k_j)}^{\{M\}}(U) = \mathcal{E}_{L^2}^{\{M\}}(U)$ implies that $m_{k_{j+1}}/m_{k_j}$ and k_{j+1}/k_j are bounded.

Remark 6.6. Liess (see Remark 4.4) works with L^2 based bounds of Roumieu type, but his arguments apply to L^p based bounds for all $1 \leq p \leq \infty$ of Roumieu and Beurling type if the weight sequence $M = (M_j)$ satisfies the properties listed in Remark 4.4. (The inclusion $\mathcal{E}_{L^p,(k_j)}^{[M]}(U) \subseteq \mathcal{E}_{L^p}^{[M]}(U)$ is continuous, by the closed graph theorem, since convergence in the left-hand side entails pointwise convergence, by the Sobolev inequality. In the Beurling case, the quantifiers for the constants in Liess's proof change but this does not affect the conclusion. The arguments also work for global classes of type $\mathcal{B}_{L^p}^{[M]}(I)$, where I is a compact interval.)

APPENDIX A.

Let $\mathfrak{M} = \{M^{(s)} : s > 0\}$ be a totally ordered family of weight sequences such that $\lim_{j\to\infty} (M_j^{(s)})^{1/j} = \infty$ for all s > 0. We claim that there exists a weight sequence $N = (N_i)$ and sequences of positive integers (k_i) and (ℓ_i) with the following properties:

- (1) $\dots < \ell_j < k_j < \ell_{j+1} < k_{j+1} < \dots,$ (2) $N_{k_j} = j^{-k_j} M_{k_j}^{(1/j)}$ for all $j \ge 1$, and (3) $N_{\ell_j} = 2^{2^{\ell_j}} M_{\ell_j}^{(j)}$ for all $j \ge 1$.

There exists $f \in \mathcal{B}_{L^{\infty}}^{\{N\}}(\mathbb{R})$ with $|f^{(j)}(0)| \geq N_j$ for all $j \geq 0$ (cf. [23, Theorem 1] or [21, Lemma 2.4]). Then (2) guarantees that $f \in \mathcal{B}_{L^{\infty},(k_j)}^{(\mathfrak{M})}(\mathbb{R})$. On the other hand, $f \notin \mathcal{B}_{L^{\infty}}^{\{\mathfrak{M}\}}(\mathbb{R})$, by (3). This shows that

(22)
$$\mathcal{B}_{L^{\infty},(k_j)}^{(\mathfrak{M})}(\mathbb{R}) \neq \mathcal{B}_{L^{\infty}}^{(\mathfrak{M})}(\mathbb{R}) \text{ and } \mathcal{B}_{L^{\infty},(k_j)}^{\{\mathfrak{M}\}}(\mathbb{R}) \neq \mathcal{B}_{L^{\infty}}^{\{\mathfrak{M}\}}(\mathbb{R}).$$

It is easy to adjust the arguments so that they give (22) for \mathbb{R} replaced with any interval $I \subseteq \mathbb{R}$.

Let us construct $N = (N_i)$, (k_i) , and (ℓ_i) with the desired properties. Let $N_0 := 1$. We will define N_j in terms of $\nu_j := N_j/N_{j-1}$ for $j \ge 1$. For simplicity of notation, we put $A_j := 2^{2^j}$.

Let us first assume that (k_i) and (ℓ_i) are arbitrary positive sequences of integers satisfying (1). We will define ν_i such that (2) and (3) hold. In the end, we discuss how (k_j) and (ℓ_j) must be chosen such that (ν_j) is increasing, i.e., $N = (N_j)$ is a weight sequence.

For $j \ge 1$, set

$$\begin{split} \nu_{k_j} &:= \Big(\frac{M_{k_j}^{(1/j)}}{j^{k_j} A_{\ell_j} M_{\ell_j}^{(j)}}\Big)^{1/(k_j - \ell_j)}, \\ \nu_{\ell_{j+1}} &:= \Big(\frac{j^{k_j} A_{\ell_{j+1}} M_{\ell_{j+1}}^{(j+1)}}{M_{k_j}^{(1/j)}}\Big)^{1/(\ell_{j+1} - k_j)}, \\ \nu_k &:= \nu_{k_j}, \quad \text{for } \ell_j + 1 \le k \le k_j, \\ \nu_k &:= \nu_{\ell_{j+1}}, \quad \text{for } k_j + 1 \le k \le \ell_{j+1}. \end{split}$$

Additionally,

$$\nu_1 = \dots = \nu_{\ell_1} := (A_{\ell_1} M_{\ell_1}^{(1)})^{1/\ell_1}.$$

By construction, (2) and (3) are satisfied. Taking ℓ_1 large enough we have $\nu_1 \geq 1$ (since $(M_j^{(1)})^{1/j} \to \infty$). Next let us check that we may choose $k_1 < \ell_2 < k_3 < \cdots$ in such a way that (ν_i) is increasing. More precisely, we have to make sure that

$$\nu_{\ell_j} \le \nu_{k_j} \le \nu_{\ell_{j+1}}$$
 for all $j \ge 1$

It is easy to see that taking $k_1 > \ell_1$ sufficiently large gives $\nu_{\ell_1} \leq \nu_{k_1}$ (since $(M_j^{(1)})^{1/j} \to \infty$). Next $\nu_{k_j} \leq \nu_{\ell_{j+1}}$ for $j \geq 1$ amounts to

(23)
$$\left(\frac{M_{k_j}^{(1/j)}}{j^{k_j}A_{\ell_j}M_{\ell_j}^{(j)}}\right)^{1/(k_j-\ell_j)} \le \left(\frac{j^{k_j}A_{\ell_{j+1}}M_{\ell_{j+1}}^{(j+1)}}{M_{k_j}^{(1/j)}}\right)^{1/(\ell_{j+1}-k_j)}.$$

If $\ell_j < k_j$ are already chosen, then this can be achieved by taking $\ell_{j+1} > k_j$ sufficiently large since $(M_k^{(j+1)})^{1/k} \to \infty$ as $k \to \infty$: if the left-hand side L of (23) satisfies $L \leq 1$, then it is clear; otherwise $L^{\ell_{j+1}-k_j} \leq L^{\ell_{j+1}}$ and

$$L \le \left(\frac{j^{k_j} A_{\ell_{j+1}} M_{\ell_{j+1}}^{(j+1)}}{M_{k_j}^{(1/j)}}\right)^{1/\ell_{j+1}}$$

visibly can be achieved. Finally, $\nu_{\ell_j} \leq \nu_{k_j}$ for $j \geq 2$ means

(24)
$$\left(\frac{(j-1)^{k_{j-1}}A_{\ell_j}M_{\ell_j}^{(j)}}{M_{k_{j-1}}^{(1/(j-1))}}\right)^{1/(\ell_j-k_{j-1})} \le \left(\frac{M_{k_j}^{(1/j)}}{j^{k_j}A_{\ell_j}M_{\ell_j}^{(j)}}\right)^{1/(k_j-\ell_j)}.$$

Here we assume that $k_{j-1} < \ell_j$ are already chosen and we choose $k_j > \ell_j$ sufficiently large such that (24) holds. Similarly as for (23), we see that this is possible because $(M_k^{(1/j)})^{1/k} \to \infty$ as $k \to \infty$. This ends the construction of $N = (N_j)$, (k_j) , and (ℓ_j) .

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