Symbolic Dynamics of Quadratic Polynomials
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Abstract. We describe various combinatorial invariants of iterated quadratic polynomials $p_c(z) = z^2 + c$ and discuss their relations: external angles, itineraries, kneading sequences, internal addresses, Hubbard trees and others. Among others, we cover the following topics:

- Necessary and sufficient criteria for the existence of quadratic polynomials realizing given combinatorics, in particular a characterization of complex admissible kneading sequences;
- A forcing order on the space of Hubbard trees; this leads to a combinatorial model which describes the tree structure of the Mandelbrot set;
- The admissible kneading sequences in $\{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$ have positive mass with respect to $\frac{1}{2}$-$\frac{1}{2}$ product measure;
- Most points on Julia sets J_c, $c \neq -2$, resp. the Mandelbrot set $\mathcal{M}$ (in the sense of harmonic measure ω_c resp. ω, or in the sense of Hausdorff dimension) are not biaccessible;
- For ω-almost every $c \in \partial\mathcal{M}$, the critical point $0 \in J_c$ is typical in the sense of the Birkhoff Ergodic Theorem applied to harmonic measure ω_c.

We also collect old and new algorithms and show how to turn these combinatorial invariants into each other.

Contents

I. Introduction

1. Introduction

Our understanding of the dynamics of polynomials in the complex plane, and of the Mandelbrot set, builds largely on the seminal work of Douady and Hubbard, the *Orsay notes* [**DH1**]. They showed that the topology, geometry and dynamics of polynomial Julia sets can be understood in terms of combinatorics and symbolic dynamics. The underlying reason is that complex differentiable families of maps $p\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are very rigid: while there is a huge set of continuous maps from $\overline{\mathbb{C}}$ to itself, complex differentiable maps are automatically rational maps of finite degree, which are described by a finite number of complex coordinates. If in addition $p^{-1}(\infty) = \{\infty\}$, then p is a polynomial. In the same spirit, iteration of p yields a dynamical system which becomes very rigid when p is known to be complex differentiable.

$$\left\{ \begin{array}{c} \text{topology} \\ \text{dynamics} \end{array} \right\} \overset{\longleftarrow\!\longrightarrow}{\text{complex structure}} \left\{ \begin{array}{c} \text{combinatorics} \\ \text{symbolic dynamics} \end{array} \right\}$$

When investigating a family of holomorphic maps, such as $p_c\colon z \mapsto z^2 + c$ or $E_c\colon z \mapsto e^z + c$ depending on a complex parameter c, great interest lies in distinguishing and describing different types of dynamics; a plan of investigation could proceed as follows:

(1) distinguish different types of dynamics in combinatorial terms;
(2) subdivide the parameter plane (in our examples, the complex c-plane) into sets with combinatorially equivalent dynamics ("combinatorial classes");
(3) describe the combinatorial possibilities and their locations in parameter space;
(4) describe the set of maps within each combinatorial class: show that either the class consists of a single map, or describe how the various maps differ;
(5) give a topological or geometric model for the dynamics on the Julia set in a given combinatorial class, and discuss which properties of the Julia set are preserved in the model.

There are at least three fundamental combinatorial concepts to describe polynomial dynamics: *external angles*, *Hubbard trees*, and *itineraries/kneading sequences*, each with their particular advantages. At least for quadratic polynomials, the first and third problems have been investigated quite successfully in terms of these concepts, and they help to deal with the second question (a combinatorial subdivision of parameter space). There is also substantial progress on the fourth question: the fact that a combinatorial class consists of a single map is known as *combinatorial rigidity* (or *triviality of a fiber* of the Mandelbrot set). In cases of failing rigidity, one investigates the (topological or quasiconformal) deformation space consisting of (topologically or quasiconformally) conjugate maps; see e.g. [**MS**]. For all combinatorial classes, there are several ways to build "nice" topological models of the Julia set which usually are homeomorphic to the actual Julia set only if the latter is locally connected [**D3**].

In the list above, the first three questions are combinatorial ones, while the remaining two transfer the combinatorial results back into the dynamical or parameter spaces of complex dynamics. This article focuses on the combinatorial side: we discuss the three different concepts of symbolic dynamics named above and explore their properties and interrelations. We will exclusively restrict to the case of quadratic polynomials because it is the simplest, and because we have now a rather complete picture in the quadratic case; for the case of "unicritical polynomials" $z \mapsto z^d + c$ with an integer $d \geq 2$, some generalizations have been obtained ([**LS**, Section 12] and recently [**Kau2**]).

We begin by describing three prototypical quadratic polynomials (Figure 1.1). Let $p_c\colon z \mapsto z^2 + c$ be a (monic and centered) quadratic polynomial with filled-in Julia set K_c and Julia set $J_c = \partial K_c$. The *Mandelbrot set* $\mathcal{M}$ is the *quadratic connectedness locus*, that is the set of all $c \in \mathbb{C}$ for which J_c (or equivalently K_c) is connected.

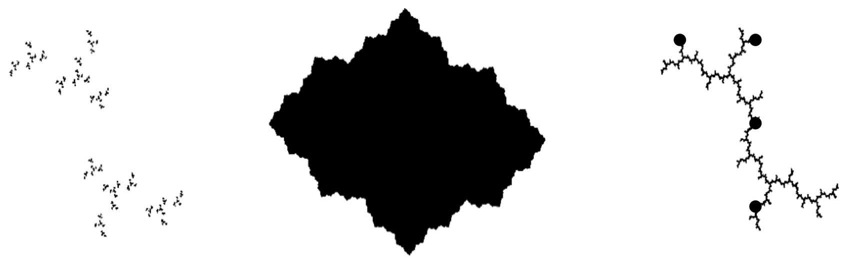

FIGURE 1.1. (1): A totally disconnected quadratic Julia set, which is a Cantor set ($c = 0.1 + i$); (2): a Julia set which is a topological circle ($c = -0.5 + 0.1i$); (3): the Julia set of a quadratic polynomial in which the critical orbit is strictly preperiodic ($c = i$); the postcritical points are marked by heavy dots.

In the first picture, the $J_c = K_c$ is totally disconnected, which happens if and only if $c \notin \mathcal{M}$; since Julia sets of polynomials are always compact and perfect (i.e. they contain no isolated points), the Julia set is a Cantor set. The *equipotential of the critical point* is the set

$$\{z \in \mathbb{C}\colon |p_c^{\circ n}(z)/p_c^{\circ n}(0)| \to 1 \text{ as } n \to \infty\} \ .$$

This is a lemniscate homeomorphic to the figure $\mathtt{8}$, and J_c is contained in the two bounded complementary components. If we call these two bounded components $U_\mathtt{0}$ and $U_\mathtt{1}$ such that $c \in U_\mathtt{1}$, then $J_c \subset U_\mathtt{0} \cup U_\mathtt{1}$ and every $z \in J_c$ gets an *itinerary* in $\Sigma := \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$ describing which components the orbit of z visits in order. It is quite easy to see that this itinerary map is a homeomorphism from J_c onto Σ which conjugates the dynamics on J_c to the shift map on Σ: this is one prototypical example of how a simple symbolic dynamical system completely describes the topology and dynamics of a polynomial Julia set completely, up to homeomorphism.

The second picture displays a Julia set J_c which is homeomorphic to a circle, so that the dynamics of p_c of topologically conjugate to angle doubling on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ (or to the squaring map on $\partial\mathbb{D} \subset \mathbb{C}$). Again, we have an easy model dynamical system which completely describes topology and dynamics of the Julia set. The circle $\mathbb{S}^1$ will be the space of external angles, on which the natural dynamics is the doubling map.

The third picture shows a dendrite Julia set: it is simply path connected, i.e. every pair of points in J_c is connected by a unique path within J_c (up to reparametrization). The critical orbit is finite (marked by heavy dots) and spans a finite subtree within the Julia set (the Hubbard tree). It turns out that the topology and dynamics of the Julia set are completely described by the Hubbard tree with its dynamics: if $T \subset J_c$ is the Hubbard tree, then $J_c = \overline{\cup_{k\geq 0} p_c^{-k}(T)}$.

As described, the three methods (itineraries, Hubbard trees, angle doubling on the circle) are natural to different kinds of Julia sets. We will now discuss the three methods in more detail, pointing out how they make sense for more general quadratic Julia sets. Since all disconnected quadratic Julia sets have topologically (and even quasiconformally) conjugate dynamics, most of the interest lies in the connected case, hence in the case $c \in \mathcal{M}$ and thus in the Mandelbrot itself as the parameter space of connected quadratic Julia sets. We will restrict most of the discussion to this case.

External Rays and External Angles: A rather general method to understand the dynamics on the Julia set in terms of angle doubling on $\mathbb{S}^1$ applies whenever the Julia set is connected, i.e. $c \in \mathcal{M}$ (as in our second and third examples): there is a unique conformal isomorphism (a Riemann map) $\varphi_c\colon \overline{\mathbb{C}} \setminus K_c \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with $\varphi_c(z)/z \to 1$ as $z \to \infty$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. It turns out that φ_c conjugates the dynamics outside of K_c to the complex squaring map on $\mathbb{C} \setminus \overline{\mathbb{D}}$: $\varphi_c(p_c(z)) = (\varphi_c(z))^2$. The *external ray* at *external angle* $\vartheta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is $R_c(\vartheta) := \varphi_c^{-1}(\{re^{2\pi i\vartheta}\colon r > 1\})$. In the context of Julia sets, we will speak of *dynamic rays* instead of external rays in order to distinguish them from *parameter rays* of the Mandelbrot set (see below). Every dynamic ray $R_c(\vartheta)$ is an analytic curve which connects K_c to ∞, and $p_c(R_c(\vartheta)) = R_c(2\vartheta)$. The dynamics of p_c outside of K_c is thus easy to understand. The goal is to use dynamic rays to extend this information to the Julia set, which is the locus of the interesting dynamics.

We say that the dynamic ray $R_c(\vartheta)$ *lands at* $z \in K_c$ if $z = \lim_{r\searrow 1} \varphi_c^{-1}(re^{2\pi i\vartheta})$. Not every dynamic ray necessarily lands, but the set of external angles ϑ such that $R_c(\vartheta)$ lands for given c has full measure. If J_c is locally connected, then every ray lands at a unique point $z(\vartheta) \in J_c$, the landing point $z(\vartheta)$ depends continuously on ϑ, and every $z \in J_c$ is the landing point of at least one $R_c(\vartheta)$. In this case, the inverse Riemann map extends continuously to a map $\varphi_c^{-1}\colon \overline{\mathbb{C}} \setminus \mathbb{D} \to (\overline{\mathbb{C}} \setminus K_c) \cup J_c$, and restriction gives a continuous surjection $\varphi_c^{-1}\colon \mathbb{S}^1 \to J_c$ (the *Carathéodory loop*) which semiconjugates the dynamics: $\varphi_c^{-1}(e^{2\pi i(2\vartheta)}) = p_c(\varphi_c^{-1}(e^{2\pi i\vartheta}))$. This helps to understand the dynamics of p_c on the Julia set, and we get a topological model for the Julia set as a quotient of $\mathbb{S}^1$ under the equivalence relation:

$$\vartheta \sim \vartheta' \quad \text{if and only if} \quad R_c(\vartheta) \text{ and } R_c(\vartheta') \text{ land at the same point .}$$

In fact, this dynamic quotient of $\mathbb{S}^1$ is locally connected, and it provides a homeomorphic model for J_c if and only if J_c is also locally connected. In a similar spirit, one can get a model for the filled-in Julia set K_c by an appropriate quotient of $\overline{\mathbb{D}}$: this is the *pinched disk model* of K_c, see [**D3**]. It turns out that topology and dynamics of these models can be constructed in terms of only one external angle corresponding to a dynamic ray landing at the critical value (with a slight modification if the critical value is in the interior of K_c, so that no ray can land there). This point of view has been pioneered by Thurston [**Th**] and expanded by Keller [**Ke**].

The main point for us is that the topology and dynamics on J_c (for $c \in \mathcal{M}$) can be described in terms of one of the simplest cases of symbolic dynamics: the doubling map on $\mathbb{S}^1$.

Itineraries, Kneading Sequence, and Internal Address: One fundamental idea of symbolic dynamics is to subdivide phase space into a number of disjoint components U_i and code the orbit of a point by the sequence of components U_i which are visited in order by this orbit. Depending on the chosen partition, similar symbolic spaces can encode the same dynamical system in different ways. This works best if the U_i form a *Markov partition*: i.e. if the image of every U_i is the union of some U_j (i.e. if the image contains every U_j which it intersects), and p restricted to each U_i is injective. A prototypical example was given in the example above of a Cantor quadratic Julia set: we had $J_c \subset U_{\mathtt{0}} \cup U_{\mathtt{1}}$ and $p_c(J_c \cap U_{\mathtt{0}}) = p_c(J_c \cap U_{\mathtt{1}}) = J_c$. Much of the complication in generalizing itineraries to non-Cantor Julia sets arises from the fact that we often do not have a Markov partition.

For quadratic polynomials, itineraries have interesting similarities to external angles (or to their dyadic expansions, see below), sometimes confusingly so. However, there are striking and important differences.

The dyadic expansion of an external angle $\vartheta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the sequence of binary digits $\vartheta = 0.x_1x_2x_3\ldots$ with $\vartheta = \sum_{k\geq 1} x_k 2^{-k}$, where $x_k \in \{0,1\}$. This dyadic expansion can be read off from the following symbolic dynamic system: set $\vartheta_k := 2^{k-1}\vartheta$ (the orbit of ϑ under angle doubling on $\mathbb{S}^1$) and set $U_0 = [0,1/2)$ and $U_1 = [1/2,1)$. Then $x_k = 0$ iff $\vartheta_k \in U_0$ and $x_k = 1$ iff

$\vartheta_k \in U_1$, so the dyadic expansion of the external angle is the itinerary with respect to this Markov partition. Note that another partition with the same property is $U_0' = (0, 1/2]$ and $U_1' = (1/2, 1]$; the itineraries of ϑ with respect to these two partitions differ if and only if some $2^k\vartheta = 0$ in $\mathbb{S}^1$, i.e. if and only if ϑ is a dyadic rational number $a/2^k$: this exactly reflects the ambiguity in the binary representation of dyadic rational numbers.

In the context of real quadratic polynomials, the Julia set restricted to $\mathbb{R}$ is an interval or a Cantor set. It is natural to subdivide this real Julia set at the critical point (which is the point of symmetry) and write the itinerary of an arbitrary orbit as a sequence of symbols L and R, coding whether an orbit is to the left or to the right of the critical point. A natural generalization to the complex case $p_c\colon z \mapsto z^2 + c$ if $c \notin \mathcal{M}$ was described above. If $c \in \mathcal{M}$, i.e. K_c is connected, one can proceed as follows: *supposing* there is a dynamic ray $R_c(\vartheta)$ which lands at the critical value c, then the two dynamic rays $R_c(\vartheta/2)$ and $R_c((\vartheta + 1)/2)$ land at the critical point 0 and divide $\mathbb{C}$ into two open complementary parts $U_{\mathtt{0}}$ and $U_{\mathtt{1}}$, i.e. $\mathbb{C} = U_{\mathtt{0}} \cup U_{\mathtt{1}} \cup R_c(\vartheta) \cup R_c((\vartheta+1)/2) \cup \{0\}$. Choose labels so that $c = p_c(0) \in U_{\mathtt{1}}$. For any $z \in \mathbb{C}$, the *itinerary* is the sequence of labels in $\{\mathtt{0}, \mathtt{1}, \star\}^{\mathbb{N}^*}$ describing whether the $k-1$-st image $p_c^{\circ(k-1)}(z)$ is in $U_{\mathtt{0}}$, in $U_{\mathtt{1}}$, or on the common boundary.[1]

The same partition associates an itinerary also to every dynamic ray $R(\varphi)$. In fact, this itinerary can be read off purely from symbolic dynamics on $\mathbb{S}^1$: given ϑ, cut $\mathbb{S}^1$ at the two preimages of ϑ into the two arcs $A_{\mathtt{1}} = (\vartheta/2, (\vartheta+1)/2)$ and $A_{\mathtt{0}} = ((\vartheta + 1)/2, \vartheta/2)$ (in the positive orientation, so that $\vartheta \in A_1$ and $0 \in A_0$), and set $A_\star := \{\vartheta/2, (\vartheta + 1)/2\}$. Then the itinerary of $R(\varphi)$ with respect to the partition formed by dynamic rays can be read off from the itinerary of φ under the angle doubling map in $\mathbb{S}^1$. This construction is quite similar to the binary expansion of the external angle, but in this case the Markov property fails.

Different dynamic rays have different external angles, so dynamic rays can be distinguished by their external angles. To a point $z \in J_c$ one can associate the external angles of all dynamic rays (if any) which land at z, but there is no preferred choice if several rays land at the same point. It is not obvious to tell which rays land together.

Itineraries are intrinsically defined for every $z \in K_c$. A dynamic ray $R_c(\vartheta)$ can land at $z \in J_c$ only if ray and prospective landing point have identical itineraries; in good cases, this is an "if and only if" condition[2]. In particular,

[1]If there is an attracting or parabolic orbit, the critical point is in the interior of the filled-in Julia set, but one can adapt the construction.

[2]"Good cases" are those in which the Julia set is locally connected, and all periodic points are repelling: in the presence of attracting and parabolic orbits, the construction can successfully be

two dynamic rays land together if and only if they have the same itineraries. However, it can no longer be possible to distinguish different dynamic rays by their itineraries, and it is also not obvious which itineraries in $\{\mathtt{0}, \mathtt{1}, \star\}^{\mathbb{N}^*}$ occur for dynamic rays and points in J_c.

Note that there is a choice involved in the construction of itineraries in case that several dynamic rays land at the critical value: one has to be consistent for any given dynamical system, of course; it turns out that this choice does not affect the itinerary of any point on the Hubbard tree, and in particular it does not affect the kneading sequence defined below.

The most important itinerary is that of the critical value: this is called the *kneading sequence* of p_c. This name comes historically from symbolic dynamics of interval maps, but it proves very useful in the complex case as well. The kneading sequence alone determines the dynamics of a quadratic polynomial essentially uniquely, at least in the "good cases" mentioned above (up to certain symmetries that can be easily described). Here is the idea if $c \in J_c$. Different points in J_c have different itineraries, so one can describe J_c as the space of those itineraries which occur, appropriately topologized; the dynamics on the itineraries is simply the shift map. To see which itineraries can occur, recall that $c \in J_c$, so ν is not periodic. If some $z \in J_c$ maps after finitely many steps onto the critical point 0 with itinerary $\star\nu$, then the itinerary of z is a finite string over $\{\mathtt{0}, \mathtt{1}\}$, followed by $\star\nu$. Otherwise, the itinerary of z is in $\Sigma = \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$. Since $\star\nu$ must be the only immediate preimage of ν (which describes the critical value), the itineraries $\mathtt{0}\nu$ and $\mathtt{1}\nu$ are forbidden, and so are the backwards orbits of those. (Penrose [**Pen1**] uses this as the starting point for his construction of Julia sets as glueing spaces, starting from the topological space Σ, glueing $\mathtt{0}\nu$ to $\mathtt{1}\nu$, and continuing this glueing along backwards orbits so as to obtain a topological dynamical system.)

The *internal address* is a strictly monotone integer sequence (finite or infinite) which recodes the kneading sequence in a compact and readable form, displaying the most important data. For example, the more important periodic points in J_c (including periodic branch points) have periods that occur in the internal address. In the Mandelbrot set, the internal address gives the

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modified; polynomial Julia sets with Cremer points are never locally connected, and if there are Siegel disks, then all points on the boundary of the Siegel disk have the same periodic itinerary unless they map onto the critical point after finitely many iterations. Local connectivity has been established for Julia sets without indifferent periodic orbits as long as they are not infinitely renormalizable [**HY**], and even for certain infinitely renormalizable ones [**Ly2**]. When local connectivity fails, there is still a locally connected model dynamics (the dynamic quotient of the circle) which in the case of Cremer points models a Siegel disk. In the other cases we still get a continuous map from the Julia set to the locally connected model; this map is many-to-one, and all points in the Julia set which correspond to the same model point have the same itinerary.

periods of the hyperbolic components of lowest period on the path in $\mathcal{M}$ from 0 to c, which help to describe the location of parameters within $\mathcal{M}$. Symmetries leading to different dynamical systems with identical kneading sequences can conveniently be described by additional information called *angled internal addresses.*

Hubbard Trees: These trees were introduced in [**DH1**] for polynomials for which all critical orbits are finite; in this case, the filled-in Julia set K is connected and locally connected. There is a finite topological tree $T \subset K$ which connects all points on the critical orbits, which is forward invariant under the dynamics, and which is minimal in the sense that no subtree has the same properties. (In the presence of a superattracting orbit, an additional condition is needed to make this tree unique.) Douady and Hubbard showed that the topological type of this tree (that is, an equivalence class of homeomorphic trees with conjugate dynamics, with a cyclic order at the branch points specified) suffices to encode the Julia set completely. Later, Hubbard trees for postcritically finite polynomials have been classified by Poirier [**Poi**] using another fundamental theorem of Thurston [**DH2**] and extending earlier work [**BFH**].

Douady [**D2**] writes that he *"finds it much more convenient to draw [the] Hubbard tree than to give [the] coordinates"* when he wants to talk about a specific [postcritically finite] quadratic polynomial. Indeed, the Hubbard tree is a sketch of the essential topological features of the Julia set from which most other relevant data can be reconstructed quite easily; the converse, finding the Hubbard tree associated e.g. to some external angle or to a kneading sequence, is far more difficult and one of the main themes of this paper: turning the different combinatorial descriptions of quadratic polynomials into the others. Since Hubbard trees contain the dynamical information in the most accessible way, most of our discussions are based on them.

For us, it turns out that the discussion becomes easier by considering a slight generalization of Hubbard trees: we consider finite topological trees as abstract topological spaces, without embeddings into the complex plane. There are less data to deal with, and there is an easy criterion which trees are realized by actual polynomials: those without "evil" orbits. (Readers familiar with Thurston's classification of rational maps may see similarities with "obstructions" in that classification: in both cases, an obstruction makes it impossible for a dynamical system resp. Hubbard tree to be realized by a holomorphic map, such as a quadratic polynomial. Most of the work consists in showing that this is the only possible obstruction.) Different polynomials with the same kneading sequence have Hubbard trees which are topologically conjugate, but differently embedded into the plane.

Each of these three combinatorial concepts leads to a description of the Mandelbrot set $\mathcal{M}$ in a different way. Douady and Hubbard showed that the Mandelbrot set is compact, connected, and full (i.e. $\mathbb{C} \setminus \mathcal{M}$ is connected), and there is a unique conformal isomorphism $\Phi\colon \overline{\mathbb{C}} \setminus \mathcal{M} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with $\lim_{c\to\infty} \Phi(c)/c \to 1$. Therefore, we have external rays for the Mandelbrot set, too (to be called *parameter rays*). If we suppose that the Mandelbrot set is locally connected, Φ^{-1} extends to the boundaries as a continuous surjection, and we get a Carathéodory loop for $\mathcal{M}$ like for filled-in Julia sets above. Identifying points on $\partial\mathbb{D}$ if and only if the corresponding parameter rays land together, we get a pinched disk model for the Mandelbrot set which is locally connected and it is a homeomorphic model for $\mathcal{M}$ if and only $\mathcal{M}$ is also locally connected. A particular point $c \in \partial\mathcal{M}$ can be described by the external angle of any parameter ray $R(\vartheta)$ landing at c (this might involve a choice again).

Different connected quadratic Julia sets can (up to certain symmetries as mentioned above) be distinguished by their associated kneading sequences ν. The Mandelbrot set can thus be modeled as the space of kneading sequences with an appropriate topology. The internal address associated to a kneading sequence describes where in the Mandelbrot set this particular kneading sequence may be realized.

However, some kneading sequences occur for several different maps, and others do not occur at all. The first issue, originating from dynamical symmetries, can be dealt with by adding extra combinatorial information ("angled internal addresses"). The second issue is more involved; so far, it has been an open question to describe which kneading sequences do or do not occur (which ones are "complex admissible"). We give a complete answer in Section 5: we describe an "obstruction" which explains the non-existence ("evil orbits"), we show how to extend the Mandelbrot set so as to contain all kneading sequences (complex admissible or not), and we show where the non-admissible subsets in the extension branch off from $\mathcal{M}$ (Section 6).

This extended Mandelbrot set is based on a tree structure which is a parameter space analog of Hubbard trees. It is related to a space constructed by Penrose [**Pen1**] as "abstract abstract Mandelbrot set".

Our discussion in the initial Sections 3, 4, 5 and the first half of Section 6 focuses on the case that the critical orbit is finite, i.e. periodic or preperiodic. These cases are technically the simplest, and it turns out that the main things to look at are periodic points for which Hubbard trees with periodic critical points are sufficient. The general case can be treated by passing to a limit along a sequence of approximating kneading sequences of high periods.

We have now assembled various ways to describe a polynomial $p_c \in \mathcal{M}$ (see Figure 1.2): by its kneading sequence, by its internal address, by its Hubbard tree, by its external (parameter) angle, or by the complex parameter $c \in \mathbb{C}$. The first four notions are purely combinatorial, while the complex parameter is a piece of analytic information. (However, if c describes a postcritically finite polynomial in which 0 has preperiod $l \geq 0$ and period $k \geq 1$, then c is a solution of $p_c^{\circ(l+k)}(0) = p_c^{\circ l}(0)$, which is a polynomial in c with integer coefficients, hence with finitely many roots. The solutions for fixed l and k are apparently algebraically indistinguishable.)

Figure 1.2 shows this in more detail. The left column gives three equivalent ways of describing the combinatorial information of a quadratic Julia set; these make sense whether or not the Hubbard tree can be embedded into the plane, or whether or not the kneading sequence is complex admissible. The concepts in the middle column specify the additional information needed to embed the dynamics into the complex plane, i.e. to choose a complex analytic representative for the symbolic dynamics from the left column. This is not always possible; if it is, it need not be unique (and might therefore involve a choice). The right column specifies additional possibilities in case that the dynamics is realized by a complex polynomial.

These combinatorial tools give a surprisingly large amount of information about the dynamics. For example, an important analytic information is the "Collet-Eckmann-condition" which says that the derivatives along the critical orbit grow exponentially; it has significant geometric consequences (for example, all Fatou components are *Hölder* [**GSm**], i.e. they are images of the unit disk under a Hölder diffeomorphism). The Collet-Eckmann condition is in fact equivalent to several geometric or topological properties and can also can be formulated in combinatorial terms [**PR, PRS**] which are determined by the kneading sequence alone. Another example is the prevalence of certain types of (often geometric, see below) behavior of f_c among parameters in $c \in \partial\mathcal{M}$. To make this more precise, we equip J_c and $\partial\mathcal{M}$ with harmonic measure. Harmonic measure can be defined for the boundary of an arbitrary domain in $\mathbb{C}$ in several equivalent ways. In our context, we define the harmonic measure ω_c on J_c as the push-forward of Lebesgue measure on the circle of external angles under the extended Riemann map [**Pom**], i.e.

$$\omega_c(A) = \mathrm{Leb}\{\vartheta : \text{ the dynamic ray } R_c(\vartheta) \text{ lands in } A\}.$$

In the dynamical plane, ω_c is f_c-invariant and is the (unique) measure of maximal entropy [**Ly1**]. In the parameter plane we define harmonic measure ω on $\partial\mathcal{M}$ by

$$\omega(A) = \mathrm{Leb}\{\vartheta : \text{ the parameter ray } R(\vartheta) \text{ lands in } A\}.$$

Let us list in more detail the goals and structure of this paper.

Algorithmic translations: between Hubbard tree, external angle, kneading sequence and internal address, as indicated in Figure 1.2 and discussed above.

Fundamental algorithms, often in the form of algorithmic definitions, are given in the text as needed, beginning with Section 2. In Sections 15 and 13, we collect further algorithms which were not needed earlier, or which are more complicated to describe.

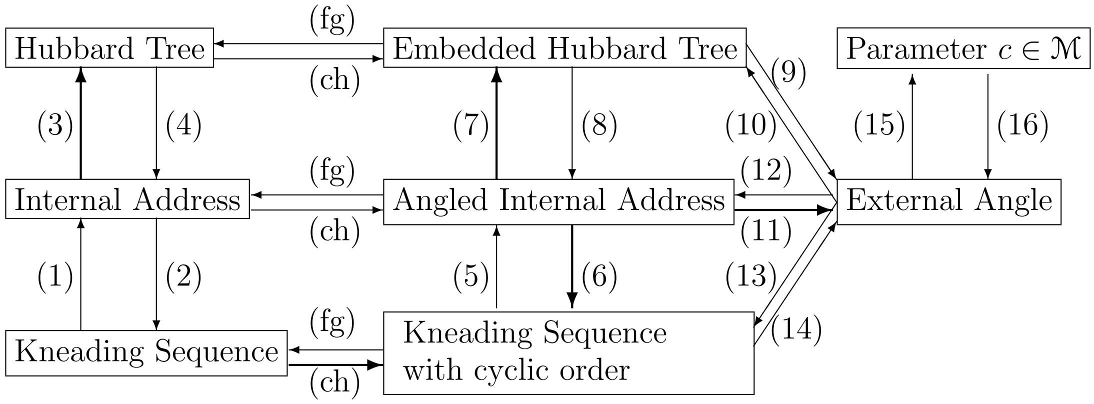

FIGURE 1.2. Schematic diagram of algorithms; heavy arrows indicate more difficult combinatorial algorithms:

(1): Def. 2.2;
(2): Algorithm 2.3;
(3): Algorithm 15.2; the composition (3) $\circ$ (1) is Theorem 3.10;
(4): Cutting Times Algorithm 13.8; the composition (2) $\circ$ (4) is Def. 3.5;
(5): [**LS**, Lemma 6.3/Prop. 6.5];
(6): (13) $\circ$ (11) or (13) $\circ$ (9) $\circ$ (7);
(7): Algorithm 15.2 (with embedding information as in Corollary 4.11);
(8): Cutting Times Algorithm 13.8, together with the combinatorial rotation number at the periodic branch points;
(9): Algorithm 13.6;
(10): Algorithm 13.5;
(11): (9) $\circ$ (7) or "Growing of Trees" as in [**LS**, Theorem 9.3];
(12): (5) $\circ$ (13);
(13): Def. 2.1 (retaining information on cyclic order); the composition (fg) $\circ$ (13) is Def. 2.1;
(14): Algorithm 13.9;
(15): Spider Algorithm [**HS**];
(16): Inverse Spider Algorithm 13.10;
(fg): forgetful maps;
(ch): arbitrary choice (possible only if the Hubbard tree has no evil orbits, or if the kneading sequence is admissible; in this case, the allowed choices for an embedding of the tree, or for the angles in the angled internal address, are easy to specify, while the possible cyclic orders for kneading sequences are not so obvious to single out).

The used terms are defined at the following places:

Hubbard Tree: Definition 3.2; with embedding: Proposition 4.10;
Internal Address: Definition 2.2; angled internal address: Definition 11.9;
Kneading Sequence: Definition 2.1; with cyclic order: for a periodic angle $\vartheta \in \mathbb{S}^1$ of period n, retain the cyclic order of $2^k\vartheta$ for $k = 0, 1, \ldots, n-1$, as well as the preperiodic preimage of ϑ; compare Algorithm 13.9.
External Angle: Angle $\vartheta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Include long internal addresses here?

Existence and uniqueness of Hubbard trees: (Section 3). Viewed as a subset of a filled-in Julia set, the existence of a Hubbard tree is immediate (provided J_c is arcwise connected). However, we take an arbitrary kneading sequence ν as starting point and prove that there exists a Hubbard tree (not embedded into the plane) with kneading sequence ν. This tree and its dynamics are unique up to a semi-conjugacy off certain "marked points".

Branch points of the Hubbard tree: (Section 4). It turns out that branch points (on Hubbard trees as well as in Julia sets) are always (pre)periodic or precritical. We show that every periodic orbit (unless it is an endpoint of the Julia set) contains a preferred *characteristic point* which is on the arc between critical point and critical value. We show that branch points come in two kinds, "tame" and "evil" as follows: a branch point z is

- **tame** if the first return map to z permutes the local arms of z in one cycle;
- **evil** if the first return map to z fixes the local arm towards the critical point and permutes the remaining local arms of z in one cycle.

We show that a Hubbard tree can be embedded into the plane so that its dynamics respects the embedding if and only if it has no evil periodic orbit. A kneading sequence is *(complex) admissible* if the associated Hubbard tree can be so embedded. Estimates on the number of non-homotopic embeddings are given as well (see Corollary 4.11 and Lemma 16.11).

Types of branch points: (Section 5). For any periodic orbit on the Hubbard tree, we describe how to read off from its itinerary whether this orbit is tame or evil and how many arms it has, in particular whether or not it is a branch point (Propositions 5.19 and 5.13). The itinerary also tells the order of the characteristic periodic points on the Hubbard tree.

(Complex) admissibility: (Sections 5 and 18). A kneading sequence is defined to be complex admissible if the associated Hubbard tree has no evil orbits. We show that evil orbits are the only obstruction to the existence of an external angle which generates this kneading sequence in the sense of Definition 2.2 (Corollary 14.2). Furthermore, we give a purely combinatorial characterization telling whether or not a given kneading sequence is complex admissible (Definition 5.1). Analogous conditions for real admissibility were found in the early 1970's. We also answer the question whether the set of admissible kneading sequences has positive measure in the affirmative (Theorem 18.4).

Internal addresses: (Section 6). Defined as a compact recoding of the kneading sequence, it comprises a lot of geometric information: it describes the position of a complex parameter $c \in \mathcal{M}$ in terms of hyperbolic components of lowest periods on the parameter path from 0 to c (Algorithm 11.3), it enumerates periodic points of lowest periods in the Julia set between 0 and c (critical point and value), and it works similarly with precritical points. These properties come from [**LS**] and are collected in Proposition 6.8 and Algorithm 13.8.

Order structure in the Mandelbrot set: (Section 6). We define an order relation for kneading sequences with the following property: if (T, f) and (T', f') are two Hubbard trees with kneading sequences $\nu > \nu'$, then all orbits in (T', f'), as described by their itineraries (periodic or not), also occur in (T, f). In particular, $\nu > \nu'$ implies that the topological entropy of (T, f) is no less than that of (T', f'). These ideas are related to work of Penrose [**Pen1**].

This order relation can be read off from the kneading sequence alone and endows the space of kneading sequences with the structure of an ordered tree. The admissible kneading sequences form a subtree on which this order generates the topology of the Mandelbrot set (again, up to symmetries encoded in the angled internal address). We show how the various non-admissible subtrees are attached to the admissible subtree.

Biaccessibility: (Sections 16 and 17). A point $z \in J_c$ is called *biaccessible* if at least two dynamical rays land at z (so that $J_c \setminus \{z\}$ has at least two connected components). For example, understanding which points in J_c are biaccessible is crucial in the mating construction of Julia sets, see Douady [**D1**], Rees [**Re**], Shishikura [**Shi**] and Tan Lei [**Ta2**].

We show that the geometric notion of biaccessibility leads to a combinatorial characterization, both in the dynamical and parameter plane. Using this, we show that the Hausdorff dimension of the set of external angles corresponding to biaccessible points in a Julia set is strictly less than 1, except for the polynomial $z \mapsto z^2 - 2$; in particular ω_c-a.e. $z \in J_c$ is not biaccessible (Theorem 16.9). Furthermore, the parameter angles of biaccessible parameters in $\mathcal{M}$ are a countable union of sets of Hausdorff dimension less than 1, and hence for ω-a.e. $c \in \partial\mathcal{M}$, c is not biaccessible (Theorem 16.13).

These results involve estimates of how much the map from external angles to itineraries and to kneading sequences can distort Hausdorff dimensions.

Section 16 deals with the combinatorial aspects of biaccessibility and contains the dimension estimates. In Section 17 we show that these combinatorial predictions are (under local connectivity assumptions) realized in the actual topological spaces given by Julia sets and Mandelbrot set.

Ergodic Theory and Collet-Eckmann Maps: (Section 19). Given a continuous observable $\varphi : \mathbb{C} \to \mathbb{R}$, we show in Theorem 19.1 that for ω-a.e. $c \in \partial\mathcal{M}$, the critical values c satisfies Birkhoff's Ergodic Theorem:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f_c^{\circ i}(c)) = \int \varphi \, d\omega_c,$$

This results extends to the $L^1(\omega_c)$-observable $\varphi(z) = \log|f'(z)|$, which implies that for ω-a.e. $c \in \partial\mathcal{M}$, f_c satisfies the Collet-Eckmann condition, and more precisely (see Corollary 19.2)

$$\lim_{n\to\infty} \frac{1}{n} \log|Df_c^{\circ n}(c)| = \int \log|f_c'|\, d\omega_c > 0.$$

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2. Fundamental Concepts

In this section, we define some important concepts of symbolic dynamics for quadratic polynomials; some of these definitions come with algorithms.

Let $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ be the space of external angles and let $\mathbb{N}^* = \{1, 2, 3, \dots\}$ be the set of positive integers. We say that a sequence $\nu = \nu_1\nu_2 \dots$ has period n if $\nu_{k+n} = \nu_k$ for all k; this allows the case that ν is periodic with period n/k for some k dividing n. The minimal n for which ν is periodic is called the *exact period* of ν (this is also known as the *prime period* or *minimal period*).

2.1. **Definition (Itinerary and Kneading Sequence of External Angle)**
Given an external angle $\vartheta \in \mathbb{S}^1$, we associate to any $\varphi \in \mathbb{S}^1$ its itinerary $\nu_\vartheta(\varphi) = \nu_1\nu_2 \dots$ *with $\nu_i \in \{\mathtt{0}, \mathtt{1}, \star\}$ by setting:*

$$\nu_i := \begin{cases} \mathtt{0} & \textit{if } (\vartheta+1)/2 < 2^{i-1}\varphi < \vartheta/2; \\ \mathtt{1} & \textit{if } \vartheta/2 < 2^{i-1}\varphi < (\vartheta+1)/2; \\ \star & \textit{if } 2^{i-1}\varphi \in \{\vartheta/2, (\vartheta+1)/2\}, \end{cases}$$

where the inequalities are interpreted with respect to cyclic order. The kneading sequence *$\nu(\vartheta)$ of ϑ is its itinerary with respect to itself: $\nu(\vartheta) = \nu_\vartheta(\vartheta)$. See Figure 2.1.*

Note that in the definition of the kneading sequence, a change of ϑ amounts to a change of the orbit for which we take the itinerary, as well as of the partition itself.

A kneading sequence ν contains a $\star$ at position n if and only if ϑ is periodic with period n; the exact period of ϑ may divide n. (There are non-periodic angles that yield periodic kneading sequences without $\star$; see Theorem 14.10.) We say that a sequence ν is *$\star$-periodic of period n* if $\nu = \overline{\nu_1 \dots \nu_{n-1}\star}$ with $\nu_1 = \mathtt{1}$ and $\nu_i \in \{\mathtt{0}, \mathtt{1}\}$ for $1 < i < n$; $\star$-periodic sequences are a special case of periodic sequences. Let

$$\begin{aligned} \Sigma &:= \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}, \\ \Sigma^1 &:= \{\nu \in \Sigma \colon \text{the first entry in } \nu \text{ is } \mathtt{1}\} \\ \Sigma^\star &:= \Sigma^1 \cup \{\text{all } \star\text{-periodic sequences}\}, \\ \Sigma^{\star\star} &:= \{\nu \in \Sigma^1 \colon \nu \text{ is non-periodic}\} \cup \{\text{all } \star\text{-periodic sequences}\}. \end{aligned}$$

In order to avoid silly counterexamples, $\overline{\star}$ is not considered to belong to $\Sigma^\star$. All sequences in $\Sigma^\star$ will be called *kneading sequences*, whether or not they occur as the image of an angle $\vartheta \in \mathbb{S}^1$. They all begin with $\mathtt{1}$.

2.2. **Definition (ρ-Function and Internal Address)**
For a sequence $\nu \in \Sigma^\star$, define

$$\rho_\nu : \mathbb{N}^* \to \mathbb{N}^* \cup \{\infty\}, \quad \rho_\nu(n) = \inf\{k > n : \nu_k \neq \nu_{k-n}\}.$$

We usually write ρ for ρ_ν. For $k \geq 1$, we call

$$\operatorname{orb}_\rho(k) := k \to \rho(k) \to \rho^{\circ 2}(k) \to \rho^{\circ 3}(k) \to \dots$$

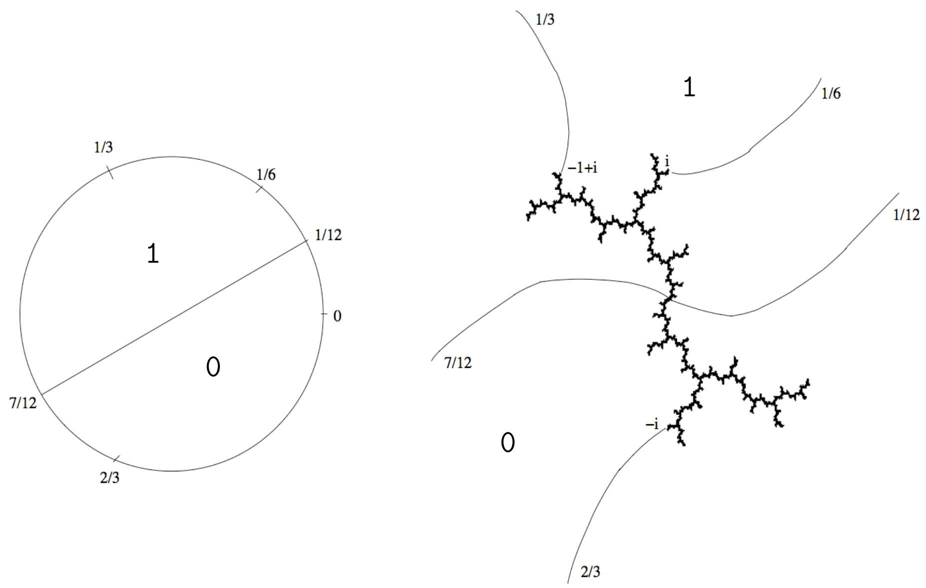

Figure 2.1. Left: the kneading sequence of an external angle ϑ (here $\vartheta = 1/6$) is defined as the itinerary of the orbit of ϑ under angle doubling, where the itinerary is taken with respect to the partition formed by the angles $\vartheta/2$, and $(\vartheta+1)/2$. Right: in the dynamics of a polynomial for which the ϑ-ray lands at the critical value, an analogous partition is formed by the dynamic rays at angles $\vartheta/2$ and $(\vartheta+1)/2$, which land together at the critical point.

the ρ-orbit of k. The case $k = 1$ is the most important one; we call

$$\operatorname{orb}_\rho(1) = 1 \to \rho(1) \to \rho^{\circ 2}(1) \to \rho^{\circ 3}(1) \to \dots$$

the internal address *of ν. If $\rho^{\circ k+1}(1) = \infty$, then we say that the internal address is finite: $1 \to \rho(1) \to \dots \to \rho^{\circ k}(1)$; as a result, the orbit* orb_ρ *is a finite or infinite sequence that never contains ∞.*

We have $\rho(n) = \infty$ if and only if the sequence ν is periodic and its period divides n. If $\nu \in \Sigma^1$ is periodic of exact period n, the number n may or may not appear in the internal address (with the first $n-1$ entries in ν fixed, these two possibilities can be realized by putting `0` or `1` at the n-th position). If it does, then the internal address stops there; otherwise, it is infinite, as will become clear in Lemma 20.2.

The ρ-function is of fundamental importance in the work of Penrose [**Pen1**] under the name of *non-periodicity function*; the internal address is called *principal non-periodicity function*.

The map from kneading sequences in Σ^1 to internal addresses is injective. In fact, the algorithm of this map (originally from [**LS**, Algorithm 6.2]) can easily be inverted:

2.3. **Algorithm (From Internal Address to Kneading Sequence)**
The following inductive algorithm turns internal addresses into kneading sequences in Σ^1: the internal address $S_0 = 1$ has kneading sequence $\overline{1}$, and given the kneading sequence ν^k associated to $1 \to S_1 \to S_2 \to \ldots \to S_k$, the kneading sequence associated to $1 \to S_1 \to S_2 \to \ldots \to S_k \to S_{k+1}$ consists of the first $S_{k+1} - 1$ entries of ν^k, then the opposite to the entry S_{k+1} in ν (switching `0` and `1`), and then repeating these S_{k+1} entries periodically.

Proof. The kneading sequence $\overline{1}$ has internal address 1. If ν^k has internal address $1 \to S_1 \to S_2 \to \ldots \to S_k$ and ν is the internal address of period S_{k+1} as constructed in the algorithm, then the internal address of ν clearly starts with $1 \to S_1 \to S_2 \to \ldots \to S_k$, and $\rho_\nu(S_k) = S_{k+1}$, so the internal address of ν is $1 \to S_1 \to S_2 \to \ldots \to S_k \to S_{k+1}$. □

The space of internal addresses is the space of all strictly increasing sequences of integers (finite or infinite) starting with 1; we have just shown that the internal address map is a bijection from Σ^1 to this space. Of course, every $\star$-periodic kneading sequence of period n also yields a finite internal address ending in n, and there is a bijection between $\star$-periodic kneading sequences and finite internal addresses.

Remarks about real quadratic dynamics. Kneading sequences describe the position of the critical orbit on $\mathbb{R}$ with respect to the critical point; they seem to have appeared for the first time in [**MSS**]. A description of those kneading sequences which are realized by a real quadratic map (*real admissibility*) has been known for a long time. We recommend Milnor and Thurston [**MiT**]; they also show that for real quadratic polynomials the topological entropy depends monotonically on the parameter (see also [**CoE**] for an exposition and more references, and the remark after Corollary 6.3 for the complex case). In the real context, the internal address has been known as the sequence of *cutting times* and was first used by Hofbauer; see *e.g.* [**Ho**] or [**Bru1**] for an exposition, and Section 6 for further analogies. A complex version of cutting times can be found in Algorithm 13.8.

The map from external angles to kneading sequences, as defined in Definition 2.1, is neither injective nor surjective and difficult to understand. One reason to introduce internal addresses in [**LS**] was to describe it geometrically in the general case. The non-injectivity can be understood in terms of *angled internal addresses* (compare [**LS**, Section 6] and **??**). One major result of the present paper is to classify the range of the kneading sequence map (Theorem 5.2 and Corollary 14.2).

REMARK. Our notation differs from that in [**LS**]: for us, all sequences in $\Sigma^\star$ are called kneading sequences, whether or not they are admissible, while in [**LS**], only admissible sequences were called "kneading sequences", using the term "abstract kneading sequences" for arbitrary sequences in $\Sigma^\star$.

II. Hubbard Trees

3. Hubbard Trees

In this section, we define Hubbard trees as abstract trees with dynamics and show their most fundamental properties. Our trees do not necessarily come with an embedding into the complex plane. We show that for every $\star$-periodic or preperiodic kneading sequence, there is a unique Hubbard tree (for trees which are embedded in the plane, both existence and uniqueness are false in general; see Section 4).

3.1. **Definition (Trees, Arms, Branch Points and Endpoints)**
A tree T *is a finite connected graph without loops. For a point* $x \in T$*, the* (global) arms *of* x *are the connected components of* $T \setminus \{x\}$*. A* local arm *at* x *is an intersection of a global arm with a sufficiently small neighborhood of* x *in* T*. The point* x *is an* endpoint *of* T *if it has only one arm; it is a* branch point *if it has at least three arms.*

To be more precise, a graph is the union of finitely many edges, each homeomorphic to $[0, 1]$, and disjoint except that different edges may have common endpoints. Such a graph inherits its topology as the quotient of its set of edges with endpoints identified. A loop is a subset homeomorphic to a circle.

Between any two points x, y in a tree, there exists a unique closed arc connecting x and y; we denote it by $[x, y]$ and its interior by (x, y).

3.2. **Definition (The Hubbard Tree)**
A Hubbard tree *is a tree* T *equipped with a map* $f\colon T \to T$ *and a distinguished point, the* critical point*, satisfying the following conditions:*

(1) $f : T \to T$ *is continuous and surjective;*
(2) *every point in* T *has at most two inverse images under* f*;*
(3) *at every point other than the critical point, the map* f *is a local homeomorphism onto its image;*
(4) *all endpoints of* T *are on the critical orbit;*
(5) *the critical point is periodic or preperiodic, but not fixed;*
(6) *(expansivity) if* x *and* y *with* $x \neq y$ *are branch points or points on the critical orbit, then there is an* $n \geq 0$ *such that* $f^{\circ n}([x, y])$ *contains the critical point.*

We denote the critical point by $c_0 = 0$ and its orbit by $\mathrm{orb}_f(0) = \{0, c_1, c_2, \dots\}$. The *critical value* c_1 is the image of the critical point. We use a standing assumption that $c_1 \neq c_0$ in order to avoid having to deal with counterexamples when the entire tree is a single point. The branch points and the points on the critical orbit (starting with c_0) will be called *marked points.* Notice that the set of marked points is finite and forward invariant, because the number of arms at any point can decrease under f only at the critical point.

Two Hubbard trees (T, f) and (T', f') are *equivalent* if there is a bijection between their marked points which is respected by the dynamics, and if the edges of the tree connect the same marked points. This is weaker than a topological conjugation. In particular, we do not care about details of the dynamics between marked points; there may be intervals of periodic points, attracting periodic points, and so on. (This is related to an equivalence class of branched covers in the sense of Thurston as in [**HS, DH2**].)

3.3. **Lemma (Basic Properties of the Hubbard Tree)**
The critical value c_1 is an endpoint, and the critical point 0 divides the tree into at most two parts. Each branch point is periodic or preperiodic, it never maps onto the critical point, and the number of arms is constant along the periodic part of its orbit. Any arc which does not contain the critical point in its interior maps homeomorphically onto its image.

PROOF. Suppose that c_1 has at least two arms. The points c_2, c_3, $\ldots$ also have at least two arms as long as f is a local homeomorphism near this orbit. If this is no longer the case at some point, then the orbit has reached the critical point, and the next image is c_1 again. In any case, all points on the critical orbit have at least two arms. This contradicts the assumption that all endpoints of a Hubbard tree are on the critical orbit. Hence c_1 has exactly one arm, and 0 has at most two arms (or its image would not be an endpoint).

Since near every non-critical point, the dynamics is a local homeomorphism onto the image, every branch point maps onto a branch point with at least as many arms. Since the critical point has at most two arms, it can never be the image of a branch point. The tree and thus the number of branch points is finite, so every branch point is preperiodic or periodic and its entire orbit consists of branch points; the number of arms is constant along the periodic part of the orbit.

Let γ be an arc within the tree. Since f cannot be constant on γ and there is no loop in the tree, the subtree $f(\gamma)$ has at least two endpoints. If an endpoint of $f(\gamma)$ is not the image of an endpoint of γ, then it must be the image of the critical point since f is a local homeomorphism elsewhere, and the critical point 0 must be in the interior of γ. $\square$

Hubbard trees were first defined by Douady and Hubbard in [**DH1**]. Their definition uses the filled-in Julia set $K(p)$ of a polynomial p: this is the set of points $z \in \mathbb{C}$ whose orbits under p are bounded. It is well known [**Mi1**] that if all critical points of p are periodic or preperiodic, then $K(p)$ is path-connected.

3.4. **Definition (The Douady-Hubbard Tree)**
Let p be a quadratic polynomial with periodic or preperiodic critical orbit and filled-in Julia set $K(p)$. The Hubbard tree *of p is a minimal tree $T \subset K(p)$ connecting the critical orbit so that the intersection of T with any component U of $\mathring{K}(p)$ is (part of) a geodesic with respect to the hyperbolic metric of U.*

If p has preperiodic critical orbit, then $K(p)$ is a dendrite and any two points are connected by a unique arc; in this case, it suffices to define T as a minimal tree in $K(p)$ connecting the critical orbit. However, if the critical orbit is periodic, then every component U of $\mathring{K}(p)$ is conformally isomorphic to the unit disk $\mathbb{D}$; if a conformal isomorphism $\varphi_U \colon U \to \mathbb{D}$ is chosen which sends the unique precritical point in U to 0, then the geodesic condition means that $\varphi^{-1}(T \cap U)$ is a radius or diameter of $\mathbb{D}$.

The main difference in our definition is that we do not specify an embedding of T into the plane. In Section 4, we will investigate which trees can be embedded into the plane in a dynamically plausible way, and if so, in how many different ways. It is quite easy to see that every Hubbard tree in the sense of Douady and Hubbard satisfies our Definition 3.2.

We have seen that $T \setminus \{0\}$ consists of at most two components. Let us denote them by T_0 and T_1 so that $c_1 \in T_1$ (with $c_1 \neq 0$ by definition); T_0 may be empty.

3.5. **Definition (Itinerary and Kneading Sequence)**
The itinerary *of a point $z \in T$ on a Hubbard tree is the infinite sequence $e(z) = e_1(z)e_2(z)e_3(z)\ldots$ with*

$$e_i(z) = \begin{cases} \mathtt{0} & \text{if } f^{\circ(i-1)}(z) \in T_0, \\ \star & \text{if } f^{\circ(i-1)}(z) = 0, \\ \mathtt{1} & \text{if } f^{\circ(i-1)}(z) \in T_1. \end{cases}$$

The itinerary $e(c_1) =: \nu = \nu_1\nu_2\nu_3\ldots$ of c_1 is called the kneading sequence *of the Hubbard tree.*

Obviously $e \circ f(z) = \sigma \circ e(z)$ where σ denotes the left shift. The expansivity condition of Definition 3.2 means that no two marked points have the same itinerary.

3.6. **Lemma (Same Itinerary on Connected Subtree)**
Suppose that z and z' are two points on a Hubbard tree such that $e_i(z) = e_i(z')$ for all $i < n$ (for some $n \leq \infty$). Then all $w \in [z, z']$ have $e_i(w) = e_i(z) = e_i(z')$ for $i < n$.

Proof. We can assume that $z \neq z'$. Since 0 is the only point whose itinerary starts with $\star$, z and z' lie in the same component of $T \setminus \{0\}$. Therefore $[z, z']$ is mapped homeomorphically onto $f([z, z'])$ by f. Since $e_2(z) = e_2(z')$, the arc $f([z, z'])$ is contained in a single component of $T \setminus \{0\}$. By induction, we obtain that $f^{\circ i}([z, z'])$ is contained in a single component of $T \setminus \{0\}$ for each $i < n$. The claim follows. □

3.7. **Lemma (α-Fixed Point on Hubbard Tree)**
There is a unique fixed point in T_1; it lies in $(0, c_1)$.

Proof. Since c_1 is an endpoint, the intersection $[0, c_1] \cap f([0, c_1])$ is a non-degenerate arc $[c_1, x]$, *i.e.*, $x \neq c_1$. If $x = 0$, then f maps $[0, c_1]$ over itself in an orientation reversing way, so there is a fixed point on $[0, c_1]$.

We may thus assume that $x \in (0, c_1)$. If $f([0, c_1]) \subset [0, c_1]$, then as above we have a fixed point in (c_1, x). Otherwise $f([0, c_1])$ branches off from $[0, c_1]$ at x. Let $y = f(x)$. Then y cannot be on $(0, x)$ because the path $f([0, c_1])$ starts at c_1 and branches off at x before reaching $(0, x)$. If $y \in [x, c_1]$, then f maps $[x, c_1]$ over $[y, c_2]$, and f has a fixed point in $[x, y]$.

The last possibility is that $y \in (x, c_2]$, and we show this does not occur. Let T' be the connected component of $T \setminus \{x\}$ containing y. The image $f(T') \not\ni x$, because one of the two inverse images of x is on $[0, x]$, and the other is separated from x by the critical point. Since x maps into T' and no point in T' maps onto x, all of T' maps strictly into itself under f. But this violates the expansivity condition: T' has an endpoint x' other than x, and the forward orbits of x and the endpoint are never separated by 0.

Now we have a fixed point in T_1; call it α. Suppose that it is not unique. Since f maps T_1 homeomorphically onto its image, f must fix a component G of $T_1 \setminus \{\alpha\}$. This is not the component with 0 as boundary point, because α separates 0 from $c_1 = f(0)$. Let z be an endpoint of this fixed component G. Then α and z both are marked points with the same itinerary $\overline{\mathtt{1}}$, and this contradicts the expansivity condition. $\square$

Remark. The unique fixed point in T_1 is usually called α . The component $T_\mathtt{0}$ can contain more fixed points, but by Lemma 3.6, they are all contained in a connected subtree of constant itinerary $\overline{\mathtt{0}}$. If there is an endpoint with itinerary $\overline{\mathtt{0}}$, it is called β ; it exists on the Hubbard tree if and only if the kneading sequence terminates in an infinite string of symbols $\mathtt{0}$. A generalization of Lemma 3.7 will be proved in Lemma 5.18.

A point $z \in T$ is *(pre)periodic* if $f^{\circ l}(z) = f^{\circ(l+m)}(z)$ for some $l \geq 0, m \geq 1$. We take l and m minimal with this property. Then m is the (exact) *period* of z and l the *preperiod.*

3.8. **Lemma (Preperiod and Period)**
The exact preperiod and period of any marked (pre)periodic point are equal to the exact preperiod and period of its itinerary.

Proof. Suppose z is periodic with period m and let m' be the period of $e(z)$ (under the shift). Obviously m' divides m. If $m' \neq m$, then z and $f^{\circ m'}(z)$ are different marked points with the same itinerary. This contradicts expansivity. The same argument works in the preperiodic case. $\square$

3.9. **Lemma (Periodic Points and Itineraries)**
If a Hubbard tree contains a point with periodic itinerary τ, then it contains a periodic point p with itinerary τ such that the exact periods of p and τ coincide.

Proof. Let $T' \subset T$ be the set of all points with itinerary τ. By Lemma 3.6, T' is connected, so it is a connected subtree (possibly not closed).

Let n be the period of τ. Then $f^{\circ n}$ maps T' homeomorphically onto its image in T'. Since marked points in T have different itineraries, T' can contain at most one branch point of T. If it contains one, then it must be fixed by $f^{\circ n}$, so its exact period is n. Otherwise, T' is either a single point (and we are done), or T' is homeomorphic to an interval. If $f^{\circ n}$ sends T' to itself reversing the orientation, we get a unique fixed point in the interior of T', and we are done again.

Now suppose that $f^{\circ n}$ preserves the orientation of T'. If $f^{\circ n}\colon T' \to T'$ is not surjective, then for at least one endpoint x, say, $f^{\circ n}(x)$ is in the interior of T'. If x is a branch point or an endpoint of T, then it is marked and we are done. Otherwise, x has a neighborhood in T which is homeomorphic to an open interval, but only a one-sided neighborhood has itinerary τ. This implies that x maps to the critical point after finitely many iterations. Again T' contains a marked point, which must be fixed by $f^{\circ n}$.

The last case is that $f^{\circ n}$ maps T' homeomorphically onto itself, preserving the orientation and fixing both endpoints. Then the claim is satisfied by any endpoint of T' which does not hit the critical point on its forward orbit. If both endpoints do, say after k and m iterations with k and m minimal and $k < m$, then $f^{\circ(k+1)}(T')$ and $f^{\circ(m+1)}(T')$ are both intervals with c_1 as endpoints and not containing branch points of T, and $m-k < n$. Hence $f^{\circ(m-k)}$ must map T' onto itself reversing the orientation, so it fixes some point in T' which must have an itinerary with period dividing n. This is a contradiction. □

The main results of this section are existence and uniqueness of Hubbard trees.

3.10. **Theorem (Existence and Uniqueness of Hubbard Trees)**
Every $\star$-periodic or preperiodic kneading sequence is realized by a Hubbard tree; this tree is unique (up to equivalence).

Sometimes, it is useful to extend Hubbard trees by finitely many more marked points with periodic or preperiodic itineraries. The basic properties of such extended Hubbard trees remain the same as in Definition 3.2, except for the requirement that all endpoints of the tree need to belong to the critical orbit and that the dynamics on the tree is surjective.

3.11. **Corollary (Extended Hubbard Tree)**
For every $\star$-periodic or preperiodic kneading sequence and every finite collection S of periodic or preperiodic sequences in $\{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$, there exists a tree (T,f), unique up to equivalence, which satisfies all properties of a Hubbard tree (except the requirements that all endpoints of the tree must be on the critical orbit and the dynamics on the tree is surjective), and so that for every sequence in S, there is a periodic or preperiodic point in T which has this itinerary.

We start with the uniqueness part of the proof; existence (together with Corollary 3.11) requires a number of preparations and is postponed to Section 20. We start with the uniqueness part of the proof; existence (together with Corollary 3.11) requires a number of preparations and is postponed to Section 20 (see also Section 15 for an explicit construction of Hubbard trees). We describe the branch structure of a Hubbard tree in terms of *triods*, and then show that the symbolic dynamics of the kneading sequence allows to reconstruct the branch structure of the tree.

3.12. **Definition (Triod)**
A triod *is a connected compact set homeomorphic to a subset of the letter* Y. *It is* degenerate *if it is homeomorphic to an arc or a point.*

For a sequence $\nu \in \Sigma^\star$, let $\star\nu$ be the symbol $\star$ followed by ν and define $S(\nu) := \{\star\nu, \nu, \sigma(\nu), \sigma^{\circ 2}(\nu), \dots\}$ (the orbit of $\star\nu$ under the shift). Then $\sigma(S(\nu)) \subset S(\nu)$. Note that $\mathtt{0}\nu \in S(\nu)$ or $\mathtt{1}\nu \in S(\nu)$ would imply that ν was periodic but not $\star$-periodic; thus $S(\nu) \cap \{\mathtt{0}\nu, \mathtt{1}\nu\} = \emptyset$ for $\star$-periodic and preperiodic ν.

3.13. **Definition (Formal Triod)**
Any triple of pairwise different sequences $s, t, u \in S(\nu) \cup \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$ is called a formal triod $[s, t, u]$.

3.14. **Definition (The Formal Triod Map)**
If $\{s, t, u\} \cap \{\mathtt{0}\nu, \mathtt{1}\nu\} = \emptyset$, then we define the formal triod map as follows:

$$\varphi[s,t,u] := \begin{cases} [\sigma(s), \sigma(t), \sigma(u)] & \textit{if } s_1 = t_1 = u_1 \in \{\mathtt{0}, \mathtt{1}\}; \\ \mathtt{stop} & \textit{if } \{s_1, t_1, u_1\} = \{\mathtt{0}, \mathtt{1}, \star\}; \\ [\sigma(s), \sigma(t), \nu] & \textit{if } s_1 = t_1 \neq u_1 \ ; \\ [\sigma(s), \nu, \sigma(u)] & \textit{if } s_1 = u_1 \neq t_1 \ ; \\ [\nu, \sigma(t), \sigma(u)] & \textit{if } t_1 = u_1 \neq s_1 \ . \end{cases} \tag{1}$$

By construction, the only sequence which starts with $\star$ is $\star\nu$, so at most one of s, t, u can start with $\star$. If one of them does, then the other two sequences either have first entries which are different from each other (and we are in line 2), or the other two first entries are equal and we are in lines 3–5. Therefore, the list covers all possible cases.

In all cases other than the `stop` case, $\varphi[s, t, u]$ returns three sequences in $S(\nu) \cup \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$. These form another formal triod, i.e. all three image sequences are different: in line 1, this is clear; in the other lines, this follows because s, t, u are all different from $\mathtt{0}\nu$ and 1ν by assumption. In the last three cases, we say that u, t, s (respectively) are *chopped off* from the triod (and replaced by $\star\nu$). If s, t or u equals $\star\nu$; then this sequence is chopped off and replaced by itself, so formally the outcome is the same as it would be in line 1, but we record the chopping.

3.15. **Proposition (Uniqueness of Hubbard Trees)**
Any two Hubbard trees with the same $\star$-periodic or preperiodic kneading sequence are equivalent. The same is true for extended Hubbard trees (from Corollary 3.11) with the same kneading sequence and the same set of additional sequences.

FIGURE 3.1. Left: A Hubbard tree with a triod defined by three sequences s, t, u (the triod is indicated by heavy lines). Right: the same tree with the image triod marked: since $\sigma(u)$ is on the other side of the critical point than $\sigma(s)$ and $\sigma(t)$, the image triod is *chopped off* at the critical point $\star$ and is defined by the three points $\sigma(s)$, $\sigma(t)$, and $\star$. Note that the triod $[\sigma(s), \sigma(t), \sigma(u)]$ is non-degenerate, so the branch point is on the same side as the majority of its three endpoints, *i.e.,* the side of the two points $\sigma(s)$ and $\sigma(t)$ (and the same is true for the triod $[\sigma(s), \sigma(t), \star]$).

PROOF. Given three marked points x, y, z on a Hubbard tree, we denote the triod that they form by $[x, y, z]$; so $[x, y, z] = [x, y] \cup [y, z] \cup [z, x]$. For any two Hubbard trees T and T' with the same kneading sequence, we prove that any pair of triods $[c_k, c_l, c_m]$ and $[c'_k, c'_l, c'_m]$ are both non-degenerate or both degenerate in the same way.

We decide whether a triod is degenerate by iterating it; assume that c_k, c_l and c_m are pairwise different.

(1) If the triod $[c_k, c_l, c_m]$ does not contain 0 in its interior, then it maps homeomorphically onto its image; we take the image.
(2) If 0 belongs to the interior of $[c_k, c_l, c_m]$ and $0 \notin \{c_k, c_l, c_m\}$, then we take the component of $[c_k, c_l, c_m] \setminus \{0\}$ containing two of the three points c_k, c_l, c_m, and take the closure of its image as new triod (we "chop off" the arc from 0 to the isolated endpoint of the triod and map only the rest).
(3) If 0 belongs to the interior of $[c_k, c_l, c_m]$, and 0 is equal to one of the three points, say $0 = c_k$, then the algorithm terminates. The triod is degenerate, and c_k is an interior point of $[c_k, c_l, c_m]$.

We iterate this procedure. Since the critical orbit is finite, the algorithm either terminates or eventually reaches a loop. If the algorithm never terminates, then at least two endpoints must be chopped off during the iteration of the triod: otherwise, at least two endpoints must have identical itineraries (if ν is $\star$-periodic, then we must exclude the case that the triod iteration involves a triod with endpoint $\mathtt{0}\nu$ or $\mathtt{1}\nu$; but this is clear). If each of the three points of the triod is chopped off at some step, the triod must be non-degenerate. If exactly one endpoint of the triod is never chopped off, then the triod is degenerate with the latter endpoint in the middle.

The key observation is that the type of the triod can be read off from the itineraries of its endpoints in terms of the form triod map: the triod $[c_k, c_l, c_m]$ is represented by the triple $(\sigma^{\circ(k-1)}(\nu), \sigma^{\circ(l-1)}(\nu), \sigma^{\circ(m-1)}(\nu))$. Then the image triod has endpoints $\varphi(\sigma^{\circ(k-1)}(\nu), \sigma^{\circ(l-1)}(\nu), \sigma^{\circ(m-1)}(\nu))$:

(1) If the first entries of $\sigma^{\circ(k-1)}(\nu), \sigma^{\circ(l-1)}(\nu), \sigma^{\circ(m-1)}(\nu)$ are the same, where $\star$ counts as $\mathtt{0}$ (resp. $\mathtt{1}$) if the other two first entries are $\mathtt{0}$ (resp. $\mathtt{1}$), then the shifted triple represents the image triod $[c_{k+1}, c_{l+1}, c_{m+1}]$.
(2) If the first entries of $\sigma^{\circ(k-1)}(\nu), \sigma^{\circ(l-1)}(\nu), \sigma^{\circ(m-1)}(\nu)$ are $\mathtt{0}$ (say twice) and $\mathtt{1}$ (say once), then we take the shift of the sequences starting with $\mathtt{0}$ and replace the remaining sequence by ν. This represents the chopping off one arm of the triod.
(3) If the first entries of $\sigma^{\circ(k-1)}(\nu), \sigma^{\circ(l-1)}(\nu), \sigma^{\circ(m-1)}(\nu)$ are $\{0, \star, 1\}$, then we do not define $\varphi(\sigma^{\circ(k-1)}(\nu), \sigma^{\circ(l-1)}(\nu), \sigma^{\circ(m-1)}(\nu))$: the iteration terminates, the triod is degenerate and the sequence starting with $\star$ represents an interior point of the triod.

The kneading sequence fully describes the behavior of φ and thus determines which points on the critical orbit are between which others on the tree, and which are endpoints.

For any non-degenerate triod, the iteration of φ also encodes the itinerary of the interior branch point: this itinerary is constructed by a “majority vote” from the first entries of the sequences of the triple at every step. The branch points have itineraries different from $\mathtt{0}\nu$ and $\mathtt{1}\nu$ because they are marked points, and their images are different from the critical value. Therefore, the argument above can also be applied to triods whose endpoints are arbitrary marked points (branch points or points on the critical orbit), and this implies that any two Hubbard trees with the same kneading sequence are equivalent. □

4. Periodic Orbits on Hubbard Trees

In this section, we discuss periodic points of Hubbard trees, in particular branch points, and show that they come in two kinds: *tame* and *evil*. This determines whether or not Hubbard trees and kneading sequences are admissible: they are if and only if there is no evil orbit.

4.1. **Lemma (Characteristic Point)**
Let (T, f) be the Hubbard tree with kneading sequence ν. Let $\{z_1, z_2, \ldots, z_n = z_0\}$ be a periodic orbit which contains no endpoint of T. If the critical orbit is preperiodic, assume also that the itineraries of all points z_k are different from the itineraries of all endpoints of T.

Then there are a unique point $z \in \{z_k\}_{k=1}^n$ and two different components of $T \setminus \{z\}$ such that the critical value is contained in one component and 0 and all other points $z_k \neq z$ are in the other one.

4.2. **Definition (Characteristic Point)**
The point z in the previous lemma is called the characteristic point *of the orbit $\{z_k\}$; we will always relabel the orbit cyclically so that the characteristic point is z_1.*

Proof of Lemma 4.1. Note first that every $z_k \neq 0$ (or $z_{k+1} = c_1$ would be an endpoint). For each z_k, let X_k be the union of all components of $T \setminus \{z_k\}$ which do not contain the critical point. Clearly X_k is non-empty and $f|_{X_k}$ is injective. If X_k contains no immediate preimage of 0, then f maps X_k homeomorphically into X_{k+1}. Obviously, if X_k and X_l intersect, then either $X_k \subset X_l$ or $X_l \subset X_k$. At least one set X_k must contain an immediate preimage of 0: if the critical orbit is periodic, then every endpoint of T eventually iterates onto 0, and every X_k contains an endpoint. If the critical orbit is preperiodic, we need the extra hypothesis on the itinerary of the orbit (z_k): if no X_k contains a point which ever iterates to 0, then all endpoints of X_k have the same itinerary as z_k in contradiction to our assumption.

If X_k contains an immediate preimage w of 0, then the corresponding z_k separates w from the critical point, i.e., $z_k \in [w, 0]$ and thus $z_{k+1} \in [0, c_1]$ (always taking indices modulo n), hence $c_1 \in X_{k+1}$.

Among the non-empty set of points $z_{k+1} \in [0, c_1]$, there is a unique one closest to c_1; relabel the orbit cyclically so that this point is z_1. We will show that this is the characteristic point of its orbit.

For every k, let n_k be the number of points from $\{z_i\}$ in X_k. If X_k does not contain an immediate preimage of 0, then $n_{k+1} \geq n_k$. Otherwise, n_{k+1} can be smaller than n_k, but $z_{k+1} \in [0, c_1]$; since no $z_k \in (z_1, c_1]$, we have $z_{k+1} \in [0, z_1]$ and either $z_{k+1} = z_1$ or $n_{k+1} \geq 1$.

Therefore, if $n_1 \geq 1$, then all $n_k \geq 1$; however, the nesting property of the X_k implies that there is at least one 'smallest' X_k which contains no further $X_{k'}$ and thus no $z_{k'}$; it has $n_k = 0$. Therefore, $n_1 = 0$; this means that all $z_k \neq z_1$ are in the same component of $T \setminus \{z_1\}$ as 0. Since $c_1 \in X_1$, the point z_1 is characteristic. $\square$

4.3. **Proposition (Images of Global Arms)**
Let z_1 be a characteristic periodic point of exact period m and let G be a global arm at z_1. Then either $0 \notin f^{\circ k}(G)$ for $0 \leq k < m$ (and in particular the first return map of z_1 maps G homeomorphically onto its image), or the first return map of z_1 sends the local arm in G to the local arm at z_1 pointing to the critical point or the critical value.

Proof. Let $z_k := f^{\circ(k-1)}(z_1)$ for $k \geq 1$. Consider the images $f(G)$, $f(f(G))$, etc. of the global arm G; if none of them contains 0 before z_1 returns to itself, then G maps homeomorphically onto its image under the first return map of z_1 and the claim follows. Otherwise, there is a first index k such that $f^{\circ(k-1)}(G) \ni 0$, so that the image arm at z_k points to 0; so far, the map is homeomorphic on G. If $z_k = z_0$, then the image point is z_1 and the local image arm at z_1 points to c_1. If $z_k \neq z_0$, then the local arm at z_k points to 0 and the image arm at z_{k+1} points to c_1; since z_1 is characteristic, the image arm points also to z_1. Continuing the iteration, the image arms at the image points will always point to some z_l. When the orbit finally reaches z_0, the local arm points to some $z_{l'}$. If it also points to 0, then the image at z_1 will point to c_1 as above; otherwise, it maps homeomorphically and the image arm at z_1 points to $z_{l'+1}$. By Lemma 4.1, the only such arm is the arm to the critical point. $\square$

4.4. **Corollary (Two Kinds of Periodic Orbits)**
Let z_1 be the characteristic point of a periodic orbit of branch points. Then the first return map either permutes all the local arms transitively, or it fixes the arm to 0 and permutes all the other local arms transitively.

Proof. Let n be the exact period of z_1. Since the periodic orbit does not contain the critical point by Lemma 3.3, the map $f^{\circ n}$ permutes all the local arms of z_1. Let G be any global arm at z_1. It must eventually map onto the critical point, or the marked point z_1 would have the same itinerary as all the marked points in G, contradicting the expansivity condition. By Lemma 4.3, the orbit of any local arm at z_1 must include the arm at z_1 to 0 or to c_1 or both, and there can be at most two orbits of local arms.

Consider the local arm at z_1 to 0. The corresponding global arm cannot map homeomorphically, so $f^{\circ n}$ sends this local arm to the arm pointing to 0 or to c_1. If the image local arm points to c_1, then all local arms at z_1 are on the same orbit, so $f^{\circ n}$ permutes these arms transitively. If $f^{\circ n}$ fixes the local arm at z_1 pointing to 0, then the orbit of every other local arm must include the arm to c_1, so all the other local arms are permuted transitively. $\square$

4.5. **Definition (Tame and Evil Orbits)**
A periodic orbit of branch points is called tame *if all its local arms are on the same cycle, and it is called* evil *otherwise.*

Remark. Obviously, evil orbits are characterized by the property that not all local arms have equal periods; their first return dynamics is described in Corollary 4.4. For periodic points (not containing a critical point) with two local arms, the situation is analogous: the first return map can either interchange these arms or fix them both. It will become clear below that periodic points with only two arms are less interesting than branch points; however, Proposition 4.8 shows that they have similar combinatorial properties. In Section 6, we will also call a periodic point with two arms *tame* if the first return map permutes the two local arms (but we reserve the adjective *evil* for branch points because only the latter destroy admissibility; see Proposition 4.10).

4.6. **Lemma (Global Arms at Branch Points Map Homeomorphically)**
Let z_1 be the characteristic point of a periodic orbit of period n and let $q \geq 3$ be the number of arms at each point. Then the global arms at z_1 can be labelled G_0, G_1, ..., G_{q-1} so that G_0 contains the critical point, G_1 contains the critical value, and the arms map as follows:

- *if the orbit of z_1 is tame, then the local arm $L_0 \subset G_0$ is mapped to the local arm $L_1 \subset G_1$ under $f^{\circ n}$; the global arms $G_1, \ldots, G_{q-2}$ are mapped homeomorphically onto their images in $G_2, \ldots, G_{q-1}$, respectively, and the local arm $L_{q-1} \subset G_{q-1}$ is mapped to L_0;*
- *if the orbit is evil, then the local arm L_0 is fixed under $f^{\circ n}$, the global arms $G_1, \ldots, G_{q-2}$ are mapped homeomorphically onto their images in $G_2, \ldots, G_{q-1}$, respectively, and the local arm $L_{q-1} \subset G_{q-1}$ is sent to the local arm L_1; however, the global arm G_{q-1} maps onto the critical point before reaching G_1.*

In particular, if the critical orbit is periodic, then its period must strictly exceed the period of any periodic branch point.

Proof. We will use Proposition 4.3 repeatedly, and we will always use the map $f^{\circ n}$. The global arms at z_1 containing 0 and c_1 are different because $z_1 \in (0, c_1)$. If the orbit is tame, then the local arm L_0 cannot be mapped to itself; since $G_0 \ni 0$, L_0 must map to L_1. There is a unique local arm at z_1 which maps to the local arm towards 0. Let G_{q-1} be the corresponding global arm; it may or may not map onto 0 under $f^{\circ k}$ for $k \leq n$. All the other global arms are mapped onto their images homeomorphically. They can be labelled so that G_i maps to G_{i+1} for $i = 1, 2, \ldots, q-2$. This settles the tame case.

In the evil case, the local arm L_0 is fixed, and the other local arms are permuted transitively. Let L_{q-1} be the arm for which $f^{\circ n}(L_{q-1})$ points to the critical value. Then all other global arms map homeomorphically and can be labelled $G_1, G_2, \ldots, G_{q-2}$ so that G_i maps homeomorphically into G_{i+1} for $i = 1, 2, \ldots, q-2$.

If $f^{\circ k}(G_{q-1}) \not\ni 0$ for all $k \le n$, then the entire cycle $G_1, \ldots, G_{q-1}$ of global arms would map homeomorphically onto their images, and all their endpoints would have identical itineraries with z_1. This contradicts the expansivity condition for Hubbard trees. □

4.7. **Corollary (Itinerary of Characteristic Point)**
In the Hubbard tree for the $\star$-periodic kneading sequence ν, fix a periodic point z whose orbit does not contain the critical point. Let m be the period of z; if z is not a branch point, suppose that the itinerary of z also has period m. Then if the first $m-1$ entries in the itinerary of z are the same as those in ν, the point z is characteristic.

There is a converse if z is a branch point: if z is characteristic, then the first m entries in its itinerary are the same as in ν.

Proof. If z is not characteristic, then by Lemma 4.1, the arc $[z, c_1]$ contains the characteristic point of the orbit of z; call it z'. The itineraries of z and z' differ at least once within the period (or the period of the itinerary would divide the period of z; for branch points, this would violate the expansivity condition, and otherwise this is part of our assumption). If the itinerary of z coincides with ν for at least $m-1$ entries, then the same must be true for $z' \in [z, c_1]$ by Lemma 3.6. Since the number of symbols $\mathtt{0}$ must be the same in the itineraries of z and z', then z and z' must have identical itineraries, and this is a contradiction.

Conversely, if z is the characteristic point of a branch orbit, then by Lemma 4.6, $[z, c_1]$ maps homeomorphically onto its image under $f^{\circ m}$ without hitting 0, and the first m entries in the itineraries of z and c_1 coincide. □

The following result allows to distinguish tame and evil branch points just by their itineraries.

4.8. **Proposition (Type of Characteristic Point)**
Let z_1 be a characteristic periodic point. Let τ be the itinerary of z_1 and let n be the exact period of z_1. Then:

- *if n occurs in the internal address of τ, then the first return map of z_1 sends the local arm towards 0 to the local arm toward c_1, and it permutes all local arms at z_1 transitively;*
- *if n does not occur in the internal address of τ, then the first return map of z_1 fixes the local arm towards 0 and permutes all other local arms at z_1 transitively.*

In particular, a characteristic periodic branch point of period n is evil if and only if the internal address of its itinerary does not contain n.

Proof. The idea of the proof is to construct certain precritical points $\zeta'_{k_j} \in [z_1, 0]$ so that $[z_1, \zeta'_{k_j}]$ contains no precritical points ζ' with $\mathrm{Step}(\zeta') \le \mathrm{Step}(\zeta'_{k_j})$. Using these points, the mapping properties of the local arm at z_1 towards 0 can be investigated. We also need a sequence of auxiliary points w_i which are among the two preimages of z_1.

Let $\zeta'_1 = 0$ and $k_0 = 1$ and let w_1 be the preimage of z_1 that is contained in T_1 and let $k_1 \ge 2$ be maximal such that $f^{\circ(k_1-1)}|_{[z_1,w_1]}$ is homeomorphic. If $k_1 < \infty$, then there exists a unique point $\zeta'_{k_1} \in (z_1, w_1)$ such that $f^{\circ k_1 - 1}(\zeta'_{k_1}) = 0$. All points on $[z_1, \zeta'_{k_1})$ have itineraries which coincide for at least $k_1 - 1$ entries. If $k_1 < n$ then the interval $[w_2, f^{\circ(k_1-1)}(z_1)]$ is non-degenerate and contained in $f^{\circ(k_1-1)}((\zeta'_{k_1}, z_1])$, where w_2 denotes the preimage of z_1 that is not separated from $f^{\circ(k_1-1)}(z_1)$ by 0. Let $y_{k_1} \in (\zeta'_{k_1}, z_1)$ be such that $f^{\circ(k_1-1)}(y_{k_1}) = w_2$. Next, let $k_2 > k_1$ be maximal such that $f^{\circ(k_2-1)}|_{[z_1,y_{k_1}]}$ is homeomorphic. If $k_2 < \infty$, then there exists a point $\zeta'_{k_2} \in (z_1, \zeta'_{k_1})$ such that $f^{\circ k_2 - 1}(\zeta'_{k_2}) = 0$, and the points on $[z_1, \zeta'_{k_2})$ have the same itineraries for at least $k_2 - 1$ entries. If $k_2 < n$ then, as above, the interval $[w_3, f^{\circ(k_2-1)}(z_1)]$ is non-degenerate (where again w_3 is an appropriate preimage of z_1) and there is a $y_{k_2} \in (\zeta'_{k_2}, z_1)$ such that $f^{\circ(k_2-1)}(y_{k_2}) = w_3$. Continue this way while $k_j < n$.

Note that the ζ'_{k_j} are among the precritical points on $[z_1, w_1]$ closest to z_1 (in the sense that for each ζ'_{k_j}, there is no $\zeta' \in (z_1, \zeta'_{k_j})$ with $\mathrm{Step}(\zeta') \le \mathrm{Step}(\zeta'_{k_j})$; compare also Definition 5.5), but the ζ'_{k_j} are *not* all precritical points closest to z_1; in terms of the cutting time algorithm, the difference can be described as follows: starting with $[z_1, w_1]$, we iterate this arc forward until the image contains 0; when it does after some number k_j of iterations, we cut at 0 and keep only the closure of the part containing $f^{\circ k_j}(z_1)$ at the end. The usual cutting time algorithm would continue with the entire image arc after cutting, but we cut additionally at the point $w_{j+1} \in f^{-1}(z_1)$.

The point of this construction is the following: let ρ_τ be the ρ-function with respect to τ, i.e., $\rho_\tau(j) := \min\{i > j \ : \ \tau_i \ne \tau_{i-j}\}$. Then $k_1 = \rho_\tau(1)$ and $k_{j+1} = \rho_\tau(k_j)$ (if $k \ne 1$, then the exact number of iterations that the arc $[z_1, z_k]$ can be iterated forward homeomorphically is $\rho_\tau(k) - 1$ times). Therefore we have constructed a sequence ζ'_{k_j} of precritical points on $[z_1, 0]$ so that (for entries less than n) $k_0, k_1, \dots = \mathrm{orb}_{\rho_\tau}(1)$, which is the internal address associated to τ.

Recall that n is the exact period of z_1. If n belongs to the internal address, then there exists $\zeta'_n \in [z_1, 0]$ and $f^{\circ n}$ maps $[z_1, \zeta'_n]$ homeomorphically onto $[z_1, c_1]$. Therefore $f^{\circ n}$ sends the local arm towards 0 to the local arm towards c_1. By Lemma 4.6, $f^{\circ n}$ permutes all arms at z_1 transitively.

On the other hand, assume that n does not belong to the internal address. Let m be the last entry in the internal address before n. Then $f^{\circ(m-1)}$ maps $[z_1, \zeta'_m]$ homeomorphically onto $[z_m, 0]$, and the restriction to $[z_m, w_j] \subset [z_m, 0]$ survives another $n - m$ iterations homeomorphically (maximality of m). There is a point $y_m \in [z_1, 0]$

so that $f^{\circ m-1}([z_1, y_m]) \to [z_m, w_j]$ is a homeomorphism, so $f^{\circ n}([z_1, y_m]) \to [z_1, z_{n-m+1}]$ is also a homeomorphism. The local arm at z_1 to 0 maps under $f^{\circ n}$ to a local arm at z_1 to z_{n-m+1}, and since z_1 is characteristic, this means that the local arm at z_1 to 0 is fixed under the first return map. The other local arms at z_1 are permuted transitively by Lemma 4.6. □

4.9. **Definition (Admissible Kneading Sequence and Internal Address)**
We call a $\star$-periodic kneading sequence and the corresponding internal address admissible *if the associated Hubbard tree contains no evil orbit.*

This definition is motivated by the fact, shown in Proposition 14.1, that a kneading sequence is admissible if and only if there is an external angle which generates this kneading sequence in the sense of Definition 2.1. Another equivalent statement is the following.

4.10. **Proposition (Embedding of Hubbard Tree)**
A Hubbard tree (T, f) can be embedded into the plane so that f respects the cyclic order of the local arms at all branch points if and only if (T, f) has no evil orbits.

Proof. If (T, f) has an embedding into the plane so that f respects the cyclic order of local arms at all branch points, then clearly there can be no evil orbit (this uses the fact that no periodic orbit of branch points contains a critical point).

Conversely, suppose that (T, f) has no evil orbits, so all local arms at every periodic branch point are permuted transitively. First we embed the arc $[0, c_1]$ into the plane, for example on a straight line. Every cycle of branch points has at least its characteristic point p_1 on the arc $[0, c_1]$, and it does not contain the critical point. Suppose p_1 has q arms. Take $s \in \{1, \ldots, q-1\}$ coprime to q and embed the local arms at p_1 in such a way that the return map $f^{\circ n}$ moves each arc over by s arms in counterclockwise direction. This gives a single cycle for every $s < q$ coprime to q. Furthermore, this can be done for all characteristic branch points independently.

We say that two marked points x, y are adjacent if (x, y) contains no further marked point. If a branch point x is already embedded together with all its local arms, and y is an adjacent marked point on T which is not yet embedded but $f(y)$ is, then draw a line segment representing $[x, y]$ into the plane, starting at x and disjoint from the tree drawn so far. This is possible uniquely up to homotopy. Embed the local arms at y so that $f\colon y \to f(y)$ respects the cyclic order of the local arms at y; this is possible because y is not the critical point of f.

Applying the previous step finitely many times, the entire tree T can be embedded. It remains to check that for every characteristic branch point p_1 of period m, say, the map $f\colon p_1 \to f(p_1) =: p_2$ respects the cyclic order of the local arms. By construction, the forward orbit of p_2 up to its characteristic point p_1 is embedded before embedding p_2, and $f^{\circ(m-1)}\colon p_2 \to p_1$ respects the cyclic order of the embedding. If the orbit of p_1

is tame, the cyclic order induced by $f\colon p_1 \to p_2$ (from the abstract tree) is the same as the one induced by $f^{\circ(m-1)}\colon p_2 \to p_1$ used in the construction (already embedded in the plane), and the embedding is indeed possible. $\square$

Remark. It is well known that once the embedding respects the cyclic order of the local arms and their dynamics, then the map f extends continuously to a neighborhood of T within the plane, and even to a branched cover of the entire plane with degree 2. See for example [**BFH**]. This implies that the kneading sequence of T is generated by an external angle (Corollary 13.7 or Proposition 14.1) and even that T occurs as the Hubbard tree of a quadratic polynomial (see again [**BFH**] or **??**).

The previous proposition can be sharpened: we consider two embeddings of a Hubbard tree into the plane as equal if the cyclic order of all the arms at each branch point is the same: then Algorithm 13.6 associates the same external angles to them, and the extensions of the map to the plane are equivalent in the sense of Thurston [**DH2**]. Different embeddings yield different external angles and different Thurston maps.

4.11. **Corollary (Number of Embeddings of Hubbard Tree)**
Let (T, f) be a Hubbard tree without evil branch points, and let $q_1, q_2, \dots$ be the number of arms of the different characteristic branch points. Then there are $\prod \varphi(q_i)$ different ways to embed T into the plane such that f extends to a two-fold branched covering. Here $\varphi(q)$ is the Euler function counting the integers $1 \le i < q$ that are coprime to q.

Proof. Follow the proof of Proposition 4.10: for each of the characteristic branch points z_i there are $\varphi(q_i)$ cyclic orders in which the q_i arms can be embedded. The choices can be made independently from all other characteristic branch points. This gives the asserted number. $\square$

Remark. We show in Lemma 16.11 that a Hubbard tree with critical orbit of period n has less than n different embeddings into the plane that are compatible with the dynamics.

4.12. **Lemma (Characteristic Points have Two Inverse Images)**
For every Hubbard tree (T, f) with $\star$-periodic or preperiodic kneading sequence, every characteristic periodic point $p \in T \setminus \{c_1\}$ with itinerary $\tau(p) \neq \overline{1}$ has two different preimages $\{p_0, -p_0\} = f^{-1}(p) \subset T$; these are separated by 0, i.e. $0 \in (-p_0, p_0)$.

Proof. Since $f((\alpha, 0)) = (\alpha, c_1) \ni p$, there is always a preimage of p in $(\alpha, 0) \subset T_1$.

Since $\tau(p) \neq \overline{1}$, there is a $p_i \in \mathrm{orb}(p)$ such that $p_i \in T_0$. Then $[p_i, 0) \subset T_0$ is mapped homeomorphically onto $[p_{i+1}, c_1)$. Since p is characteristic, we have $p \in [p_{i+1}, c_1)$, and there is a point $p_0 \in [p_i, 0) \subset T_0$ with $f(p_0) = p$. $\square$

5. The Admissibility Condition

In this section we derive the nature of periodic branch points of the Hubbard tree from the kneading sequence (Propositions 5.13 and 5.19). We also prove a condition (*admissibility condition*) on the kneading sequence which decides whether there are evil orbits: Proposition 5.12 shows that an evil orbit violates this condition, and Proposition 5.13 shows that a violated condition leads to an evil orbit within the Hubbard tree. Since a Hubbard tree can be embedded in the plane whenever there is no evil orbit (Proposition 4.10), we obtain a complete classification of admissible kneading sequences (Theorem 5.2).

5.1. **Definition (The Admissibility Condition)**
A kneading sequence $\nu \in \Sigma^\star$ fails the admissibility condition for period m *if the following three conditions hold:*

1. *the internal address of* ν *does not contain* m*;*
2. *if* $k < m$ *divides* m*, then* $\rho(k) \le m$*;*
3. $\rho(m) < \infty$ *and if* $r \in \{1, \ldots, m\}$ *is congruent to* $\rho(m)$ *modulo* m*, then* $\operatorname{orb}_\rho(r)$ *contains* m*.*

A kneading sequence fails the admissibility condition *if it does so for some* $m \ge 1$*.* *An internal address* fails the admissibility condition *if its associated kneading sequence does.*

If a kneading sequence fails the admissibility condition for period m, then it follows that $\rho(m)$ is in the associated internal address (Lemma 20.1).

This definition applies to all sequences in $\Sigma^\star$, i.e., all sequences in $\{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$ and all $\star$-periodic sequences, provided they start with $\mathtt{1}$. However, in this section and the next we will only consider $\star$-periodic and preperiodic kneading sequences because these are the ones for which we have Hubbard trees. The main result in this section is that this condition precisely describes admissible kneading sequences in the sense of Definition 4.9 (those for which the Hubbard tree has no evil orbits):

5.2. **Theorem (Evil Orbits and Admissibility Condition)**
A Hubbard tree contains an evil orbit of exact period m *if and only if its kneading sequence (from Definition 3.5) fails the Admissibility Condition 5.1 for period* m*.*

Equivalently, a Hubbard tree can be embedded into the plane so that the dynamics respects the embedding if and only if the associated kneading sequence does not fail the Admissibility Condition 5.1 for any period.

The proof of the first claim will be given in Propositions 5.12 and 5.13, and the second is equivalent by Proposition 4.10.

While this theorem only deals with the postcritically finite case, we will show in Corollary 14.2 that a $\star$-periodic or non-periodic kneading sequence is admissible in the sense of Definition 5.1 if and only if it is generated from an external angle by the algorithm in Definition 2.1. Note that the fact whether or not a sequence fails the admissibility condition for period m is determined by its first $\rho(m)$ entries.

5.3. **Example (Non-Admissible Kneading Sequences)**
The internal address $1 \to 2 \to 4 \to 5 \to 6$ with kneading sequence $\overline{\mathtt{101\,10}\star}$ (or any address that starts with $1 \to 2 \to 4 \to 5 \to 6 \to$) fails the admissibility condition for $m = 3$, and the Hubbard tree indeed has a periodic branch point of period 3 that does not permute its arms transitively, as can be verified in Figure 5.1. This is the simplest and best known example of a non-admissible Hubbard tree; see [**LS, Ke, Pen1**].

More generally, let $\nu = \overline{\nu_1 \dots \nu_{m-1}\star}$ be any $\star$-periodic kneading sequence of period m so that there is no k dividing m with $\rho(k) = m$ (for kneading sequences in the Mandelbrot set, this means that ν is not a bifurcation from a sequence of period k; see **??**). This clearly implies $\rho(k) < m$ for all k dividing m. Let $\nu_m \in \{\mathtt{0}, \mathtt{1}\}$ be such that m does not occur in the internal address of $\overline{\nu_1 \dots \nu_m}$. Then for any $s \geq 2$, every sequence starting with

$$\overline{\underbrace{\nu_1 \dots \nu_m \dots \nu_1 \dots \nu_m}_{s-1 \text{ times}} \nu_1 \dots \nu_{m-1}\nu'_m}$$

(with $\nu'_m \neq \nu_m$) fails the admissibility condition for m. The example $1 \to 2 \to 4 \to 5 \to 6$ above with kneading sequence $\overline{\mathtt{10110}\star}$ has been constructed in this way, starting from $\overline{\mathtt{10}\star}$. We show after Proposition 6.7 that every non-admissible kneading sequence is related to such an example.

While $1 \to 2 \to 4 \to 5 \to 6$ is not admissible, the internal address $1 \to 2 \to 4 \to 5 \to 11$ is admissible; its Hubbard tree is shown in Figure 5.2. This shows that the Translation Principle from [**LS**, Conjecture 8.7] does not hold: the address $1 \to 2 \to 4 \to 5 \to 11$ is realized in the $\frac{1}{3}$ and $\frac{2}{3}$-sublimbs of the real period 5 component $1 \to 2 \to 4 \to 5$ of the Mandelbrot set. The Translation Principle would predict that $1 \to 2 \to 4 \to 5 \to 6$ should exist within the $\frac{1}{2}$-sublimb, but no such hyperbolic component exists (the same counterexample was found independently by V. Kauko [**Kau1**]).

Remark. The three conditions in the admissibility condition are independent: here are examples of kneading sequences where exactly two of the three conditions are satisfied.

- $\nu = \overline{101\star}$ $(1 \to 2 \to 4)$, $m = 2$: condition 1 is violated; ν is admissible.
- $\nu = \overline{111\star}$ $(1 \to 4)$, $m = 2$: condition 2 is violated; ν is admissible.
- $\nu = \overline{101\star}$ $(1 \to 2 \to 4)$, $m = 3$: condition 3 is violated; ν is admissible.

These conditions can be interpreted as follows: the first condition picks a candidate period for an evil orbit, taking into account that a branch point is always tame when its period occurs on the internal address (Proposition 5.12); the second condition assures that the period m of the evil orbit is the exact period, and the third condition makes

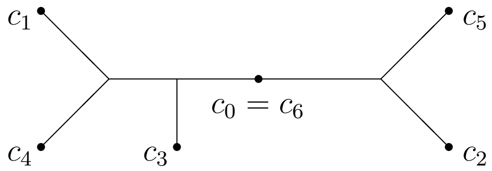

FIGURE 5.1. The Hubbard tree for $1 \to 2 \to 4 \to 5 \to 6$ contains an evil orbit of period 3.

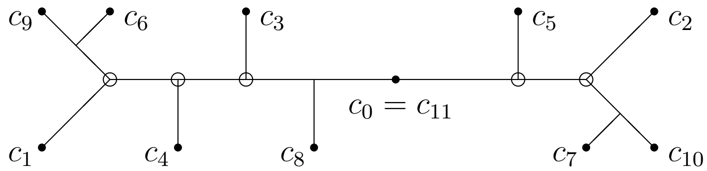

FIGURE 5.2. The Hubbard tree for $1 \to 2 \to 4 \to 5 \to 11$ is admissible. There is a tame periodic orbit of branch points of period 5 (indicated by $\circ$'s). The other branch points are preperiodic.

the periodic orbit evil by assuring that the first return map of the characteristic point maps a different local arm than the one pointing to 0 onto the local arm to the critical value.

5.4. **Lemma (Bound on Failing the Admissibility Condition)**
If a $\star$-periodic kneading sequence of period n fails the admissibility condition for period m, then $m < n$.

PROOF. Since n occurs in the internal address, we may suppose $m \neq n$. If $m > n$, then $\rho(m) < m + n$ (because one of the entries between m and $m + n$ is a $\star$), hence $r < n$ and $\mathrm{orb}_\rho(r)$ terminates at n, so $m \notin \mathrm{orb}_\rho(r)$. □

A different way to interpret Lemma 5.4 is to say that a $\star$-periodic kneading sequence fails the admissibility condition for period m if and only if the associated Hubbard tree has an evil branch point of period m (Theorem 5.2), and the period of a branch point is bounded by the period of the kneading sequence (Lemma 4.6).

One of the main tools are closest precritical points.

5.5. **Definition (Precritical Points)**
A point $x \in T$ is called precritical *if $f^{\circ k}(x) = c_1$ for some $k \geq 1$; the least such index k is called* STEP(x). *The point x is called a* closest precritical point *and denoted ζ_k if $f^{\circ j}([c_1, x]) \not\ni c_1$ for all $j \in \{1, \dots, k-1\}$.*

The critical point is always ζ_1; if the critical point is periodic of period n, then $\zeta_n = c_1$ and there is no closest precritical point x with $\textsc{Step}(x) > n$. Closest precritical points are those which are "visible from c_1" in the sense of [**LS**, Section 8]: the idea is that a precritical point ζ blocks the view of all ζ' behind ζ with $\textsc{Step}(\zeta') \geq \textsc{Step}(\zeta)$ (figuratively speaking, ζ is so big that the smaller point ζ' cannot be seen if it is behind ζ). We say that ζ is the *earliest* precritical point on an arc (x, y) (or $[x, y]$ etc.) if it is the one with the lowest $\textsc{Step}$.

5.6. **Lemma (Closest Precritical Points Unique)**
A Hubbard tree contains at most one closest precritical point ζ_k for every index k.

Proof. If for some k, there are two closest precritical points ζ_k and ζ'_k, then $f^{\circ(k-1)}$ maps $[\zeta_k, \zeta'_k]$ homeomorphically onto its image, but both endpoints map to the critical point 0. This is a contradiction. □

5.7. **Lemma (Elementary Properties of ρ)**
If $\zeta_k \neq c_1$, then the earliest precritical point on $(\zeta_k, c_1]$ is $\zeta_{\rho(k)}$. For $k \geq 1$, the earliest precritical point on $[c_{1+k}, c_1)$ is $\zeta_{\rho(k)-k}$.

If $\zeta_k \neq c_1$, then $[c_1, \zeta_k]$ contains those and only those closest precritical points ζ_m for which $m \in \operatorname{orb}_\rho(k)$. In particular, $\zeta_m \in [0, c_1]$ if and only if m belongs to the internal address.

Proof. The first two statements follow immediately from the definition of ρ, using the idea of *cutting times* (see Algorithm 13.8); note that the arc $[c_1, \zeta_k)$ can be iterated homeomorphically for at least k iterations, and $f^{\circ k}([c_1, \zeta_k)) = [c_{1+k}, c_1)$. The first time that $f^{\circ(m-1)}([c_1, \zeta_k))$ hits 0 is for $m = \rho(k)$ by definition, and the earliest precritical point on $[c_1, \zeta_k)$ takes exactly $\rho(k)$ steps to map to c_1. The claim now follows by induction. The statement about the internal address follows because $\zeta_1 = 0$. □

5.8. **Lemma (Images of Closest Precritical Points)**
If $k < k' \leq \rho(k)$, then $f^{\circ k}(\zeta_{k'})$ is the closest precritical point $\zeta_{k'-k}$.

Proof. Let $x := f^{\circ k}(\zeta_{k'})$. Then the arc $[c_1, \zeta_{k'}]$ maps under $f^{\circ k}$ homeomorphically onto $[c_{k+1}, x]$, and there is no precritical point $\zeta \in (c_{k+1}, x)$ with $\textsc{Step}(\zeta) \leq k' - k$. If $\rho(k) > k'$ then by Lemma 5.7 there is no such precritical point $\zeta \in [c_1, c_{k+1}]$ either, and hence none on $(x, c_1]$. Since $\textsc{Step}(x) = k' - k$, the point x is indeed the closest precritical point $\zeta_{k'-k}$. Finally, if $\rho(k) = k'$, then $\zeta_{\rho(k)-k} = \zeta_{k'-k}$ is the earliest precritical point on $[c_1, c_{1+k}]$ (Lemma 5.7). But since x also has $\textsc{Step}(x) = k' - k$ and $[x, \zeta_{k'-k}]$ contains no point of lower $\textsc{Step}$, we have $x = \zeta_{k'-k}$. □

5.9. **Lemma (Precritical Points Near Periodic Points)**
Let z_1 be a characteristic periodic point of period m such that $f^{\circ m}$ maps $[z_1, c_1]$ homeomorphically onto its image. Assume that ν is not $\star$-periodic of period less than m. If z_1 has exactly two local arms, assume also that the first return map of z_1 interchanges them. Then

(1) *the closest precritical point ζ_m exists in the Hubbard tree, $z_1 \in [\zeta_m, c_1]$ and $\zeta_{\rho(m)} \in [c_1, z_1]$;*
(2) *if ζ is a precritical point closest to z_1 with* $\textsc{Step}(\zeta) < m$ *in the same global arm of z_1 as ζ_m, then $\zeta_m \in [z_1, \zeta]$;*
(3) *if z_1 is a tame branch point, then $\zeta_m \in [0, z_1]$ and m occurs in the internal address;*
(4) *if z_1 is an evil branch point, then $\zeta_m \in G_{q-1}$ (where global arms are labelled as in Lemma 4.6) and m does not occur in the internal address.*

Proof. (1) First we prove the existence of ζ_m in T. Let $G_0, G_1, \ldots, G_{q-1}$ be the global arms of z_1 with $0 \in G_0$ and $c_1 \in G_1$. (Note that $q = 2$ is possible.) Let $L_0, \ldots, L_{q-1}$ be the corresponding local arms. Let j be such that $f^{\circ m}(L_j) = L_1$.

If $j = 0$, then $0 = \zeta_1 \in G_j$. If $j \neq 0$, then $q \geq 3$ by assumption, so $j = q - 1$ by Lemma 4.6 and there is an $i < m$ so that $f^{\circ i}(G_j)$ contains 0. Therefore, in both cases there exists a unique $\zeta_k \in G_j$ with $k \leq m$ maximal, and it satisfies $z_1 \in (\zeta_k, c_1)$. We want to show that $k = m$.

If $k < m$, then $f^{\circ k}$ maps (z_1, ζ_k) homeomorphically onto $(z_{k+1}, c_1) \ni z_1$. By maximality of k, the restriction of $f^{\circ m}$ to (z_1, ζ_k) is a homeomorphism with image $(z_1, c_{m-k+1}) \subset G_1$, and it must contain $f^{\circ(m-k)}(z_1) = z_{1+m-k}$ in contradiction to the fact that z_1 is characteristic. Hence $k = m$, ζ_m exists and $z_1 \in [c_1, \zeta_m]$.

Clearly $f^{\circ m}$ maps $[z_1, \zeta_m]$ homeomorphically onto $[z_1, c_1]$. By Lemma 5.7, $\zeta_{\rho(m)} \in [c_1, \zeta_m]$. If $\zeta_{\rho(m)} \in [z_1, \zeta_m]$, then $f^{\circ m}(\zeta_{\rho(m)}) \in [c_1, z_1] \subset [c_1, \zeta_{\rho(m)}]$, but then $\zeta_{\rho(m)}$ would not be a closest precritical point. Hence $\zeta_{\rho(m)} \in [z_1, c_1]$.

(2) For the second statement, let $k :=$ $\textsc{Step}(\zeta) < m$. We may suppose that (z_1, ζ) contains no precritical point ζ'' with $\textsc{Step}(\zeta'') < m$ (otherwise replace ζ by ζ''). Clearly $\zeta \notin [z_1, \zeta_m]$. Assume by contradiction that $\zeta_m \notin [z_1, \zeta]$ so that $[z_1, \zeta, \zeta_m]$ is a non-degenerate triod. Since both $[z_1, \zeta]$ and $[z_1, \zeta_m]$ map homeomorphically under $f^{\circ m}$, the same is true for the triod $[z_1, \zeta, \zeta_m]$.

Under $f^{\circ k}$, the triod $[z_1, \zeta, \zeta_m]$ maps homeomorphically onto the triod $[z_{k+1}, c_1, \zeta']$ with $\textsc{Step}(\zeta') = m - k$, and $z_{k+1} \neq z_1$. Then $z_1 \in (z_{k+1}, c_1)$, so the arc (z_{k+1}, ζ') contains either z_1 or a point at which the path to z_1 branches off. Under $f^{\circ(m-k)}$, the triod $[z_{k+1}, c_1, \zeta'] \ni z_1$ maps homeomorphically onto $[z_1, c_{m-k+1}, c_1] \ni z_{m-k+1}$. Therefore (z_1, c_1) contains either the point z_{m-k+1} or a branch point from which the path to z_{m-k+1} branches off. Both are in contradiction to the characteristic property of z_1.

(3) If z_1 is tame, then $j = 0$, so $\zeta_m \in G_0$. By the previous statement, $\zeta_m \in [z_1, 0]$, and Lemma 5.7 implies that m belongs to the internal address.

(4) Finally, if z_1 is evil, then $\zeta_m \in G_{q-1}$ and m does not occur in the internal address by Lemma 5.7. □

The following lemma is straightforward, but helpful to refer to in longer arguments.

5.10. **Lemma (Translation Property of ρ)**
If $\rho(m) > km$ for $k \geq 2$, then $\rho(km) = \rho(m)$.

Proof. Let ν be a kneading sequence associated to ρ. Then $\rho(m) > km$ says that the first m entries in ν repeat at least k times, and $\rho(m)$ finds the first position where this pattern is broken. By definition, $\rho(km)$ does the same, omitting the first k periods. □

5.11. **Lemma (Bound on Number of Arms)**
Let $z_1 \in [c_1, \zeta_m]$ be a characteristic point of period m with q arms. Assume that ν is not $\star$-periodic of period less than m. If z_1 has exactly two local arms, assume also that the first return map of z_1 interchanges them. If z_1 is evil, then $(q-2)m < \rho(m) \leq (q-1)m$; if not, then $(q-2)m < \rho(m) \leq qm$.

Proof. Let $G_0 \ni 0$, $G_1 \ni c_1, \dots, G_{q-1}$ be the global arms at z_1. By Lemma 5.9, $\zeta_{\rho(m)} \in [z_1, c_1]$. The lower bound for $\rho(m)$ follows from Lemma 4.6.

First assume that z_1 is evil, so $q \geq 3$. Lemma 5.10 implies $\rho((q-2)m) = \rho(m)$. Assume by contradiction that $\rho(m) > (q-1)m$. Then $r := \rho(m) - (q-2)m > m$. By Lemma 5.8, $\zeta_r = f^{\circ((q-2)m)}(\zeta_{\rho(m)})$ is a closest precritical point. It belongs to the same arm G_{q-1} as ζ_m, but $\zeta_m \notin [z_1, \zeta_r]$. As $\zeta_{\rho(m)}$ is the earliest precritical point on $[c_1, \zeta_m)$, we cannot have $\zeta_r \in [z_1, \zeta_m]$ either. Therefore $[z_1, \zeta_r, \zeta_m]$ is a non-degenerate triod within $\overline{G}_{q-1}$; let $y \in G_{q-1}$ be the branch point, see Figure 5.3. Obviously

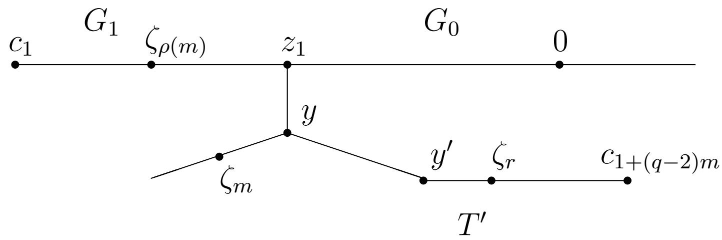

Figure 5.3. Subtree with an evil branch point z_1 of a Hubbard tree.

$f^{\circ m}(y) \in [z_1, c_1]$ and since $f^{\circ((q-2)m)}$ maps G_1 homeomorphically into G_{q-1}, we find $y' := f^{\circ((q-1)m)}(y) \in [z_1, c_{1+(q-2)m}]$, see Figure 5.3. If $y' \in [z_1, y]$, then $f^{\circ((q-1)m}$ maps $[z_1, y]$ homeomorphically into itself. This contradicts expansivity of the tree. Therefore $y' \in (y, c_{1+(q-2)m}]$. Let T' be the component of $T \setminus \{y\}$ containing $c_{1+(q-2)m}$. Since $\zeta_{\rho(m)} \in [z_1, c_1]$ and $f^{\circ(q-2)m}(\zeta_{\rho(m)}) = \zeta_r$, T' contains ζ_r but not ζ_m. Now $f^{\circ((q-1)m)}$ maps T' homeomorphically into itself (otherwise, there would be an earliest precritical point

$\zeta \in T'$ with $\text{STEP}(\zeta) < m$, but then $\zeta_m \in [z_1, \zeta]$ by Lemma 5.9 (2)). Again, expansivity of the tree is violated. Thus indeed $\rho(m) \leq (q-1)m$.

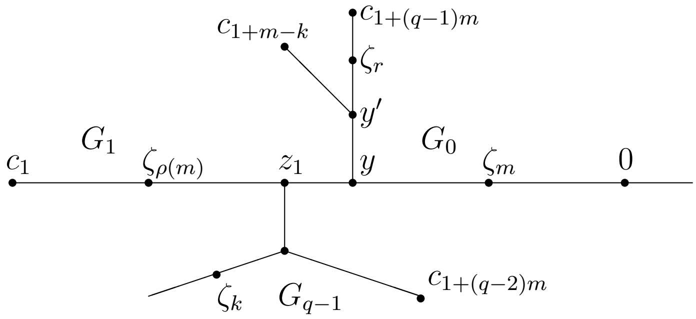

FIGURE 5.4. Subtree with a tame branch point z_1 of a Hubbard tree.

Now assume that z_1 is not evil (and maybe not even a branch point). Assume by contradiction that $\rho(m) > qm$. We repeat the above argument with $r := \rho(m) - (q-1)m > m$, conclude that $\rho((q-1)m) = \rho(m)$ and find the closest precritical point $\zeta_r \in G_0$, so ζ_r is in the same global arm at z_1 as ζ_m. As before, $[z_1, \zeta_r, \zeta_m]$ is a non-degenerate triod with branch point y and $y' := f^{\circ qm}(y)$ lies on $[y, c_{1+(q-1)m}]$. Let T' be the component of $T \setminus \{y\}$ containing $c_{1+(q-1)m}$. We claim that $f^{\circ qm}$ is homeomorphic on T'.

It follows as above, using Lemma 5.9, that $f^{\circ m}$ maps T' homeomorphically into G_1, and $f^{\circ (q-2)m}$ maps G_1 homeomorphically into G_{q-1}. Let $T'' = f^{\circ (q-1)m}(T')$ and assume by contradiction that $f^{\circ m}$ is not homeomorphic on T''. Then T'' contains a closest precritical point ζ_k for some $k < m$. Take $k < m$ maximal. Then $f^{\circ m}$ is homeomorphic on $[z_1, \zeta_k]$, and since $f^{\circ k}([z_1, \zeta_k]) = [z_{k+1}, c_1] \ni z_1$, it follows that $f^{\circ m}([z_1, \zeta_k]) = [z_1, c_{1+m-k}]$ contains z_{1+m-k}. But since $\zeta_k \in T''$, hence $f^{\circ (q-1)m}(y) \in [z_1, \zeta_k]$, we also have $y' \in [z_1, c_{1+m-k}]$. As a result, $[z_1, c_{1+m-k}] \subset [z_1, y] \cup T'$.

If $z_{1+m-k} \in [z_1, y]$, then $f^{\circ m}$ maps $[z_1, z_{1+m-k}]$ homeomorphically onto its image. This is a contradiction: both endpoints are fixed, but the image must be in G_1. Therefore, $z_{1+m-k} \in T'$. By Lemma 5.9 (2) again, there can be no precritical point with STEP less than m on $[z_1, z_{1+m-k}]$, and we get the same contradiction.

We can conclude as above that $f^{\circ qm}$ maps T' homeomorphically into itself as claimed. But this is a contradiction to expansivity of the tree. $\square$

5.12. **Proposition (Evil Orbit Fails Admissibility Condition)**
If a Hubbard tree has an evil orbit of exact period m and ν is not $\star$-periodic of period less than m, then the kneading sequence fails the admissibility condition for period m.

Proof. Let z_1 be the characteristic point of the evil orbit of period m and let $G_0, \ldots, G_{q-1}$ be the global arms labelled as in Lemma 4.6. The corresponding local arms will be labelled $L_0, \ldots, L_{q-1}$.

We know from Lemma 5.9 that m is not in the internal address, so the first part of the admissibility condition is already taken care of.

Since $[z_1, c_1]$ maps homeomorphically for m steps, the first m entries in the itineraries of z_1 and c_1 coincide, and $e(z_1) = \overline{\nu_1 \ldots \nu_m}$. Let $k < m$ be a divisor of m. Suppose by contradiction that $\rho(k) > m$. Then $e(z_1)$ has period k. By Lemma 3.8, the period of z_1 is then k as well, contradicting the assumption. This settles the second condition of Definition 5.1

Let $r := \rho(m) - (q-2)m$. By Lemma 5.11, $0 < r \le m$. By Lemma 5.8, $f^{\circ(q-2)m}(\zeta_{\rho(m)}) = \zeta_{\rho(m)-(q-2)m} = \zeta_r \in G_{q-1}$.

By Lemma 5.9 (4) and (2), we have $\zeta_m \in G_{q-1}$ and then $\zeta_m \in [z_1, \zeta_r]$. Now Lemma 5.7 shows that $m \in \mathrm{orb}_\rho(r)$. Hence ν fails the admissibility condition for period m. ☐

In Propositions 5.13 and 5.19, we will determine the exact number of arms at all branch points, and determine from the internal address which branch points a Hubbard tree has.

5.13. **Proposition (Number of Arms at Evil Branch Points)**
Suppose a kneading sequence ν fails the Admissibility Condition 5.1 for period m, and that ν is not $\star$-periodic of period less than m. Then the Hubbard tree for ν contains an evil branch point of exact period m; the number of its arms is $q := \lfloor \rho(m)/m \rfloor + 2 \ge 3$.

Proof. Write $\rho(m) = (q-2)m + r$ for $r \in \{1, 2, \ldots, m\}$ and $q \ge 3$. Then $\rho((q-2)m) = \rho(m)$ by Lemma 5.10 and the earliest precritical point on $[c_{1+(q-2)m}, c_1)$ is ζ_r by Lemma 5.7. Since ν fails the admissibility condition for m, this implies in particular $m \in \mathrm{orb}_\rho(r)$, hence by Lemma 5.7 $\zeta_m \in [\zeta_r, c_1] \subset [c_{1+(q-2)m}, c_1]$.

Consider the connected hull

$$H := [c_1, c_{1+m}, c_{1+2m}, \ldots, c_{1+(q-3)m}, \zeta_m] .$$

Since $\rho(km) = \rho(m) > (q-2)m$ for $k = 2, 3, \ldots, q-3$ by Lemma 5.10, the map $f^{\circ m}$ sends the arc $[c_1, c_{1+km}]$ homeomorphically onto its image, and the same is obviously true for $[c_1, \zeta_m]$. We thus get a homeomorphism $f^{\circ m} \colon H \to H'$ with

$$H' = [c_{1+m}, c_{1+2m}, c_{1+3m}, \ldots, c_{1+(q-2)m}, c_1] .$$

Since $\zeta_m \in [c_{1+(q-2)m}, c_1]$, we have $H \subset H' \subset H \cup [\zeta_m, c_{1+(q-2)m}]$. Moreover, $\zeta_r \in [c_{1+km}, c_{1+(q-2)m}]$ for $k = 0, 1, \ldots, q-3$: the first difference between the itineraries of $c_{1+(q-2)m}$ and c_1 occurs at position r, while c_1 and c_{1+km} have at least m identical entries. Since $\zeta_m \in [\zeta_r, c_1]$, it follows similarly that $\zeta_m \in [\zeta_r, c_{1+km}] \subset [c_{1+(q-2)m}, c_{1+km}]$ for $k \le q-3$. Therefore, $H' \setminus H = (\zeta_m, c_{1+(q-2)m}]$.

Among the endpoints defining H, only $c_{1+(q-3)m}$ maps outside H under $f^{\circ m}$, so $c_{1+(q-3)m}$ is an endpoint of H and thus also of H'. It follows that $c_{1+(q-4)m}$ is an endpoint of H and thus also of H' and so on, so $c_1, \ldots, c_{1+(q-2)m}$ are endpoints of H. Finally, also ζ_m is an endpoint of H (or c_1 would be an inner point of H'). As a result, H and H' have the same branch points.

If $q = 3$, then H is simply an arc which is mapped in an orientation reversing manner over itself, and hence contains a fixed point of $f^{\circ m}$. Otherwise H contains a branch point. Since $f^{\circ m}$ maps H homeomorphically onto $H' \supset H$, it permutes the branch points of H. By expansivity there can be at most one branch point, say z_1, which must be fixed under $f^{\circ m}$. Since $f^{\circ m}\colon [z_1, c_1] \to [z_1, c_{1+m}]$ is a homeomorphism with $[z_1, c_1] \cap [z_1, c_{1+m}] = \{z_1\}$, the arc $(z_1, c_1]$ cannot contain a point on the orbit of z_1, so z_1 is characteristic.

If z_1 is a tame branch point, then $\zeta_m \in [z_1, 0]$ by Lemma 5.9, and m occurs in the internal address in contradiction to the failing admissibility condition. If z_1 has exactly two arms, these are interchanged by $f^{\circ m}$, and $\zeta_m \in G_0$, the global arm containing 0. By Lemma 5.9 (2), $\zeta_m \in [0, z_1]$ and m occurs in the internal address, again a contradiction. Hence z_1 is an evil branch point.

Now H has exactly $q-1$ endpoints, and these are contained in different global arms of z_1. The corresponding local arms are permuted transitively by $f^{\circ m}$. Since z_1 is evil, it has exactly q arms. □

This also concludes the proof of Theorem 5.2. □

5.14. **Definition (Upper and Lower Kneading Sequences)**
If ν is a $\star$-periodic kneading sequence of exact period n, we obtain two periodic kneading sequences $\nu_{\mathtt{0}}$ and $\nu_{\mathtt{1}}$ by consistently replacing every $\star$ with `0` *(respectively with* `1`*); both sequences are periodic with period n or dividing n, and exactly one of them contains the entry n in its internal address. The one which does is called the* upper kneading sequence associated to *ν and denoted $\mathcal{A}(\nu)$, and the other one is called the* lower kneading sequence associated to *ν and denoted $\overline{\mathcal{A}}(\nu)$.*

5.15. **Lemma (Itinerary Immediately Before c_1)**
When $x \to c_1$ in a Hubbard tree for the $\star$-periodic kneading sequence ν, the itinerary of x converges (pointwise) to $\overline{\mathcal{A}}(\nu)$.

Proof. Let τ be the limiting itinerary of x as $x \to c_1$ and let n be the period of ν. Then τ is clearly periodic with period (dividing) n and contains no $\star$, so $\tau \in \{\mathcal{A}(\nu), \overline{\mathcal{A}}(\nu)\}$. Let m be the largest entry in the internal address of ν which is less than n. Then there is a closest precritical point $\zeta_m \in [0, c_1)$ (Lemma 5.7) and $f^{\circ n}$ maps $[\zeta_m, c_1]$ homeomorphically onto its image. Since $f^{\circ m}$ sends $(\zeta_m, c_1) \ni x$ onto $(c_1, c_{1+m}) \ni f^{\circ m}(x)$, which is sent by $f^{\circ(n-m)}$ onto (c_{1+n-m}, c_{1+n}), we get $\tau_1 \ldots \tau_{n-m} =$

$\nu_1 \dots \nu_{n-m} = \tau_{m+1} \dots \tau_n$. Hence $\rho_\tau(m) > n$; since m occurs in the internal address of τ, the number n does not. □

5.16. **Proposition (Exact Period of Kneading Sequence)**
For every $\star$-periodic kneading sequence of period n, the associated upper kneading sequence $\mathcal{A}(\nu)$ has exact period n.

Proof. Let $\tau := \mathcal{A}(\nu)$ be the upper kneading sequence associated to ν and suppose by contradiction that the exact period of τ is $m < n$. Then ν fails the admissibility condition for period m: since $\rho_\tau(m) = \infty$ and n is in the internal address of τ by assumption, m cannot occur on the internal address of τ and hence neither on the internal address of ν. If $\rho_\tau(k) \geq m$ for a proper divisor k of m, then the exact period of τ would be less than m, a contradiction. Hence $\rho_\nu(k) = \rho_\tau(k) < m$. The third part of the admissibility condition is clear because $r = m$.

Thus by Theorem 5.2 the Hubbard tree for ν, say (T, f), has an evil orbit with period m. Let z_1 be its characteristic point; it has itinerary τ. Then $f^{\circ n}$ sends $[z_1, c_1]$ homeomorphically onto itself, so all points on $[z_1, c_1)$ have itinerary τ. By Lemma 5.15, it follows that τ is the lower kneading sequence associated to ν, a contradiction. □

A different proof for this result, based entirely on the combinatorics of kneading sequences, is given in Lemma 20.2.

Remark. The exact period of $\overline{\mathcal{A}}(\nu)$ can a proper divisor of n; see Lemma 6.12.

5.17. **Lemma (Characteristic Points and Upper Sequences)**
Let z be a characteristic point with itinerary τ and exact period n. Then exactly one of the following two cases holds:

(1)
- *all local arms are permuted transitively, i.e. z is tame,*
- *the internal address of τ contains the entry n,*
- *the exact period of τ equals n,*
- *$\tau = \mathcal{A}(\nu)$ for some $\star$-periodic kneading sequence ν of exact period n.*

(2)
- *the local arm towards 0 is fixed, all others are permuted transitively,*
- *the internal address of τ does not contain the entry n,*
- *if the exact period of z and τ coincide then $\tau = \overline{\mathcal{A}}(\nu)$ for some $\star$-periodic kneading sequence ν of exact period n.*

For any $\star$-periodic sequence ν, there is at most one tame periodic point in T such that $\tau(p) = \mathcal{A}(\nu)$.

Proof. By Corollary 4.4 either all local arms at z are permuted transitively or the local arm pointing to 0 is fixed and all others are permuted transitively. By Proposition 4.8, n is contained in the internal address of τ if and only of the local arm towards 0 is not fixed.

Let n' be the exact period of τ. Then $n = kn'$ for some $k \geq 1$ and $\rho_\tau(n) = \infty$. Therefore, if n is contained in the internal address of τ then $k = 1$ by Lemma 20.2.

The last remaining property of the first case follows immediately from $n = n'$, the definition of upper and lower kneading sequences and Proposition 5.16. Similarly, the third property in the second case follows from these results.

For the last statement, let us assume that there are two tame periodic points p, q with itinerary $\mathcal{A}(\nu)$. Then they have both exact period n' and $f^{\circ n'}([p,q]) = [p,q]$. Thus not all local arms of p, q are permuted transitively and neither p nor q is tame, a contradiction. □

An immediate corollary of the preceding lemma is that if the period of z and of τ coincide, then the type of z (tame or not) is completely encoded in τ.

In the second case however, if the exact period of z and τ do not coincide, then τ may equal the upper or the lower kneading sequence of some $\star$-periodic kneading ν of exact period n'.

5.18. **Lemma (Periodic Point behind Closest Precritical Point)**
Let $\zeta \in [0, c_1)$ be a precritical point with $\textsc{Step}(\zeta) = m$ *so that $f^{\circ m} : [\zeta, c_1] \to [c_1, c_{1+m}]$ is homeomorphic. Then the arc (ζ, c_1) contains a characteristic periodic point z with exact period m. The first return map of z fixes no local arm at z.*

Proof. First we show that (c_1, ζ) contains a periodic point of period m. Assume by contradiction that this is not the case. By Theorem 20.12 we can extend the construction of the Hubbard tree in Theorem 3.10 so as to include an m-periodic point p with itinerary $\tau = \overline{\nu_1 \ldots \nu_m}$, where ν is the kneading sequence of the Hubbard tree. (If c_1 is periodic of period $n < m$, then we will use the itinerary $\tilde{\nu}$ of a point x very close to c_1; so $\tau = \overline{\tilde{\nu}_1 \ldots \tilde{\nu}_m}$.) By the choice of τ, ζ does not separate p from c_1. The triod $H_0 = [c_1, p, \zeta]$ maps under $f^{\circ m}$ homeomorphically onto $[c_{1+m}, p, c_1]$, see

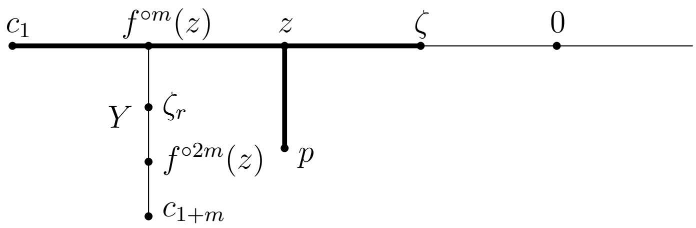

Figure 5.5. Subtree $H_0 = [c_1, p, \zeta]$ (bold lines) with its image under $f^{\circ m}$.

Figure 5.5. If H_0 is degenerate, then it must necessarily have c_1 in the middle. But then, $f^{\circ m}([c_1, p, \zeta]) = [c_{1+m}, p, c_1]$ is degenerate with c_{1+m} in the middle: we have $\zeta \in [0, c_1]$, $c_1 \in [\zeta, p]$ and $c_{1+m} \in [c_1, p]$, hence $c_1 \in [0, c_{1+m}]$ in contradiction to the fact

that c_1 is an endpoint of the Hubbard tree (we cannot have $c_{1+m} = c_1$ because then $\zeta = c_1$).

Hence there is a branch point, say z, in the interior of H_0. Since $z \in (p, \zeta)$, we have $f^{\circ m}(z) \in (p, c_1) \subset [z, p) \cup [z, c_1)$. The possibility $f^{\circ m}(z) = z$ contradicts our initial assumption.

If $f^{\circ m}(z) \in (z, p)$, then $f^{\circ m}$ maps $[z, p]$ homeomorphically into itself, so all points on $[z, p]$ have the same itinerary. This contradicts either expansivity or finiteness of the orbit of the branch point z.

Therefore, $f^{\circ m}(z) \in (c_1, z)$. In this case, $f^{\circ m}([c_1, z])$ branches off from $[c_1, \zeta]$ at $f^{\circ m}(z)$; it belongs to an arm Y at $f^{\circ m}(z)$, and $f^{\circ 2m}(z) \in Y$. By expansivity, $f^{\circ m}$ cannot map Y homeomorphically into itself, so there exists a closest precritical point $\zeta_k \in Y$ with $k < m$. By Lemma 5.7, $m \notin \text{orb}_\rho(k)$. There is a unique $s \in \text{orb}_\rho(k)$ with $s < m < \rho(s)$. Then $f^{\circ m}$ maps the triod $[c_1, \zeta_s, z]$ homeomorphically onto the image triod $[c_{1+m}, c_{1+m-s}, f^{\circ m}(z)]$ with branch point in $(f^{\circ m}(z), c_{1+m}) \subset Y$. Therefore, $c_{1+m-s} \in Y$. Now let ζ' be the earliest precritical point on $[c_1, c_{1+m-s}]$. By Lemma 5.7, $\textsc{Step}(\zeta') = \rho(m - s) - (m - s)$. The first assertion of Lemma 20.2 [1] states that $m \in \text{orb}_\rho(\rho(m - s) - (m - s))$, so $\zeta_m \in [c_1, c_{1+m-s}]$. Therefore, $\zeta_m \neq \zeta$ (the points ζ and ζ_m are in different arms at $f^{\circ m}$), and this is a contradiction.

We have now proved the existence of a periodic point $z \in (c_1, \zeta)$ with itinerary τ and period m. It is characteristic: if not, let $z_1 \in (z, c_1)$ be the characteristic point; then $f^{\circ m}(z_1, \zeta) = (z_1, c_1)$ and $z \in (z_1, \zeta)$, which is a contradiction.

Let $k|m$ be the exact period of z. By Lemma 4.6, $f^{\circ k}$ sends the local arm at z to 0 either to itself or to the local arm to c_1. The first case is excluded by the fact that $f^{\circ m} \colon [z, \zeta] \to [z, c_1]$ is a homeomorphism. In the second case, $f^{\circ k} \colon [z, \zeta] \to [z, f^{\circ k}(\zeta)] \subset [z, c_1]$ is a homeomorphism. If $k < m$, then $f^{\circ k}(\zeta) \in (z, c_1)$ and $f^{\circ m}$ could not be a homeomorphism on $[\zeta, c_1]$. Hence $k = m$ is the exact period of z, and no local arm at z is fixed by $f^{\circ m}$. □

For any $m \geq 1$, let $r \in \{1, 2, \ldots, m\}$ be congruent to $\rho(m)$ modulo m, and define

$$q(m) := \begin{cases} \frac{\rho(m)-r}{m} + 1 & \text{if } m \in \text{orb}_\rho(r) , \\ \frac{\rho(m)-r}{m} + 2 & \text{if } m \notin \text{orb}_\rho(r) . \end{cases} \tag{2}$$

5.19. **Proposition (Number of Arms at Tame Branch Points)**
If z_1 is a tame branch point of exact period m, then m occurs in the internal address, and the number of arms is $q(m)$. Conversely, for any entry m in the internal address with $q(m) \geq 3$, there is a tame branch point of exact period m with $q(m)$ arms (unless the critical orbit has period m).

---

[1]This lemma is part of an analysis of the structure of ρ-orbits for arbitrary `0`-`1`-sequences; it is independent of earlier results, so this forward reference does not introduce a circular reference.

PROOF. Let z_1 be the characteristic point of an orbit of tame branch points with exact period m, and let $q' \geq 3$ be the number of arms at z_1. By Lemma 4.6, the critical value cannot be periodic with period less than m. By Lemma 5.9 (3) and (1), m occurs in the internal address, $\zeta_m \in (0, z_1) \subset G_0$, and $\zeta_{\rho(m)} \in [z_1, c_1]$. Let $r' := \rho(m) - (q'-2)m$. By Lemma 5.11, $0 < r' \leq 2m$. Therefore, by Lemma 5.10, $\rho((q'-2)m) = \rho(m)$, so by Lemma 5.8, $f^{\circ(q'-2)m}(\zeta_{\rho(m)}) = \zeta_{r'}$ is a closest precritical point and by Lemma 4.6, $\zeta_{r'} = f^{\circ(q'-2)m}(\zeta_{\rho(m)}) \in G_{q'-1}$. Since $\zeta_m \in G_0$, we have $\zeta_m \notin [c_1, \zeta_{r'}]$ and thus $m \notin \operatorname{orb}_\rho(r')$ (Lemma 5.7).

If $r' \leq m$, then $r = r'$ and we are in the case $q(m) = \frac{\rho(m)-r}{m} + 2 = \frac{\rho(m)-r'}{m} + 2 = q'$.

If $r' > m$, then $r = r' - m$ and $f^{\circ m}(\zeta_{r'}) = \zeta_r$ by Lemma 5.8. Then $f^{\circ m}$ maps $[z_1, \zeta_{r'}]$ homeomorphically onto $[z_1, \zeta_r]$, hence $\zeta_r \in G_0$. By Lemma 5.9 (2) we find that either $r = m$ or $r < m$ and $\zeta_m \in [z_1, \zeta_r]$, so in both cases $m \in \operatorname{orb}_\rho(r)$. Therefore, $q(m) = \frac{\rho(m)-r}{m} + 1 = q'$. Again $q(m) = q'$.

For the converse, let m be an entry in the internal address. By Lemma 5.7, the closest precritical point ζ_m exists on $[0, c_1]$. By Lemma 5.18, ζ_m gives rise to a characteristic point $z_1 \in [\zeta_m, c_1]$ of exact period m and the first return map of z_1 fixes no local arm. By Lemma 5.9 (1), $\zeta_{\rho(m)} \in [z_1, c_1]$.

If z_1 is a branch point, then no local arm of z_1 is fixed by $f^{\circ m}$, so z_1 is tame. By the first assertion of the lemma, the number of arms is $q(m)$.

Finally, suppose that z_1 has only two arms $G_0 \ni 0$ and $G_1 \ni c_1$. If the critical orbit is periodic and m is an entry in the internal address, the period of the critical orbit is at least m, and equality is excluded by hypothesis. By Lemma 5.11, we have $\rho(m) \leq 2m$ and $r' := \rho(m) - m = r$. Then $f^{\circ m}(\zeta_{\rho(m)}) = \zeta_{r'} \in G_0$. Lemma 5.9 (2) then gives that $\zeta_m \in [\zeta_r, z_1]$ and hence $m \in \operatorname{orb}_\rho(r)$. It follows that $q(m) = \frac{\rho(m)-r}{m} + 1 = 2$. $\square$

Together, Propositions 5.13 and 5.19 describe all branch points in all Hubbard trees.

In [**LS**], a special sufficient condition for admissibility of kneading sequences was derived and used to determine the Galois groups of periodic points: see Section 11. We show that this condition follows easily from our general Admissibility Condition 5.1.

5.20. **Corollary (Sufficient Condition for Admissibility)**
If ν is a kneading sequence such that $\nu_{\rho(n)} = \mathtt{0}$ for all $n \geq 1$, then ν is admissible.

PROOF. Let $S_1 \to \ldots \to S_k \to \ldots$ be the internal address associated to ν. Comparing $\nu_{S_k+1}\nu_{S_k+2}\ldots$ with $\nu_1\nu_2\ldots$, the first difference occurs when comparing $\nu_{S_{k+1}} = \nu_{\rho(S_k)}$ with $\nu_{S_{k+1}-S_k}$, so we have $\nu_{S_{k+1}} = \mathtt{0}$ and thus $\nu_{S_{k+1}-S_k} = \mathtt{1}$ for all k.

Now we claim that

$$S_k < r < S_{k+1} \text{ implies } \rho(r) \leq S_{k+1} \ . \tag{3}$$

To prove this, assume by contradiction that $\rho(r) > S_{k+1}$. Together with $\rho(S_k) = S_{k+1}$ this yields

$$\nu_1 \dots \nu_{S_{k+1}-r} = \nu_{r+1} \dots \nu_{S_{k+1}} = \nu_{r-S_k+1} \dots \nu'_{S_{k+1}-S_k},$$

where $\nu' = \mathtt{1}$ if $\nu = \mathtt{0}$ and vice versa. This implies $\rho(r - S_k) = S_{k+1} - S_k$ and $\nu_{\rho(r-S_k)} = \nu_{S_{k+1}-S_k} = \mathtt{1}$, contrary to the hypothesis of the lemma, so (3) holds.

Now assume that ν fails the admissibility condition for period m; then m is different from all S_k. Write $\rho(m) = am + r$ with $r \in \{1, 2, \dots, m\}$ and $a \geq 1$; then $m \in \mathrm{orb}_\rho(r)$. Let k be maximal so that $S_k < r$. Since m is on the ρ-orbit of r but not on the internal address, (3) implies $m < S_{k+1}$ and $\rho(m) \leq S_{k+1}$.

Choose q so that $(q-1)S_k < m \leq qS_k$. Part (2) of the Admissibility Condition 5.1 implies that S_k cannot divide m, hence $m < qS_k$. We have $S_k < r \leq am + r - m = \rho(m) - m$, hence $qS_k = (q-1)S_k + S_k < m + \rho(m) - m = \rho(m) \leq S_{k+1}$. Therefore ν starts with at least q repetitions of $\nu_1 \dots \nu_{S_k}$. Since $0 < m - (q-1)S_k < S_k$, we have

$$\nu_{m-(q-1)S_k+1} \dots \nu_{S_k} = \nu_{m+1} \dots \nu_{qS_k} \ .$$

Now $\rho(m) > qS_k$ implies $\rho(m - (q-1)S_k) > S_k$. This contradicts (3) and completes the proof. $\square$

Remark. The condition in Corollary 5.20 is equivalent to the condition that no shift $\sigma^k(\nu)$ exceeds ν with respect to lexicographic ordering; see Lemma 12.5 or [**LS**, Corollary 10.8].

III. Parameter Space

6. Orbit Forcing and Internal Addresses

In this section, we compare the Hubbard trees for different kneading sequences. We define a partial order relation on the set of $\star$-periodic kneading sequences so that trees for greater sequences contain all periodic orbits of trees for smaller sequences. Periodic kneading sequences of fixed maximal periods form a finite connected tree which is related to the Mandelbrot set. We identify where in this tree the non-admissible sequences sit.

Recall that a kneading sequence ν can contain a $\star$ only if it is periodic of some period n and contains a $\star$ exactly at positions n, $2n$, $3n$, $\dots$; such a kneading sequence ν is called $\star$-periodic. In Definition 5.14 we associated upper and lower kneading sequences $\mathcal{A}(\nu)$ and $\overline{\mathcal{A}}(\nu)$ to it by consistently replacing every $\star$ by $\mathtt{0}$ or by $\mathtt{1}$ so that $\mathcal{A}(\nu)$ contains n in its internal address. We will also need an inverse $\mathcal{A}_n^{-1}(\tau)$ which turns an n-periodic kneading sequence τ without $\star$ (upper or lower) into a $\star$-periodic kneading sequence of period n (note that $\mathcal{A}_n^{-1}$, $\mathcal{A}_{2n}^{-1}$, etc. can be applied to τ, but the results are different).

6.1. **Definition (Order on $\star$-Periodic Kneading Sequences)**
Let ν and ν' be two $\star$-periodic kneading sequences. We say that $\nu > \nu'$ if the Hubbard tree for ν contains a characteristic periodic point with itinerary $\mathcal{A}(\nu')$.

Remark. Trivially, $\nu > \overline{\star}$ for every $\star$-periodic $\nu \neq \overline{\star}$ because the tree for ν has the α-fixed point with itinerary $\overline{\mathtt{1}} = \mathcal{A}(\overline{\star})$ (Lemma 3.7).

In the tree for ν, the characteristic periodic point with itinerary $\mathcal{A}(\nu')$ is not on the critical orbit. If it is a branch point, then it is tame by Proposition 4.8. If it is not a branch point (and hence not marked), then its period might be a proper multiple of the period of the itinerary. However, by Lemma 3.9, there is always a characteristic periodic point with itinerary $\mathcal{A}(\nu')$ which has the same period as $\mathcal{A}(\nu')$.

Remark. As we will see below, *this order is not linear*, i.e. there are sequences ν and ν' for which neither $\nu > \nu'$ nor $\nu' > \nu$ (for instance, the sequences $\overline{\mathtt{11}\dots\mathtt{1}\star}$ and $\overline{\mathtt{111}\dots\mathtt{1}\star}$ are not comparable as soon as their periods are different). It is easy to see that $\nu \not> \nu$ for every ν (i.e., the order is irreflexive): if the tree for ν has a characteristic periodic point p with itinerary $\mathcal{A}(\nu)$ and ν has period n, then by Lemma 5.15, the arc (p, c_1) must contain a precritical point ζ with $\mathrm{Step}(\zeta) = n$ so that $f^{\circ n}$ sends $[\zeta, c_1]$ homeomorphically onto itself, but both endpoints map to c_1; this is a contradiction. In Corollary 6.3, we will show that *the order is transitive* (justifying the term *order*). By irreflexivity, it follows that we never have $\nu > \nu'$ and $\nu' > \nu$ (i.e., the order is asymmetric).

This order is naturally related to the tree structure of the Mandelbrot set: every hyperbolic component W of the Mandelbrot set of some period n has a unique center parameter in which the critical orbit is periodic and has an associated $\star$-periodic kneading sequence, also of period n; we associate this kneading sequence to W. Hyperbolic components are partially ordered: we say that W is greater than W' (or: W is behind W') if $W' \neq W$ and the arc in the Mandelbrot set from the origin to W' contains W (or equivalently if the wake of W' is contained in the wake of W); see Figure 6.1 and Section 11.

Our goal in this section is to extend and simultaneously simplify the structure of the Mandelbrot set so as to obtain a natural space that contains all kneading sequences and Hubbard trees, admissible or not: this is a natural generalization of the combinatorial data of the Mandelbrot set in a simpler setting. From a combinatorial point of view, the admissibility condition adds difficulties; moreover, every kneading sequence occurs only once in our space, while symmetries of the Mandelbrot set entail that many sequences occur more than once (see the discussion of angled internal addresses in Section 11). Our space is related to a construction of Penrose [**Pen1**]; see the discussion after Definition 7.4.

FIGURE 6.1. Figure Needed!

If W is greater than W', then the associated kneading sequences satisfy $\nu > \nu'$ (Proposition 11.21); this proposition has a converse as follows: if $\nu'' > \nu$ for an admissible $\star$-periodic kneading sequence ν'', then there exists a hyperbolic component W'' with kneading sequence ν'' so that W'' is behind W. There may be several hyperbolic components with the same kneading sequence, and not all can be greater than W (compare Figure 6.1); they are related by certain symmetries of the Mandelbrot set which are distinguished by angled internal addresses: see Definition 11.9 and [**LS**, Section 9].

6.2. **Theorem (Orbit Forcing)**
Let ν and $\tilde\nu$ be two $\star$-periodic kneading sequences realized by Hubbard trees (T, f) and $(\tilde T, \tilde f)$. Suppose that one of the following is true:

(1) *there are characteristic periodic points $p \in T$ and $\tilde p \in \tilde T$ with identical itineraries;*
(2) *there is a characteristic periodic point $\tilde p \in \tilde T$ with itinerary $\mathcal{A}(\nu)$ or $\overline{\mathcal{A}}(\nu)$; set $p := c_1$;*
(3) *there is a characteristic periodic point $p \in T$ with itinerary $\mathcal{A}(\tilde\nu)$ or $\overline{\mathcal{A}}(\tilde\nu)$; set $\tilde p := \tilde c_1$;*

then for every characteristic periodic point $p' \in [0, p)$ of T such that (p', p) contains a precritical point,[1] *there is a characteristic periodic point $\tilde p' \in [0, \tilde p)$ in $\tilde T$ such that p' and $\tilde p'$ have the same itinerary, the same number of arms and the same type (tame or evil), and so that the exact period of $\tilde p'$ equals that of its itinerary.*

Of course, in case (2) the condition that there is a point $\tilde p$ with itinerary $\mathcal{A}(\nu)$ is equivalent to $\tilde\nu > \nu$ ("forcing to a larger tree"), and similarly, in case (3) the condition

[1] and, in case (3), such that $\tau' \neq \overline{\mathcal{A}}(\tilde\nu)$

that there is a point p with itinerary $\mathcal{A}(\tilde\nu)$ is equivalent to $\tilde\nu < \nu$ ("forcing to a smaller tree").

Proof. Since the proofs of the three statements are essentially the same in slightly different contexts, we treat the first case in detail and indicate the necessary changes for the two other cases in [2].

Let n' be the exact period of p' and let τ and τ' be the itineraries of p and p'. Let p_0 and $-p_0$ be the periodic and preperiodic inverse images of p; they exist by Lemma 4.12. Let T_p be the subtree spanned by the orbit of p minus the interval $I := (-p_0, p_0)$. Let $\pm p_0$ be the element of $\{p_0, -p_0\}$ in the component of $T \setminus \{0\}$ containing c_1.[3]

Then the map $f\colon T \to T$ restricts to a map $f\colon T_p \to (T_p \cup I)$. Since p' is characteristic and $p' \in [0, p]$, no point on the forward orbit of p' is in I, so the entire forward orbit of p' is contained within T_p. The arc $I_0 := [\pm p_0, p]$ contains p' and maps homeomorphically onto $I_1 := [p, f(p)]$ containing $f(p')$. If $[p, f(p)]$ intersects I, then it contains I; in this case, let instead I_1 be the connected component of $f(p')$ in $[p, f(p)] \setminus \mathring{I}$. [4]

This step can be repeated: map the interval I_k forward homeomorphically, remove I if necessary [5] and let I_{k+1} be the part containing $f^{\circ(k+1)}(p')$. Repeat this step infinitely often (after n' times the point p' is mapped back to itself). Each forward image $f^{\circ k}(p')$ is thus contained in an arc I_k with endpoints in the finite set $\{-p_0, p, f(p), f(f(p)), \dots, p_0\}$, and $f(I_k)$ maps homeomorphically onto an arc containing I_{k+1}. Each arc I_k is contained in [6] a connected component of $T \setminus I$, so for every k the first k entries in the itinerary of any point in the interior of I_0 whose orbit visits the interiors I_1, I_2, …, I_k are the same as in τ', and the set of such points is non-empty.

Now consider the Hubbard tree $(\tilde T, \tilde f)$ that contains a characteristic periodic point $\tilde p$ with itinerary τ.[7] Construct points $\tilde p_0$ and $-\tilde p_0$ and the interval $\tilde I = (-\tilde p_0, \tilde p_0)$ as before and let $\tilde I_k \subset \tilde T$ be the closed arcs bounded by the corresponding points in $\{-\tilde p_0, \tilde p, \tilde f(\tilde p), \tilde f(\tilde f(\tilde p)), \dots, \tilde p_0\}$ as above. None of their endpoints is in the interior of $\tilde I$.[8]

We still have that $\tilde f$ maps $\tilde I_k$ homeomorphically onto its image, which contains $\tilde I_{k+1}$: the itineraries of the endpoints of $\tilde I_k$ encode whether or not $\tilde f(\tilde I_k)$ contains 0 or equivalently $\tilde I$; if it does not, then $\tilde f(\tilde I_k) = \tilde I_{k+1}$, and if it does, $\tilde I_{k+1}$ has one endpoint in common with $\tilde f(I_k)$ and the other one with $\tilde I$, so $\tilde I_{k+1} \subset \tilde f(\tilde I_k)$. Each arc $\tilde I_k$ is contained

[2] footnotes

[3] In case (2), let τ be the itinerary of $\tilde p$, let $p_0 = -p_0 = \pm p_0$ be the critical point, $I := \{0\}$ and $T_p := T \setminus \{0\}$.

[4] In case (2), if $0 \in [p, f(p)]$, then cut at 0 and retain only $[p, 0]$ or $[0, f(p)]$ so as to keep $f(p')$.

[5] or cut at 0 in case (2)

[6] the closure of

[7] In case (3), the itinerary of $\tilde p = \tilde c_1$ is $\tilde \nu$.

[8] In case (2), for any interval I_k which ends at $p_0 = -p_0$ in the tree for ν, we have an ambiguity in the tree for $\tilde\nu$: in the latter, we let the corresponding interval $\tilde I_k$ end at $\tilde p_0$ or $-\tilde p_0$ in such a way that it does not contain the critical point. In case (3), let $\tilde p_0 = -\tilde p_0 := 0$, $\tilde I := \{0\}$ and $\tilde I_k$ as in case (1).

in [9] a connected component of $\tilde{T} \setminus \tilde{I}$, and the itinerary of the sequence of arcs $(\tilde{I}_k)$ is the same as before, that is τ'. Let $J := \{x \in \tilde{I}_0 \colon \tilde{f}^{\circ k}(x) \in \tilde{I}_k \text{ for all } k \geq 0\}$. This is a nested intersection of non-empty compact connected intervals, so J is non-empty, connected and compact, and every point in the interior of J has itinerary τ'.

If J has non-empty interior, then by Lemma 3.9 there is a $\tilde{p}' \in J$ which is fixed by $\tilde{f}^{\circ n'}$ and whose exact period equals that of τ'. If J is a single point whose itinerary contains no $\star$, then the itinerary equals τ' and the same argument works once more.

The last case is that $J = \{\tilde{p}'\}$ is a singleton, and its itinerary contains a $\star$. This can occur only in case (3): in the other cases, all points in J have itinerary τ'.[10]

To justify the statement about the number of arms at $\tilde{p}'$ and its type, we use the triod argument from the uniqueness proof of Hubbard trees (Proposition 3.15). Since $p' \in [0, p]$, its arms towards critical point and critical value are in the subtree T_p; by the mapping properties of arms as in Corollary 4.4, all its global arms contain points on the orbit of p, so the number of arms and their type can be determined from the itineraries of p and p' alone using the triod argument, and the same holds for $\tilde{p}$ and $\tilde{p}'$.

The same argument shows that $\tilde{p}'$ is characteristic: this fact can be determined from the triod argument applied to $\tilde{p}$ and the orbit of $\tilde{p}'$.[11] □

6.3. **Corollary (Order on Kneading Sequences Transitive)**
The order on $\star$-periodic kneading sequences is transitive: if $\nu > \nu'$ and $\nu' > \nu''$, then $\nu > \nu''$.

Proof. By hypothesis $\nu' > \nu''$, the tree for ν' contains a characteristic periodic point with itinerary $\mathcal{A}(\nu'')$. Since $\nu > \nu'$, such a characteristic periodic point exists also in the tree for ν by Theorem 6.2 (2), and $\nu > \nu''$. □

6.4. **Corollary (Order and Admissibility)**
If $\tilde{\nu} > \nu$ are two $\star$-periodic kneading sequences and $\tilde{\nu}$ is admissible, then ν is also admissible.

Proof. By Theorem 6.2 (2), an evil branch point in the tree for ν would imply the existence of an evil branch point in the tree for $\tilde{\nu}$. □

Remark. The topological entropy of Hubbard trees is monotone with respect to the order of kneading sequences: all periodic orbits of the smaller tree (identified by their

[9] the closure of

[10] If we are in case (3) and the itinerary contains a $\star$, then $\tilde{p}'$ is a precritical point and points arbitrarily close to $\tilde{p}'$ have itineraries coinciding with τ' for arbitrarily long time; therefore, $\tilde{p}'$ must actually be periodic and hence on the critical orbit.

[11] In case (3), if $\tilde{p}'$ is on the critical orbit, then this argument shows that $\tilde{p}'$ is characteristic, hence equal to the critical value, and $\tau' \in \{\mathcal{A}(\tilde{\nu}), \overline{\mathcal{A}}(\tilde{\nu})\}$. Now Lemma 5.15 implies that $\tau' = \overline{\mathcal{A}}(\tilde{\nu})$, and this case was excluded in the hypothesis.

itineraries) reappear in the larger tree; in fact, the proof of Theorem 6.2 (2) extends to show that every itinerary (periodic or not) which is realized by an orbit of (T, f) is also realized by an orbit of $(\tilde{T}, \tilde{f})$.

6.5. **Lemma (Linearly Ordered Subset)**
For any $\star$-periodic kneading sequence ν, the set of $\star$-periodic kneading sequences less than ν is linearly ordered: if $\nu' < \nu$ and $\nu'' < \nu$ with $\nu'' \neq \nu'$, then either $\nu' < \nu''$ or $\nu'' < \nu'$.

Proof. Let $T(\nu)$ be the Hubbard tree for kneading sequence ν and let p be its critical value. Since $\nu' < \nu$ and $\nu'' < \nu$, the tree $T(\nu)$ contains characteristic periodic points with itineraries $\mathcal{A}(\nu')$ and $\mathcal{A}(\nu'')$; denote these periodic points by p' and p''. Then $\{p', p''\} \subset [0, p]$ and we may suppose that $p'' \in [0, p']$ (possibly by switching p' and p''). By Theorem 6.2 (3), we get $\nu' > \nu''$. □

The Parameter Tree for Finite Periods. For a positive integer n, take the $\star$-periodic kneading sequences of periods up to n (including $\overline{\star}$) as vertices. Connect two vertices ν and ν' by an oriented edge $\nu' \to \nu$ if and only if $\nu' < \nu$ and there is no kneading sequence ν'' of period up to n with $\nu' < \nu'' < \nu$. Then we get a finite connected oriented tree called the *parameter tree of period* n, denoted $\mathcal{T}_n$. To see this, note that connectivity follows from $\nu > \overline{\star}$ for all $\star$-periodic $\nu \neq \overline{\star}$, and the tree structure follows from Lemma 6.5. These trees for different n are related: increasing n requires extending the tree and refining its edges. In where? , we will construct a single parameter tree $\mathcal{T}$ which contains all $\star$-periodic and all non-periodic sequences.

6.6. **Lemma (Abstract Lavaurs' Lemma)**
If $\nu > \nu'$ are two $\star$-periodic kneading sequences of the same period, then there is another $\star$-periodic kneading sequence ν'' of lower period with $\nu > \nu'' > \nu'$.

Proof. Let n be the period of ν. By definition, the tree $T(\nu)$ contains a characteristic periodic point p' with itinerary $\mathcal{A}(\nu')$. Since $\nu \neq \nu'$, these two sequences must differ first at some position $m < n$, so there is a precritical point $\zeta \in (p', c_1)$ with $\text{Step}(\zeta) = m$. Now Lemma 5.15 implies that there is a characteristic periodic point $p'' \in (\zeta, c_1)$ with exact period m and so that its first return map fixes no local arm. By Proposition 4.8 (and Definition 5.14), there is a $\star$-periodic kneading sequence ν'' of period m so that the itinerary of p'' is $\mathcal{A}(\nu'')$, hence $\nu'' < \nu$. Since $p'' \in (p', c_1)$, Theorem 6.2 (2) implies that we also have $\nu' < \nu''$. □

Recall from Definition 4.9 that we call a $\star$-periodic kneading sequence *admissible* if its Hubbard tree contains no evil orbit. An immediate consequence of Theorem 6.2 (2) is that if a kneading sequence ν is non-admissible and $\nu' > \nu$, then ν' is also non-admissible, so the set of admissible kneading sequences forms a connected subtree of every parameter tree $\mathcal{T}_n$; see also Lemma 7.3.

The following result answers the “vanishing point conjecture” of Kauko [**Kau2**, § 5.29] in the affirmative.

6.7. **Proposition (Branch Points of Non-Admissible Subtrees)**
For every $\star$-periodic non-admissible kneading sequence ν, there is a $\star$-periodic admissible kneading sequence ν_* such that a $\star$-periodic kneading sequence $\nu' < \nu$ is admissible if and only if $\nu' < \nu_*$. Every kneading sequence $\nu' < \nu_*$ has $\nu' < \nu$, but $\nu_* \not< \nu$ (compare Figure 6.2). The exact periods of ν_* and $\overline{\mathcal{A}}(\nu_*)$ coincide.

The Hubbard tree for ν has an evil orbit with the same period as ν_*, and the itinerary of its characteristic point is $\tau_* = \overline{\mathcal{A}}(\nu_*)$.

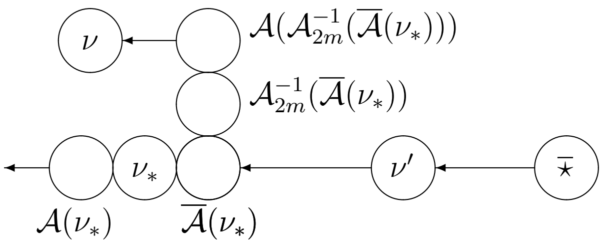

Figure 6.2. A picture with a $\star$-periodic kneading sequence ν_* where a non-admissible sequence ν branches off, with the upper and lower sequences of ν_* indicated (ν_* fails the admissibility condition for period m). Disks represent periodic sequences, arrows indicate increasing order.

Proof. The number of evil periodic orbits in the Hubbard tree of ν is finite. Their characteristic points are on $[0, c_1]$, so there is a unique one closest to 0; call it p_*, let τ_* be its itinerary and n_* be its exact period. Since evil periodic points are marked, the exact period of p_* equals the exact period of its itinerary τ_* by Lemma 3.8. Let $\nu_* := \mathcal{A}_{n_*}^{-1}(\tau_*)$. Then ν_* is $\star$-periodic with exact period n_*, and it is admissible: if there was an evil orbit in the tree for ν_*, then by Theorem 6.2 (2), an evil characteristic periodic point with the same itinerary would show up in the tree for ν on $(0, p_*)$, in contradiction to minimality of p_*. Hence every $\star$-periodic $\nu' < \nu_*$ is admissible.

Conversely, fix some admissible $\star$-periodic $\nu' < \nu$. Then in the Hubbard tree for ν there is a characteristic periodic point p' with itinerary $\mathcal{A}(\nu')$. If $p_* \in [0, p']$, then the ν'-tree has an evil orbit by Theorem 6.2 (3) in contradiction to admissibility of ν'. Since $p', p_* \in [0, c_1]$ and $p' \neq p_*$, we get $p' \in [0, p_*]$ and $\nu' < \nu_*$, again by Theorem 6.2 (3).

Pick any $\star$-periodic kneading sequence $\nu' < \nu_*$. By definition, there is a characteristic periodic point with itinerary $\mathcal{A}(\nu')$ in the Hubbard tree for ν_*, and by Theorem 6.2 (2) also in the tree of ν, so $\nu' < \nu$.

Before showing $\nu_* \not< \nu$, we observe that the last statement is clear now: the tree for ν has the evil characteristic periodic point p_* with the same exact period n_* as ν_*, and Proposition 4.8 (together with Definition 5.14) implies $\tau_* = \overline{\mathcal{A}}(\nu_*)$. In particular, the exact period of τ_* equals n_*.

Now suppose by contradiction that the tree $T(\nu)$ for ν has a characteristic periodic point p with itinerary $\mathcal{A}(\nu_*)$. From Lemma 4.1, we know $\{p, p_*\} \subset [0, c_1]$, so $p \in [p_*, c_1]$ or $p \in [0, p_*]$. In the first case, Lemma 4.6 says that $[p_*, c_1]$ maps after n_* iterations homeomorphically onto an arc $[p_*, c_{n_*+1}]$ in a different global arm at p_* than $[p_*, c_1]$, so $p \neq f^{\circ n_*}(p)$; but since p has itinerary $\mathcal{A}(\nu_*)$ with period n_*, the points p and $f^{\circ n_*}(p)$ have identical itineraries, while $p_* \in [p, f^{\circ n_*}(p)]$ has a different itinerary, in contradiction to Lemma 3.6.

The second case is $p \in [0, p_*]$. From the itineraries of p_* and p, it follows that the arc $[p_*, p]$ maps in $n_* - 1$ iterations homeomorphically onto an interval containing 0. It follows that $f^{\circ n_*}$ sends the local arm at p_* towards p and 0 to the local arm at p_* towards c_1, which is in contradiction to the fact that p_* is evil (Lemma 4.6). Hence there can be no characteristic periodic point p with itinerary $\mathcal{A}(\nu_*)$ at all in $T(\nu)$, and indeed $\nu_* \not< \nu$. □

Remark. Every admissible $\star$-periodic kneading sequence ν_* of some period m for which $\overline{\mathcal{A}}(\nu_*)$ has exact period m is the root point of a non-admissible subtree, i.e. ν_* is associated to some non-admissible ν as in Proposition 6.7: for every $s \geq 2$, the $\star$-periodic kneading sequence $\nu := \mathcal{A}_{sm}^{-1}(\overline{\mathcal{A}}(\nu_*))$ fails the admissibility condition for period m, so it has an evil orbit of period m with itinerary $\overline{\mathcal{A}}(\nu_*)$. This is the underlying geometry in parameter space of Example 5.3.

A hyperbolic component W of period n in the Mandelbrot set has an associated $\star$-periodic kneading sequence ν_* of period n. The lower sequence $\overline{\mathcal{A}}(\nu_*)$ has exact period n if and only if W is a primitive component; otherwise W bifurcates from some component W' of exact period n' properly dividing n, and the exact period of $\overline{\mathcal{A}}(\nu_*)$ is n' (see **??**). In the words of the conjecture of Kauko [**Kau2**], this means that "vanishing points" of non-admissible kneading sequences are exactly the roots of primitive hyperbolic components; here the vanishing point ν_* of a non-admissible sequence ν is the greatest sequence $\nu_* \in [0, \nu]$ so that all sequences in $[0, \nu_*)$ are admissible.

Internal Addresses. There are various ways to associate an internal address to the Hubbard tree for a $\star$-periodic kneading sequence ν of some period n. All of them yield a finite strictly monotone sequence of positive integers starting with `1`; we will show below that they all coincide, which gives us useful information about Hubbard trees

Insert Figure 3.5 of Alex' thesis here.

FIGURE 6.3. Illustration of the locus of non-admissible sequences: the base plane is the complex plane containing the Mandelbrot set and all admissible kneading sequences; the non-admissible sequences (indicated as grey "Mandelbrot sets") are attached to the Mandelbrot set at roots of primitive hyperbolic components.

and parameter space. The last two definitions refer to the Hubbard tree associated to ν with critical point 0 and critical value c_1.

In Definition 4.5, we called a branch point *tame* if the first return map permutes all local arms transitively; in the last definition below, we will generalize this and call any periodic point with at least two arms tame if all its local arms are permuted transitively. By Lemma 4.1, every such orbit has a unique characteristic point, and by Proposition 4.8, the characteristic periodic point is tame in this sense if and only if the itinerary τ of p is an upper sequence (compare Definition 5.14).

(1) The *internal address associated to the kneading sequence* ν has already been defined in Definition 2.2, using the ρ_ν-map.
(2) The *internal address in parameter space* of ν starts with $S_0 = 1$ corresponding to the kneading sequence $\mu^0 = \overline{\star}$. Given $S_0, \ldots, S_{k-1}$, let S_k be the lowest period of any $\star$-periodic kneading sequence μ^k with $\mu^{k-1} < \mu^k \leq \nu$; this procedure stops when S_k is equal to the period of ν. (By Lemma 6.6, the sequence μ^k is always well-defined.) In Proposition 6.8 below we will give another characterization of the sequences μ^k.
(3) The *internal address of closest precritical points* is the sequence of indices k of closest precritical points $\zeta_k \in [c_1, 0]$. (This generalizes *cutting times* as used in the theory of unimodal interval maps; see Section 2 Mention this in Sec 2!)
(4) The *internal address of closest tame characteristic periodic points* is the sequence of exact periods S_k of tame characteristic periodic points p_k such that there is no tame periodic point of lower or equal period on $[p_k, c_1]$ (including tame periodic points with 2 arms). In this definition, the critical value counts as the final closest tame characteristic periodic point. By Lemma **??**, all these tame characteristic periodic points have different itineraries.

Compare also Section 11 for internal addresses in the Mandelbrot set.

6.8. **Proposition (Interpretations of Internal Address)**
Let ν be a $\star$-periodic kneading sequence of period n with its associated Hubbard tree.

(1) The given definitions 1.–4. of internal addresses coincide.

(2) If ν' is a $\star$-periodic kneading sequence whose internal address is an extension of ν, then $\nu' > \nu$.

(2') Let the internal address of ν be $1 \to S_1 \to \ldots \to S_k$ with $S_k = n$, and let ν^m be the $\star$-periodic kneading sequences corresponding to $1 \to \ldots \to S_m$ (for $0 \le m \le k$, so that $\nu^0 = \overline{\star}$ and $\nu^k = \nu$). Then $\nu^0 < \nu^1 < \nu^2 < \ldots < \nu^k$, and these equal the kneading sequences μ^k defining the internal address in parameter space.

(3) For $m \ge 1$, let ζ_{S_m} be the closest precritical points which take S_m steps to map to c_1, and let p_m be the closest tame characteristic periodic points of period S_m (with $S_0 = 1$). Then these points are on the arc $[0, c_1]$ in the order $\zeta_{S_0} = 0$, $p_0 = \alpha, \zeta_{S_1}$, p_1, ζ_{S_2}, p_2, $\ldots$, $\zeta_{S_{k-1}}$, p_{k-1}, $\zeta_{S_k} = p_k = c_1$ as in Figure 6.4.

(4) For $m < k$, the itinerary of p_m differs from ν after position S_m. Conversely, every tame characteristic periodic point p for which the itinerary differs from ν after one period of p is one of the p_m, and the period of p is less than n.

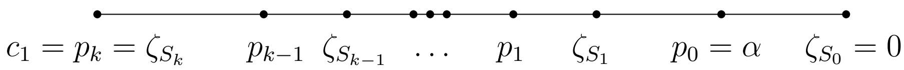

FIGURE 6.4. The order of the points in Proposition 6.8 (3).

PROOF. The equivalence between the internal address in *parameter space* and that of *closest tame characteristic periodic points* is built into the definition of $<$: there is a bijection between $\star$-periodic kneading sequences $\nu' < \nu$ and tame characteristic periodic points in the Hubbard tree of ν, and this bijection preserves periods and the order (of kneading sequences respectively of points on $[0, c_1]$).

The equivalence of the internal addresses associated to the *kneading sequence* and to *closest precritical points* has been shown in Lemma 5.7.

Now we show the equivalence of internal addresses of *closest precritical points* and of *closest tame characteristic periodic points*. Let p be a closest tame characteristic periodic point of some period $S < n$. If the itineraries of p and c_1 differed at position S or before, there would be a closest precritical point $\zeta' \in [p, c_1]$ with $\textsc{Step}(\zeta') \le S$ and hence, by Lemma 5.18, a periodic point on $(p, c_1]$ with period at most S, so p was not closest.

Let $[p, \zeta] \subset [p, 0]$ be the largest arc which can be iterated homeomorphically S times. Then clearly ζ is a precritical point, and we claim that $\textsc{Step}(\zeta) = S$. Suppose not, then $S' := \textsc{Step}(\zeta) < S$ (maximality of ζ excludes $S' > S$), and $f^{\circ S'}([p, \zeta]) = [f^{\circ S'}(p), c_1] \ni p$, hence $f^{\circ S}([p, \zeta]) \ni f^{\circ(S-S')}(p)$. But $f^{\circ S}$ restricted to $[p, \zeta]$ is a homeomorphism, and by assumption $f^{\circ S}$ sends the local arm at p towards 0 to the local arm at p towards c_1, which contradicts the fact that p is characteristic. It follows in particular that $\zeta \ne 0$ if $S > 1$.

Since the first S entries in the itineraries of c_1 and p coincide, there can be no earlier precritical point than ζ on $[c_1, \zeta]$, and ζ is the closest precritical point ζ_S.

Conversely, for each closest precritical point $\zeta_{S_m} \in [0, c_1]$, there is a tame characteristic periodic point $p_m \in [\zeta_{S_m}, c_1]$ of exact period S_m by Lemma 5.18. If there was a tame periodic point $p' \in [p_m, c_1]$ with period $n' < S_m$, then by the previous paragraph there would be a precritical point $\zeta' \in [p', 0]$ with $\text{Step}(\zeta') = n'$. If $\zeta' \in [p', p_m] \subset [c_1, \zeta_{S_m}]$, then ζ_{S_m} is not closest precritical, while if $\zeta' \in [p_m, 0]$, then $p_m \in (p', \zeta')$, so the map $f^{\circ n'}$ sending $[p', \zeta']$ homeomorphically onto $[p', c_1]$ would map p_m to an image in $[p', c_1]$ and p_m could not be characteristic. Therefore p_m is the tame periodic point of lowest period on $[p_m, c_1]$. This finishes the proof of (1).

We already know the first half of *statement (4)*. For the converse, pick a characteristic periodic point p as in the claim. Its period is less than n because ν has a $\star$ at position n. If p is not the tame characteristic periodic point of minimal period on $[p, c_1]$, then let q be this point and let $\zeta \in [0, q]$ be the corresponding precritical point constructed above. Then $p \in [\zeta, q]$ (the alternative $\zeta \in [p, q]$ is excluded because the itineraries of p and c_1 coincide for more than a period of q), so the first return map of q sends $[q, \zeta] \ni p$ onto $[q, c_1]$ and p is not characteristic, a contradiction.

For *statement (3)* of the proposition, we have $p_m \in [\zeta_{S_m}, c_1]$, and it remains to show that $\zeta_{S_m} \in [p_{m-1}, c_1]$ for all $m \geq 2$. We know $\{\zeta_{S_m}, p_{m-1}\} \subset [0, c_1]$, and (3) could only be false if $p_{m-1} \in [\zeta_{S_m}, c_1]$. Now $p_{m-1} \in [p_m, c_1]$ would contradict the choice of p_m, and $p_{m-1} \in [\zeta_{S_m}, p_m]$ implies that an appropriate iterate sends $[p_m, \zeta_{S_m}]$ to $[p_m, c_1]$ containing a point on the orbit of p_{m-1}, which is impossible because p_{m-1} is characteristic.

For *statement (2)*, let $1 \to S_1 \to \ldots \to n$ and $1 \to S_1 \to \ldots \to n \to n'$ be the internal addresses of ν and ν'. By statement (1), the tree $T(\nu')$ contains a closest tame characteristic point p of period n, and by statement (4), the point p has itinerary $\tau(p) = \overline{\nu'_1 \ldots \nu'_n}$. The first $n-1$ entries of ν and ν' coincide, so we have $\tau(p) \in \{\mathcal{A}(\nu), \overline{\mathcal{A}}(\nu)\}$. The internal address of ν' has an entry n, so $\tau(p) = \mathcal{A}(\nu)$ by Definition 5.14. This is just the definition of $\nu < \nu'$.

For *statement (2')*, we now know that $\overline{\star} = \nu^0 < \nu^1 < \ldots < \nu^k = \nu$. Let μ^m be the kneading sequences from the definition of internal addresses in parameter space. Clearly $\mu^0 = \overline{\star} = \nu^0$. Suppose by induction that $\mu^m = \nu^m$. If the periods of ν and μ^{m+1} (and necessarily also of ν^{m+1}) coincide, then $\mu^{m+1} = \nu = \nu^{m+1}$. Otherwise, we know $\mu^m < \mu^{m+1} < \nu$ (definition) and $\nu^m < \nu^{m+1} < \nu$ (statement (2)), so μ^{m+1} and ν^{m+1} are comparable. If $\mu^{m+1} \neq \nu^{m+1}$, then $\mu^{m+1} < \nu^{m+1}$ or $\nu^{m+1} < \mu^{m+1}$, and by the Abstract Lavaurs Lemma 6.6, there must be a sequence of lower period in between, contradicting the definition of the μ^m. □

Remark. The fact that the internal address records only tame periodic points (as in definition (4) above) makes it so difficult to find evil orbits and hence to tell non-admissible kneading sequences apart.

6.9. **Corollary (Order Near $\star$-Periodic Kneading Sequences)**
Let ν be a $\star$-periodic kneading sequence of period n. If some $\star$-periodic kneading sequence $\nu' \neq \nu$ coincides with ν for $n-1$ entries, then $\nu' > \nu$ if and only if the n-th entry in ν' equals that of $\mathcal{A}(\nu)$.

Remark. As written, this result requires that the period of ν' is greater than n. In Corollary 7.6, we will generalize it to arbitrary sequences ν'.

Proof. The internal address of $\mathcal{A}(\nu)$ contains an entry n by definition. If the n-th entry in ν' equals that of $\mathcal{A}(\nu)$, then the internal address of ν' contains n as well, and is then a continuation of the internal address of ν. By Proposition 6.8 (2), we have $\nu' > \nu$.

For the converse, suppose that $\nu' > \nu$. We will work in the Hubbard tree for ν'. It contains a tame characteristic periodic point p with itinerary $\mathcal{A}(\nu)$. By assumption that $\mathcal{A}(\nu)$ and ν' coincide for $n-1$ entries, there is no precritical point $\zeta \in (c_1, p)$ with $\mathrm{Step}(\zeta) < n$. If there was a precritical point $\zeta \in (c_1, p)$ with $\mathrm{Step}(\zeta) = n$, then Lemma 5.18 would supply a periodic point $p' \in (\zeta, c_1) \subset (p, c_1)$ with exact period n for which the first return map fixes no local arm; its itinerary would then be $\mathcal{A}(\nu)$ by Proposition 4.8. But this is also the itinerary of p, which is a contradiction to the existence of $\zeta \in (p, p')$. Hence the first n entries in $\mathcal{A}(\nu)$ coincide with the itinerary of c_1, which is ν'. □

Now we state one special case how to extend an admissible internal address so as to yield another admissible internal address, modeling bifurcations in the Mandelbrot set; see [**LS**, Proposition 5.4]: at every hyperbolic component W of some period S and every $q \geq 2$, there are bifurcating hyperbolic components W_q of period qS attached to W. The number of such components is $\varphi(q)$ (the number of positive integers in $\{1, 2, \ldots, q\}$ coprime to q), but all have the same kneading sequence. In terms of internal addresses, the internal address of W_q equals that of W with an appended entry qS.

6.10. **Lemma (Bifurcating Kneading Sequence)**
If the internal address $1 \to S_1 \to S_2 \to \ldots \to S_k$ is admissible and S_{k+1} is a proper multiple of S_k, then $1 \to S_1 \to S_2 \to \ldots \to S_k \to S_{k+1}$ is also admissible.

Proof. Let ν be the $\star$-periodic kneading sequence associated to $1 \to S_1 \to S_2 \to \ldots \to S_k \to S_{k+1}$ and let ν^k be the same for $1 \to S_1 \to S_2 \to \ldots \to S_k$. Then $\nu^k < \nu$ by Proposition 6.8 (2). If T is the Hubbard tree associated to ν, then T contains a characteristic periodic point p with itinerary $\mathcal{A}(\nu^k)$. Comparing itineraries of p and c_1, it follows that the arc $[p, c_1]$ maps homeomorphically onto itself after S_{k+1} iterations. If ν was not admissible, then it would have an evil orbit of branch points with characteristic point p', say, and the period of p' would be less than S_{k+1}. Since ν^k is admissible, it would follow that $p' \in (p, c_1)$, but there is no periodic point of period less than S_{k+1} on (p, c_1): the period of p' must be a multiple of that of p (otherwise, we easily get a contradiction to the fact that p is characteristic, considering iterates of the arc $[p, p']$), and then it must be a multiple of S_{k+1} (all other iterates are in wrong global arms of p). $\square$

6.11. **Lemma (Admissible Periodic and ⋆-Periodic Sequences)**
A ⋆-periodic kneading sequence ν is admissible if and only if $\mathcal{A}(\nu)$ is.

Proof. Let n be the period of ν, choose an integer $s \geq 2$ and let ν' be the ⋆-periodic sequence of period sn which coincides for $sn-1$ entries with $\mathcal{A}(\nu)$, so that the internal address of ν' is that of ν (or equivalently $\mathcal{A}(\nu)$) with sn added after the last n. We have $\nu' > \nu$ by Proposition 6.8 (2).

Suppose that ν is admissible. By Lemma 6.10, ν' is admissible, so the n-word formed by the first n entries of $\mathcal{A}(\nu)$ is admissible. By Lemma 14.9, $\mathcal{A}(\nu)$ is admissible as well.

Conversely, suppose that ν fails the admissibility condition for period m; then clearly $m < n$. Let $m' := \rho_{\mathcal{A}(\nu)}(m)$: this is finite because n is the exact period of $\mathcal{A}(\nu)$ by Proposition 5.16. Choose ν' as above, with s large enough so that $sn > \rho_{\mathcal{A}(\nu)}(m)$. The sequence ν' fails the admissibility condition for the same period m because $\nu' > \nu$ (both are ⋆-periodic). But since $\rho_{\mathcal{A}(\nu)}(m) = \rho_{\nu'}(m)$, this implies that $\mathcal{A}(\nu)$ fails the admissibility condition for the same m. $\square$

Remark. The analogous statement for $\overline{\mathcal{A}}(\nu)$ is false: the sequences $\nu = \overline{\texttt{101 10}\star}$ and $\mathcal{A}(\nu) = \overline{\texttt{101 100}}$ both have internal address $1 \to 2 \to 4 \to 5 \to 6$ and are both non-admissible (Example 5.3), while $\overline{\mathcal{A}}(\nu) = \overline{\texttt{101 101}} = \overline{\mathcal{A}}(\overline{\texttt{10}\star})$ has internal address $1 \to 2 \to 4 \to 5 \to 7 \to 8 \ldots$ and is admissible (compare Lemma 6.12).

We conclude this section with a discussion of periodic kneading sequences without ⋆.

6.12. **Lemma (Periodic Kneading Sequences Without ⋆)**
Let ν_* be a ⋆-periodic kneading sequence of period n, and let n' be the exact period of $\nu := \overline{\mathcal{A}}(\nu_*)$. Then exactly one of the following holds:

- *$n' = n$ and the internal address of ν is infinite;*
- *n' strictly divides n, and there is a ⋆-periodic kneading sequence ν'_* of period n' with $\nu = \mathcal{A}(\nu'_*)$ and $\nu'_* < \nu_*$; admissibility of ν_* and of ν'_* are equivalent;*
- *n' strictly divides n, and there is a ⋆-periodic kneading sequence ν'_* of period n' with $\nu = \overline{\mathcal{A}}(\nu'_*)$; ν_* is not admissible.*

The exact periods of ν_* and $\mathcal{A}(\nu_*)$ coincide, and both have finite internal addresses terminating with n.

We will see in **??** that for admissible sequences, the first case corresponds to a primitive component of the Mandelbrot set, while the second corresponds to non-primitive (satellite) components (compare [**LS**, Corollary 5.5]).

Proof. The statements for the upper sequence $\mathcal{A}(\nu_*)$ are already clear: both have finite internal addresses terminating with n (for $\mathcal{A}(\nu_*)$ this is part of the Definition 5.14), so their internal addresses must coincide. The exact period of ν is n by Proposition 5.16.

Now we turn attention to $\nu = \overline{\mathcal{A}}(\nu_*)$. The three cases are clearly mutually exclusive because ν'_* is uniquely determined by ν and n'.

To see that one of the three cases always holds, suppose first that $n' = n$. If the internal address of ν is finite, then the last entry must be the exact period n' by Lemma 20.2; but then ν is a upper sequence by Definition 5.14.

If $n' \neq n$, then $n' < n$ must be a proper divisor of n, so $n = sn'$ for some $s \geq 2$. Let ν'_* be the $\star$-periodic kneading sequence of period n' which coincides with ν for $n' - 1$ entries. Then $\nu = \mathcal{A}(\nu'_*)$ or $\nu = \overline{\mathcal{A}}(\nu')$.

If $\nu = \mathcal{A}(\nu'_*)$, then the internal address associated to ν_* has an entry n' (this is clearly so for ν'_* and hence for ν, and thus also for ν_*); thus the internal address of ν_* equals that of ν'_* with an appended entry n, and $\nu'_* < \nu_*$ by Proposition 6.8 (2). Admissibility of ν_* thus implies admissibility of ν'_*, and the converse is Lemma 6.10.

If $\nu = \overline{\mathcal{A}}(\nu'_*)$, then ν_* has exactly the form of Example 5.3 of non-admissible sequences: take a $\star$-periodic sequence of some period n' so that the lower sequence $\overline{\mathcal{A}}(\nu'_*)$ has exact period n', repeat for some number $s \geq 2$ of times, and replace the final entry by a $\star$. $\square$

7. The Parameter Tree for Non-Periodic Sequences and Infinite Addresses

In this section, we endow the space of all kneading sequences with the topological structure of an infinite tree. We discuss which sequences are branch points and which are endpoints.

In Section 6, we discussed only $\star$-periodic kneading sequences. Many properties carry over to non-periodic sequences simply by continuity. Every non-periodic kneading sequence ν has an associated infinite internal address $1 \to S_1 \to \ldots \to S_k \to \ldots$ defined as $\mathrm{orb}_\rho(1)$, and each S_k comes with a $\star$-periodic kneading sequence ν^k of period S_k (associated to $1 \to S_1 \to \ldots \to S_k$) which first differs from ν at position S_k (compare Proposition 6.8 (2')).

7.1. **Definition (Order for Non-Periodic Kneading Sequences)**
Let ν be a non-periodic kneading sequence with internal address $1 \to S_1 \to \ldots \to S_k \to \ldots$. For $k \geq 0$, let ν^k be the $\star$-periodic kneading sequence associated to the finite truncation $1 \to S_2 \to \ldots \to S_k$. Then we define that $\nu > \nu^k$ for all k. Conversely, if a $\star$-periodic kneading sequence ν' has $\nu' > \nu^k$ for all k, then we say that $\nu' > \nu$. This order is then extended to its transitive hull on the set of $\star$-periodic and non-periodic kneading sequences.

Remark. It is not difficult to check that the order defined here does not lead to any new relations for $\star$-periodic sequences, so it gives a strict partial order on the set $\Sigma^{\star\star}$ of all $\star$-periodic and non-periodic kneading sequences.

The extension of the order to non-periodic sequences was defined as above in order to yield a closed order relation.

One can extend the order to periodic sequences without $\star$ in the following way: if $\nu = \mathcal{A}(\nu_\star)$ for a $\star$-periodic sequence ν_*, then we define $\nu > \nu_\star$, and ν compares to all sequences $\nu' \neq \nu_\star$ just like $\nu_\star$ does. If $\nu = \overline{\mathcal{A}}(\nu_\star)$, then $\nu < \nu_\star$, and $\nu > \nu'$ iff $\nu_* > \nu'$. However, $\nu < \nu'$ if either $\nu_\star < \nu'$ or $\nu_* = \nu'$ or the kneading sequence ν' is non-admissible and the associated tree has an evil branch point with itinerary $\overline{\mathcal{A}}(\nu_\star)$: compare Figure 6.2. In terms of Proposition 6.7 and the remark thereafter, this means that evil orbits are "created" at $\star$-periodic sequences ν_* for which $\overline{\mathcal{A}}(\nu_*)$ has the same period as ν_*, and the subtree of non-admissible sequences which have such orbits branches off "just before" the sequence $\nu_\star$; the sequence $\overline{\mathcal{A}}(\nu_\star)$ is the maximum of those sequences "just before" $\nu_\star$. We will usually work with the space $\Sigma^{\star\star}$, excluding periodic sequences without $\star$ (see Definition 7.4 of the parameter tree and the remarks thereafter). An extension to periodic kneading sequences without $\star$ is straightforward, but it would not add any new types of dynamics.

We can now extend Lemma 6.5 to kneading sequences in $\Sigma^{\star\star}$: recall that these are $\star$-periodic or non-periodic sequences starting with `1`.

7.2. **Lemma (Linearly Ordered Subset, General Case)**
For every kneading sequence $\nu \in \Sigma^{\star\star}$, the set of all kneading sequences $\nu' \in \Sigma^{\star\star}$ with $\nu' < \nu$ is linearly ordered.

Moreover, if ν is non-periodic and $\nu' < \nu$ for any ν', then $\nu' < \nu^k$ for some k, where ν^k are the approximating $\star$-periodic kneading sequences of ν as in Definition 7.1.

Proof. The only sequences that are defined to be less than ν are the ν^k; any other relation $\nu' < \nu$ must follow by transitivity, so we must have $\nu' < \nu^k < \nu$ for some k. This shows the second claim.

It follows from Lemma 6.5 that the set of all $\star$-periodic kneading sequences less than ν is linearly ordered because any two of them are less than some ν^k.

Now consider a non-periodic kneading sequence $\nu' < \nu$ and let $(\nu')^k < \nu'$ be its approximating kneading sequences. Every $\star$-periodic $\nu_* < \nu$ can be compared to all $(\nu')^k$. If $\nu_* > (\nu')^k$ for all k, then $\nu_* > \nu'$ by definition; otherwise, $\nu_* < (\nu')^k$ for some k, and $\nu_* < \nu'$. Hence every non-periodic $\nu' < \nu$ can be compared to all periodic $\nu_* < \nu$.

Finally, let $\nu'' < \nu$ be non-periodic with $\nu'' \neq \nu'$. We will show that there is a $\star$-periodic kneading sequence ν_* such that $\nu'' < \nu_* < \nu'$ (or conversely), so ν' and ν'' can be compared.

Let n_* be the position of the first difference between ν' and ν'' and let ν_* be the $\star$-periodic kneading sequence of period n_* with coincides with ν' and ν'' for exactly $n_* - 1$ positions. Possibly by switching ν' and ν'', suppose that ν' is such that its first n_* entries coincide with $\mathcal{A}(\nu_*)$; then the first n_* entries of ν'' coincide with $\overline{\mathcal{A}}(\nu_*)$. For k sufficiently large, the first n_* entries of $(\nu')^k$ must coincide with ν' and hence with $\mathcal{A}(\nu_*)$, so $(\nu')^k > \nu_*$ by Corollary 6.9. Hence also $\nu' > \nu_*$. On the other hand, by the same reasoning, we cannot have $(\nu'')^k > \nu_*$ for large k. Since $(\nu'')^k < \nu'' < \nu$ and $\nu_* < (\nu')^k < \nu$, it follows that $(\nu'')^k$ and ν_* are comparable, hence $(\nu'')^k < \nu_*$ for all large k and thus for all k. Therefore, $\nu'' < \nu_* < \nu'$. □

7.3. **Lemma (Admissible Subtree Connected)**
If ν is admissible, then every $\nu' < \nu$ is also admissible (for arbitrary $\nu, \nu' \in \Sigma^{\star\star}$).

Proof. If ν is admissible and non-periodic, this means by definition that ν does not fail the admissibility condition for any m, and then no ν^k can fail the admissibility condition (with ν^k as in Definition 7.1) because the failing of the admissibility condition for period m is determined by the first $\rho(m)$ entries. Since $\nu' < \nu$ implies $\nu' < \nu^k$ for some k by Lemma 7.2, we may as well suppose that ν is $\star$-periodic.

If ν' is $\star$-periodic, the result is Corollary 6.4. If ν' is non-periodic, then $(\nu')^k < \nu' < \nu$, so every $(\nu')^k$ is admissible, and the claim follows again. □

7.4. **Definition (The Parameter Tree $\mathcal{T}$)**
The set of all kneading sequences in $\Sigma^{\star\star}$ with the order from Definition 7.1 will be called the parameter tree $\mathcal{T}$.

This space is closely related to the Mandelbrot set (compare the remark after Definition 6.1, Section 11 and [**LS**], but now the non-periodic sequences have been filled in by continuity). Closely related is work of Penrose [**Pen1**]: he constructs a “glueing space” (called “the abstract abstract Mandelbrot set”) as the quotient of the set of all sequences in Σ by the smallest closed equivalence relation generated by glueing $\mathcal{A}(\nu)$ and $\overline{\mathcal{A}}(\nu)$ for all $\star$-periodic ν. He obtains a connected infinite tree (a dendrite), and he studies especially the dynamics for non-periodic kneading sequences. In contrast, we focus on the $\star$-periodic sequences and fill in the non-periodic ones by continuity. His and our spaces differ near periodic ones: we distinguish the $\star$-periodic sequences which are bifurcations from each other (as in Lemma 6.10), and our space is disconnected at such points. In order to obtain a path connected space, Penrose identifies such bifurcating sequences.

The Mandelbrot set, which is modeled by both symbol spaces, is connected, but it does not identify bifurcating sequences: instead, every $\star$-periodic sequence is realized by an open subset of $\mathbb{C}$ (a hyperbolic component). Our main interest here is in the combinatorics of the parameter tree and its order, not on its topology.

There are two fundamental differences between the Mandelbrot set on the one side and our parameter tree and Penrose's abstract abstract Mandelbrot set on the other side (besides the fact that the Mandelbrot set is a rich topological space embedded in $\mathbb{C}$, while the others are just combinatorial models with a topological structure inherited from the order): the first difference is that the combinatorial spaces contain non-admissible kneading sequences (which do not occur for the Mandelbrot set); the second one is that admissible sequences may occur several times in the Mandelbrot set (they are distinguished by *angled* internal addresses; see Definition 11.9).

In the following, it will be convenient to set $\mathcal{A}(\nu) = \overline{\mathcal{A}}(\nu) := \nu$ for non-periodic ν.

7.5. **Lemma (Intermediate Kneading Sequence)**
Let $\nu' < \nu$ be any two kneading sequences. Suppose that $\mathcal{A}(\nu') \neq \overline{\mathcal{A}}(\nu)$. Let n_* be the position of the first difference between $\mathcal{A}(\nu')$ and $\overline{\mathcal{A}}(\nu)$. Then the unique $\star$-periodic kneading sequence ν_* of period n_* which coincides with $\mathcal{A}(\nu')$ and $\overline{\mathcal{A}}(\nu)$ for $n_* - 1$ entries satisfies $\nu' < \nu_* < \nu$.

If $\mathcal{A}(\nu') = \overline{\mathcal{A}}(\nu)$, then there exists no $\star$-periodic kneading sequence with $\nu' < \nu_* < \nu$.

Proof. If ν is not $\star$-periodic, then we replace it by one of its approximating $\star$-periodic kneading sequences $\nu^k < \nu$, for k large enough so that ν and ν^k differ only after their n_*-th entries. We proceed similarly for ν'. If we show that $(\nu')^k < \nu_* < \nu^k$ for all large k, then $\nu_* < \nu$ follows by transitivity, and $\nu_* > \nu'$ follows by definition. We may thus assume that ν and ν' are $\star$-periodic to begin with.

Since $\nu' < \nu$, there is a tame characteristic periodic point p' with itinerary $\mathcal{A}(\nu')$ in the Hubbard tree for ν. The limiting itinerary for points x near c_1 is $\overline{\mathcal{A}}(\nu)$ by Lemma 5.15.

Suppose that $\mathcal{A}(\nu') \neq \overline{\mathcal{A}}(\nu)$. Then there is an earliest precritical point $\zeta_* \in (p', c_1)$, it maps to c_1 after exactly n_* steps and its itinerary coincides with $\mathcal{A}(\nu')$ and $\overline{\mathcal{A}}(\nu)$ for n_*-1 entries. By Lemma 5.18, there is a tame characteristic periodic point $p_* \in [\zeta_*, c_1]$ with exact period n_* such that its itinerary coincides with that of ζ_* for $n_* - 1$ entries. We claim that this itinerary is $\mathcal{A}(\nu_*)$: for the first $n_* - 1$ entries, this is clear, and for the n_*-th entry it follows from Proposition 4.8. But by definition, this implies $\nu_* < \nu$. Since $p_* \in (p', c_1)$, Theorem 6.2 (3) implies $\nu' < \nu_*$.

If $\mathcal{A}(\nu') = \overline{\mathcal{A}}(\nu)$ and there is a $\star$-periodic kneading sequence ν_* with $\nu' < \nu_* < \nu$, then there is a tame characteristic periodic point $p_* \in (p', c_1)$ with itinerary $\mathcal{A}(\nu_*)$; this itinerary must coincide with $\mathcal{A}(\nu')$. Since the exact periods of ν_* and ν' coincide with those of $\mathcal{A}(\nu_*)$ and $\mathcal{A}(\nu')$ by Proposition 5.16, we have $\nu_* = \nu'$. □

The following is a more general variant of Corollary 6.9.

7.6. **Corollary (Order Near $\star$-Periodic Kneading Sequences)**
Let ν be a $\star$-periodic kneading sequence of some period n and let $\nu' \in \Sigma^{\star\star}$ be such that $\overline{\mathcal{A}}(\nu')$ coincides with ν for $n-1$ entries. Then $\nu' > \nu$ if and only if the n-th entry in $\overline{\mathcal{A}}(\nu')$ equals that of $\mathcal{A}(\nu)$.

REMARK. If the n-th entries differ, then ν' and ν may or may not be comparable.

PROOF. We will use Corollary 6.9 several times; it settles the case that ν' is periodic and its period exceeds n.

If ν' is non-periodic, then take some approximating $\star$-periodic kneading sequence $(\nu')^k < \nu'$, where k is large enough so that the period of $(\nu')^k$ is greater than n. If the n-th entries in $\mathcal{A}(\nu)$ and $\overline{\mathcal{A}}(\nu') = \nu'$ coincide, then $\nu < (\nu')^k < \nu'$. Conversely, if $\nu < \nu'$, then $\nu < (\nu')^k$ for sufficiently large k, and the claim follows.

It remains the case that ν' is $\star$-periodic with some period $n' < n$. If $\nu' > \nu$, then the Hubbard tree for ν' has a tame characteristic periodic point p with itinerary $\mathcal{A}(\nu)$, and it has is no precritical point ζ on (p, c_1) with $\text{STEP}(\zeta) \leq n$. Therefore, the first n entries in $\mathcal{A}(\nu)$ and in $\overline{\mathcal{A}}(\nu')$ coincide by Lemma 5.15.

For the converse, let $1 \to S_1 \to \ldots \to S_k \to n'$ be the internal address of ν'. Since the first $n-1 \geq n'$ entries of ν and $\overline{\mathcal{A}}(\nu')$ coincide, the internal address of ν must start with $1 \to S_1 \to \ldots \to S_k \to S_{k+1} \ldots$ with $S_{k+1} > n'$. Since $\nu^k < \nu'$ (Proposition 6.8 (2)) and the kneading sequence of lowest period between ν^k and ν is ν^{k+1} of period $S_{k+1} > n'$, Lemma 7.5 shows that ν^{k+1} is also the kneading sequence of lowest period between ν^k and ν', hence $\nu^{k+1} < \nu'$. If $\nu = \nu^{k+1}$, then we are through. Otherwise, $S_{k+1} < n$ and the argument can be repeated: let $S_{k+2} \leq n$ be the next entry in the internal address of ν, so that $\nu^{k+1} < \nu^{k+2} \leq \nu$. Again by Lemma 7.5, $\nu^{k+1} < \nu^{k+2} < \nu'$, and a finite repetition of the argument until $\nu^{k+s} = \nu$ shows $\nu < \nu'$. □

7.7. **Corollary (Internal Address and Truncated Kneading Sequence)**
For a $\star$-periodic or non-periodic kneading sequence ν and $s \geq 1$, let ν^s be the $\star$-periodic sequence of period s which coincides with ν for $s-1$ positions; if ν is $\star$-periodic of some period n, suppose $s < n$. Then $\nu^s < \nu$ if and only if s occurs in the internal address of ν.

Proof. Suppose first that ν is $\star$-periodic. Since ν coincides with ν^s for $s-1$ entries, it follows that $\nu > \nu^s$ if and only if the s-th entry in ν is the same as in $\mathcal{A}(\nu^s)$ (Corollary 6.9), which holds if and only if s occurs in the internal address of ν (because the internal address of $\mathcal{A}(\nu^s)$ ends with s).

If ν is non-periodic, we have $\nu > \nu^S$ for every S on the internal address of ν by definition, and the converse follows by replacing ν with ν^S for sufficiently large S. $\square$

7.8. **Definition (Endpoints of Parameter Tree)**
A kneading sequence ν is called an endpoint *of the parameter tree $\mathcal{T}$ if there is no $\star$-periodic kneading sequence ν' with $\nu' > \nu$.*

Remark. Endpoints which are admissible can also be characterized in the following way: suppose that the Mandelbrot set is locally connected (or use a locally connected model). Then an *admissible* ν is an endpoint of the parameter tree if and only if it describes points in the Mandelbrot set which are landing points of exactly one external ray each, while admissible non-endpoints are sequences corresponding to landing points of at least two external rays each (see Section 17).

Remark. If ν is an endpoint of the parameter tree, then there can be no $\nu' > \nu$ even for non-periodic ν' (otherwise, by Lemma 7.2, we would have $(\nu')^k > \nu$ for some of the approximating $\star$-periodic sequences $(\nu')^k$ of ν').

No $\star$-periodic kneading sequence ν can be an endpoint of the parameter tree: if ν corresponds to $1 \to S_1 \to \ldots \to S_k$, where S_k is the period of ν, then the kneading sequence corresponding to $1 \to S_1 \to \ldots \to S_k \to S_k+1$ is greater by Proposition 6.8 (2).

However, there are endpoints which are preperiodic or which are not (pre-)periodic; examples are $1\overline{0}$ (which is preperiodic and associated to $1 \to 2 \to 3 \to 4 \to \ldots$) and the kneading sequence associated to $1 \to 3 \to 9 \to 27 \to \ldots$ which is not (pre-)periodic. Examples of kneading sequences which are not endpoints are $1 \to 3 \to 6 \to 9 \to 12 \to \ldots$ (preperiodic) or $1 \to 2 \to 4 \to 8 \to \ldots$ (not (pre-)periodic).

7.9. **Proposition (Endpoints of Parameter Tree)**
A non-periodic kneading sequence ν is an endpoint of the parameter tree if and only if, for every positive integer s, the ρ-orbits of 1 and of s eventually meet.

More generally, for any non-periodic kneading sequence ν and $s \geq 1$, let ν^s be the unique $\star$-periodic kneading sequence of period s which coincides with ν for $s-1$ entries. Then for any s, we have $\nu^s > \nu$ if and only if the ρ-orbits of 1 and of s never meet.

Proof. Most of the work is in showing the second statement. Since ν is non-periodic, $\rho(s) < \infty$ for every $s \geq 1$, and the internal address $1 \to S_1 \to \ldots \to S_k \to \ldots$ of ν is infinite with $S_k = \rho^{\circ k}(1)$. For typographical reasons, we will write $\nu(s)$ for ν^s.

The strategy of the proof will be as follows: we have $\overline{\star} = \nu(1) < \nu(S_1) < \cdots < \nu(S_k) < \cdots < \nu$ by Proposition 6.8 (2'); this describes the positions of $\nu(s)$ for $s \in \mathrm{orb}_\rho(1)$. For an arbitrary $s > 1$, it will turn out that $\nu(s) > \nu$ implies $\nu(s) > \nu(\rho(s)) > \nu(\rho^{\circ 2}(s)) > \cdots > \nu$ and the ρ-orbits of 1 and s are disjoint, while $\nu(s) \not> \nu$ implies that there is an m with $\nu(s) > \nu(\rho(s)) > \cdots > \nu(\rho^{\circ m}(s))$ and $\nu(\rho^{\circ(m+1)}(s)) < \nu(\rho^{\circ(m+2)}(s)) < \cdots < \nu$ and the ρ-orbits of 1 and s meet eventually.

The case that s is on the internal address of ν is equivalent to $\nu(s) < \nu$, and we already know that $\nu(s) < \nu(\rho(s)) < \nu(\rho(\rho(s))) < \cdots < \nu$. Therefore, we may assume that s is not on the internal address, hence $\nu(s) \not< \nu$. Then ν and $\overline{\mathcal{A}}(\nu(s))$ coincide for at least s entries (this is clear for the first $s-1$ entries, and for the last it follows from Corollary 7.6 because $\nu(s) \not< \nu$). We claim that the first difference is at position $\rho(s)$ using a standard argument: if $\sigma^s(\nu)$ and $\overline{\mathcal{A}}(\nu(s))$ differ first at position s', say, then ν and $\overline{\mathcal{A}}(\nu(s))$ differ first at position $s'+s > s'$; the position of the first difference between ν and $\sigma^s(\nu)$ is by definition at position $\rho(s) - s$, hence $\rho(s) - s = s'$.

If $\nu(s) > \nu$, then $\nu(s) > \nu(\rho(s)) > \nu$ by Lemma 7.5, and inductively $\nu(s) > \nu(\rho(s)) > \nu(\rho(\rho(s))) > \cdots > \nu$. It follows that the ρ-orbits of s and 1 are disjoint.

Finally, suppose that $\nu(s)$ and ν are not comparable. Exactly one of ν and $\overline{\mathcal{A}}(\nu(s))$ coincides with $\mathcal{A}(\nu(\rho(s)))$ for $\rho(s)$ positions. If this is ν, then $\nu > \nu(\rho(s))$ (and only then); then $\rho(s) \in \mathrm{orb}_\rho(1)$, and the ρ-orbits of s and 1 eventually meet. Otherwise $\nu(s) > \nu(\rho(s))$ (Corollary 7.6), and we continue inductively with $\rho(s)$ in place of s.

If there is an m with $\nu(\rho^{\circ(m)}) < \nu$, then the ρ-orbits of s and 1 eventually meet. If there is no such m, then $\nu(s) > \nu(\rho(s)) > \cdots > \nu(\rho^{\circ m}(s)) > \nu(\rho^{\circ(m+1)}(s)) > \ldots$ for all m. In this case, for every entry S_k in the internal address of ν, all sufficiently large m satisfy $\nu(s) > \nu(\rho^{\circ m}(s)) > \nu(S_k)$ (because we have $\nu > \nu(S_k)$ and the first sufficiently many entries in $\nu(\rho^{\circ m}(s))$ and ν coincide). Since this is true for all S_k, it follows $\nu(s) > \nu$ by definition, and we are in the first case above. This proves the second claim.

Now the rest is easy: if ν is an endpoint of the parameter tree, then for every $s \geq 1$, we cannot have $\nu(s) > \nu$, so the ρ-orbits of 1 and s eventually meet.

Conversely, if ν is not an endpoint of the parameter tree, then there is a $\star$-periodic kneading sequence ν' of some period s with $\nu' > \nu$. We may assume that there is no kneading sequence of lower period between ν and ν'. Then $\nu' = \nu(s) > \nu$, and the ρ-orbit of s is disjoint from the orbit of 1. $\square$

Remark. More generally, we can define "branches" at a non-periodic kneading sequence ν as follows, again writing $\nu(s)$ for ν^s. Start by partitioning the set of positive integers into equivalence classes so that $s \sim 1$ unless $\nu(s) > \nu$; if $\nu(s_1) > \nu$ and

$\nu(s_2) > \nu$, then $s_1 \sim s_2$ if and only if there are $m_1, m_2 > 0$ with $\nu(\rho^{\circ m_1}(s_1)) < \nu(s_2)$ and $\nu(\rho^{\circ m_2}(s_2)) < \nu(s_1)$. The proof just given extends to show that two positive integers are equivalent if and only if their ρ-orbits eventually meet, and the number of equivalence classes equals the number of disjoint ρ-orbits for ν. Every equivalence class of positive integers gives then rise to a branch at ν: all $\nu' \in \Sigma^{\star\star}$ with $\nu' \not> \nu$ form one branch, and every other branch is the set of all ν' with $\nu' > \nu(s)$ for some s from any given equivalence class.

It can be shown that these branches partition $\Sigma^{\star\star} \setminus \{\nu\}$ into finitely many branches: the number of branches is 1 if and only if ν is an endpoint, and it can exceed 2 only if ν is eventually periodic: i.e., branch points of the parameter tree are periodic or preperiodic. We do not give a proof here. See also Section 8 and Theorem 9.13.

Remark. Proposition 6.7 extends easily to arbitrary sequences in $\Sigma^{\star\star}$: every non-admissible sequence sits in a non-admissible subtree which branches off from the subtree of admissible kneading sequences at a $\star$-periodic sequence.

Finiteness of the number of branches at any non-periodic kneading sequence follows like this: if there are at least three arms, then the sequence is preperiodic by the Branch Theorem above, and the number of arms there is the same as in the dynamic plane. This needs to be discussed in context of Alex' theorem. In any case, the last two paragraphs need to be checked.

Further facts to be proved: whatever is left of translation and correspondence principles, even for non-existing trees (what can we say about them???)

8. The Branch Theorem

In this section we show that for the space $(\Sigma^{\star\star}, <)$ branching occurs only at preperiodic or $\star$-periodic kneading sequences. This result is based on a study of the dynamical plane: motivated by the correspondence principle of dynamic and parameter rays, we investigate the arrangement of periodic points in Hubbard trees and extend Theorem 6.2.[12]

8.1. **Definition (Attracting Dynamics)**
A Hubbard tree (T, f) with n-periodic critical point has attracting dynamics *if for each $c_k \in \operatorname{orb}(0)$, there is a neighborhood U_k of c_k such that for all $x \in U_k$, $f^{\circ jn}(x) \to c_k$ as $j \to \infty$.*

The next lemma identifies all exact periods that periodic points with given itinerary τ might have. Lemma 8.3 summarizes some properties of Hubbard trees with attracting dynamics. Proposition 8.4 then shows that every equivalence class of Hubbard trees with periodic critical point contains a representative which has attracting dynamics.

The following statement is an easy but useful observation: suppose that $f|_{[x,y]}$ is injective and there are three consecutive iterates $p, f(p),\ f^{\circ 2}(p) \in [x, y]$ such that $f(p) \in (p, f^{\circ 2}(p))$. Then $f^{\circ 2}(p) \in (f(p), f^{\circ 3}(p))$.

8.2. **Lemma (Length of Periodic Orbits)**
Let $\tau \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$ be a periodic itinerary of exact period n. Then there is an $l > 0$ such that the exact period of any periodic point in T with itinerary τ is either n or ln.

Proof. Let $T_\tau = \{p \in T : \tau(p) = \tau\}$ and assume that this set is non-empty (otherwise the claim is trivially true). By expansivity, T_τ contains at most one branch point, and there is at most one periodic point in T_τ of exact period n whose local arms are not all fixed. By the above observation, for any periodic point $p \in T_\tau$ with exact period ln and $l > 1$, there is an n-periodic point $z \in (p, f^{\circ n}(p))$ and $f^{\circ n}$ does not fix the local arms at z pointing to p and $f^{\circ n}(p)$ (z is a branch point for $l > 2$ because otherwise the orbit of z would be infinite).

Therefore, if for every periodic point in T_τ of exact period n all its local arms are fixed under $f^{\circ n}$, then all periodic points in T_τ have exact period n. Now suppose that there is a periodic point $z \in T_\tau$ of exact period n that has a cycle C of local arms of length $l > 1$. By Corollary 4.4, there is at most one such cycle at z. Let G be a global arm of z whose local arm L is contained in the cycle C. If $p \in G \cap T_\tau$ is periodic, then p must be fixed under the first return map $f^{\circ ln}$ of L: if not, the reasoning above shows that $G \cap T_\tau$ contains a periodic (inner) point p' of exact period ln whose local arms are permuted by $f^{\circ ln}$. This contradicts that $f^{\circ ln}([z, p']) = [z, p']$. If z has an arm G' such that the associated local arm is fixed under $f^{\circ n}$, then all periodic points in

[12] The results of this section have been obtained in [**Kaf2, Kaf1**]. Our discussion stays fairly close to these manuscripts.

$G' \cap T_\tau$ have exact period n; for otherwise $G' \cap T_\tau$ would contain another n-periodic point whose local arms are not all fixed, which is impossible. Thus, the period of any periodic point in T_τ equals n or ln. □

8.3. **Lemma (Consequences of Attracting Dynamics)**
Let (T, f) be a Hubbard tree with periodic critical point and attracting dynamics. Then the following are true:

(1) *The set of all periodic points with itinerary τ is closed.*
(2) *There is a neighborhood U of c_1 such that $f^{\circ j}|_U$ is injective for all $j \geq 0$.*

Proof. For the first claim, observe that if τ contains a $\star$, then the claim is trivially true because there is at most one point with this itinerary. Otherwise suppose that p is the limit point of periodic points p_n with $\tau(p_n) = \tau$. By Lemma 8.2, there is an $m \in \mathbb{N}$ such that all p_n have (not necessarily exact) period m. By continuity, $f^{\circ m}(p) = p$, *i.e.,* p is periodic and its period divides m. Moreover, p has either itinerary τ or is on the critical orbit. But since T has attracting dynamics, no point on the critical orbit can be the limit point of periodic points.

For the second claim, let n be the period of c_1. There is a neighborhood U of c_1 such that U contains no precritical point unequal to c_1 of Step at most n. Since (T, f) has attracting dynamics, we can choose U so small that for all $p \in U$, $f^{\circ n}(p) \in (p, c_1)$. If there was a precritical point $\xi \in U$ with $\xi \neq c_1$, then there is a j_0 such that $f^{\circ j_0}(\xi)$ is precritical of Step at most n and by the choice of U, we have $f^{\circ j_0}(\xi) \in U$, a contradiction. □

8.4. **Proposition (Representative with Attracting Dynamics)**
Every equivalence class of Hubbard trees with periodic critical point contains a representative with attracting dynamics.

Proof. For any given equivalence class pick a Hubbard tree (T, f) and suppose that the critical orbit is not attracting. We are going to define a new dynamics $\tilde{f}$ on the topological tree T such that any point on the critical orbit is locally attracting. To achieve this, it suffices to change f locally at the critical point 0. Let V be the set of marked points of T, let n be the exact period of 0 and let G be the unique global arm of 0 whose associated local arm is fixed under $f^{\circ n}$. Choose $y_0 \in G$ such that the interval $I := (0, y_0)$ has the following properties: $f^{\circ n}|_I$ is a homeomorphism onto its image, $I \cap V = \emptyset = f^{\circ n}(I) \cap V$ and $f^{\circ i}(I) \cap I = \emptyset$ for all $0 < i < n$. Without loss of generality, we can assume that there is a $z \in I$ such that $f^{\circ n}(z) = z$: if such a point does not exist then $p \in (0, f^{\circ n}(p))$ for all $p \in I$, and we can pick $z, y, y' \in I$ with $0 < y < z < f^{\circ n}(z) < y'$ (here $<$ denotes the natural order on $\overline{I}$ with 0 as the smallest element). There is a homeomorphism $h' : T \to T$ such that $h'|_{T \setminus [y,y']} \equiv \text{id}$ and

$h'(f^{\circ n}(z)) = z$. Set $f' := h' \circ f$. Then $(f')^{\circ n}(z) = z$ and the two Hubbard trees (T, f) and (T, f') are equivalent.

So we can assume that there is a $z \in I$ which is fixed under $f^{\circ n}$. Pick any homeomorphism $\varphi : [0, z] \to [0, 1]$ with $\varphi(0) = 0$, $\varphi(z) = 1$, and consider the function $h : [0, 1] \to [0, 1]$, $x \mapsto x^2$. It induces a map

$$\tilde{h} : T \longrightarrow T,\ p \mapsto \begin{cases} (\varphi^{-1} \circ h \circ \varphi)(p) & \text{if } p \in [0, z] \\ p & \text{otherwise.} \end{cases}$$

Let f_{-n} be the inverse branch of $f^{\circ n}|_I$ that maps $[0, z]$ onto itself and define

$$g : T \longrightarrow T,\ p \mapsto \begin{cases} (\tilde{h} \circ f_{-n})(p) & \text{if } p \in [0, z] \\ p & \text{otherwise.} \end{cases}$$

The map $\tilde{f} := g \circ f : T \to T$ is a continuous surjection and $(T, \tilde{f})$ is a Hubbard tree. Moreover, it is equivalent to the given one and has attracting dynamics: for the first part, observe that $V \subset T \setminus f^{-1}((0, z)) =: S$ and $\tilde{f}|_S = g \circ f|_S = f|_S$; to prove attracting dynamics, it suffices to show that $\tilde{f}^{\circ jn}(x) \to 0$ as $j \to \infty$ for all $x \in (0, z)$. For any $x \in (0, z)$, we have

$$\tilde{f}^{\circ n}(x) = \left((g \circ f) \circ (\mathrm{id} \circ f)^{\circ(n-1)}\right)(x) = (g \circ f^{\circ n})(x) = (\tilde{h} \circ f_{-n} \circ f^{\circ n})(x) = \tilde{h}(x)$$

and $\tilde{h}(x) \in (0, z)$. Thus, $\tilde{f}^{\circ jn}(x) = \tilde{h}^{\circ j}(x)$ for all $j \in \mathbb{N}$ and all $x \in (0, z)$, and by definition $\tilde{h}^{\circ j}(x) \to 0$ as $j \to \infty$. □

8.5. **Definition (Minimal Hubbard Trees)**
A Hubbard tree (T, f) is minimal *if it satisfies the following two properties:*

(1) *there are no two periodic points $p \neq p'$ in T with $\tau(p) = \tau(p')$;*
(2) *if 0 is periodic then (T, f) has attracting dynamics.*

Remark. This definition implies that minimal Hubbard trees do not contain two preperiodic points which have the same itinerary either. Recall that expansivity implies that there are no (pre-)periodic marked points with the same itinerary. Thus one could consider minimal Hubbard trees to be Hubbard trees with all (pre-)periodic points marked. Minimal Hubbard trees have the property that the exact period and preperiod of any (pre-)periodic point p equals the exact period and preperiod of its itinerary $\tau(p)$ (cf. Lemma 3.9).

8.6. **Lemma (Properties of Minimal Hubbard Trees)**
Let (T, f) be a minimal Hubbard tree and $z \in T$ be any periodic point which is disjoint from the critical orbit.

(1) *Let G be a global arm of z and $I := \{p \in G : p$ has the same itinerary as $z\}$. If I is not a singleton then $I = [z, x)$ such that $x \in \mathrm{orb}(0)$ and 0 is periodic.*

(2) *The periodic point z is repelling, i.e., there is a neighborhood U of z such that for all $p \in U$, $p \in (z, f^{\circ jn}(p))$, where n is the exact period of p and j is the period of the local arm at z pointing to p.*
(3) *If the Hubbard tree (T, f) generates the $\star$-periodic kneading sequence ν then there is a characteristic point z such that $\tau(z) = \overline{\mathcal{A}}(\nu)$.*

PROOF. For the first statement, recall that by expansivity, I is an interval. The boundary point $x \neq z$ is periodic: note first that I contains no precritical point nor an image of 0 (by minimality) and that there is an integer N such that the local arm pointing to x is fixed. Suppose first that x is precritical. Then $f^{\circ N}(x) \notin I$ and if $f^{\circ N}(x) \neq x$, then continuity implies that not all points in I have itinerary $\tau(z)$. Thus in this case, $f^{\circ N}(x) = x$. On the other hand, if x is not precritical, then $\tau(x) = \tau(z)$ and $f^{\circ N}(x) = x$ by continuity and maximality of I. This contradicts that (T, f) is minimal.

To show the second statement, let k equal the period of a non-fixed local arm at z if such a local arm exists, and set $k = 1$ otherwise. Furthermore, let $U \subset T$ be a neighborhood of z such that $f^{\circ kn}|_U$ is a homeomorphism onto its image. For any global arm G of z, set $L := G \cap U$. Since z is the only periodic point in T with itinerary $\tau(z)$, either $f^{\circ jn}(p) \in (z, p)$ for all $p \in L$ or $p \in (z, f^{\circ jn}(p))$ for all $p \in L$, where $j \in \{1, k\}$ is the period of the local arm associated to G. Suppose that the first case holds. Then the property that $f^{\circ jn}(p) \in (z, p)$ extends to all $p \in L' := \{x \in G : \tau(x) = \tau(z)\} \supset L$. The hypothesis implies that the set L' is an interval and its boundary point $p_0 \neq z$ is a periodic point contained in $\operatorname{orb}(0)$ by statement (1). By construction, all points in L' (except for z) are repelled from p_0 by $f^{\circ jn}$. This contradicts that (T, f) has attracting dynamics.

For statement (3), let n be the period of c_1 and let $U \subset T$ be the maximal connected set with $c_1 \in \overline{U}$ such that U contains no precritical point and such that for all $p \in U$, $f^{\circ jn}(p)$ converges to c_1 as $j \to \infty$. By Lemma 8.3, $U \neq \emptyset$ and since each branch point in T has finite orbit, U is an interval contained in $(0, c_1)$. By continuity, $f^{\circ n}(\overline{U}) \subset \overline{U}$, $\operatorname{orb}(0) \cap \overline{U} = \{c_1\}$ and if $z \in \partial U$ with $z \neq c_1$ then z is not precritical. If $f^{\circ n}(z) \neq z$ then there is a $y \notin [z, c_1]$ close to z such that $[y, z]$ contains no precritical point of STEP at most n and $f^{\circ n}(y) \in (z, c_1)$. Thus $f^{\circ n}([y, z]) \subset U$, $[y, z]$ contains no precritical point and $f^{\circ jn}(p)$ converges to c_1 for all $p \in [y, z]$, in contradiction to maximality of U. Therefore $f^{\circ n}(z) = z$, and since all points in $[z, c_1)$ have the same itinerary, which equals $\overline{\mathcal{A}}(\nu)$ by Lemma 5.15, the claim is proven. □

8.7. **Proposition (Minimal Representative)**
Every equivalence class of Hubbard trees contains a minimal representative.

PROOF. Given any equivalence class of Hubbard trees, pick a representative (T, f) so that (T, f) has attracting dynamics if 0 is periodic. For any periodic itinerary τ, let

$X_\tau \subset T$ be the smallest connected subset of T which contains all periodic points of itinerary τ. By Lemma 8.3 (1) and expansivity, X_τ is a closed (possibly degenerate) n-od. We define the following equivalence relation on T:

$$x \sim y \quad :\Longleftrightarrow \quad \begin{array}{l} x = y \text{ or } \tau(x) = \tau(y) \text{ and there is an } n > 0 \text{ such that} \\ f^{\circ n}(x) \text{ and } f^{\circ n}(y) \in X_\tau \text{ for some periodic itinerary } \tau. \end{array}$$

Observe that an equivalence class is either a singleton, of the form X_τ or a non-periodic iterated preimage of some X_τ. Thus all equivalence classes are closed connected subsets of T. Let $\tilde{f}$ be the dynamics induced by f on the quotient $\tilde{T} := T/_\sim$ and $\pi : T \longrightarrow \tilde{T}$ the natural projection. It suffices to show that $(\tilde{T}, \tilde{f})$ is a Hubbard tree: it is minimal by construction and equivalent to (T, f) because we have not changed the mutual location of marked points (marked points and in particular branch points are preserved under π by expansivity of (T, f)). Since all equivalence classes are closed and connected, $\sim$ is a closed equivalence relation. It follows that $\tilde{T}$ is Hausdorff, and since all equivalence classes are connected, $\tilde{T}$ is a tree. The map $\tilde{f}$ is continuous and locally injective on $\tilde{T} \setminus \{\pi(0)\}$, and since $f^{-1}(X_\tau)$ splits into at most two equivalence classes, $\tilde{f}$ is at most 2-to-1. Furthermore, $(\tilde{T}, \tilde{f})$ meets the expansivity condition. Putting everything together, $(\tilde{T}, \tilde{f})$ is a Hubbard tree. □

In the remainder of the section, we only regard minimal Hubbard trees.

8.8. **Definition (Bifurcation Itinerary)**
Suppose $\tau \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$ is periodic of exact period n. Then we define for any $q > 0$ $B^q(\tau) := \overline{(\tau_1 \cdots \tau_n)^{q-1} \tau_1 \cdots \tau'_n}$, where $\tau'_n = 1 - \tau_n$.

8.9. **Lemma (Points with Bifurcation Itinerary)**
Let (T, f) be a Hubbard tree and z be a characteristic point of period n with itinerary τ. If z' is periodic with itinerary $B^k(\tau)$ for some $k > 0$, then there is a $0 \le j \le k$ such that $f^{\circ jn}(z')$ is the characteristic point of $\operatorname{orb}(z')$.

Moreover, $f^{\circ jn}(z') \in (z, c_1)$ unless $j = 0$, $\tau(z) = \mathcal{A}(\nu)$ and $\tau(z') = \overline{\mathcal{A}}(\nu)$ for some $\star$-periodic kneading sequence ν.

Proof. Let ζ be the precritical point in (z, z') of lowest Step, which equals kn. Assume first that z' is characteristic. If $k > 1$ and $z \in (z', c_1)$, then the local arm of z pointing to 0 is fixed under $f^{\circ n}$. On the other hand, $f^{\circ kn}([\zeta, z]) = [z, c_1]$, a contradiction. If $k = 1$ and $z \in (z', c_1)$, it follows that the local arm of z' pointing to c_1 is fixed and therefore the local arm of z' pointing to 0 is fixed, too. Thus $\tau(z)$ equals the upper kneading sequence and $\tau(z')$ the lower kneading sequence of a $\star$-periodic kneading sequence μ as claimed.

Now suppose that z' is not characteristic and let $f^{\circ i_0}(z')$ be the characteristic point of $\operatorname{orb}(z')$, where $0 < i_0 < kn$. Since $\tau_i(p) = \tau_i(z)$ for all $p \in [z, z']$ and all $0 < i \le kn$, minimality implies that $\operatorname{orb}(z) \cap (z', z) = \emptyset$ and $\operatorname{orb}(z') \cap (z', z) = \emptyset$. Thus $f^{\circ i_0}(z') \in$

(z, c_1). Now suppose that $i_0 \neq jn$ for all $0 < j < k$. Then $z \in (f^{\circ i_0}(z), f^{\circ i_0}(z'))$ and $f^{\circ(kn-i_0)}(z) \in f^{\circ(kn-i_0)}((f^{\circ i_0}(z), f^{\circ i_0}(z'))) = (z, c_1] \cup (z', c_1]$. Now z characteristic implies that $f^{\circ(kn-i_0)}(z) \in (z, z')$, contradicting that (z, z') contains no iterate of z. $\square$

8.10. **Proposition (Existence of Dynamical Bifurcation Point)**
Let (T, f) be a Hubbard tree with itinerary ν and $z \in (0, c_1)$ be a characteristic n-periodic point with itinerary τ. Then there is a characteristic point $z' \in (z, c_1]$ such that (z, z') contains no further characteristic point. Moreover, either $z' \in (z, c_1)$ and z' has itinerary $B^Q(\tau)$, or $z' = c_1$ and $\nu = \mathcal{A}_{Qn}^{-1}(\tau)$, where Q is the period of the local arm at z that points to c_1.

By Corollary 4.4, Q or $Q + 1$ equals the number of arms at z according to whether z is tame or not.

Proof. We first consider the case that 0 is periodic. Let G be the global arm of z containing the critical value c_1 and set $N := Qn$. Moreover, let H be the connected component of the set $\{p \in \overline{G} : f^{\circ N}(p) \in \overline{G}\}$ that contains z. Pick any non-empty interval $I := [z, y) \subset \overline{H}$ such that $f^{\circ N}|_I$ is a homeomorphism and $\mathring{I}$ contains no marked point, and such that for all $p \in \overline{I}$, $p \in (z, f^{\circ N}(p))$ (this is guaranteed by Lemma 8.3 (3)).

Let ξ denote the precritical point in H of lowest Step. We claim that $\mathrm{Step}(\xi) = N$. Note that there is a $0 < j < N$ such that $0 \in f^{\circ j}(H)$: if such an iterate does not exist, then $H = \overline{G}$. But then $f^{\circ N}(G) = G$ and $0 \notin f^{\circ j}(G)$ for all $j \in \mathbb{N}$, contradicting expansivity. Therefore, there is a smallest integer $j_0 < N$ with $0 \in f^{\circ j_0}(H)$. By Lemma 4.6, $j_0 \neq kn - 1$ for $kn < N$ and thus $j_0 < N - 1$ implies that $f^{\circ(j_0+1)}(H) \ni c_1$ contains z in its interior and $f^{\circ(N-j_0-1)}(z) \in f^{\circ N}(H) \subset G$, contradicting that z is characteristic. Hence, $j_0 = N - 1$.

As a consequence, $f^{\circ N}|_{\overline{H}}$ is at most 2-to-1 and since $f^{\circ N}(z) = z$, ∂H contains at most one non-periodic N-th preimage of z, denote it by z_-. Now continuity implies that $\overline{G \setminus H}$ consists of at most one connected component. Consider the set $H' := H \setminus I$ and the continuous map

$$g : H' \longrightarrow H', p \mapsto r \circ f^{\circ N}(p),$$

where $r : \overline{G} \longrightarrow H'$ is the unique retraction. Since the tree H' has the fixed point property, there is a point $z' \in H'$ with $g(z') = z'$. If $g(z') \neq f^{\circ N}(z')$, then $z' \in \partial H' \setminus G = \{y, z_-\}$. But $g(z_-) = r \circ f^{\circ N}(z_-) = r(z) = y$ and by the choice of I, $g(y) \neq y$. Thus $g(z') = f^{\circ N}(z')$, which shows the existence of an N-periodic point in H. By minimality, $\xi \in (z, z')$ (ξ as defined above). Consequently, $\tau(z') = B^Q(\tau)$, or $\nu = \mathcal{A}_{Qn}^{-1}(\tau)$ if $z' = c_1$.

If $z' \neq c_1$, then Lemmas 8.9 and 4.6 imply that $z' \in (z, c_1)$ is characteristic.

The second possibility is that 0 is preperiodic. By minimality, there is a precritical point $\xi \in (z, c_1)$. Let $\zeta \in (z, c_1)$ be the precritical point of lowest Step. By

Missing picture here!

FIGURE 8.1. Subsets and points considered in the proof of the existence of dynamical bifurcation points. The subtree H' is drawn in thick lines.

Lemma 5.18, there is a tame characteristic point $x \in (\zeta, c_1)$. Let ν' be $\star$-periodic such that $\tau(x) = \mathcal{A}(\nu')$. Then Theorem 6.2 implies that there is a characteristic point $z_{\nu'}$ with $\tau(z_{\nu'}) = \tau$ in the tree $T(\nu')$ associated to ν'. By case (1), $(z_{\nu'}, c_1] \subset T(\nu')$ contains a characteristic periodic point $z'_{\nu'}$ with itinerary $B^Q(\tau)$ or $\nu' = \mathcal{A}^{-1}_{Qn}(\tau)$. Now we apply again Theorem 6.2 to see that there is a characteristic point $z' \in T$ with $\tau(z') = B^Q(\tau)$. Since $z, z_{\nu'}$ are of the same type and have the same number of arms, this proves the claim. $\square$

This proposition says in particular that if T contains an inner characteristic point z such that its two arms are fixed, then the next characteristic point z' has the same period as z and all its arms are permuted transitively, that is, z' is tame.

8.11. **Corollary (Precritical Point Behind Characteristic Point)**
Suppose that $z \in T$ is a characteristic point with exact period n and q arms. Then there is a precritical point $\zeta \in (z, c_1]$ such that if $N = \text{STEP}(\zeta)$, then $N = qn$ if z is tame and $N = (q-1)n$ otherwise. More precisely, if $z' \in (z, c_1]$ is the characteristic point constructed in the previous proposition, then ζ is the precritical point of lowest STEP *contained in $(z, z']$.* $\square$

The following lemma extends Theorem 6.2 (3): we drop the requirement that the point $p \in T$ has to be characteristic.

8.12. **Lemma (Forcing of Characteristic Points)**
Let (T, f) be a Hubbard tree and $p \in T$ be a periodic non-precritical point of exact period n and itinerary τ such that $[0, c_1] \cap [0, p] =: [0, b]$ is non-empty. Set $\tilde{\nu} := \mathcal{A}_n^{-1}(\tau)$ and let $(\widetilde{T}, \tilde{f})$ be the Hubbard tree associated to the kneading sequence $\tilde{\nu}$. If $z \in (0, b)$ is a characteristic point, then there is a characteristic point $\tilde{z} \in \widetilde{T}$ such that z and $\tilde{z}$ have the same itinerary and are of the same type. If additionally (z, b) contains a characteristic point, then z and $\tilde{z}$ have the same number of arms.

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FIGURE 8.2. Location of points in the proof of the modified Orbit Forcing Lemma 8.12.

PROOF. In order to tell points in the two Hubbard trees apart, all points in $\widetilde{T}$ are marked by $\sim$. We can assume that T contains both preimages $p_0, -p_0$ of p: if it does not contain the preimage, say, p_0, then we attach an arc $[0, p_0]$ to the tree T such that $[0, p_0]$ is mapped homeomorphically onto $[c_1, p]$. This extended tree is not a Hubbard tree in the strict sense anymore because not all of its endpoints are on the critical orbit. However, all other properties of Hubbard trees are preserved under this extension. We assume that $p_0 \in T_1$, that is in the component of $T \setminus \{0\}$ containing c_1. Observe that if $p \in [-p_0, p_0]$ then there is no characteristic point in $[0, b]$ and the statement is empty.

Analogously to the proof of Theorem 6.2, we define iteratively closed intervals I_k with endpoints in $P = \{-p_0, p_0, p, f(p), \ldots, f^{\circ(n-1)}(p)\}$ such that $f^{\circ k}(z) \in I_k$ and $f|_{I_k}$ is a homeomorphism. Whenever $0 \in f|_{I_k} =: [x_k, y_k]$ we chop the interval; that is, if $f^{\circ k}(z) \in [x_k, 0]$ we set $I_{k+1} = [\pm p_0, y_k]$, where $\pm p_0$ it the preimage of p such that $0 \notin I_{k+1}$. Since $z \in (0, b)$ is characteristic, $\operatorname{orb}(z) \cap [-p_0, p_0] = \emptyset$ and $f^{\circ(k+1)}(z) \in I_{k+1}$. (Contrary to the intervals defined in Theorem 6.2, it is possible that $I_{k+1} \not\subset f(I_k)$ because p is not characteristic.) Let $\tilde{I}_k \subset \widetilde{T}$ be the closed interval with endpoints in $\{0, \tilde{c}_1, \ldots, f^{\circ(n-1)}(\tilde{c}_1)\}$ which correspond to the endpoints of I_k ($0 \in \widetilde{T}$ corresponds both to p_0 and $-p_0$ in T). Since p and $\tilde{c}_1$ have the same itinerary (modulo $\star$), $f|_{\tilde{I}_k}$ is a homeomorphism onto its image and $\tilde{I}_{k+1} \subset \tilde{I}_k$ because in $\widetilde{T}$ we replace the chopped off point by 0. Therefore the set $\widetilde{S} := \{\tilde{x} \in [0, \tilde{c}_1] : \tilde{f}^{\circ k}(\tilde{x}) \in \tilde{I}_k \text{ for all } k \geq 0\}$ is a non-empty, compact (possibly degenerate) interval. Each $\tilde{x}$ in the interior of $\widetilde{S}$ has itinerary $\tau(z)$. By Lemma 3.9, $\widetilde{T}$ contains a periodic point $\tilde{z}$ with itinerary $\tau(z)$ unless $\widetilde{S} = \{\tilde{x}\}$ and $\tau(\tilde{x})$ contains a $\star$, in which case $\tilde{x} \in \operatorname{orb}(0)$ (compare the proof of Theorem 6.2). We claim that this situation cannot occur for minimal Hubbard trees. Let us assume the contrary. Then the construction of the set $\widetilde{S}$ implies that $\tilde{x}$ is the limit of precritical points. But $\tilde{x}$ is contained in the critical orbit and thus is locally attracting; in particular $\tilde{x}$ is not the limit of precritical points, a contradiction. Thus we showed the existence of a periodic point $\tilde{z} \in (0, \tilde{c}_1)$ of itinerary $\tau(z)$.

A triod argument shows that $\tilde{z}$ is characteristic. Let m be the period of $\tilde{z}$ and suppose that $\tilde{z}$ was not characteristic. Then there is an $l \leq m$ such that $f^{\circ(l-1)}(\tilde{z}) =: \tilde{z}_l$, $\tilde{z}$ and $\tilde{p} = \tilde{c}_1$ form a degenerate triod with $\tilde{z}_l$ in the middle while in the Hubbard tree T, the points z_l, z, p form a degenerate triod with z in the middle. Iterate the triods $Y, \widetilde{Y}$ under $f, \tilde{f}$ so that whenever 0 is contained in their interiors the endpoint which is chopped off is replaced by p_0 or $-p_0$ in T and 0 in $\widetilde{T}$. Since no image of z is contained in $[-p_0, p_0]$ chopping does not change the mutual location of the three points that generate the triods. This process eventually yields a contradiction to the fact that the itineraries of p, z and $\tilde{c}_1, \tilde{z}$ coincide (modulo $\star$).

The two points z and $\tilde{z}$ are of the same type because the type is completely encoded in the internal address by Lemma 5.17.

To prove the last statement of the lemma, let z be a characteristic m-periodic point such that (z, b) contains a further characteristic point y, and let q, $\tilde{q}$ be the number of arms at z, $\tilde{z}$, respectively. We are going to show that $q = \tilde{q}$. By Proposition 8.10, there are characteristic points $z^q \in (z, y] \subset (z, b) \subset T$ and $\tilde{z}^{\tilde{q}} \in \widetilde{T}$ with itinerary $B^q(\tau(z))$ and $B^{\tilde{q}}(\tau(z))$ (or $\mathcal{A}^{-1}_{\tilde{q}m}(\tau(z))$). If $q \neq \tilde{q}$, then there is a characteristic point $\tilde{z}^q \in \widetilde{T}$ with itinerary $B^q(\tau(z))$ as we just proved and $\tilde{z}^q \in (\tilde{z}, \tilde{c}_1)$. Let us assume that $\tilde{z}^q \in (\tilde{z}, \tilde{z}^{\tilde{q}})$, the case that $\tilde{z}^{\tilde{q}} \in (\tilde{z}, \tilde{z}^q)$ works exactly the same way. The precritical point $\zeta_{qm} \in [\tilde{z}, \tilde{z}^q]$ of smallest Step has $\textsc{Step}(\zeta_{qm}) = qm$, and the one in $[\tilde{z}, \tilde{z}^{\tilde{q}}]$, denote it by $\zeta_{\tilde{q}m}$, has $\textsc{Step}(\zeta_{\tilde{q}m}) = \tilde{q}m$. Therefore we must have that $q > \tilde{q}$ and $\zeta_{\tilde{q}m} \in (\tilde{z}^q, \tilde{z}^{\tilde{q}})$. By minimality, we have for all $\tilde{x} \in (\tilde{z}, \zeta_{\tilde{q}m})$ that $\tilde{x} \in (\tilde{z}, f^{\circ\tilde{q}m}(\tilde{x})) \subset (\tilde{z}, \tilde{c}_1)$; this is in particular true for $\tilde{z}^q$, which consequently cannot be characteristic. □

Remark. Let ν' be $\star$-periodic. If T contains no periodic point p with itinerary $\mathcal{A}(\nu')$, then we can extend T to a tree T' such as to contain the orbit of p (see Corollary 20.12). The claim of Lemma 8.12 also holds in this situation. Observe first that there are several possibilities for the location of p in T':

If $p \in T_0$ then the statement is empty. If p is contained in a subtree branching off from $(0, c_1)$, then we have exactly the same situation as in Lemma 8.12. The last possibility is that $c_1 \in (0, p)$. In this case, $0 \in T'$ is a branch point and $f|_{f(I_k)}$ might be injective although $0 \in f(I_k)$. As a consequence, the criterion for chopping is now that $f|_{f(I_k)}$ is not injective so that there might be intervals I_k which contain the critical point 0. (This always holds if we have to chop the interval $f(I_{k-1})$ to get I_k.) So the proof of Lemma 8.12 carries over to all characteristic points $z \in (0, c_1)$, and thus Lemma 8.12 extends to the case of $c_1 \in (0, p)$.

8.13. **Lemma (Characteristic Points Not Forced)**
Let (T, f) be a Hubbard Tree and define $p, b \in T$ and $\tilde{\nu}$ as in Lemma 8.12. Suppose that $z \in (b, c_1)$ is a characteristic point such that either there is a characteristic point

$y \neq p$ in $[b, z)$ or b is the limit of characteristic points in $(0, b)$. Then there is no characteristic point $\tilde{z}$ with itinerary $\tau(z)$ in the Hubbard tree $\widetilde{T}$ of $\tilde{\nu}$.

PROOF. We state the proof for the case that there is a characteristic point $y \in [b, z)$. For the reasoning of the second case we refer to the footnotes. By way of contradiction, we assume that there is a characteristic point $\tilde{z} \in \widetilde{T}$ such that $\tau(z) = \tau(\tilde{z})$.

The triod $[p, y, z] \subset T$ is degenerate and contains y in the middle. The point $\tilde{z} \in \widetilde{T}$ forces a characteristic point $\tilde{y} \in \widetilde{T}$ with itinerary $\tau(\tilde{y}) = \tau(y)$ by Theorem 6.2. Thus the triod $[\tilde{y}, \tilde{z}, \tilde{p}] \subset \widetilde{T}$ is degenerate with $\tilde{z}$ in the middle (where $\tilde{p} = \tilde{c}_1$).[13] Iterate the triods so that a chopped off point in T is replaced by the appropriate preimage $\pm z_0$ of z and a chopped off point in $\widetilde{T}$ by $\pm\tilde{z}_0$. There is a iterate k where the images of y, z (under the chopping map) are separated by 0. At this moment, the images of p, z are separated by 0 while the images $\tilde{p}, \tilde{z}$ are not, contradicting that corresponding points have equal itineraries (modulo $\star$). Note that chopping causes no problems because no image of the characteristic point y is contained in $[-z_0, z_0]$[14]. □

8.14. **Lemma (Periodic Points Behind c_1)**
Suppose (T, f) and $(\widetilde{T}, \tilde{f})$ are two Hubbard trees that generate the $\star$-periodic kneading sequences $\nu \neq \tilde{\nu}$. If necessary, extend T such as to contain a periodic point p with itinerary $\mathcal{A}(\tilde{\nu})$ and $\widetilde{T}$ such as to contain a periodic point $\tilde{q}$ with $\tau(\tilde{q}) = \mathcal{A}(\nu)$. Then it is not possible that both $c_1 \in (0, p)$ and $\tilde{c}_1 \in (0, \tilde{q})$.

PROOF. By way of contradiction assume that $c_1 \in (0, p)$ and $\tilde{c}_1 \in (0, \tilde{q})$. By Lemma 8.6 (3), there is a characteristic point $z \in (0, c_1) \subset T$ with itinerary $\overline{\mathcal{A}}(\nu)$ and by the remark after Lemma 8.12, p forces a characteristic point $\tilde{z} \in (0, \tilde{c}_1) \subset \widetilde{T}$ with $\tau(\tilde{z}) = \overline{\mathcal{A}}(\nu)$. Thus, if n equals the period of ν, then the precritical point $\tilde{\zeta}$ of lowest step in $(\tilde{z}, \tilde{q})$ has $\text{Step}(\tilde{\zeta}) = n$. Therefore $\tilde{f}^{\circ n}$ maps $[\tilde{z}, \tilde{\zeta}]$ homeomorphically onto $[\tilde{z}, \tilde{c}_1]$ and $[\tilde{\zeta}, \tilde{q}]$ homeomorphically onto $[\tilde{c}_1, \tilde{q}]$. If $\tilde{c}_1 \in (\tilde{z}, \tilde{\zeta})$, then $\tilde{f}^{\circ n}(\tilde{c}_1) \in (\tilde{c}_1, \tilde{z})$ and $|\text{orb}(\tilde{c}_1)| = \infty$. Otherwise, since $\tilde{\nu} \neq \nu$ and thus $\tilde{c}_1 \neq \zeta$, $\tilde{c}_1 \in (\tilde{\zeta}, \tilde{q})$ and $\tilde{f}^{\circ n}(\tilde{c}_1) \in (\tilde{c}_1, \tilde{q})$. But this contradicts the fact that $\text{orb}(\tilde{c}_1)$ spans the (not extended) Hubbard tree T.□

8.15. **Definition (Combinatorial Arc)**
For two pre- or $\star$-periodic kneading sequences $\nu \leq \nu'$, we define

$$[\nu, \nu'] := \{\mu \in \Sigma^\star : \mu \textit{ pre- or } \star\textit{-periodic such that } \nu \leq \mu \leq \nu'\}.$$

We call $[\nu, \nu']$ the combinatorial arc *between ν and ν'.*

[13] If there is no characteristic point in $[b, z)$, we set $y = b$. Any characteristic point in $[0, b) \subset T$ is forced in $\widetilde{T}$ by Lemma 8.12. Hence there is a limit point $\tilde{b} \in \widetilde{T}$, which is either the critical value or has the same itinerary as b. Because of the existence of $\tilde{z}$, the second case must hold. We consider the degenerate triod $[\tilde{b}, \tilde{z}, \tilde{p}]$.

[14] Since b is the limit of characteristic points it follows that no image of b is contained in $[-z_0, z_0]$.

Missing picture here!

FIGURE 8.3. The extended Hubbard tree $\widetilde{T}$ with the periodic points $\tilde{z}, \tilde{q}$, where $\tau(\tilde{z}) = \overline{\mathcal{A}}\nu$ and $\tau(\tilde{q}) = \mathcal{A}(\nu)$.

8.16. **Theorem (Branch Theorem)**
Let ν and $\tilde{\nu}$ be two $\star$-periodic kneading sequences. Then there is a unique kneading sequence μ such that exactly one of the following holds:

(1) $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = [\overline{\star}, \mu]$, *where μ is either $\star$-periodic or preperiodic.*
(2) $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = [\overline{\star}, \mu] \setminus \{\mu\}$, *where μ is $\star$-periodic such that the exact period of μ and $\overline{\mathcal{A}}(\mu)$ are equal.*

In both cases, it is possible that $\mu = \nu$ or $\mu = \tilde{\nu}$. Observe that the case (2) can only occur if at least one of the two given kneading sequences ν and $\tilde{\nu}$ is non-admissible.

PROOF. Note first that $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = \{\mu' \star\text{- or preperiodic: } \mu' \leq \nu \text{ and } \mu' \leq \tilde{\nu}\}$. Thus finding the kneading sequence μ of the Branch Theorem is equivalent to finding the supremum of the set $\{\mu' \star\text{-periodic: } \mu' \leq \nu \text{ and } \mu' \leq \tilde{\nu}\}$. Let T be the Hubbard tree of ν and $\widetilde{T}$ the one of $\tilde{\nu}$.

If $\nu = \tilde{\nu}$, there is nothing to show. If T (or $\widetilde{T}$) contains a characteristic point with itinerary $\mathcal{A}(\tilde{\nu})$ (or $\mathcal{A}(\nu)$), then $\nu > \tilde{\nu}$ (or $\tilde{\nu} > \nu$), and $\mu := \tilde{\nu}$ (or $\mu := \nu$) satisfies the theorem. This is in particular true if one of the given kneading sequences equals $\overline{\star}$. For the remaining cases we can assume that T contains a periodic point p with itinerary $\mathcal{A}(\tilde{\nu})$ (by possibly enlarging T) and by Lemma 8.14 that $c_1 \notin [0, p]$ (interchange T and $\widetilde{T}$ if necessary). Let $b \in T$ be such that $[0, p] \cap [0, c_1] = [0, b]$. The point b is either a branch point or equals p. Let

$$P := \{z \in [0, b) \, : z \text{ is a characteristic point}\}.$$

Suppose that there is a kneading sequence $\nu' = \overline{1 \dots 1\star}$ such that $\nu' < \nu$ and $\nu' < \tilde{\nu}$ (otherwise $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = \{\overline{\star}\}$). Then by Lemma 8.13, P contains the α-fixed point and hence is not empty. Set $a := \sup(P)$. We distinguish whether $a = b$ or not:

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FIGURE 8.4. Location of points in the proof of the Branch Theorem 8.16.

- Let us first assume that $a = b$. Since b is the limit of characteristic points, the local arm L of b pointing towards 0 is fixed and all iterates of b are contained in the closure of the global arm associated to L. So if b is periodic then it is a non-tame characteristic point. Lemma **??** implies that $\tau(b) \neq \mathcal{A}(\widetilde{\nu}) = \tau(p)$ and $b \neq p$. Thus b is a preperiodic or periodic branch point, and in the latter case, b is characteristic and evil.

  We first consider the case that b is preperiodic. Let $\mu := \tau(b)$ be the preperiodic kneading sequence generated by b. We claim that $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = [\overline{\star}, \mu]$. Let $1 \to S_1 \to \cdots \to S_k \to \cdots$ be the (infinite) internal address of μ, and let μ^k be the unique $\star$-periodic sequence associated to $1 \to S_1 \to \cdots \to S_k$ for each $k > 0$. We claim that b is the limit point of characteristic points $p_{S_k} \in (0, b)$ with itinerary $\tau(p_{S_k}) = \mathcal{A}(\mu^k)$. By assumption, b is the limit point of a sequence of characteristic points $z_k \in (0, b)$ of period n_k. We can assume that for all $k \geq l$, the itineraries of z_k have the same first n_l entries as μ. Fix an entry S_k of the internal address of μ. Then there are $n_{k_1}, n_{k_2} \in \mathbb{N}$ such that $n_{k_1} < S_k \leq n_{k_2}$. Since the itinerary τ_{k_2} of z_{k_2} coincides with μ for the first n_{k_2} entries, S_k is contained in the internal address of z_{k_2}. Then by Proposition 6.8, there is a characteristic point p'_{S_k} with internal address $1 \to S_1 \to \cdots \to S_k$ in the Hubbard tree of $\mathcal{A}^{-1}_{n_{k_2}}(\tau(z_{k_2}))$. This forces a characteristic point $p_{S_k} \in T$ with $\tau(p_{S_k}) = \mathcal{A}(\mu^k)$. By induction on k, we get a sequence of characteristic points $p_{S_k} \in T$, which has to converge to b because $p_{S_k} \in [z_{k_1}, z_{k_2}]$ for all k. Since for all $k > 0$, T contains a characteristic point with itinerary $\mathcal{A}(\mu^k)$ and since these points are contained in the arc $[0, b)$, we have $\nu > \mu^k$, $\tilde{\nu} > \mu^k$ for all $k > 0$ and thus, $\nu > \mu$ and $\tilde{\nu} > \mu$.

  To finish this case, it remains to prove that μ is the largest sequence with these properties. If μ' is $\star$-periodic such that $\nu > \mu'$ and $\tilde{\nu} > \mu'$, then there is a characteristic point $q' \neq b$ with itinerary $\mathcal{A}(\mu')$. Since b is the limit of characteristic points, Lemma 8.13 implies that $q' \in [0, b)$, and since $p_{S_k} \to b$, there is a k_0 such that $q' \in (0, p_{S_{k_0}})$. By Theorem 6.2, the Hubbard tree of μ^{k_0} contains a characteristic point with itinerary $\mathcal{A}(\mu')$ and hence $\mu' < \mu^{k_0} < \mu$.

Now suppose that b is periodic. Let $\mu = \mathcal{A}^{-1}(\tau(b))$. Then the exact periods of $\overline{\mathcal{A}}(\mu)$ and μ coincide, and since b is an evil branch point, $\tau(b) = \overline{\mathcal{A}}(\mu)$. Then $\mu \not< \nu$ but all $\mu' < \mu$ satisfy $\mu' < \nu$ by Proposition 6.7. We know that b is the limit point of characteristic points $z_n \in P$. We may assume that $0 < z_n < z_{n+1} < b$. Hence, there is a sequence of corresponding characteristic points $\tilde{z}_n \in \widetilde{T}$. By compactness of the Hubbard tree and since $0 < \tilde{z}_n < \tilde{z}_{n+1} < \tilde{c}_1$, this sequence has to converge to a point $\tilde{b} \in (0, \tilde{c}_1) \subset \widetilde{T}$ ($\tilde{b} \neq \tilde{c}_1$ by Lemma 8.3 (2)). Since $\tau(\tilde{z}_n) \to \overline{\mathcal{A}}(\mu)$, the itinerary of $\tilde{b}$ is $\overline{\mathcal{A}}(\mu)$ by continuity. We claim that $\tilde{b}$ is a characteristic point. Since $\tilde{b}$ is the limit of characteristic points, all iterates $f^{\circ i}(\tilde{b})$ are contained in the closure of the global arm of $\tilde{b}$ containing 0. Let m be the period of μ. If $\tilde{b}$ was not m-periodic, then there is a $k \in \mathbb{N}$ such that $\tilde{z}_k \in (f^{\circ m}(\tilde{b}), \tilde{b})$ and $\tau(\tilde{z}_k) = \tau(\tilde{b}) = \overline{\mathcal{A}}(\mu)$, which is not true.

We claim that either $\mu \not< \tilde{\nu}$ and $\mu' < \nu$ for all $\mu' < \mu$ or $\mu \leq \tilde{\nu}$, so that no sequence smaller than μ is the supremum of $\{\mu' \ \star\text{-periodic: } \mu' \leq \nu \text{ and } \mu' \leq \tilde{\nu}\}$. If $\tilde{b}$ is an inner point, then Proposition 8.10 guarantees the existence of a characteristic point with itinerary $\mathcal{A}(\mu)$ or μ and hence $\mu \leq \tilde{\nu}$. If $\tilde{b}$ is an evil branch point, then again by Proposition 6.7, $\mu \not< \tilde{\nu}$ and $\mu' < \tilde{\nu}$ for all $\mu' < \mu$.

There is no $\mu' \not< \mu$ such that both $\mu' < \nu$ and $\mu' < \tilde{\nu}$. Indeed, if $\mu' < \nu$ and $\mu' \not< \mu$, then there is a characteristic periodic point $z \in (b, c_1] \subset T$ with itinerary $\mathcal{A}(\mu')$. Since b is characteristic itself, Lemma 8.13 says that there is no characteristic point in $\widetilde{T}$ with itinerary $\tau(z) = \mathcal{A}(\mu')$, and so $\mu' \not< \tilde{\nu}$. Consequently, μ satisfies the theorem.

- The second possibility is that $a \in (0, b)$. We claim that in this case $a \in P$. To show this observe first that the arc (a, b) contains a precritical point because a and b have different itineraries. Let I be the set of points in $[0, b]$ with itinerary $\tau(b)$. The reasoning of Lemma 8.6 (1) can be applied to show that $I = (\zeta, b]$ where ζ is a precritical point. So, if $a \in I$, then there is an open neighborhood of a which is also contained in I and hence $P \cap I \neq \emptyset$, a contradiction to minimality.

  Suppose that there are infinitely many points in P (otherwise $a \in P$ trivially holds). Let ζ be the precritical point in (a, b) such that there is no precritical point in (a, b) with lower Step and set $k := \text{Step}(\zeta)$. Since a is a limit point of characteristic points, we have that for all $l \in \mathbb{N}$, $f^{\circ l}(a) \notin (a, c_1]$. Hence, $f^{\circ k}([a, \zeta])$ covers $[a, \zeta]$ homeomorphically.

Therefore, there is a $z \in [a, \zeta)$ with $z = f^{\circ k}(z)$. If $z = a$, we are done. Otherwise, let $w \in [a, \zeta]$ be the periodic point with lowest period. If $w \neq a$ (otherwise we are done), then the point w is characteristic. If it is not, then there is an $s < k$ such that $f^{\circ s}([a, w])$ covers $[a, w]$ homeomorphically. But this yields a periodic point $\tilde{w} \in (a, w)$ of period smaller than the one of w, a contradiction to the choice of w. Hence $w \in P$ and $a \neq \sup(P)$, which is false. This proves that a is periodic and as a limit point of characteristic points it is characteristic itself.

Let τ be the itinerary of a, n be its period and let μ' be the $\star$-periodic kneading sequence such that $\tau = \mathcal{A}(\mu')$ or $\tau = \overline{\mathcal{A}}(\mu')$ according as a is tame or not. Let q be the number of arms at a and Q be the period of the local arm at a pointing to c_1. By Lemma 8.12, there is a characteristic point $\tilde{a} \in \widetilde{T}$ which has itinerary τ and is of the same type as a. Let $\tilde{q}$ be the number of arms at $\tilde{a}$ and $\tilde{Q}$ the period of the local arm at $\tilde{a}$ pointing to $\tilde{c}_1$. Then $q = \tilde{q}$ if and only if $Q = \tilde{Q}$. Proposition 8.10 implies that T contains a characteristic point x with itinerary $B^Q(\tau)$ (it is not possible that $x = c_1$) and $\widetilde{T}$ contains a characteristic point $\tilde{x}$ that has itinerary $B^{\tilde{Q}}(\tau)$ or $\mathcal{A}^{-1}_{\tilde{Q}n}(\tau) = B^{\tilde{Q}}(\mu')$. In the last paragraph of the proof of Lemma 8.12, we have seen that a Hubbard tree cannot contain two characteristic points with itinerary $B^Q(\tau)$ and $B^{\tilde{Q}}(\tau)$ for $Q \neq \tilde{Q}$.

Since there is no characteristic point in (a, x), it follows that $x \in [b, c_1)$. Therefore Lemma 8.13 yields that if $q \neq \tilde{q}$ and μ'' is $\star$-periodic with $\mu'' < \nu$ and $\mu'' < \tilde{\nu}$, then $\mu'' \leq \mu'$. If $\tau = \mathcal{A}(\mu')$, then $\mu' < \nu$ and $\mu' < \tilde{\nu}$. If $\tau = \overline{\mathcal{A}}(\mu')$, then the exact period of μ' and $\overline{\mathcal{A}}(\mu')$ are equal. If a (or $\tilde{a}$) is an inner point, then $\mu' < \nu$ ($\mu' < \tilde{\nu}$). If a is a branch point then $\mu' \not< \nu$ and $\mu'' < \nu$ for all $\mu'' < \mu'$. Similarly, if $\tilde{a}$ is a branch point, then $\mu' \not< \tilde{\nu}$ but $\mu'' < \tilde{\nu}$ for all $\mu'' < \mu'$. In all cases, $\mu = \mu'$ is the claimed kneading sequence.

If $q = \tilde{q}$, then $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = [\overline{\star}, B^Q(\mu')]$, so that $\mu = B^Q(\mu')$ satisfies the theorem. Let μ'' be $\star$-periodic such that $\mu'' < \nu$ and $\mu'' \not\leq B^Q(\mu')$. Then there is a characteristic point $y \in T$ with $\tau(y) = \mathcal{A}(\mu'')$ and $y \in (x, c_1)$. Since $p \neq x$, Lemma 8.13 yields that there is no characteristic point in $\widetilde{T}$ with itinerary $\tau(y)$ and thus, $\mu'' \not< \tilde{\nu}$. $\square$

Remark. The statement of Theorem 8.16 extends to all kneading sequences in $\Sigma^{\star\star}$ by continuity.

Douady and Hubbard show in [**DH1**] that branching in the Mandelbrot set can only occur at postcritically finite parameters. Since admissible kneading sequences form a combinatorial model for the Mandelbrot set (compare Section **??**), Theorem 8.16 extends Douady's and Hubbard's result to the space $\Sigma^{\star\star}$, which comprises all admissible and non-admissible kneading sequences that are $\star$-periodic or non-periodic.

In the remark after Definition 7.1 the order "$<$" on $\Sigma^{\star\star}$ is extended to the set $\Sigma^{\star}$ of all kneading sequences. For $\Sigma^{\star}$, the branch theorem reads as follows: *Given any two kneading sequences ν, ν' then either $\nu \le \nu'$ or $\nu' \le \nu$ or there is a largest third kneading sequence μ such that $\nu > \mu$ and $\nu' > \mu$ and μ is either preperiodic or periodic but not $\star$-periodic.* In particular, in item (2) of Theorem 8.16, the branch point now exists and equals the lower kneading sequence $\overline{\mathcal{A}}(\mu)$ of μ, where μ is such that the exact periods of $\overline{\mathcal{A}}(\mu)$ and μ coincide. Figure 6.2 illustrates the local structure near a $\star$-periodic kneading sequence.

Remark. In the remark after Proposition 7.9, for any non-periodic kneading sequence ν its "branches" are defined as equivalence classes, where two kneading sequences μ, μ' are equivalent if the following holds: let $\nu(s)$ denote the $\star$-periodic kneading sequence whose $(s-1)$-st symbols coincide with those of ν. Then $\mu \equiv \mu'$ if there are s, s' such that $\mu > \nu(s) > \nu$ and $\mu' > \nu(s') > \nu$ and there are $m, m' > 0$ with $\nu(\rho^{\circ m}(s) < \nu(s')$ and $\nu(\rho^{\circ m'}(s')) < \nu(s)$. Let us denote this branches by *combinatorial branches.*

8.17. **Proposition** (Title!)
Let $\nu, \mu, \mu' \in \Sigma^{\star\star}$. Then μ, μ' lie in two different combinatorial branches of ν that are both unequal to the one containing the origin if and only if μ and μ' cannot be compared (with respect to $<$ on $\Sigma^{\star\star}$) and ν is the branch point of μ, μ' as defined via (the extension to $\Sigma^{\star\star}$ of) Theorem 8.16.

Proof. Since non-periodic kneading sequences can be approximated arbitrarily closely by $\star$-periodic ones, we can assume that μ, μ' are $\star$-periodic.

Let us first consider the case that ν is the branch point of μ, μ'. Suppose that μ, μ' were contained in the same combinatorial branch. Thus there is an t such that $\nu < \nu(t) \le \mu, \mu'$.

If ν is $\star$-periodic, then there are s, s' such that $\nu < \nu(s) \le \mu$ and $\nu < \nu(s') \le \mu'$ (s, s' are distinct multiples of the period of ν). Since there is no element of $\Sigma^{\star\star}$ contained in $]\nu, \nu(s)[$, it follows that $\nu(s) \le \nu(t)$ (recall that the set of sequences smaller than a given one is linearly ordered), and thus $\nu(s) \le \mu'$ and $\mu' \not> \nu(s')$, contradicting our assumption.

If ν is preperiodic, then $\nu < \nu(\rho(t)) < \nu(t)$ and there is a sequence of $\star$-periodic kneading sequences $\tau^n < \nu$ converging to ν. Consider the Hubbard tree associated to μ and extend it such as to contain a periodic point p' with itinerary $\mathcal{A}(\mu')$. Then there is a $b \in T$ such that $]0, c_1] \cap]0, p'] =]0, b]$. Let z_ν be the characteristic point in T with itinerary $\mathcal{A}(\nu)$, and $z, \hat{z}$ the characteristic points with itineraries $\mathcal{A}(\nu(t))$, $\mathcal{A}\nu(\rho(t))$. We are going to show that the $\hat{z} \in]z_\nu, b[$ and thus by Lemma 8.12, ν is not the branch point of μ, μ'. If $b = c_1$ this statement is trivial, otherwise suppose that $\hat{z} \in [b, c_1]$. Then Lemma **??** implies that there is no characteristic point z' in the Hubbard tree of μ' with itinerary $\mathcal{A}(\nu(t))$ and thus $\nu(t) \not< \mu'$, contradicting our assumption again.

To prove the other direction, suppose that there is a $\nu' > \nu$ such that $\nu' < \mu$ and $\nu' < \mu'$. If ν is $\star$-periodic, then the existence of ν' implies that there is a t such that $\nu(t) \leq \mu, \mu'$ and thus μ and μ' are in the same combinatorial branch of ν. If ν is preperiodic then there is a t such that $\mu > \nu(t) > \nu(\rho^{\circ i}(t)) > \nu(\rho^{\circ(i+1)}(t)) > \nu$ for all $i > 0$. Since all sequences smaller than a given one are totally ordered, there is a j such that $\nu' > \nu(\rho^{\circ j}(t))$. Pick $s := \rho^{\circ j}(t)$, $s' := \rho^{\circ(j+1)}(t)$ and $m := 2$, $m' = 1$. Then $\mu > \nu(s) > \nu(\rho^{m'}(s'))$ and $\mu' > \nu(s') > \nu(\rho^{\circ m}(s))$ and thus by definition, μ, μ' are in the same combinatorial branch. $\square$

It seems that sometimes we denote open intervals by (a, b) and sometimes by $]a, b[$. We need to check for consistency! Also: several references don't work or lemmas don't exist. Changed use of macros: $\overline{\mathcal{A}}$, STEP

IV. The Mandelbrot Set and Algorithms

9. The Mandelbrot Set

In this section, we review and prove some important results about combinatorics and topology of the Mandelbrot set. Most of these results are "folklore" and go back to Douady and Hubbard and their Orsay Notes [**DH1**] at least implicitly; other useful references include [**Bra, Mi2, Sch2, Eb, Sch3**]. However, a number of these results are difficult to find with proofs in the literature, even though many of them are "well known among those who know them well". Therefore we try to provide complete proofs for all results that have not yet been formally published.

9.1. Rays and Ray Pairs. For every parameter $c \in \mathbb{C}$, iteration of the polynomial $p_c(z) = z^2 + c$ defines a dynamical system, and every quadratic polynomial is affinely conjugate to a unique p_c. The *filled-in Julia set* K_c is defined as

$$K_c = \{z \in \mathbb{C}\colon \text{ the orbit of } z \text{ under iteration of } p_c \text{ is bounded}\}\ ;$$

we also define the Julia set $J_c := \partial K_c$. The Mandelbrot set $\mathcal{M}$ is defined as the set of parameters c for which the J_c or equivalently K_c are connected. For these parameters, there is a preferred conformal isomorphism $\Phi_c\colon \overline{\mathbb{C}} \setminus K_c \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with $\Phi_c(\infty) = \infty$ and $\lim_{z\to\infty} \Phi_c(z)/z = 1$ [**DH1, Mi1**]. The map Φ_c conjugates the dynamics of p_c on $\overline{\mathbb{C}} \setminus K_c$ to the dynamics of $z \mapsto z^2$ on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$: we have

$$\Phi_c(p_c(z)) = (\Phi_c(z))^2 \qquad \text{on} \qquad \overline{\mathbb{C}} \setminus K_c\ . \tag{4}$$

We define *dynamic rays* $R_c(\vartheta) := \Phi_c^{-1}\left(e^{2\pi i\vartheta}(1,\infty)\right)$ for $\vartheta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ (Figure 9.1). Equation (4) implies that $p_c(R_c(\vartheta)) = R_c(2\vartheta)$: the dynamic ray $R_c(\vartheta)$ is periodic (as a set) under p_c if and only if ϑ is periodic (as an element of $\mathbb{S}^1$) under multiplication by 2, i.e., ϑ has the form $a/(2^n - 1)$ for some positive integers a and n.

A dynamic ray $R_c(\vartheta)$ *lands* at a point $z \in \mathbb{C}$ if $\lim_{r\searrow 1} \Phi_c^{-1}\left(re^{2\pi i\vartheta}\right)$ exists and equals z. If a ray $R_c(\vartheta)$ lands at $z \in K_c$, then $R_c(2\vartheta)$ lands at $p_c(z)$, and conversely: if $R_c(2\vartheta)$ lands at $p_c(z)$, then $R_c(\vartheta)$ lands at one of the two p_c-preimages of $p_c(z)$, i.e. at z or at $-z$. If two dynamic rays $R_c(\vartheta)$ and $R_c(\vartheta')$ land at a common point z, we call $P_c(\vartheta,\vartheta') := R_c(\vartheta) \cup R_c(\vartheta') \cup \{z\}$ a *dynamic ray pair*. Then $\mathbb{C} \setminus P_c(\vartheta,\vartheta')$ consists of two components, say X_1 and X_2, and we say that the ray pair *separates* two points if one of them is in X_1 and the other in X_2.

If $c \notin \mathcal{M}$, there is still a preferred conformal isomorphism Φ_c in a neighborhood of ∞ satisfying (4): more precisely, there is a neighborhood $U(c)$ of ∞ containing c on which Φ_c is defined so that $\Phi_c(U(c)) = \{z \in \overline{\mathbb{C}}\colon |z| > r\}$ for some $r > 1$. The map Φ_c allows to define restrictions of dynamic rays within $U(c)$, and these can be extended by pulling back under the dynamics unless the critical value interferes: $\Phi_c^{-1}(e^{2\pi i 2\vartheta}(r,\vartheta))$

[1] The angles are, in counterclockwise order, $\frac{1}{32}$, $\frac{1}{16}$, $\frac{17}{240}$, $\frac{19}{240}$, $\frac{23}{240}$, $\frac{1}{8}$, $\frac{31}{240}$, $\frac{9}{56}$, $\frac{1}{6}$, $\frac{11}{56}$, $\frac{5}{24}$, $\frac{1}{4}$, $\frac{15}{56}$, $\frac{289}{992}$, $\frac{297}{992}$, $\frac{299}{992}$, $\frac{315}{992}$, $\frac{319}{992}$, $\frac{43}{128}$, $\frac{11}{32}$, $\frac{3}{8}$, $\frac{13}{32}$, $\frac{7}{16}$ (upper half), $\frac{16}{31}$, $\frac{8}{15}$, $\frac{17}{31}$, $\frac{4}{7}$, $\frac{18}{31}$, $\frac{9}{15}$, $\frac{38}{63}$, $\frac{19}{31}$, $\frac{20}{31}$, $\frac{41}{63}$, $\frac{2}{3}$, $\frac{21}{31}$, $\frac{22}{31}$, $\frac{5}{7}$, $\frac{46}{63}$, $\frac{11}{15}$, $\frac{23}{31}$, $\frac{24}{31}$, $\frac{12}{15}$, $\frac{23}{31}$, $\frac{26}{31}$, $\frac{53}{63}$, $\frac{6}{7}$, $\frac{13}{15}$, $\frac{27}{31}$, $\frac{28}{31}$, $\frac{14}{15}$, $\frac{29}{31}$, $\frac{59}{63}$, $\frac{60}{63}$, $\frac{30}{31}$, $\frac{61}{63}$, $\frac{62}{63}$ (lower half).

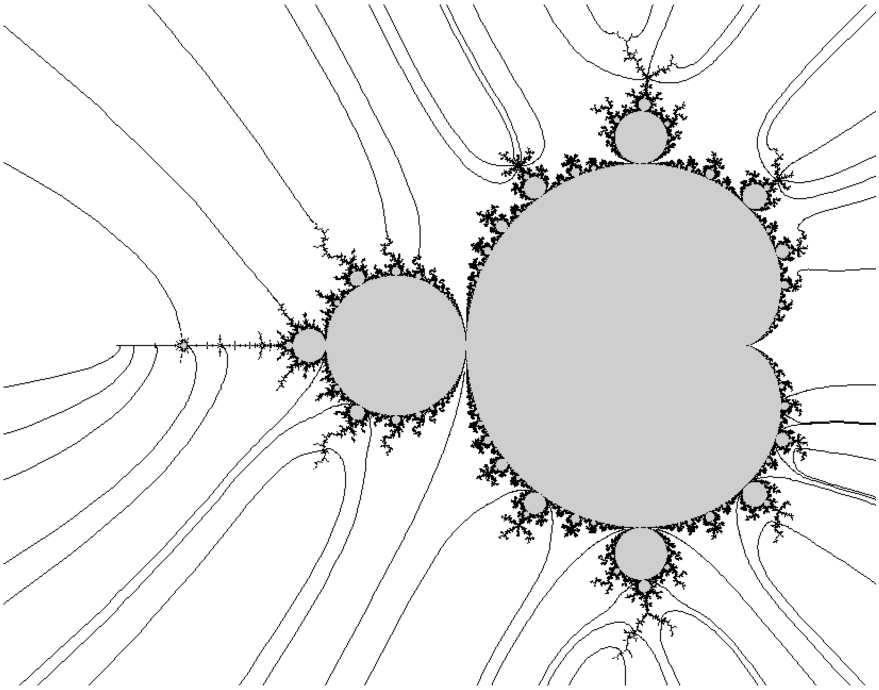

Figure 9.1. The Mandelbrot set with a number of parameter rays drawn in: the 23 parameter rays in the upper half are at preperiodic angles, the 33 rays in the lower half are at periodic angles[1].

is well-defined for sufficiently large $r > 1$ and defines $R_c(2\vartheta)$ near infinity; if it does not contain the critical value, then $p_c^{-1}(\Phi_c^{-1}(e^{2\pi i 2\vartheta}(r, \vartheta)))$ consists of two connected components; one of them contains $\Phi_c^{-1}(e^{2\pi i\vartheta}(r, \vartheta))$ and thus extends the definition of $R_c(\vartheta)$. Countable repetition of the argument defines $R_c(\vartheta)$ for all $\vartheta \in \mathbb{S}^1$ with at most countably many exceptions (compare [**GM**, Appendix A] and Section **??**).

It is well known [**Mi1**] that for p_c with $c \in \mathcal{M}$ every dynamic ray at a periodic angle lands at a periodic point which is on a repelling or parabolic orbit; and conversely, every repelling or parabolic periodic point is the landing point of at least one periodic dynamic rays. If $c \notin \mathcal{M}$, then the dynamic ray at any periodic angle ϑ either fails to exist (because $c = \Phi_c^{-1}(e^{2\pi i 2^k \vartheta})$ for some $k \geq 1$), or it lands at a repelling periodic point (the accumulation set of the ray is a connected subset of the Julia set, and the latter is totally disconnected). All rays landing at the same periodic point have the same period: the common period of the rays is a (possibly proper) multiple of the period of their landing point; therefore one distinguishes the *ray period* from the *orbit period.* As a consequence, only finitely many rays can land at any periodic point. Analogous results follow for preperiodic rays and preperiodic points on repelling or parabolic orbits.

A dynamic ray pair $P_c(\vartheta,\vartheta')$ is called *characteristic* if it separates the critical value from all rays $R_c(2^k\vartheta)$ and $R_c(2^k\vartheta')$ for all $k\geq 1$ (except of course from those on the ray pair $P_c(\vartheta,\vartheta')$ itself)[2]. Every cycle of periodic ray pairs has a unique characteristic ray pair with angles in $\bigcup_{k\geq 0}\{2^k\vartheta, 2^k\vartheta'\}$ (see Lemma 9.3 below).

As described earlier, $\Phi_c(c)$ is defined for every $c\in\mathbb{C}\setminus\mathcal{M}$ (because $c\in U(c)$). This defines a map $\Phi\colon\mathbb{C}\setminus\mathcal{M}\to\mathbb{C}\setminus\overline{\mathbb{D}}$ via $\Phi(c):=\Phi_c(c)$. Douady and Hubbard [**DH1, DH0**] showed that $\Phi\colon\mathbb{C}\setminus\mathcal{M}\to\mathbb{C}\setminus\overline{\mathbb{D}}$ is a conformal isomorphism. This implies that $\mathcal{M}$ is connected, and it gives rise to *parameter rays* of the Mandelbrot set via $R(\vartheta):=\Phi^{-1}\left(e^{2\pi i\vartheta}(1,\infty)\right)$ for $\vartheta\in\mathbb{S}^1$. If two parameter rays $R(\vartheta)$ and $R(\vartheta')$ land at a common parameter $c\in\mathcal{M}$, then we say that $P(\vartheta,\vartheta')=R(\vartheta)\cup R(\vartheta')\cup\{c\}$ is a *parameter ray pair*. It will be convenient to say that the parameter rays $R(0)$ and $R(1)$ form a ray pair as well (although $0=1$ in $\mathbb{S}^1$). As above, we say that a parameter ray pair $P(\vartheta,\vartheta')$ separates two points $c,c'\in\mathbb{C}$ if c and c' are in different components of $\mathbb{C}\setminus P(\vartheta,\vartheta')$.

The following theorem is one of the fundamental results of Douady and Hubbard. It is the basis of all results on the combinatorial structure of the Mandelbrot set.

9.1. **Theorem (Parameter Ray Pairs at Periodic Angles)**
For every period n, the parameter rays of the Mandelbrot set at angles with exact period n land in pairs at parabolic parameters. Every parabolic parameter is the landing point of exactly two parameter rays at periodic angles, and both angles have equal period.[3] *No parameter ray at non-periodic angle lands at a parabolic parameter.*

Suppose $R(\vartheta)$ and $R(\vartheta')$ are the two parameter rays landing at a parabolic parameter c, where ϑ and ϑ' are periodic angles of the same period n. Then the dynamic rays $R_c(\vartheta)$ and $R_c(\vartheta')$ land together at a point of the parabolic orbit of p_c and form a characteristic ray pair. Conversely, if p_c has a parabolic orbit, then at least two dynamic rays land at each point of the parabolic orbit[1]*; if $P_c(\vartheta,\vartheta')$ is the characteristic ray pair of the dynamic rays landing at the parabolic orbit, then the parameter rays $R(\vartheta)$ and $R(\vartheta')$ both land at c.* □

Most of this was shown in the Orsay Notes [**DH1**]; more recent proofs can be found in [**Mi2, PetR, Sch2**]. The fact that parameter rays at non-periodic angles cannot land at parabolic parameters additionally requires a deeper local study of the Mandelbrot set near parabolic parameters; see [**HY**], [**Ta4**] or [**Sch3**].

This theorem implies that there is an involution on the set of periodic angles of exact period n, associating to any n-periodic angle ϑ the unique angle ϑ' such that

[2]This definition is sufficient for periodic ray pairs. For non-periodic rays, we allow a characteristic ray pair to contain the critical value: a ray pair $P_c(\vartheta,\vartheta')$ is characteristic if $\mathbb{C}\setminus P_c(\vartheta,\vartheta')$ consists of two components X_0, X_1 so that $\overline{X}_0$ contains all rays $R_c(2^k\vartheta)$ and $R_c(2^k\vartheta')$ for $k\geq 1$ and $\overline{X}_1$ contains the critical value.

[3]For $n=1$, there is only a single parameter ray $R(0)=R(1)$, and only a single dynamic ray $R_c(0)=R_c(1)$; the statement still holds if we count these rays separately.

$P(\vartheta, \vartheta')$ forms a ray pair. We say that ϑ and ϑ' are *conjugate angles* and provide two algorithms to calculate one from the other: one is known as Lavaurs' algorithm and determines the pairs of conjugate angles systematically in the order of increasing periods (Algorithm 13.4); the other finds conjugate angles directly using the dynamics of ray pairs (Algorithm 13.3). In Figure 9.2, we sketch the ray pairs of periods up to 4, and we draw the corresponding parameter rays of the Mandelbrot set (compare also Figure 13.2).

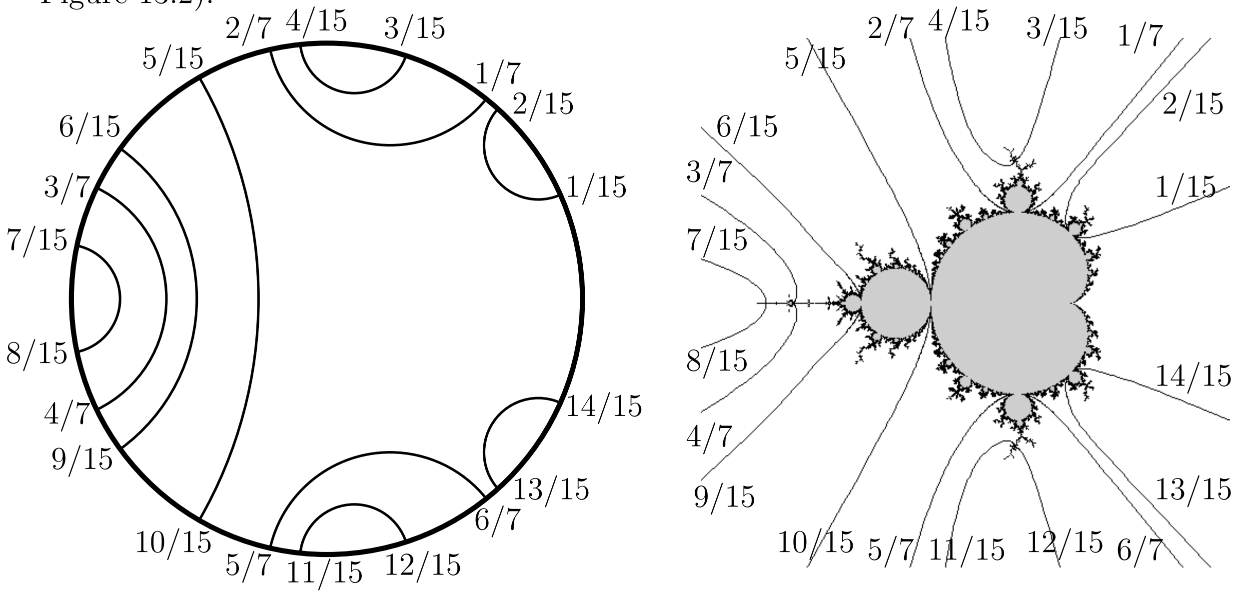

FIGURE 9.2. Left: Pairs of conjugate angles of periods up to 4 (schematically). Right: the corresponding parameter ray pairs of the Mandelbrot set.

We will also need parameter rays at preperiodic angles. For these, we use the following definition: a *Misiurewicz-Thurston parameter* is a parameter $c \in \mathcal{M}$ for which the critical orbit is strictly preperiodic[4]. The following result is due to Douady and Hubbard's Orsay Notes [**DH1**] with extensions by Yoccoz [**HY**] or Tan Lei [**Ta3**]; a simplified approach is in [**Sch2**, Theorem 1.1].

9.2. **Theorem (Parameter Rays at Preperiodic Angles)**
Every parameter ray $R(\vartheta)$ at a preperiodic angle ϑ lands at a Misiurewicz-Thurston parameter, and every Misiurewicz-Thurston parameter is the landing point of a finite positive number of preperiodic parameter rays.

A Misiurewicz-Thurston parameter c is the landing point of a parameter ray $R(\vartheta)$ if and only if, in the dynamical plane of p_c, the dynamic ray $R_c(\vartheta)$ at the same angle

[4]The original definition of a Misiurewicz parameter is one where the critical orbit is non-recurrent. In complex dynamics, this term is often used in the much stricter sense when the critical orbit is strictly preperiodic (and hence non-recurrent). For clarity, we use the term Misiurewicz-Thurston parameter.

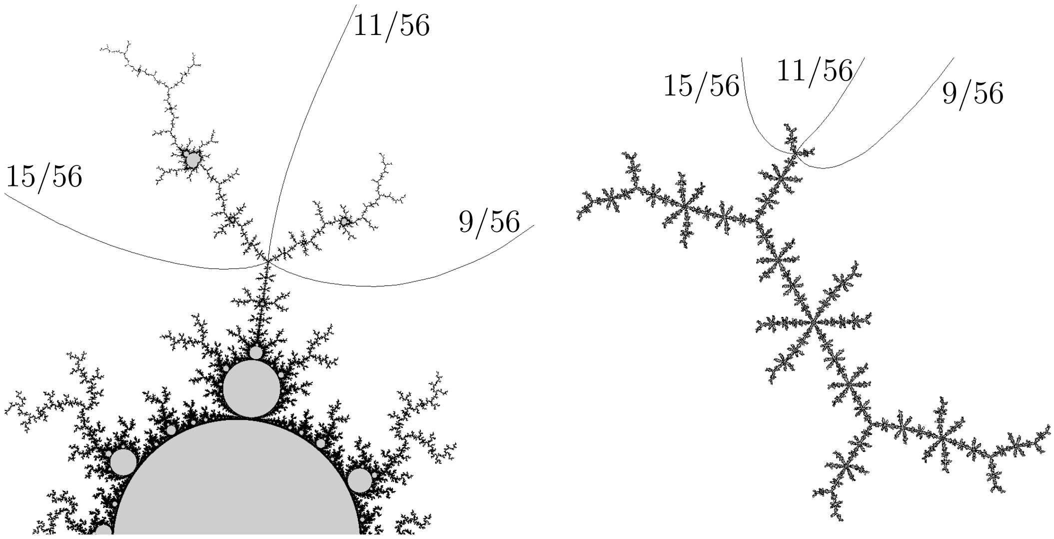

FIGURE 9.3. Illustration of Theorem 9.2. Left: there is a Misiurewicz-Thurston parameter $c \in \mathcal{M}$ which is the landing point of the parameter rays $R(9/56)$, $R(11/56)$ and $R(15/56)$; right: in the dynamical plane of this polynomial p_c, the critical value is the landing point of the dynamic rays $R_c(9/56)$, $R_c(11/56)$ and $R_c(15/56)$.

lands at the critical value. All such rays have preperiodic angles; all angles have equal preperiods and equal periods. □

The following dynamical facts are often useful; the first part is [**Mi2**, Lemma 2.11] (the "smallest sector"), the second one [**Mi2**, Lemma 2.7], and (4) is analogous to Lemma 4.6. Check Milnor: Lemma 2.7 or 2.11?

9.3. **Lemma (Permutation by First Return Map and Characteristic Ray Pairs)**
(1) Every orbit of periodic dynamic ray pairs has a unique characteristic ray pair; the forward orbit of any preperiodic dynamic ray pairs has at least one characteristic ray pair.
(2) If $q \geq 3$ dynamic rays land at the same periodic point z of a quadratic polynomial, then the first return map of z permutes its rays transitively. If $q = 2$, then the first return map of z may or may not permute its two rays.
(3) If two rays $R_c(\vartheta)$ and $R_c(\vartheta')$ land a common periodic point z of period n, then a ray $R_c(\vartheta'')$ lands at z if and only if $\vartheta'' = 2^{kn}\vartheta$ or $\vartheta'' = 2^{kn}\vartheta'$ for some $k \geq 0$.
(4) If z is the landing point of $q \geq 3$ rays, including the characteristic ray pair, then the connected components of $\mathbb{C} \setminus \{z\}$ with these q rays removed can be labelled $V_1, \ldots, V_q$ so that V_1 contains the critical value and the first return map of z maps V_j conformally

onto V_{j+1} for $j = 1, 2, \dots, q-2$, and V_q contains the entire orbit of z (except the point z itself).
(5) If the critical value is strictly preperiodic, then the dynamic rays landing at the critical value subdivide $\mathbb{C}$ into finitely many sectors so that one of them contains the entire postcritical orbit.

Proof. Suppose z is a repelling periodic point at which a characteristic periodic ray pair $P_c(\vartheta, \vartheta')$ lands. Let $z_i := p_c^{\circ(i-1)}(z)$ for $i = 1, 2, 3, \dots$, and let n be the period of z. Each z_i is the landing point of the same number $q \geq 2$ of periodic dynamic rays, and these divide $\mathbb{C}$ into q open sectors at z_i. The map p_c is a bijection from the rays at z_i to the rays at z_{i+1}, and this bijection preserves the cyclic order of the rays at z_i and z_{i+1} because p_c is a local homeomorphism between neighborhoods of z_i and z_{i+1}.

If a sector V does not contain the critical point, then p_c maps V conformally onto a unique sector at z_{i+1}. Otherwise, there is a unique sector V' at z_{i+1} so that $p_c\colon V \to \mathbb{C}$ covers V' twice and $\mathbb{C} \setminus \overline{V'}$ once, and V' contains the critical value (each sector at z_i other than V has a unique image sector at z_{i+1}). In this case, we call V' the image sector of V; in fact, p_c maps the restriction to V of a sufficiently small neighborhood of z_i to a neighborhood of z_{i+1} in V'.

If a sector V is bounded by $R_c(\vartheta)$ and $R_c(\vartheta')$, we define its *width* $w(V)$ as the length of the interval in $\mathbb{S}^1$ bounded by $\{\vartheta, \vartheta'\}$ that contains all angles of rays in V, so that always $w(V) \leq 1$. With respect to the sector dynamics defined above, every sector is periodic (because its boundary rays are), and every cycle of sectors must contain at least one sector containing the critical point (a sector V does not contain the critical point if and only if $w(V) \leq 1/2$; in this case, the image sector has width $2w(V)$).

Among all the qn sectors at all the n points z_i, let V_1 be one of minimal width, and let V be the unique sector whose image sector is V_1. Then V must contain the critical point (because otherwise $w(V) = w(V_1)/2$ would contradict the choice of V_1), so V_1 contains the critical value but cannot contain any z_i. It follows that V_1 is the unique sector of minimal width, and the rays bounding V_1 form the characteristic ray pair. Uniqueness of the characteristic ray pair is clear; this proves (1) in the periodic case. Thus any preperiodic ray pair has a characteristic periodic ray pair on its forward orbit, and this is all we need to prove (1) also in the preperiodic case.

For items (2), (3) and (4) there is nothing to show if $q = 2$, so we may suppose that $q \geq 3$. It is easy to find a point z_k at which a single sector, say V'_k, contains all z_i for $i \neq k$ (start with some z_i; if at least two sectors contain points $z_{i'}$, then one sector contains a point z_j with a sector that contains more points $z_{i'}$ than any sector at z_i).

Suppose that some sector V contains a point z_i. Then either V maps onto its image sector homeomorphically, so the image sector contains z_{i+1}, or the image sector contains the critical value and hence V_1. In this case, the image sector either equals V_1 or it contains z_1. We claim that every forward orbit of sectors must include V_1: the orbit must include a critical sector, hence a sector containing the critical value; this is either V_1 (and we are done) or a sector containing z_1. In the latter case, the orbit of sectors either includes V_1, or all sectors on this periodic orbit must contain some points z_i. Therefore the orbits of at least $q-1$ sectors at z_k must contain V_1; but since the sector dynamics preserves the cyclic order of sectors at the z_i and $q \geq 3$, this implies that every sector at z_k, and thus every sector at every z_i eventually maps to V_1, so all sectors at all z_i are on a single orbit. This proves claim (2) and (3).

The sector V_1 contains no point z_i, and as soon as some sector contains some point z_i, then all further image sectors will contain some point z_i until V_1 is reached again. Therefore the sector orbit of V_1 first visits all sectors that contain no z_i. Since there exists a point z_k so that a single sector contains all $z_i \neq z_k$, the point z_1 must have the same property.

Label the sectors at z_1 by $V_1, \ldots, V_q$ in the order of the first return dynamics at z_1. Since $V_1, \ldots, V_{q-1}$ contain no point z_i, it follows that $p_c^{\circ n}\colon V_i \to V_{i+1}$ is a conformal isomorphism for $i = 1, 2, \ldots, q-2$. This proves claim (4).

For claim (5), let $z_1, \ldots, z_n$ be the periodic orbit that the critical orbit falls into, and suppose again that each z_i is the landing point of $q \geq 2$ periodic rays, so that z_1 is the characteristic point of the orbit. The sectors $V_1, \ldots, V_{q-1}$ at z_1 contain no point on the forward orbit of z_1. Let z_0 be one of the two preimages of z_1; then $q-1$ of the q sectors at z_0 contain no point on the forward orbit of z_1. Continue this pull-back argument until the critical value is reached, and the claim follows. ☐

We will need the following lemma from [**GM**, Lemma B.1].

9.4. **Lemma (Stability of Rays Landing at Repelling Orbit)**
Suppose a quadratic polynomial p_{c_0} has the property that the periodic dynamic ray $R_{c_0}(\vartheta)$ is defined and lands at a repelling periodic point. Then there is a neighborhood U of c_0 so that for all $c \in U$, the dynamic ray $R_c(\vartheta)$ is defined and lands at a repelling periodic point $z(c)$ that depends holomorphically on c. ☐

The following result is fundamental for the transfer of many combinatorial results between dynamical planes and parameter space; see Figure 9.4 for an illustration.

9.5. **Theorem (Correspondence Theorem for (Pre-)Periodic Ray Pairs)**
For every parameter $c \in \mathbb{C}$, there is a bijection between

- *ray pairs $P(\vartheta, \vartheta')$ in parameter space at periodic or preperiodic angles $\vartheta, \vartheta' \in \mathbb{S}^1$ which separate 0 from c (the parameter), and*
- *characteristic ray pairs $P_c(\vartheta, \vartheta')$ in the dynamical plane of $p_c(z) = z^2 + c$ at periodic or preperiodic angles $\vartheta, \vartheta' \in \mathbb{S}^1$ which land at repelling periodic or preperiodic points and which separate 0 from c (the critical value);*

this bijection preserves external angles.

Proof. Consider a polynomial p_{c_1} and suppose there is a characteristic ray pair $P_{c_1}(\vartheta, \vartheta')$ landing at a repelling periodic point, say z_1. Let $U \subset \mathbb{C}$ be the set of

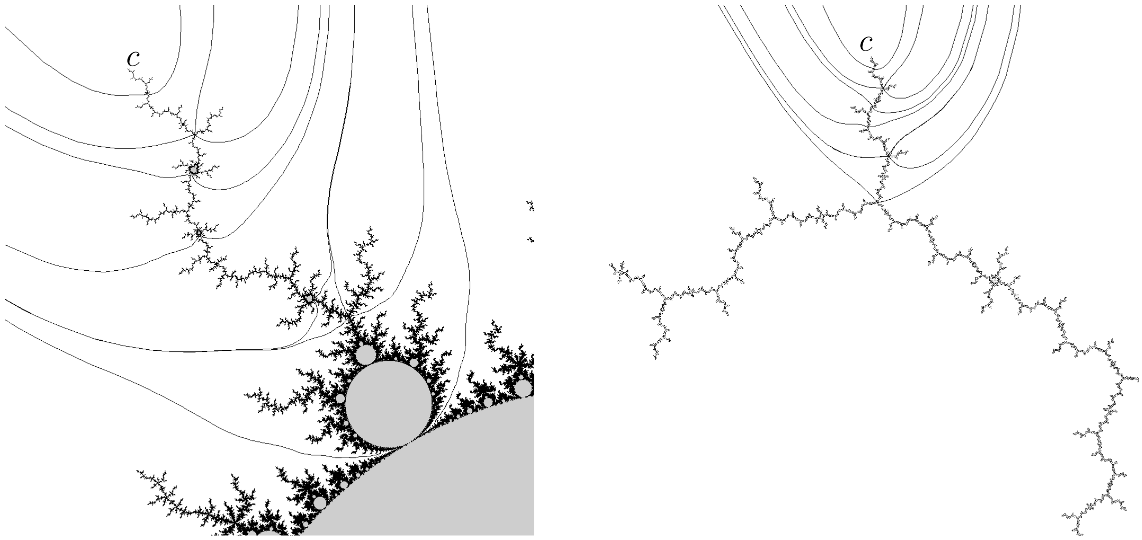

FIGURE 9.4. Illustration of Theorem 9.5. Left: a parameter $c \in \mathbb{C}$ is marked and several parameter ray pairs separating 0 and c are shown: (beginning with the ray pairs closest to c) $P(\frac{23}{63}, \frac{24}{64})$, $P(\frac{365}{1008}, \frac{389}{1008})$ (lands together with $R(\frac{359}{1008})$), $P(\frac{11}{31}, \frac{12}{31})$, $P(\frac{45}{127}, \frac{50}{127})$, $P(\frac{90}{255}, \frac{101}{255})$, $P(\frac{1423}{4032}, \frac{1597}{4023})$ (lands together with $R(\frac{1411}{4032})$), $P(\frac{22}{63}, \frac{25}{63})$; right: in the dynamical plane of p_c, the dynamic ray pairs at the same angles separate 0 and c.

parameters c that have a characteristic ray pair $P_c(\vartheta, \vartheta')$ landing at a repelling periodic point, say $z(c)$. By Lemma 9.4, U is open, and for $c \in \partial U$, one of the dynamic rays $R_c(\vartheta)$, $R_c(\vartheta')$ is either not defined or, if it is, then it does not land at a repelling orbit. This implies that if $c \in \partial U \cap \mathcal{M}$, then $R_c(\vartheta)$ or $R_c(\vartheta')$ lands at a parabolic periodic point, while if $c \in \partial U \setminus \mathcal{M}$, then the ray $R_c(\vartheta)$ or $R_c(\vartheta')$ must fail to exist. The latter condition means the critical value must be contained in one of the dynamic rays $R_c(2^k\vartheta)$, $R_c(2^k\vartheta')$ for $k \geq 1$, and the fundamental formula $\Phi(c) := \Phi_c(c)$ implies that the parameter c must be on one of the finitely many parameter rays $R(2^k\vartheta)$, $R(2^k\vartheta')$ for $k \geq 1$.

There are only finitely many parameters c with parabolic orbits at which dynamic rays of given period land because by Theorem 9.1 each of these parameters must be the landing point of a parameter ray at a periodic angle of the same period.

We have $c_1 \in U$ and $0 \notin U$, so $\partial U \neq \emptyset$ and it follows that ∂U consists of finitely many parameter rays plus finitely many parabolic parameters, and in particular that U intersects $\mathbb{C} \setminus \mathcal{M}$. But for a parameter $c \in \mathbb{C} \setminus \mathcal{M}$, the definition of the characteristic ray pair $P_c(\vartheta, \vartheta')$ implies that the critical value c must be on a dynamic ray at external angle $\tilde{\vartheta} \in (\vartheta, \vartheta')$. Therefore, ∂U consists exactly of the two parameter rays $R(\vartheta)$ and $R(\vartheta')$ together with their common landing point (in fact, this argument shows that

$R(\vartheta)$ and $R(\vartheta')$ must land; this is a key argument in Milnor's proof of Theorem 9.1). It follows that the parameter c_1 is separated from the origin by the parameter ray pair $P(\vartheta, \vartheta')$.

The argument in the preperiodic case is similar, with the following difference: the landing point $z(c)$ of the dynamic ray pair $P_c(\vartheta, \vartheta')$ is preperiodic, and a parameter $c \in \partial U \cap \mathcal{M}$ can either be parabolic as above, or it can be such that the critical value intersects the forward orbit of $z(c)$, which means that the critical value c is preperiodic with an upper bound on period and preperiod. Such parameters c satisfy a polynomial equation of fixed degree, so there are only finitely many such parameters and we conclude as above that U is bounded by the two preperiodic parameter rays $R(\vartheta)$ and $R(\vartheta')$ and their common landing point, which must necessarily be a Misiurewicz-Thurston parameter.

For the converse, suppose there is a parameter ray pair $P(\vartheta, \vartheta')$ at periodic angles, let c_0 be their common landing point and let U be the domain separated from the origin by this ray pair. By Theorem 9.1, the polynomial p_{c_0} has a parabolic orbit and the rays $R_{c_0}(\vartheta)$ and $R_{c_0}(\vartheta')$ form a characteristic ray pair landing at the parabolic orbit. We only need to know that there exists a polynomial p_{c_1} as above: this would prove as in the first part that every $c \in U$ has the property that p_c has a characteristic dynamic ray pair $P_c(\vartheta, \vartheta')$ landing at a repelling orbit. The existence of c_1 is shown by Milnor [**Mi2**, Lemma 2.9].

If the parameter ray pair $P(\vartheta, \vartheta')$ has preperiodic angles, then the landing point is a Misiurewicz-Thurston parameter c_0 by Theorem 9.2, and for p_{c_0} the dynamic rays $R_{c_0}(\vartheta)$ and $R_{c_0}(\vartheta')$ land together at the critical value, which is a repelling preperiodic point. We may assume that $\vartheta < \vartheta'$ are such that (ϑ, ϑ') contains no further preperiodic angle ϑ'' for which $R(\vartheta'')$ lands at c_0. By Lemma 9.3 (5), (ϑ, ϑ') contains no angle $2^k\vartheta$ or $2^k\vartheta'$ for $k \geq 0$.

By Lemma 9.4, there is a neighborhood V of c_0 in parameter space where the dynamic rays $R_c(\vartheta)$ and $R_c(\vartheta')$ land together at a repelling preperiodic point $z(c)$. The map $c \mapsto z(c) - c$ is holomorphic and has an isolated zero at c_0, and it follows that there is a parameter $c' \in V$ for which the ray pair $P_{c'}(\vartheta, \vartheta')$ separates the critical value from the origin. By the first half of the theorem as shown above, this implies that the parameter c' is separated from the origin by $P(\vartheta, \vartheta')$. Since the interval (ϑ, ϑ') contains no angle $2^k\vartheta$, it follows that the ray pair $P_{c'}(\vartheta, \vartheta')$ is characteristic, and every parameter c' in the domain bounded by $P(\vartheta, \vartheta')$ contains this characteristic preperiodic ray pair $P(\vartheta, \vartheta')$. □

Note that a ray pair $P_c(\vartheta, \vartheta')$ at preperiodic angles may have several characteristic ray pairs on its forward orbit: for instance, $P_c(\vartheta, \vartheta')$ could be characteristic, and the periodic angles on the forward orbit of ϑ and ϑ' form another periodic ray pair.

9.6. **Lemma (Characteristic Angles Determine All Angles)**
If $P_c(\vartheta, \vartheta')$ is the characteristic ray pair of a periodic point z, then the angles of all rays landing at z are determined uniquely by $\{\vartheta, \vartheta'\}$.

Proof. Let Θ be the set of angles of all rays landing at z. By Lemma 9.3, there are only two possibilities: either the first return map of z permutes all rays at z transitively, or z is the landing point of two rays, and both are fixed under the first return map of z.

If ϑ and ϑ' are on different orbits under multiplication by 2, then $\Theta = \{\vartheta, \vartheta'\}$, and no further ray can land at z. Otherwise, there is a $k \geq 1$ so that $2^k\vartheta = \vartheta'$. Since $R_c(\vartheta)$ and $R_c(\vartheta')$ are adjacent rays landing at z (with respect to the cyclic order at z), the map $p_c^{\circ k}$ permutes all rays landing at z transitively (the period of z might be a proper divisor of k). Therefore, the entire set Θ is determined uniquely by the two characteristic angles. □

9.2. Hyperbolic Components, Wakes, Limbs, and Bifurcations. A rational map is said to have *hyperbolic dynamics* if all critical orbits converge to attracting periodic orbits (this is equivalent to the existence of a Riemannian metric in a neighborhood of the Julia set that is uniformly expanded under the dynamics [**Mi1**, Theorem 19.1]). Every attracting orbit is the limit of a critical orbit [**Mi1**, Theorem 8.6]. Specifically for quadratic polynomials, this means that every p_c can have at most one attracting periodic orbit (other than the superattracting fixed point at ∞), and p_c is hyperbolic if and only either $c \notin \mathcal{M}$ (so the unique finite critical orbit converges to the superattracting fixed point at ∞), or $c \in \mathcal{M}$ and p_c has an attracting periodic orbit (which must attract the finite critical orbit).

A *hyperbolic component* of the Mandelbrot set is a component W of the set of polynomials p_c which have attracting periodic orbits. The period of the attracting orbit is constant on the component, and the attracting orbit has a multiplier $\mu(c) \in \mathbb{D}$. This defines a *multiplier map* $\mu_W \colon W \to \mathbb{D}$. The *center* of W is a point $c_W \in W$ with $\mu_W(c_W) = 0$, and the *root* of W is a point $r_W \in \partial W$ with $\mu(c) \to 1$ as $c \to r_W$ within W.

9.7. **Theorem (Hyperbolic Components [DH1, Mi2, Sch2])**
For every hyperbolic component W, the multiplier map is a conformal isomorphism $\mu_W \colon W \to \mathbb{D}$ which extends to a homeomorphism $\mu_W \colon \overline{W} \to \overline{\mathbb{D}}$; in particular, every hyperbolic component has a unique center and a unique root. Different hyperbolic components of equal period have disjoint closures.

Every parameter ray pair at periodic angles of period n lands at the root of a hyperbolic component of period n. Conversely, for every hyperbolic component of period n, the root is the landing point of a parameter ray pair of period n, and (for $n \geq 2$) this parameter ray pair separates W from the origin. □

A *point of bifurcation* is a parabolic parameter which is on the common boundary of two hyperbolic components, necessarily of different periods by Theorem 9.7. A hyperbolic component whose root is a point of bifurcation is called a component of *satellite type*; all other hyperbolic components are called *primitive*; see Figure 9.5.

Suppose a parabolic orbit has orbit period k, ray period $n = qk$ and multiplier μ. Then every parabolic periodic point is the landing point of q rays (if $q \geq 2$) or of 2 rays (if $q = 1$ and $k > 1$). If the combinatorial rotation number of the q rays is p/q, then $\mu = e^{2\pi i p/q}$: the dynamic rays land at parabolic periodic points through well-defined repelling directions, hence tangentially; the permutation of the dynamic rays is determined by the permutation of the repelling directions at the parabolic periodic points, and the latter has combinatorial rotation number p/q if $\mu = e^{2\pi i p/q}$. It follows that $n = k$ if and only if $\mu = 1$.

A proof of the following result can be found in [**Sch2**, Lemma 5.1].

9.8. **Lemma (Primitive and Satellite Hyperbolic Components)**
Let W be a hyperbolic component with root c. Suppose that the parabolic orbit of p_c has orbit period k and ray period n. The component W is primitive if and only if $n = k$ and the parabolic orbit has multiplier $\mu = 1$; W is of satellite type if and only if $n > k$ and $\mu \neq 1$. □

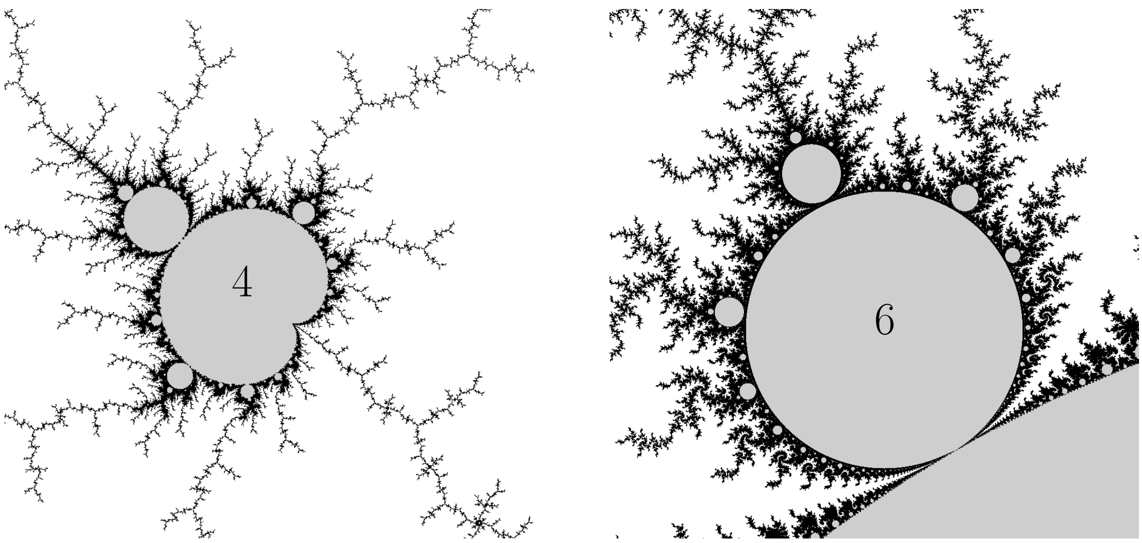

FIGURE 9.5. Left: a primitive hyperbolic component of period 4; right: a satellite type component of period 6.

9.9. **Definition (Wake and Limb, Width of Wake)**
For a hyperbolic component W of $\mathcal{M}$ of period $n \geq 2$, the wake $\mathcal{W}_W$ *of W is defined as the open subset of $\mathbb{C}$ that is separated from the parameter $c = 0$ by the parameter ray pair landing at the root r_W of W. For the unique component of period 1, we define its wake to be $\mathbb{C}\setminus[1/4, \infty)$. The* width *of the wake of W is defined as $|W| := |\vartheta' - \vartheta|$, where $P(\vartheta, \vartheta')$ is the parameter ray pair bounding the wake of W. The* limb *of a hyperbolic component W is $L_W := \mathcal{W}_W \cap \mathcal{M}$.*

9.10. **Theorem (Wakes and Dynamics)**
Consider a hyperbolic component W and let $P(\vartheta, \vartheta')$ be the two parameter rays bounding the wake $\mathcal{W}_W$.

(1) Then $\mathcal{W}_W$ is the locus of parameters c so that p_c has a repelling periodic point z_c which is the landing point of a characteristic dynamic ray pair $P_c(\vartheta, \vartheta')$.

(2) Let $q \geq 2$ be the number of dynamic rays landing at z_c. The first return map of z_c induces a cyclic permutation among these q rays with combinatorial rotation number p/q for some $p \in \{0, 1, 2, \ldots, q-1\}$.

(3) If W is primitive, then $q = 2$ and $p = 0$. If W is of satellite type, and in particular if $q \geq 3$, then p is coprime to q (and $p \neq 0$).

(4) In any case, the multiplier of the parabolic orbit at the landing point of $P(\vartheta, \vartheta')$ equals $e^{2\pi i p/q}$.

(5) For parameters $c \notin \overline{\mathcal{W}_W}$, the two dynamic rays $R_c(\vartheta)$ and $R_c(\vartheta')$ do not land at a common point (or do not land at all).

(6) The root of W is the unique parameter c for which there is a characteristic dynamic ray pair $P_c(\vartheta, \vartheta')$ landing at a parabolic periodic point.

(7) We have $W \subset \mathcal{W}_W$. Moreover, if c_0 is the root of a hyperbolic component W and $c \in W$ arbitrary, then the dynamic rays at the same angles land at a common point for p_{c_0} and for p_c.

Proof. Claim (1) is Theorem 9.5. The q rays landing at z_c are permuted cyclically by the first return map of z_c; since this first return map is a local homeomorphism near z_c, the rays have a well-defined combinatorial rotation number p/q for some $p \in \{0, 1, \ldots, q-1\}$. This is claim (2).

Let c_0 be the landing point of $P(\vartheta, \vartheta')$. This is a parabolic parameter, and by Theorem 9.1 the dynamic rays $P_{c_0}(\vartheta, \vartheta')$ land at the parabolic orbit and form its characteristic ray pair; let z_{c_0} be their landing point. By Lemma 9.6, the rays at the same angles land at z_{c_0} and at z_c for all $c \in U$. The first return map of z_{c_0} permutes these rays with combinatorial rotation number p/q, and this implies that the parabolic orbit has multiplier $\mu_0 = e^{2\pi i p/q}$. This is claim (4).

If the component W is primitive, then by Lemma 9.8 $n = k$ and $\mu_0 = 1$, hence $p = 0$; this implies $q = 2$ by Lemma 9.3. If W is of satellite type, then $\mu_0 \neq 1$

and $n > k$, hence $p \in \{1, 2, \ldots, q\}$. Lemma 9.3 implies that all q rays are permuted transitively, so p is coprime to q; this is claim (3).

Suppose for some $c \notin \overline{\mathcal{W}_W}$, there is a dynamic ray pair $P_c(\vartheta, \vartheta')$; it is then not characteristic. Let $P_c(\varphi, \varphi')$ be the associated characteristic ray pair. By definition, $(\varphi, \varphi') \subset (\vartheta, \vartheta')$ and c must be contained in the wake bounded by $P(\varphi, \varphi')$. This wake is contained in $\mathcal{W}_W$, and this contradiction proves (5).

For (6), let c_0 be the root of W and hence the landing point of $P(\vartheta, \vartheta')$. By Theorem 9.1, p_{c_0} is the unique quadratic polynomial with a parabolic periodic point that is the landing point of a characteristic dynamic ray pair $P_c(\vartheta, \vartheta')$.

Finally, we discuss (7). The fact that for any two periodic angles φ, φ' there is a periodic dynamic ray pair $P_{c_0}(\varphi, \varphi)$ if and only there is a a ray pair $P_c(\varphi, \varphi')$ (necessarily landing at a repelling orbit) follows from Corollary [**Sch2**, Corollary 5.3] or [**Mi2**, Theorem 4.1]. This implies $W \subset \mathcal{W}_W$. □

Remark. Let c_0 be the root of a hyperbolic component W. Denote the orbit period by k and the ray period by n. When the parameter is perturbed into W, the parabolic orbit splits up into one repelling orbit of period k that inherits all rays from the parabolic orbit, and one orbit of period n that is attracting throughout W (compare [**DH1**], [**Mi2**, Theorem 4.1] and [**Sch2**, Lemma 5.1 and Corollary 5.3]).

9.11. **Proposition (Bifurcations)**

(1) *Let W be a hyperbolic component of some period $n \geq 1$. Every boundary point $c_{p/q} := \mu_W^{-1}(e^{2\pi i p/q})$ (with $q \geq 2$ and p coprime to q) is a bifurcation point: it is the root of a unique hyperbolic component $W_{p/q}$; the component $W_{p/q}$ is of satellite type and has period qn.*

(2) *The closures of any two hyperbolic components intersect at most in one point; and if they do, the intersection is a bifurcation as in (1).*

(3) *If c is the point of bifurcation from a hyperbolic component of period n to a hyperbolic component of period qn, then the two parameter rays landing at c separate these two components from each other so that the origin is on the same side of the separation as the component of period n.*

(4) *Let $P(\vartheta, \vartheta')$ be a periodic parameter ray pair with $\vartheta < \vartheta'$ and let W be the hyperbolic component whose root r_W is the landing point of $P(\vartheta, \vartheta')$. Then the angles ϑ, ϑ' are on different orbits under multiplication by 2 if and only if W is primitive.*

Statements (1) and (2) can be found in [**Mi2**, Section 6] and [**Sch2**, Section 5]. Statement (4) is in [**Mi2**, Section 6]. For statement (3), let W be the hyperbolic component with root c so that W has period qn. Recall from Definition 9.9 that the wake $\mathcal{W}_W$ is bounded by the two parameter rays landing at c. We have $W \subset \mathcal{W}_W$ by Theorem 9.10 (7). Let W_0 be the component of period n so that c is the bifurcation point from W_0 to W. It follows from [**Mi2**, Lemma 4.4] that $W_0 \cap \mathcal{W}_W = \emptyset$. □

9.12. **Definition (Subwakes and Sublimbs)**
In the setting of Proposition 9.11 (1), we say that $W_{p/q}$ is a bifurcation *of W at* internal angle p/q. *Wake and limb of $W_{p/q}$ are called the p/q-*subwake *and p/q-*sublimb *of W.*

Clearly, all subwakes $W_{p/q}$ have disjoint closures.

We also define subwakes for Misiurewicz-Thurston parameters as follows: suppose c is the landing point of finitely many parameter rays, say $R(\vartheta_1), \dots, R(\vartheta_k)$ (with $k \geq 1$), where all ϑ_i are strictly preperiodic. A *subwake of c* is defined as a component of $\mathbb{C} \setminus (\{c\} \cup R(\vartheta_1) \cup \cdots \cup R(\vartheta_k))$ other than the component containing the origin. A *sublimb* is the intersection of the subwake with $\mathcal{M}$.

Different subwakes of a Misiurewicz-Thurston parameter c have different characteristic preperiodic ray pairs by Theorem 9.5: if c' is in the subwake bounded by $P(\vartheta_i, \vartheta_{i+1})$, then $P_{c'}(\vartheta_i, \vartheta_i')$ is characteristic. The dynamic rays $P_{c'}(\vartheta_i/2)$ and $P_{c'}(\vartheta_{i+1}/2)$ land together unless c' is in the subwake bounded by $P(\vartheta_i, \vartheta_{i+1})$. It follows that the Misiurewicz-Thurston subwakes can be distinguished by the patterns of which preperiodic dynamic rays land together.

Suppose two parameters $c, c' \in \mathcal{M}$ are such that $c \in \mathcal{W}_W$ implies $c' \in \mathcal{W}_W$ for every hyperbolic component W. In a combinatorial sense, this means that c' is further away from the origin than c. We say that c' is combinatorially at least as great as c and write $c' \succeq c$. In this sense, we have *monotonicity of periodic ray pairs*: for every ray pair $P_c(\varphi, \varphi')$ at periodic angles there is a corresponding ray pair $P_{c'}(\varphi, \varphi')$ (going out combinatorially only adds extra combinatorial ray pairs). The corresponding statement for preperiodic ray pairs is false: for example, a polynomial p_c has a dynamic ray pair $P_c(9/56, 11/56)$ landing at a repelling orbit if and only if c is separated from the origin by $P(1/7, 2/7)$ but not by $P(9/56, 11/56)$. This is related to the fact mentioned earlier that preperiodic ray pairs may have several characteristic ray pairs.

9.3. Combinatorial Classes. We now review a number of important theorems about the structure of the Mandelbrot set which deal with its combinatorial structure as generated by periodic or preperiodic parameter rays. In fact, the periodic parameter rays alone determine the entire combinatorial structure (every non-periodic parameter ray pair is a limit of periodic parameter ray pairs, see Theorem 9.29). Conjecturally, the entire topological structure of $\mathcal{M}$ is determined by the periodic parameter rays: this is a consequence of the famous *MLC* conjecture *the Mandelbrot set is Locally Connected*; compare Sections 9.4 and 24 make sure explanation is there! .

9.13. **Theorem (The Branch Theorem)**
Let W and W' be two hyperbolic components of the Mandelbrot set. Then exactly one of the following is true:

- *W is contained in the wake of W', or vice versa;*
- *there is a hyperbolic component W_0 so that W and W' are contained in two different sublimbs of W_0;*

- *there is a Misiurewicz-Thurston parameter c so that W and W' are contained in two different sublimbs of c.* $\Box$

This theorem is the main result of the theory of *nervures* in [**DH1**, Exposés XX–XXII] which in turn is the main motor for the proof of Douady and Hubbard [**DH1**, Exposé XXII.4] that local connectivity of $\mathcal{M}$ implies that hyperbolicity is dense in the space of quadratic polynomials (see also [**Sch3**, Theorem 3.1 and Corollary 4.6]).

9.14. **Corollary (Local Connectivity Implies Density of Hyperbolicity)**
If $\mathcal{M}$ is locally connected, then hyperbolic dynamics is open and dense in the space of quadratic polynomials.

Proof. Hyperbolic dynamics is clearly open: if all critical orbits converge to attracting orbits for a particular rational map, then this persists under sufficiently small perturbations of the map.

Every hyperbolic component W of $\mathcal{M}$ satisfies $\partial W \subset \partial\mathcal{M}$ because parabolic parameters are dense on ∂W and these are landing points of parameter rays by Theorem 9.1. Therefore each component of the interior of $\mathcal{M}$ is either hyperbolic or non-hyperbolic; since all p_c with $c \notin \mathcal{M}$ are hyperbolic, we need to prove that every component of the interior of $\mathcal{M}$ is hyperbolic.

Suppose there is a non-hyperbolic component V of the interior of $\mathcal{M}$; we have $\partial V \subset \partial\mathcal{M}$. Pick any three different parameters $c_1, c_2, c_3 \in \partial V$. If $\mathcal{M}$ is locally connected, then every c_i is the landing point of a parameter ray $R(\vartheta_i)$. Since ∂V is uncountable, we may assume that no c_i the landing point of a parameter ray at rational angle, and in particular that all ϑ_i are neither periodic nor preperiodic. Suppose without loss of generality that $\vartheta_1 < \vartheta_2 < \vartheta_3$. There are parameter ray pairs $P(\varphi_1, \varphi_1')$ and $P(\varphi_2, \varphi_2')$ at periodic angles with $\vartheta_1 < \varphi_1 < \varphi_1' < \vartheta_2 < \varphi_2 < \varphi_2' < \vartheta_3$ (for every periodic angle $\varphi_1 \in (\vartheta_1, \vartheta_2)$ there is a parameter ray pair $P(\varphi_1, \varphi_1')$ by Theorem 9.1, and it must satisfy the claim). Let W_1 and W_2 be the hyperbolic components whose roots are the landing points of the parameter ray pairs $P(\varphi_1, \varphi_1')$ and $P(\varphi_2, \varphi_2')$

By the Branch Theorem 9.13, there must be either a hyperbolic component W_0 or a Misiurewicz-Thurston parameter c so that W_1 and W_2 are in two different subwakes of W_0 or of c. In the second case, at least three parameter rays must land at c in order to define different subwakes for W_1 and W_2. The component V must be in one of the two subwakes, or outside of all subwakes of c, and c is different from c_1, c_2, c_3. In all cases, at least one of the three rays $R(\vartheta_1)$, $R(\vartheta_2)$, $R(\vartheta_3)$ must be in a different subwake than V and cannot land at V, and this is a contradiction.

In the first case, the component W_0 and the boundaries of the two subwakes containing W_1 and W_2 provide a separation between $R(\vartheta_2)$ and the two rays $R(\vartheta_1)$, $R(\vartheta_3)$, and this is a contradiction again (by construction no $R(\vartheta_i)$ lands at a parabolic parameter). $\Box$

All bifurcations at hyperbolic components occur at parabolic boundary points (i.e., at points $c \in \partial W$ with $\mu_W(c) \in \exp(2\pi i \mathbb{Q})$); this result is sometimes quoted as saying that "there are no ghost limbs": all decorations to a hyperbolic component occur at rational internal angles.

9.15. **Theorem (No Irrational Sublimbs)**
For every hyperbolic component W, we have

$$L_W \cup \{r_W\} = \overline{W} \cup \bigcup_{p/q} L_{W_{p/q}} \ ,$$

where the infinite union runs over all $p/q \in (0,1)$ in lowest terms. In words: the limb of the component W (union the root of W) is the disjoint union of the component, its boundary, and all sublimbs at rational internal angles. □

The original proof uses the Yoccoz inequality [**HY**, Theorem I]; an alternate proof can be found in [**Sch3**, Theorem 2.3].

9.16. **Corollary (Irrational Internal Angles)**
For every hyperbolic component W of $\mathcal{M}$ and every irrational (internal) angle $\varphi \in \mathbb{S}^1$, there is a unique external angle $\vartheta_\varphi \in \mathbb{S}^1$ so that the parameter ray $R(\vartheta_\varphi)$ lands at the boundary point $\mu_W^{-1}(e^{2\pi i\varphi})$. Moreover, we have

$$\mathcal{W}_W = W \cup \bigcup_{p/q} \overline{\mathcal{W}_{W_{p/q}}} \cup \bigcup_{\varphi \in \mathbb{S}^1 \, irrational} \left(R(\vartheta_\varphi) \cup \mu_W^{-1}(e^{2\pi i\varphi}) \right) \ , \tag{5}$$

where the first infinite union runs over all $p/q \in (0,1)$ in lowest terms. In particular, if a parameter ray $R(\vartheta)$ is in the wake of W but not in any subwake at rational internal angle, then $\vartheta = \vartheta_\varphi$ for some irrational $\varphi \in \mathbb{S}^1$, and $R(\vartheta)$ lands at ∂W.

Proof. For irrational $\varphi \in \mathbb{S}^1$, consider two sequences $\varphi_k \nearrow \varphi$ and $\varphi'_k \searrow \varphi$ of rational angles in $\mathbb{S}^1$. Every $c_k := \mu_W^{-1}(\varphi_k)$ and every $c'_k := \mu_W^{-1}(\varphi'_k)$ is the landing point of a parameter ray $R(\vartheta_k)$ or $R(\vartheta'_k)$ by Theorem 9.1. The sequence ϑ_k is monotonically increasing and ϑ'_k is monotonically decreasing, and we have $\vartheta_k < \vartheta'_k$ for all k. Therefore, $\Theta := \bigcap_k [\vartheta_k, \vartheta'_k]$ is non-empty, and for all $\vartheta \in \Theta$, the parameter ray $R(\vartheta)$ must land at $\mu_W^{-1}(e^{2\pi i\varphi})$ by Theorem 9.15 (all limit points of the ray must be in $\partial\mathcal{M}$, and all other points in $\partial\mathcal{M}$ are inaccessible for $R(\vartheta)$). By the Riesz theorem [**Mi1**, Theorem A.3] (or by the combinatorial estimate in Proposition 9.24), there can be only one such angle ϑ, and this proves the first claim.

The restriction of (5) to $\mathcal{M}$ is the statement of Theorem 9.15. So consider some parameter ray $R(\vartheta)$ in $\mathcal{W}_W$. If it is not in the closure of some subwake, then every limit point of $R(\vartheta)$ must be in ∂W; and since parabolic parameters are dense on ∂W and these are landing points of parameter rays by Theorem 9.1, it follows that $R(\vartheta)$ lands at a well-defined point on $c \in \partial W$, hence $c = \mu_W^{-1}(e^{2\pi i\varphi})$ for some $\varphi \in \mathbb{S}^1$. If c is not parabolic, then φ must be irrational. □

The external angles ϑ_φ that occur in this corollary have many interesting properties. For instance, these are exactly the angles that are irrational and have finite internal addresses (this follows directly from Algorithm 11.3), and this implies that the angles ϑ_φ are exactly those angles that generate periodic kneading sequences without $\star$; see also Theorem 14.10. There are interesting number theoretic relations between the internal angle φ and the associated external angle ϑ_φ; see [**BuS**] and Section **??**.

Theorem 9.15 (or equivalently Corollary 9.16) implies that the Mandelbrot set is locally connected at all boundary points of hyperbolic components, except possibly at roots of primitive components. The following stronger result from [**Sch3**, Corollary 6.3] implies local connectivity at all boundary points of hyperbolic components. For a different argument, see [**Ta4**, Theorem 1.1 and Corollary D.6].

9.17. **Theorem (Trivial Fibers of Indifferent Parameters)**
Suppose that p_c has an indifferent orbit and $c' \in \mathcal{M}$ is so that there is no hyperbolic component W of $\mathcal{M}$ with $c, c' \in \overline{W}$. Then there is a parameter ray pair at periodic angles which separates c and c'. □

9.18. **Definition (Combinatorial Class)**
A combinatorial class *in the Mandelbrot set is an equivalence class where two parameters $c, c' \in \mathcal{M}$ are equivalent if for p_c and $p_{c'}$ the dynamic rays at the same periodic angles form ray pairs, i.e., there is a dynamic ray pair $P_c(\vartheta, \vartheta')$ if and only if there is a dynamic ray pair $P_{c'}(\vartheta, \vartheta')$, for all periodic angles ϑ, ϑ'.*

9.19. **Proposition (Combinatorial Class)**
Two parameters $c, c' \in \mathcal{M}$ belong to different combinatorial classes if and only if one of the following two (mutually exclusive) conditions are satisfied:

- *they are separated by a parameter ray pair at periodic angles;*
- *there is a hyperbolic component W of $\mathcal{M}$ with $c, c' \in \overline{W}$ and at least one of c and c' is a parabolic boundary point of W other than the root of W.*

Proof. If c and c' are separated by a parameter ray pair $P(\vartheta, \vartheta')$ at periodic angles, then exactly one of them (say c) is in the wake bounded by $P(\vartheta, \vartheta')$ and by Theorem 9.5 there is a characteristic dynamic ray pair $P_c(\vartheta, \vartheta')$ landing at a repelling orbit for p_c but there is no such ray pair for $p_{c'}$; and a ray pay $P_c(\vartheta, \vartheta')$ landing at a parabolic orbit exists only at the landing point of $P(\vartheta, \vartheta')$ by Theorem 9.1.

Suppose $c, c' \in \overline{W}$ and $c \in \partial W$ is a parabolic parameter other than the root of W. Let ϑ, ϑ' be the two characteristic angles of the parabolic orbit. A parameter c'' has a characteristic periodic ray pair $P_{c''}(\vartheta, \vartheta')$ landing at a repelling orbit if and only c'' is in the wake with root c; but $\overline{W}$ is outside of this wake by Proposition 9.11 (3). The parameter c'' has such a ray pair landing at a parabolic orbit only if $c'' = c$ (because c'' must be the landing point of $R(\vartheta)$ and $R(\vartheta')$ by Theorem 9.1). Therefore, c and c' belong to different combinatorial classes.

For the converse, suppose the combinatorial classes of c and c' are different. Suppose without loss of generality that p_c has a periodic ray pair $P_c(\vartheta, \vartheta')$ that $p_{c'}$ does not have. If the ray pair $P_c(\vartheta, \vartheta')$ in p_c lands at a repelling orbit, then c is in the wake bounded by $P(\vartheta, \vartheta')$ but c' is not (Theorem 9.5), and c' cannot be in the wake boundary by Theorem 9.1; thus the parameter ray pair $P(\vartheta, \vartheta')$ separates c and c'. The other possibility is that the ray pair $P_c(\vartheta, \vartheta')$ lands at a parabolic orbit. Then c is the root of a hyperbolic component W, while c' is not in the wake of W. If c is not separated from c' by a periodic ray pair, then by Theorem 9.17, this means that c and c' are both in the closure of some hyperbolic component W', and $W' \neq W$ because $c' \in \overline{W}'$ is not in the wake of W. Therefore c is the bifurcation point between W and W'. The point c is the root of W, hence not the root of W'. ☐

9.20. **Corollary (Hyperbolic Component and Combinatorial Class)**
Every hyperbolic component is contained in a single combinatorial class: this class consists of the component, its root, and all its boundary points at irrational internal angles, but no other point. ☐

Combinatorial classes of Misiurewicz-Thurston points are easy to describe; proofs can be found in [**Ta3, Sch3**] (see also [**HY**]).

9.21. **Theorem (Fiber of Misiurewicz-Thurston Point)**
If c is a Misiurewicz-Thurston point and $c' \in \mathcal{M} \setminus \{c\}$, then there is a parameter ray at periodic angles separating c_* from c. In particular, every Misiurewicz-Thurston point is its own combinatorial class. ☐

A substantial body of work has been devoted to proving that many combinatorial classes are points; see Section 9.4.

9.4. Local Connectivity and Fibers. The principal conjecture about the Mandelbrot set is called MLC: *the Mandelbrot set is locally connected*. This would have a number of implications:

- **Topological models:** there are a number of combinatorial-topological models for the Mandelbrot set, such as the *pinched disk model* related to Thurston's *quadratic minor lamination* (see Section 24). These models are homeomorphic to $\mathcal{M}$ if and only if $\mathcal{M}$ is locally connected, so the topology of $\mathcal{M}$ would be completely described in terms of combinatorics and symbolic dynamics [**D3**].
- **Rigidity:** a related point of view has been nicknamed by Douady as *points are points*: combinatorics of periodic dynamic rays distinguishes combinatorial classes of $\mathcal{M}$. Non-hyperbolic combinatorial classes can be viewed as "combinatorial points", and MLC is equivalent to the statement that all non-hyperbolic combinatorial classes in $\mathcal{M}$ are points in $\mathbb{C}$. This is sometimes called *combinatorial rigidity* of $\mathcal{M}$. (We will introduce below the concept of fibers of $\mathcal{M}$; then MLC is equivalent to the statement that all fibers of $\mathcal{M}$ are points.)

Landing of Parameter Rays: local connectivity of $\mathcal{M}$ is equivalent to the statement that every parameter ray lands and the landing points depend continuously on the angle; then every $c \in \partial\mathcal{M}$ is the landing point of a finite positive number of rays. This gives a continuous surjection $\Psi\colon \mathbb{S}^1 \to \partial\mathcal{M}$ which is sometimes called the *Carathéodory loop* of $\mathcal{M}$. In fact, one can explicitly describe in combinatorial terms whether $\Psi(\vartheta) = \Psi(\vartheta')$[5].

Geometry: certain statements about geometric properties of $\partial\mathcal{M}$ can be made only under the hypothesis of MLC, such as statements on the Hausdorff dimension of biaccessible parameters in $\mathcal{M}$ in Section 17. There are more and better references!

Density of Hyperbolicity: local connectivity of $\mathcal{M}$ implies that hyperbolic dynamics is dense in the space of quadratic polynomials. This is one of the main results of the Orsay Notes of Douady and Hubbard [**DH1**]; see Corollary 9.14. Hyperbolic dynamical systems are particularly easy to understand, and it is part of the general philosophy of dynamical systems that while it may be difficult to analyze each individual system, one should try to understand an open dense subset (or a full measure subset) of cases.

There is some subtle confusion of notation: if $\mathcal{M}$ (or any compact connected subset of $\mathbb{C}$ so that $\mathbb{C} \setminus \mathcal{M}$ is connected) is locally connected at some point $c \in \partial\mathcal{M}$, this does not imply that c is the landing point of a parameter ray. Counterexamples are provided by comb-like structures as in Figure 9.6; see [**Sch3**] for a detailed discussion. A more appropriate notion is that of a *fiber* of a point in $\mathcal{M}$; the following discussion is taken from [**Sch3**].

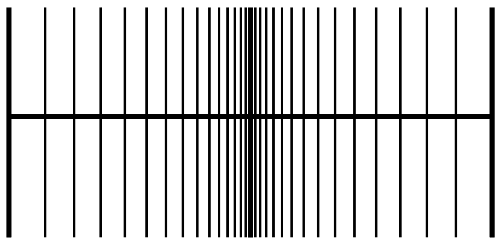

FIGURE 9.6. This compact connected and full set $K \subset \mathbb{C}$ is locally connected at the center, but no ray (or even a curve in $\mathbb{C} \setminus K$) can land at the center. No matter which external rays are used to define fibers of K, the fiber of the center must always contain the vertical interval at the center (which belongs to K by compactness).

[5]The condition is that ϑ and ϑ' are both periodic or both non-periodic, that they have the same angled internal address and, if $\vartheta \neq \vartheta'$ are non-periodic, that their internal address is infinite Section 11). The quotient $\mathbb{S}^1/\sim$ with $\vartheta \sim \vartheta'$ iff $\Psi(\vartheta) = \Psi(\vartheta')$, is an explicit topological topological model for $\partial\mathcal{M}$ (still under the condition of MLC).

9.22. **Definition (Fibers of $\mathcal{M}$)**
Two points $c, c' \in \mathcal{M}$ can be separated *if one the following two conditions is satisfied:*

(1) *there is a parameter ray pair $P(\vartheta, \vartheta')$ at periodic angles that separates c and c';*
(2) *there are two parameter rays $R(\vartheta)$ and $R(\vartheta')$ at periodic angles landing on the boundary of the same hyperbolic component W, and there is a curve $\gamma \subset W$ connecting these two landing points, so that $R(\vartheta)$, $R(\vartheta')$ and γ together separate c and c'.*

The fiber *of a point $c \in \mathcal{M}$ consists of the set of all points in $\mathcal{M}$ that cannot be separated from c.*

A fiber is trivial *if it consists of a single point.*

Fibers are compact and connected subsets of $\mathcal{M}$, and different points have either identical or disjoint fibers (the original definition involves parameter rays at periodic or preperiodic angles, but these definitions are equivalent; see [**Sch3**, Section 8]). For every $c \in \mathcal{M}$ with trivial fiber, it follows easily that $\mathcal{M}$ is locally connected at c. This is not known to be an equivalence, and this difference is important: a parameter c is necessarily the landing point of one or several parameter rays if the fiber of c is trivial (but not necessarily if $\mathcal{M}$ is locally connected at c: see Figure 9.6). Moreover, if the fiber of c is trivial, then the following holds: if the impression of a parameter ray $R(\vartheta)$ contains c, then the impression equals $\{c\}$, and in particular $R(\vartheta)$ lands at c (impressions are defined in (19) in Section 17). In general, the impression of an arbitrary parameter ray is contained in a single fiber.

Even though triviality of the fiber of $c \in \mathcal{M}$ is stronger than local connectivity, there is a converse as follows: if $\mathcal{M}$ is locally connected at all points in the fiber of c, then the fiber of c is trivial [**RS**]. In particular, $\mathcal{M}$ is locally connected everywhere if and only if all fibers of $\mathcal{M}$ are trivial.

The following statements follow directly from Section 9.3; see also [**Sch3**].

9.23. **Corollary (Fibers, Hyperbolic Components, and Combinat. Classes)**
Every point on the closure of a hyperbolic component and every Misiurewicz-Thurston point has trivial fiber. For every point which is not on the closure of a hyperbolic component, its fiber is exactly the set of parameters c' that cannot be separated from c by a periodic parameter ray pair (separation lines through hyperbolic components are not necessary in this case). A combinatorial class which does not intersect the closure of a hyperbolic component is exactly equal to a fiber.

There has been a lot of deep work to prove that many further boundary points of $\mathcal{M}$ have trivial fibers, notably by Yoccoz [**HY**], Lyubich and coauthors [**Ly2, ?**] (even though sometimes only the weaker result is stated that $\mathcal{M}$ is locally connected at that point).

9.5. More on Combinatorics of Wakes and Ray Pairs. We had defined widths of wakes in Definition 9.9. We now give a useful formula for these widths at bifurcations.

9.24. **Proposition (Width of Wake)**
If W is a hyperbolic component of some period n and $W_{p/q}$ is the bifurcation component of W at internal angle p/q in lowest terms, then the widths of the wakes W and $W_{p/q}$ are related as follows:

$$|W_{p/q}| = |W|\frac{(2^n-1)^2}{2^{qn}-1} \ .$$

In particular, the width of the wake of $W_{p/q}$ does not depend on p. Moreover,

$$|W| = \sum_{p,q} |W_{p/q}| \ .$$

In other words, almost every ray in the wake of W is contained in one of its rational subwakes (the summation is over all $p/q \in (0,1)$ with p coprime to q).

This result helps to determine periods of hyperbolic components that occur within the wake of a given component W: if k is the least period of hyperbolic components occurring in $\overline{\mathcal{W}_W}$, then k equals the least integer so that $|W| \geq 1/(2^k-1)$. To see this, recall that the width of a component of period k is at least $1/(2^k-1)$, so we must have $|W| \geq 1/(2^k-1)$. Conversely, if $|W| \geq 1/(2^k-1)$, then $\overline{\mathcal{W}_W}$ contains at least one parameter ray of period k (or dividing k) and hence one component of period k or dividing k; but a component whose period strictly divides k would have period less than k, and this is impossible.

To prove the proposition, we need the following two lemmas.

9.25. **Lemma (Summing Up Widths of Subwakes)**
For every $n \geq 1$, we have

$$\sum_{p,q} \frac{1}{2^{nq}-1} = \frac{1}{(2^n-1)^2} \ ,$$

where the sum is over all (p,q) with $q \geq 1$ and $p \in \{1,\ldots,q-1\}$ coprime to q.

Proof. This is a matter of clever rearrangement as in [**GM**]. We calculate

$$\begin{aligned}
\sum_{p,q} \frac{1}{2^{nq}-1} &= \sum_{q\geq 2}\sum_{p\perp q}\left(2^{-nq}+2^{-2nq}+2^{-3nq}+\ldots\right) = \sum_{q\geq 2}\sum_{p\perp q}\sum_{k\geq 1} 2^{-knq} \\
&\overset{\star}{=} \sum_{q\geq 2}\sum_{p=1,\ldots,q-1} 2^{-nq} = \sum_{p\geq 1}\sum_{q>p} 2^{-nq} = \sum_{p\geq 1}\frac{2^{-np}}{2^n-1} = \frac{1}{(2^n-1)^2} \ .
\end{aligned}$$

The crucial step is marked with a $\star$. The summations before, marked with $p \perp q$, involve only $p \in \{1,\ldots,q-1\}$ coprime to p. For such a pair (p,q), we associate the term 2^{-knq} to the pair (kp, kq) in the next sum, so in the second line we sum over all pairs (p, q), coprime or not. The rest is just the geometric series. □

9.26. **Lemma (Preperiodic Ray Pair Behind Characteristic Ray Pair)** *Suppose that for a parameter $c \in \mathbb{C}$, there is a periodic dynamic ray pair $P_c(\vartheta, \vartheta')$ which is characteristic and has period n. Then there is a unique preperiodic ray pair $P_c(\tilde\vartheta, \tilde\vartheta')$ with $\vartheta < \tilde\vartheta < \tilde\vartheta' < \vartheta'$ so that $2^n\colon (\vartheta, \tilde\vartheta) \to (\vartheta, \vartheta')$ and $2^n\colon (\tilde\vartheta', \vartheta') \to (\vartheta, \vartheta')$ are homeomorphisms (Figure 9.7); it satisfies $|\tilde\vartheta' - \tilde\vartheta| = (1 - 2 \cdot 2^{-n})|\vartheta' - \vartheta|$. The ray pair $P_c(\tilde\vartheta, \tilde\vartheta')$ is characteristic if and only if it separates the critical value from the origin.*

Let $U \subset \mathbb{C}$ be the component of $\mathbb{C} \setminus P_c(\vartheta, \vartheta')$ containing the critical value and let $U_1 \subset \mathbb{C}$ be the domain bounded by the ray pairs $P_c(\vartheta, \vartheta')$ and $P_c(\tilde\vartheta, \tilde\vartheta')$. Then the restriction $p_c|_{U_1}\colon U_1 \to U$ is a branched cover of degree 2. All the domains U, $p_c(U_1)$,…,$p_c^{\circ(n-1)}(U_1)$ are disjoint.

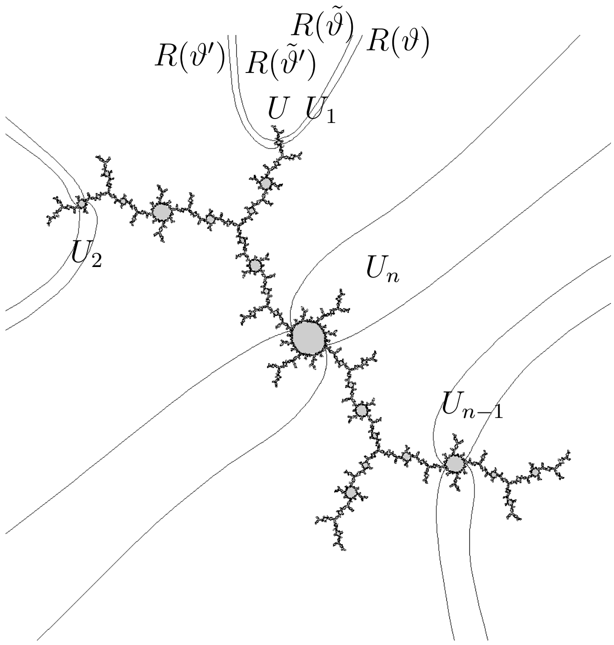

FIGURE 9.7. Illustration of Lemma 9.26, for $n = 4$: the narrow strip between $P_c(\vartheta, \vartheta')$ and $P_c(\tilde\vartheta, \tilde\vartheta')$ is U_1, and the domain separated from the origin by $P_c(\vartheta, \vartheta')$ is $U \supset U_1$; the map $p_c\colon U_1 \to U$ is a covering of degree 2.

PROOF. The domain $U_n := p_c^{-1}(U)$ is bounded by the two ray pairs $P_c(2^{n-1}\vartheta, 2^{n-1}\vartheta')$ and $P_c(2^{n-1}\vartheta + 1/2, 2^{n-1}\vartheta' + 1/2)$. Since $P_c(\vartheta, \vartheta')$ is characteristic, U is disjoint from the rays $R_c(2^k\vartheta)$ and $R_c(2^k\vartheta')$ for all $k \geq 0$, and U_n enjoys the same property.

Define recursively domains U_{n-1}, U_{n-2}, …, U_1 so that each U_k is a component of $p_c^{-1}(U_{k+1})$, choosing the branch so that ∂U_k contains $P_c(2^{k-1}\vartheta, 2^{k-1}\vartheta')$. The only obstacle for doing this would be if the critical value c was contained in U_{k+1}. But $c \in U$, and $c \in U_{k+1}$ would imply that U and U_{k+1} overlapped, and since the ray pair $P_c(2^k\vartheta, 2^k\vartheta') \in \partial U_{k+1}$ does not intersect U, this would imply $R_c(\vartheta) \in U_{k+1}$ or $R_c(\vartheta') \in U_{k+1}$; but this in turn would imply $R_c(2^{n-k-1}\vartheta) \in U_n$ or $R_c(2^{n-k-1}\vartheta') \in U_n$, a contradiction. This also shows that $U_k \cap U = \emptyset$ for $k = 2, 3, \ldots, n$, and by taking preimages it follows that $U_1, \ldots, U_n$ are disjoint.

The domain U_1 is bounded by two ray pairs; one of them is $P_c(\vartheta, \vartheta')$. Denote the other one $P_c(\tilde\vartheta, \tilde\vartheta')$; then $2^n\tilde\vartheta = \vartheta'$ and $2^n\tilde\vartheta' = \vartheta$. Since both $R_c(\vartheta)$ and $R_c(\vartheta')$ are fixed by $p_c^{\circ n}$, it follows that $U_1 \subset U$ (the domains U and U_1 have a common boundary ray pair and the map $p_c^{\circ n}$ sends U_1 onto U and fixes both rays on $P_c(\vartheta, \vartheta')$). Therefore $\vartheta < \tilde\vartheta < \tilde\vartheta' < \vartheta'$.

By construction, $p_c \colon U_n \to U$ is a degree 2 branched cover, and $p_c^{\circ(n-1)} \colon U_1 \to U_n$ is a conformal isomorphism. It follows that $|\tilde\vartheta - \vartheta| = |\vartheta' - \tilde\vartheta'| = 2^{-n}|\vartheta' - \vartheta|$ as claimed; this implies uniqueness of $\tilde\vartheta$ and $\tilde\vartheta'$. We also get $|\tilde\vartheta' - \tilde\vartheta| = (1 - 2 \cdot 2^{-n})|\vartheta' - \vartheta|$. It is clear that $P_c(\tilde\vartheta, \tilde\vartheta')$ separates $\mathbb{C}$ so that all its forward images are on the same side as the origin; therefore this ray pair is characteristic if and only if it separates the critical value from the origin. □

Proof of Proposition 9.24. Suppose we have the lower bound $|W_{p/q}| \geq |W|\frac{(2^n-1)^2}{2^{qn}-1}$. Since all wakes of $W_{p/q}$ are disjoint for all $p/q \in (0,1)$ in lowest terms, their combined widths are at least

$$|W| \geq \sum_{p/q} |W_{p/q}| \geq |W|(2^n - 1)^2 \sum_{p/q} \frac{1}{2^{qn} - 1} = |W|$$

where the last equality is Lemma 9.25. This shows that we must have equality throughout, so it suffices to prove the lower bounds.

Let $P(\vartheta, \vartheta')$ be the parameter ray pair landing at the root of W and let $\delta := |\vartheta' - \vartheta| = |W|$. For every parameter c in the wake of W, there is a characteristic periodic ray pair $P_c(\vartheta, \vartheta')$ by Theorem 9.5, and a preperiodic ray pair $P_c(\tilde\vartheta, \tilde\vartheta')$ with $\vartheta < \tilde\vartheta < \tilde\vartheta' < \vartheta'$ and $|\tilde\vartheta - \vartheta| = |\vartheta' - \tilde\vartheta'| = 2^{-n}\delta$ by Lemma 9.26. Let U' be the domain bounded by $P_c(\vartheta, \vartheta')$ and $P_c(\tilde\vartheta, \tilde\vartheta')$ (compare Figure 9.8).

For simplicity, suppose c is the center of $W_{p/q}$. Let $P(\vartheta_{p/q}, \vartheta'_{p/q})$ be the parameter ray pair bounding the wake of $W_{p/q}$. Then by Theorem 9.5, there is a characteristic dynamic ray pair $P_c(\vartheta_{p/q}, \vartheta'_{p/q})$; these are two of a total of $q \geq 2$ rays which land together at the same periodic point $z_{p/q}$ of p_c and which are permuted transitively by $p_c^{\circ n}$ (Theorem 9.10).

Now we claim that $P_c(\tilde\vartheta, \tilde\vartheta')$ does not separate c from the origin: otherwise, this dynamic ray pair would be characteristic by Lemma 9.26, so by Theorem 9.5 the

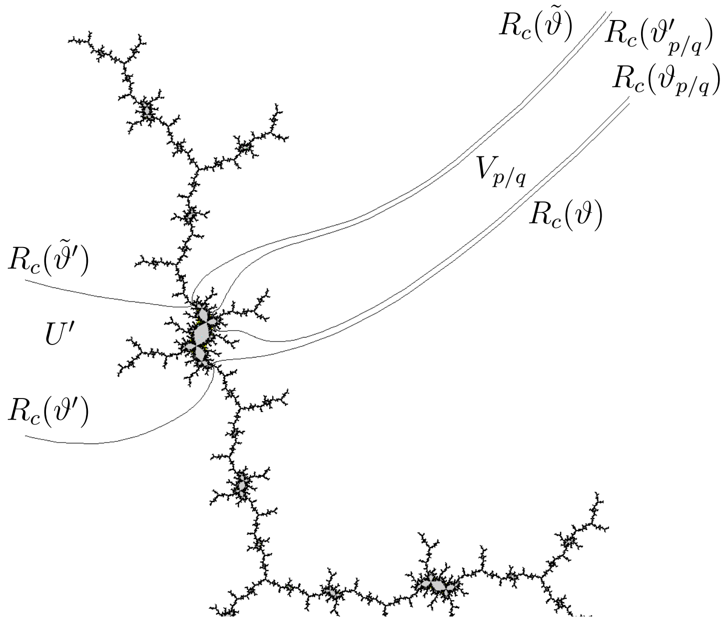

FIGURE 9.8. Dynamic ray pairs in the proof of Proposition 9.24: the characteristic ray pair $P(\vartheta, \vartheta')$, the precharacteristic ray pair $P(\tilde\vartheta, \tilde\vartheta')$ on the same orbit, and another characteristic ray pair $P(\vartheta_{p/q}, \vartheta'_{p/q})$.

parameter c would have to be separated from $P(\vartheta, \vartheta')$ by a parameter ray pair $P(\tilde\vartheta, \tilde\vartheta')$; but this is impossible because $c \in W_{p/q}$ and $P(\vartheta, \vartheta')$ lands on ∂W. However, the characteristic ray pair $P_c(\vartheta_{p/q}, \vartheta'_{p/q})$ separates the critical value c from the origin and from $P_c(\vartheta, \vartheta')$, and this implies that $P_c(\vartheta_{p/q}, \vartheta'_{p/q}) \subset U'$.

Let $V_{p/q}$ be the sector bounded by the ray pair $P_c(\vartheta_{p/q}, \vartheta'_{p/q})$. Since the ray pair is characteristic, the sector $V_{p/q}$ contains the critical value and thus, by Lemma 9.3, maps forward homeomorphically for at least $(q-2)n$ iterations. It follows that one of $p_c^{\circ kn}(V_{p/q})$ (for $k = 0, 1, \ldots, q-2$) contains $P_c(\tilde\vartheta, \tilde\vartheta')$, so $|\tilde\vartheta' - \tilde\vartheta| < 2^{kn}|\vartheta'_{p/q} - \vartheta_{p/q}| \le 2^{(q-2)n}|\vartheta'_{p/q} - \vartheta_{p/q}|$ or

$$|W_{p/q}| = |\vartheta'_{p/q} - \vartheta_{p/q}| > 2^{-(q-2)n}|\tilde\vartheta' - \tilde\vartheta| = 2^{-(q-2)n}(1 - 2 \cdot 2^{-n})\delta \ .$$

But since $\vartheta_{p/q}$ and $\vartheta'_{p/q}$ are periodic of period qn, they have the form $a/(2^{qn}-1)$ and $a'/(2^{qn}-1)$ with integers a, a', hence $|W_{p/q}| = b/(2^{qn}-1)$ for an integer b. Similarly,

$\delta = b_0/(2^n - 1)$ for another integer b_0. Therefore

$$\begin{aligned} b &> (2^{qn}-1)2^{-(q-2)n}(1-2\cdot 2^{-n})\delta = (2^n - 2^{-(q-1)n})(2^n-2)\delta \\ &= \left[(2^n-1)(2^n-1) + (-1-2^{-(q-2)n} + 2^{-(q-1)n+1}]\right)\delta \\ &= b_0(2^n-1) + \left[-1-2^{-(q-2)n}+2^{-(q-1)n+1}\right]\delta\ . \end{aligned}$$

Now b and b_0 are odd integers, while the value in the square bracket on the last line is in $(-2,0)$ and $\delta \in (0,1)$. This implies $b \geq b_0(2^n-1) = \delta(2^n-1)^2$ as claimed. □

9.27. **Lemma (Centers and Roots Accumulate in Boundary of $\mathcal{M}$)**
Let C and R be the sets of centers and roots of hyperbolic components of $\mathcal{M}$, and let MT be the set of Misiurewicz-Thurston parameters. Then $\partial\mathcal{M} = \overline{MT} = \overline{R} \subset \overline{C}$.

Proof. Let U be an open set intersecting $\partial\mathcal{M}$ and consider the sequence of polynomials P_n defined via $P_0(c) := 0$, $P_{n+1}(c) := P_n^2(c) + c$. This sequence is bounded for $c \in \mathcal{M}$ (an easy induction proves that $|P_n(c)| \leq 4$ for $c \in \mathcal{M}$) but unbounded for $c \notin \mathcal{M}$. Therefore the restrictions of P_n to U do not form a normal family. By Montel's theorem, it follows that there are infinitely many n for which there are $c \in U$ with $P_n(c) \in \{0, c, \infty\}$. Since $P_n(c) = \infty$ is impossible, we either have $P_n(c) = 0$ or $P_n(c) = c$, hence $P_{n-1}(c) = 0$. But this means there are infinitely many n for which the orbit of 0 under $z \mapsto z^2 + c$ is periodic of period n or $n-1$, so $U \cap C$ is infinite. This implies $\partial M \subset \overline{C}$.

Similarly, let $\alpha(c), \beta(c) := 1/2 \pm \sqrt{1/4 - c}$ be the two fixed points of p_c; these depend holomorphically on c for $c \in \mathbb{C} \setminus \{1/4\}$. As above, use Montel's theorem: let $U \subset \mathbb{C} \setminus \{1/4\}$ be open so that $\alpha(c)$ and $\beta(c)$ are invertible; every $c_0 \in \partial\mathcal{M}$ has such a neighborhood, possibly with finitely many exceptions for c_0 (where $\alpha(c)$ or $\beta(c)$ have critical points). For every $c \in U$ there is an automorphism $M_c\colon z \mapsto (z-\alpha(c))/(\beta(c)-\alpha(c))$ of $\mathbb{C}$. The maps $c \mapsto M_c(P_n(c))\colon U \to \mathbb{C}$ are holomorphic in c, and they cannot form a normal family if U intersects $\partial\mathcal{M}$. But $M_c(c) \in \{0,1\}$ means that $P_n(c) \in \{\alpha(c), \beta(c)\}$, so p_c is a Misiurewicz-Thurston parameter with $c \in U$. This proves that $\partial\mathcal{M} \subset \overline{MT}$. The converse is clear: every Misiurewicz-Thurston parameter c is in $\mathcal{M}$ because the critical orbit is preperiodic, and it is in $\partial\mathcal{M}$ because c is the landing point of a parameter ray by Theorem 9.2.

Finally, we discuss the set R of roots. Since $R \subset \partial\mathcal{M}$, it suffices to prove that for every $c \in \partial\mathcal{M}$ there is a sequence of roots $r_k \in R$ with $r_k \to c$. We just showed there is a sequence of centers $c_k \in C$ with $c_k \to c$. For every k, let W_k be the component with center c_k. We may as well suppose that the periods of all W_k are different, hence all W_k are disjoint. Since the total area of W_k is finite, there is a sequence of parabolic boundary points $r_k \in \partial W_k$ with $|r_k - c_k| \to 0$ (we do not claim that r_k is the root of W_k); then $r_k \to c$. □

Show that centers are simple!

Parameter ray pairs at periodic angles satisfy the following important property which is the fundamental motor of Lavaurs' Algorithm 13.4 for constructing periodic parameter ray pairs inductively by increasing periods; it was shown by Lavaurs [**Lv**].

9.28. **Lavaurs' Lemma**
If $P(\vartheta,\vartheta')$ and $P(\tilde\vartheta,\tilde\vartheta')$ are two periodic parameter ray pairs of equal period such that $P(\vartheta,\vartheta')$ separates $P(\tilde\vartheta,\tilde\vartheta')$ from the origin, then these two ray pairs are separated from each other by a ray pair of lower period.

PROOF. Suppose $P(\vartheta,\vartheta')$ and $P(\tilde\vartheta,\tilde\vartheta')$ are two ray pairs of equal period, say n, so that $P(\vartheta,\vartheta')$ separates $P(\tilde\vartheta,\tilde\vartheta')$ from the origin.

Let U be the domain separated from the origin by $P(\tilde\vartheta,\tilde\vartheta')$, and pick a Misiurewicz-Thurston parameter $c \in U$ (such a parameter exists as the landing point of a parameter ray at preperiodic angle in U). The reason why we prefer to work with a Misiurewicz-Thurston-parameter is that the filled-in Julia set K_c is a dendrite and hence uniquely arcwise connected: any two $z, z' \in K_c$ are connected by a unique (injective) arc in K_c, say $[z,z']$: by [**Mi1**, Theorem 19.7, Lemma 17.17 and Lemma 17.18], K_c is arcwise connected, and uniqueness of the arc follows because the Fatou set is connected and the Julia set has no interior.

By the Correspondence Theorem 9.5, in the dynamical plane of p_c there are two characteristic dynamic ray pairs $P_c(\vartheta,\vartheta)$ and $P_c(\tilde\vartheta,\tilde\vartheta')$; let w and $\tilde w$ be their landing points and consider the arc $[w,\tilde w] \subset K_c$ connecting them. Let $k \geq 1$ be maximal so that $p_c^{\circ k}\colon [w,\tilde w]$ is injective. It is easy to see that $k < n$: otherwise, $p_c^{\circ n}\colon [w,\tilde w] \to [w,\tilde w]$ would be a homeomorphism, and repeated pull-back of $[w,\tilde w]$ could bring this arc back to itself; but such pull-backs are contracting with respect to the hyperbolic metric in the complement (in $\mathbb{C}$) of the critical orbit; this would force $\tilde w = w$.

It follows that there is a unique point $w' \in [w,\tilde w]$ so that $p_c^{\circ(k-1)}(w') = 0$, hence $p_c^{\circ k}\colon [w,w'] \to [p_c^{\circ k}(w), c]$ is a homeomorphism. Since $P_c(\vartheta,\vartheta')$ and $P_c(\tilde\vartheta,\tilde\vartheta')$ are characteristic, it follows that $p_c^{\circ k}[w,w'] \supset [w,w']$. If $p_c^{\circ k}(w) = w$, we would have rays of period n land at a periodic point of period k strictly dividing n, but this would be a contradiction to Lemma 9.3 because the same sector at w' must contain c and $\tilde w$, and $p_c^{\circ k}\colon [w,w'] \to [w',c]$ is a homeomorphism.

Therefore $[w,w']$ contains an interior point, say z, that is fixed by $p_c^{\circ k}$, and in fact z has exact period k. Let V be a linearizing neighborhood of z. Then an appropriate branch of $p_c^{\circ(-k)}$ fixes z; this branch maps V into itself and $[w,w'] \cap V$ into itself. It follows that z is the landing point of two dynamic rays of period k, one in each component of $V \setminus [w,w']$. The period of the rays equals the period of z, so no further rays can land at z by Lemma 9.3. Let $P_c(\varphi,\varphi')$ be the ray pair landing at z. It separates $P_c(\tilde\vartheta,\tilde\vartheta')$ from $P(\vartheta,\vartheta')$. This ray pair must be characteristic because w is characteristic and $p_c^{\circ m}\colon [w,w']$ is injective for $m = 0,1,\dots,k-1$, so no $p_c^{\circ m}\colon [w,w']$ can intersect $[w,w']$ if $m \le k-1$. Again by Theorem 9.5, we obtain a parameter ray pair $P(\varphi,\varphi')$ of period $k<n$ as desired. $\square$

A different proof of Lavaurs' Lemma uses basic ideas about internal addresses as in Section 11 (it is a variant of [**LS**, Lemma 3.8]).

Proof of Lavaurs' Lemma 9.28, second variant. Suppose $P(\vartheta,\vartheta')$ and $P(\tilde\vartheta,\tilde\vartheta')$ are two parameter ray pairs of equal period, say n, that violate the result. Without loss of generality, suppose that $P(\vartheta,\vartheta')$ separates $P(\tilde\vartheta,\tilde\vartheta')$ from the origin. Let U be the wake bounded by $P(\vartheta,\vartheta')$; then for every $c\in U$, there is a characteristic dynamic ray pair $P_c(\vartheta,\vartheta')$ landing at a repelling orbit (Theorem 9.5), and $P(\tilde\vartheta,\tilde\vartheta')\subset U$.

Let $\tilde c$ be the landing point of $P(\tilde\vartheta,\tilde\vartheta')$. In the dynamics of $p_{\tilde c}$, the ray pair $P_{\tilde c}(\tilde\vartheta,\tilde\vartheta')$ must land at the parabolic orbit by Theorem 9.1, and there is a dynamic ray pair $P_{\tilde c}(\vartheta,\vartheta')$ landing at a repelling orbit. Let V and $\tilde V$ be the domains separated from the origin by $P_{\tilde c}(\vartheta,\vartheta')$ and $P_{\tilde c}(\tilde\vartheta,\tilde\vartheta')$; then $\tilde V\subset V$. Let $V_0 := p_{\tilde c}^{-1}(V)$ and $\tilde V_0 := p_{\tilde c}^{-1}(\tilde V)\subset V_0$; since the Fatou component containing the critical point must be contained in $\tilde V_0$, it follows that $P_{\tilde c}(\vartheta,\vartheta')$ does not land at any periodic Fatou component (hence not at any Fatou component at all). The kneading sequence $\nu := \nu(\vartheta)$ equals the itinerary of the orbit of $R_{\tilde c}(\vartheta)$ under the dynamics of $p_{\tilde c}$ with respect to the partition $\mathcal{P} := \mathbb{C}\setminus(V_0\cup R_{\tilde c}(\vartheta/2)\cup R_{\tilde c}((1+\vartheta)/2))$ (generating an entry $\star$ at every n-the entry); the kneading sequence $\tilde\nu := \nu(\tilde\vartheta)$ is defined similarly with respect to the partition $\widetilde{\mathcal{P}} := \mathbb{C}\setminus(\tilde V_0\cup R_{\tilde c}(\tilde\vartheta/2)\cup R_{\tilde c}((1+\tilde\vartheta)/2))$. But $\widetilde{\mathcal{P}}\subset\mathcal{P}$; this implies that the itinerary of $R_{\tilde c}(\vartheta)$ with respect to $\widetilde{\mathcal{P}}$ equals either $\mathcal{A}(\nu)$ or $\overline{\mathcal{A}}(\nu)$. We will argue below that $\tilde\nu=\nu$ and first continue the proof.

Let $P=\overline{\bigcup_{k\ge0}p_c^{\circ k}(0)}$ be the postcritical set and $W:=\mathbb{C}\setminus P$. Let z and $\tilde z$ be the landing points of $P_{\tilde c}(\vartheta,\vartheta')$ and $P_{\tilde c}(\tilde\vartheta,\tilde\vartheta')$, and let U_c be the Fatou component containing c; then $\tilde z\in U_c$. Connect z and $\tilde z$ by a smooth curve $\gamma\in W$ that does not intersect $P_{\tilde c}(\tilde\vartheta,\tilde\vartheta')$ other than at the endpoint. Pulling back this curve along the periodic branch of the backwards orbit of $P_{\tilde c}(\vartheta,\vartheta')$, we obtain a sequence of curves in W connecting z with U_c (equal itineraries assure that a consistent branches connects preimages, so that after n backwards iterations we connect the periodic point z with the periodic component U_c, rather than with preperiodic preimages). After n backwards steps, the only boundary point of U_c that can be reached from z without crossing $P_{\tilde c}(\tilde\vartheta,\tilde\vartheta')$ is $\tilde z$, so we obtain a family of smooth curves connecting z with $\tilde z$.

The continued preimages are contracting with respect to the hyperbolic metric of W, even though the hyperbolic length of γ in W is infinite. However, it is easy to see that under continued pull-backs, points on γ must converge to z and simultaneously to $\tilde z$ (the curve accesses U_c through a repelling petal). This implies that $z=\tilde z$, but one point is repelling and the other is parabolic, a contradiction.

Finally, we need to show that $\nu = \tilde{\nu}$: it is here that the assumption enters that $P(\vartheta, \vartheta')$ and $P(\tilde{\vartheta}, \tilde{\vartheta}')$ are not separated by a ray pair of lower period. The short and self-contained proof can be found in Lemma 11.2. □

An extension of Lavaurs' Lemma is true even if none of the two parameter ray pairs $P(\vartheta, \vartheta')$ and $P(\tilde{\vartheta}, \tilde{\vartheta}')$ separates the other from the origin: any two parameter rays (or two hyperbolic components) of equal period n are either separated by a parameter ray of lower period, or they are in different subwakes of a hyperbolic component of lower period. This is shown in Theorem 11.10 as *completeness of internal addresses*. An abstract version of Lavaurs' lemma can be found in Lemma 6.6.

Most of the attention so far has been on parameter ray pairs at periodic angles. In fact, these ray pairs determine the entire combinatorial structure of $\mathcal{M}$: every parameter ray pair in the Mandelbrot set can be approximated by periodic parameter ray pairs. We will show the following.

9.29. **Theorem (Periodic Lamination Suffices)**
Suppose $\vartheta < \vartheta'$ are non-periodic and $P(\vartheta, \vartheta')$ is a ray pair of the Mandelbrot set. Then there is a sequence of periodic parameter ray pairs $P(\vartheta_n, \vartheta'_n)$ with $\vartheta_n \to \vartheta$ and $\vartheta'_n \to \vartheta'$.

If ϑ and ϑ' are not even preperiodic, then there are two such sequences of periodic parameter ray pairs, one with $\vartheta_n \nearrow \vartheta$ and $\vartheta'_n \searrow \vartheta'$, and the other one with $\vartheta_n \searrow \vartheta$ and $\vartheta'_n \nearrow \vartheta'$.

The same is true if ϑ and ϑ' are preperiodic and no further parameter ray lands at the same parameter as $R(\vartheta)$ and $R(\vartheta')$.

What we will actually prove is the stronger Corollary 9.33; see below.

9.30. **Proposition (Approximation by Rational Ray Pairs, Irrational Case)**
Suppose $R(\vartheta)$ and $R(\vartheta')$ are two parameter rays at irrational angles which are not separated by a periodic or preperiodic parameter ray pair. Then either both rays land on the boundary of the same hyperbolic component, or for every $\varepsilon > 0$ there are two parameter ray pairs $P(\vartheta_1, \vartheta'_1)$ and $P(\vartheta_2, \vartheta'_2)$ at rational angles with $|\vartheta_1 - \vartheta_2| < \varepsilon$, $|\vartheta'_1 - \vartheta'_2| < \varepsilon$ and $\vartheta \in (\vartheta_1, \vartheta_2)$, $\vartheta' \in (\vartheta'_1, \vartheta'_2)$.

Proof. Suppose without loss of generality that $\vartheta < \vartheta'$. Let $\tilde{\vartheta} < \tilde{\vartheta}'$ be the infimum of angles in rational parameter ray pairs with angles in (ϑ, ϑ'), in the following sense: for every $\varepsilon > 0$ there is a rational ray pair connecting an angle in $(\tilde{\vartheta}, \tilde{\vartheta} + \varepsilon)$ with an angle in $(\tilde{\vartheta}' - \varepsilon, \tilde{\vartheta})$, and $|\tilde{\vartheta}' - \tilde{\vartheta}|$ is maximal possible with this property. We want to show that $\tilde{\vartheta} = \vartheta$ and $\tilde{\vartheta}' = \vartheta'$. Our proof is inspired by the proof that local connectivity of $\mathcal{M}$ implies density of hyperbolicity (Corollary 9.14).

Suppose not. If $\tilde{\vartheta}' = \vartheta'$ but $\tilde{\vartheta} \neq \vartheta$, then pick two periodic angles $\varphi_1 \in (\vartheta, \tilde{\vartheta})$ and $\varphi_2 \in (\tilde{\vartheta}, \vartheta')$ and let φ'_1, φ'_2 be the corresponding angles so that $P(\varphi_1, \varphi'_1)$ and $P(\varphi_2, \varphi'_2)$ are two parameter ray pairs; let W_1 and W_2 be the hyperbolic components at whose roots they land. By construction, we must have $\varphi_1' \in (\vartheta, \tilde\vartheta)$ and $\varphi_2' \in (\tilde\vartheta, \vartheta')$. Now we apply the Branch Theorem 9.13 to W_1 and W_2. The wakes of these components are disjoint, so they are separated by either a hyperbolic component W or a Misiurewicz-Thurston point. The second case would imply that a parameter ray pair landing at this Misiurewicz-Thurston point intersects a parameter ray pair at angles near $\tilde\vartheta$ and ϑ', or it separates $R(\vartheta)$ from $R(\vartheta')$, and both is a contradiction. In the first case, the wake of W contains $R(\tilde\vartheta)$ and thus also $R(\vartheta)$ and $R(\vartheta')$. They cannot be in the same subwake by choice of W and they cannot be subwake boundaries because ϑ, ϑ' are non-periodic. Since $R(\vartheta)$ and $R(\vartheta')$ are not separated from each other by periodic ray pairs, none of $R(\vartheta)$ and $R(\vartheta')$ can be in a subwake, so both parameter rays must land at ∂W by Corollary 9.16 (in fact, one easily gets a contradiction to the condition $\tilde\vartheta' = \vartheta'$). The same argument applies if $\tilde\vartheta = \vartheta$ but $\tilde\vartheta' \neq \vartheta'$.

Therefore we may suppose that $\vartheta < \tilde\vartheta < \tilde\vartheta' < \vartheta'$. As above, there are two parameter ray pairs $\varphi_1 \in (\vartheta, \tilde\vartheta)$ and $\varphi_2 \in (\tilde\vartheta', \vartheta')$ at rational angles; they form parameter ray pairs $P(\varphi_1, \varphi_1')$ and $P(\varphi_2, \varphi_2')$ landing at roots of hyperbolic components W_1 and W_2. We may again suppose that $\vartheta < \varphi_1 < \varphi_1' < \tilde\vartheta < \tilde\vartheta' < \varphi_2 < \varphi_2' < \vartheta'$ (otherwise one of the ray pairs $P(\varphi_1, \varphi_1')$ and $P(\varphi_2, \varphi_2')$ would contradict the choice of $\tilde\vartheta, \tilde\vartheta'$). As before, we apply the Branch Theorem 9.13 to W_1 and W_2, which have again disjoint wakes. As above, this yields a rational ray pair contradicting the choice of $\tilde\vartheta, \tilde\vartheta'$ unless the separation is provided by a hyperbolic component W so that neither $R(\vartheta)$ nor $R(\vartheta')$ are separated from W. In the latter case, both parameter rays must land at W by Corollary 9.16.

This proves that, unless $R(\vartheta)$, $R(\vartheta')$ land on the boundary of the same hyperbolic component, there is a sequence of parameter ray pairs at rational angles so that the angles converge to $\{\vartheta, \vartheta'\}$ from within (ϑ, ϑ'). There is a similar sequence of ray pairs whose angles converge from outside, so that one angle is in $(\vartheta - \varepsilon, \vartheta)$ and the other one in $(\vartheta', \vartheta' + \varepsilon)$; the reasoning is similar (if $R(\vartheta)$ and $R(\vartheta')$ are in the wake of a hyperbolic component W, they must either both land at ∂W as above, or they must be in the same subwake at rational internal angle by Proposition 9.24). $\square$

The same argument also works if ϑ and ϑ' are preperiodic. Since three or more parameter rays at preperiodic angles may land together, the statement has to be adapted; it becomes simpler because all preperiodic parameter rays land. Similarly, if ϑ and ϑ' are periodic and $P(\vartheta, \vartheta')$ lands at the root of a primitive component, then this parameter ray pair can be approximated from one side. In all cases, the proof remains essentially the same as above. We obtain the following statements which also prove Theorem 9.29.

9.31. **Corollary (Approximation by Rational Ray Pairs, Rational Cases)**
(1) Suppose $P(\vartheta, \vartheta')$ is a parameter ray pair at preperiodic angles so that no parameter ray $R(\vartheta'')$ with $\vartheta'' \in (\vartheta, \vartheta')$ lands at the same point. Then for every $\varepsilon > 0$ there is a

parameter ray pair $P(\vartheta_1, \vartheta_1')$ at rational angles with $|\vartheta_1 - \vartheta| < \varepsilon$, $|\vartheta_1' - \vartheta'| < \varepsilon$ and $\vartheta < \vartheta_1 < \vartheta_1' < \vartheta'$.

(2) Suppose $P(\vartheta, \vartheta')$ is a parameter ray pair at preperiodic angles so that no parameter ray $R(\vartheta'')$ with $\vartheta'' \in \mathbb{S}^1 \setminus [\vartheta, \vartheta']$ lands at the same point. Then for every $\varepsilon > 0$ there is a parameter ray pair $P(\vartheta_1, \vartheta_1')$ at rational angles with $|\vartheta_1 - \vartheta| < \varepsilon$, $|\vartheta_1' - \vartheta'| < \varepsilon$ and $\vartheta_1 < \vartheta < \vartheta' < \vartheta_1'$.

(3) Suppose $R(\vartheta)$ is a parameter ray at a preperiodic angle so that no parameter ray $R(\vartheta'')$ with $\vartheta'' \neq \vartheta$ lands at the same point. Then for every $\varepsilon > 0$ there is a parameter ray pair $P(\vartheta_1, \vartheta_1')$ at rational angles with $|\vartheta_1' - \vartheta_1| < \varepsilon$ and $\vartheta_1 < \vartheta < \vartheta_1'$.

(4) Suppose $P(\vartheta, \vartheta')$ is a parameter ray pair at periodic angles which lands at the root of a primitive hyperbolic component. Then for every $\varepsilon > 0$ there is a parameter ray pair $P(\vartheta_1, \vartheta_1')$ at rational angles with $|\vartheta_1 - \vartheta| < \varepsilon$, $|\vartheta_1' - \vartheta'| < \varepsilon$ and $\vartheta_1 < \vartheta < \vartheta' < \vartheta_1'$. □

9.32. **Lemma (Approximation of Preperiodic by Periodic Ray Pairs)**
Suppose $P(\vartheta, \vartheta')$ is a preperiodic parameter ray pair at preperiodic angles so that no parameter ray $R(\vartheta'')$ with $\vartheta'' \in (\vartheta, \vartheta')$ lands at the same point. Then for every $\varepsilon > 0$ there is a parameter ray pair $P(\vartheta_1, \vartheta_1')$ at periodic angles with $|\vartheta_1 - \vartheta| < \varepsilon$, $|\vartheta_1' - \vartheta'| < \varepsilon$ and $\vartheta < \vartheta_1 < \vartheta_1' < \vartheta'$.

Proof. By Corollary 9.31, there is a sequence of parameter ray pairs $P(\vartheta_n, \vartheta_n')$ at rational angles with $\vartheta_n \searrow \vartheta$ and $\vartheta_n' \nearrow \vartheta'$, say with monotone convergence of both sequences. We want to show that this sequence can be chosen so that all ϑ_n, and hence all ϑ_n', are periodic. For a proof by contradiction, assume that there is no such subsequence, i.e., all $(\vartheta_n, \vartheta_n')$ are preperiodic and no two parameter ray pairs $P(\vartheta_n, \vartheta_n')$ are separated by a parameter ray pair at periodic angles. Since there are only finitely many external angles with given period or preperiod, we may assume that no two ϑ_n have identical period and preperiod. That means, the dynamic rays $R_c(\vartheta_n)$ and $R_c(\vartheta_{n'})$ cannot land together for any p_c if $n' \neq n$.

Let c be the landing point of the parameter ray pair $P(\vartheta_1, \vartheta_1')$. We will work in the dynamical plane of p_c, and our contradiction will be that all $R_c(\vartheta)$ must land together. There are characteristic dynamic ray pairs $P_c(\vartheta_n, \vartheta_n')$ for all n; let V_n be the domains separated from the origin by $P_c(\vartheta_n, \vartheta_n')$, so that $V_{n+1} \supset V_n$ for all n. Let $W_n := p_c^{-1}(V_n)$; this is a neighborhood of the origin bounded by the two preimage ray pairs of $P_c(\vartheta_n, \vartheta_n')$. We will prove that all angles ϑ_n have identical associated kneading sequences $\nu(\vartheta_n)$, that therefore the landing points z_n of $P_c(\vartheta_n, \vartheta_n')$ have identical itineraries with respect to $\mathbb{C} \setminus W_1$, and that this implies that all z_n coincide; but this is a contradiction to the fact that all ϑ_n have different periods and preperiods.

The kneading sequence $\nu_n := \nu(\vartheta_n)$ is defined as the itinerary of ϑ_n on $\mathbb{S}^1 \setminus \{\vartheta_n/2, (1 + \vartheta_2)/2\}$ as in Definition 2.1; since ϑ_n is preperiodic, the orbit of ϑ_n never hits the boundary $\{\vartheta_n/2, (1 + \vartheta_2)/2\}$. The kneading sequence ν_n coincides with the itinerary of $R_c(\vartheta_n)$ on $\mathbb{C} \setminus W_n$ or, since $W_1 \subset W_n$, on $\mathbb{C} \setminus W_1$. Therefore, the point z_n has itinerary ν_n with respect to $\mathbb{C} \setminus W_1$; no point on the forward orbit of any z_n ever enters W_n (the ray pairs $R_c(\vartheta, \vartheta')$ are characteristic, and their orbits never enter $V_n = p_c(W_n)$). Since the parameter ray pairs $P(\vartheta_n, \vartheta'_n)$ are not separated by parameter ray pairs at periodic angles, all ϑ_n have identical kneading sequences ν_n (the proof of Lemma 11.2 is self-contained). But this means that all z_n have identical itineraries, and a simple hyperbolic contraction argument shows that all z_n must coincide, a contradiction as claimed. □

9.33. **Corollary (Approximation by Periodic Ray Pairs)**
All parameter ray pairs $P(\vartheta_1, \vartheta'_1)$ and $P(\vartheta_2, \vartheta'_2)$ in Proposition 9.30 and Corollary 9.31 can be chosen so that ϑ_1, ϑ'_1, ϑ_2 and ϑ'_2 are periodic.

Proof. If the approximating parameter ray pairs $P(\vartheta_1, \vartheta'_1)$ and $P(\vartheta_2, \vartheta'_2)$ in Proposition 9.30 and Corollary 9.31 have preperiodic angles, then these can in turn be approximated by periodic ray pairs by Lemma 9.32 (perhaps only from one side, but this is good enough, even if three or more preperiodic parameter rays land together). □

The following result says in particular that if three rays land together, their angles must be preperiodic.

9.34. **Corollary (Three Unseparated Rays)**
Suppose that there are three parameter rays $R(\vartheta_1)$, $R(\vartheta_2)$, $R(\vartheta_3)$ so that no two of them are separated by a parameter ray pair at a periodic angle. Then either all three rays land on the boundary of the same hyperbolic component, or all three angles are preperiodic and the three rays land at the same Misiurewicz-Thurston point.

Proof. The proof is analogous to the proof in Proposition 9.30. We may suppose that $\vartheta_1 < \vartheta_2 < \vartheta_3$. There are two hyperbolic components W_1 and W_2 so that the parameter rays pairs $P(\varphi_1, \varphi'_1)$ and $P(\varphi_2, \varphi'_2)$ landing at their roots satisfy

$$\vartheta_1 < \varphi_1 < \varphi'_1 < \vartheta_2 < \varphi_2 < \varphi'_2 < \vartheta_3 \ .$$

Once more we apply the Branch Theorem 9.13 to W_1 and W_2. The wakes of these components are disjoint, so they are separated by either a hyperbolic component W or a Misiurewicz-Thurston parameter c_0. In the first, case, none of the rays $R(\vartheta_i)$ can be in a subwake of W, so all $R(\vartheta_i)$ must land at ∂W by Corollary 9.16. In the second case, each $R(\vartheta_i)$ either lands at c_0 or is separated from c_0 by a parameter ray at periodic angles by Corollaries 9.31 and 9.33. □

10. Renormalization

We discuss renormalization

The theory of renormalization is one of the deepest and most interesting topics in the study of quadratic polynomials and the Mandelbrot set $\mathcal{M}$. One reason is that it explains the appearance of countably many homeomorphic copies of $\mathcal{M}$ embedded within itself, and in fact within all nontrivial parameter spaces of rational maps (in the words of McMullen [**Mcxxx**], *the Mandelbrot set is universal*). Another reason is that the study of quadratic polynomials — for example by the famous Yoccoz puzzle method [**HY**] — is feasible only for non-renormalizable maps, and many interesting properties of the dynamics are preserved under renormalization.

10.1. **Definition (Polynomial-Like Mapping)**
Let $U, V \subset \mathbb{C}$ be two bounded domains. A holomorphic map $f\colon U \to V$ is called a polynomial-like map *if the following conditions are satisfied:*

- *U and V are simply connected, $\overline{U}$ is compact and $\overline{U} \subset V$;*
- *$f\colon U \to V$ is holomorphic and proper.*

The filled-in Julia set *K_f is the set of all $z \in U$ which remain in U for all iterates of f.*

Recall that a map $f\colon U \to V$ is proper if the preimage of every compact set is compact; equivalently, if $z_n \to \partial U$, then $f(z_n) \to \partial V$. Every proper holomorphic map on a connected domain has a well-defined mapping degree: for every $z \in V$, the number or preimages $f^{-1}(z)$ is constant, counting multiplicities. Often, a polynomial-like map is required to have mapping degree at least 2 (for degree 1, we simply have a domain of linearization of a repelling fixed point). A polynomial-like map of degree 2 is oftten called a *quadratic-like map.*

For polynomial-like maps it is particularly important to specify the domains U and V precisely: often, restrictions of high degree polynomials (or even transcendental functions) to appropriate domains are polynomial-like of low degrees, and different restrictions might be polynomial-like of different degrees.

The fundamental result about the dynamics of polynomial-like maps is the Straightening Theorem due to Douady and Hubbard [**DHx**]. In order to state it, we need quasiconformal homeomorphisms[6]. Two polynomial-like maps $f\colon U \to V$ and $f'\colon U' \to V'$ are called *hybrid equivalent* if there exists a quasiconformal homeomorphism $h\colon V \to V'$ which conjugates f on U to f' on U', and so that $\partial h/\partial \overline{z} = 0$ almost everywhere on K_f. Check: was there a condition about generating this equivalence relation?

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[6]A homeomorphism $h\colon U \to U'$ (for $U, U' \subset \mathbb{C}$) is *quasiconformal* if there exists a $k \geq 1$ so that for every open annulus $A \subset U$, $k^{-1} \bmod (A) \leq \bmod (h(A)) \leq k \bmod (A)$, where $\bmod (A)$ denotes the modulus of an annulus. Every quasiconformal homeomorphism is differentiable almost everywhere [**Ah2**].

10.2. **Theorem (The Straightening Theorem [DHx, Theorem ???])**
For every polynomial-like map $f\colon U \to V$ of degree d there is a polynomial p of degree d, and there are two domains $U_p, V_p \subset \mathbb{C}$ so that the restriction $p\colon U_p \to V_p$ is polynomial-like of degree d and hybrid equivalent to f. The polynomial p (but not the domains U_p or V_p) are unique up to affine conjugation if and only if K_f is connected; and K_f is connected if and only if all critical points of $f\colon U \to V$ have their entire orbits in U. □

10.3. **Definition (Renormalization)**
A quadratic polynomial p_c is n-renormalizable if there exist domains $U, V \subset \mathbb{C}$ so that the restriction $p_c^{\circ n}\colon U \to V$ is quadratic-like so that the filled-in Julia set K of $p_c^{\circ n}$ is connected, or equivalently if the orbit of the unique critical point of $p_c^{\circ n}$ on U remains in U under iteration of $p_c^{\circ n}$ [7] *The set K is called the* little Julia set *of $p_c^{\circ n}\colon U \to V$.*

- *The renormalization is of* disjoint type *if all sets K, $p_c(K)$, ..., $p_c^{\circ(n-1)}(K)$ are disjoint.*
- *The renormalization if of* β-type *if there exists $k \in \{1, 2, \dots, n-1\}$ and a periodic $z \in K \cap p_c^{\circ k}(K)$ which is the landing point of periodic dynamic rays of period at most n.*[8]
- *The renormalization if of* α-type *if there exists $k \in \{1, 2, \dots, n-1\}$ and a periodic $z \in K \cap p_c^{\circ k}(K)$ which is the landing point of periodic dynamic rays of period greater than n.*

Renormalizations of disjoint or β-type are also known as simple renormalizations*; renormalizations of α-type are known as* crossed renormalizations.

Every quadratic polynomial $p_c(z) = z^2 + c$ has two fixed points: if the Julia set is connected, then the dynamic ray $R_c(0)$ lands at one of them, called the β-fixed point; the other fixed point is called α (we have $\alpha = \beta$ exactly for $z^2 + 1/4$). The α-fixed point must be the landing point of periodic dynamic rays [**Mi1**, ???], and since no fixed ray is available, there must be $q \geq 2$ dynamic rays landing at α, all of period q, and all on the same orbit of rays (this follows from Lemma 9.3 for $q \geq 3$, and for $q = 2$ there are only two rays of period 2, namely at angles 1/3 and 2/3).

If K is the little Julia set of the quadratic-like map $p_c^{\circ n}\colon U \to V$, then $p_c^{\circ n}$ has two fixed points on K; under the straightening map, these correspond to the two fixed points of a quadratic polynomial, so it makes sense to speak of the α- and β- fixed points of K.

10.4. **Lemma (Rays at Fixed Points of Little Julia Set)**
Suppose p_c is n-renormalizable on domains $U \subset V$, and let K be the little Julia set with fixed points α and β; suppose both are repelling. Then β is the landing point of

[7] The equivalence of these two conditions follows from the Straightening Theorem 10.2.
[8] We will show in where? that the rays landing at z always have exact period n.

dynamic rays of period at most n, while α is the landing point of dynamic rays whose period is a proper multiple of n.

PROOF. The quadratic-like map $p_c^{\circ n}\colon U \to V$ is hybrid equivalent to a quadratic polynomial p with connected Julia set K_p and fixed points α_p, β_p so that β_p is the landing point of the unique ray of period 1, while α_p is the landing point of $q \geq 2$ rays of period q. These q rays divide K_p into q components so that in a neighborhood of α_p, each component maps to some other component in a transitive way.

McMullen [**Mc**, Sec. 7] shows that every renormalization of a quadratic polynomial has exactly one of the types "disjoint", "β", or "α". He first shows that $K \cap p_c^{\circ k}(K)$ is either empty or a repelling periodic point z of $p_c^{\circ n}$ so that the period of z divides n [**Mc**, Theorem 7.3]. Thus z is the landing point of a periodic dynamic ray whose period is a multiple of the period of z. Since $p_c^{\circ n}\colon U \to V$ is hybrid equivalent to a quadratic polynomial p and in particular topologically conjugate to p on K, the point z corresponds to a fixed point of p. In a renormalization of β- or α-type, the point $z \in K \cap p_c^{\circ k}(K)$ corresponds after straightening to the β- or α-fixed point of p; thus the name of the renormalization type: if the fixed point z of $p_c^{\circ n}$ is the landing point of a dynamic ray of period kn for $k \geq 1$, then the conjugation provides a periodic invariant curve of period k landing at the corresponding fixed point of p, and the fixed point is then the landing point of periodic rays of the same period k which are homotopic in $\mathbb{C} \setminus K(p)$ to the ray (by Lindelöf's theorem, a point $z \in K$ that is accessible by a curve $\gamma \in \mathbb{C} \setminus K$ is also the landing point of a ray which is homotopic in $\mathbb{C} \setminus K$ to γ; see [**Ah1**, Theorem 3.5]). If p_c is n-renormalizable of non-disjoint type, then there is a single repelling periodic orbit of period at most n which contains all points of intersection $p_c^{\circ k}(K) \cap p_c^{\circ m}(K)$ [**Mc**, Theorem 7.13]: this shows that the type of renormalization cannot jump from α to β or vice versa when replacing K by $p_c(K)$.

An equivalent (and perhaps more frequent) definition of renormalization of β- and α-type goes as follows: a non-disjoint renormalization is of α-type if z disconnects K, and it is of β-type otherwise. This uses the fact that the α-fixed point of p disconnects the filled-in Julia set into as many components as there are rays landing at α (at least two rays), while the β-fixed point does not disconnect the filled-in Julia set [**Mi1**, Theorem 6.10]. Our definition has the advantage that it is more combinatorial and thus easier to verify in our context.

All renormalizable maps p_c are completely classified. The parameters $c \in \mathcal{M}$ for which p_c are simple n-renormalizable are organized in the form of finitely many compact and connected subsets $\mathcal{M}'$ of $\mathcal{M}$ which are each homeomorphic to $\mathcal{M}$: see Theorem 10.8 below and compare [**?, DH1, ?**]. The parameters $c \in \mathcal{M}$ which are crossed renormalizable are organized in homeomorphic copies of limbs of $\mathcal{M}$ [**RiSch**].

10.5. **Lemma (Simple Renormalization Associated to Periodic Ray Pair)** *For a polynomial p_c with $c \in \mathcal{M}$, consider a characteristic periodic dynamic ray $P_c(\vartheta, \vartheta')$ of some period $n \geq 2$ and let $P_c(\tilde{\vartheta}, \tilde{\vartheta}')$ be the associated precharacteristic dynamic ray pair from Lemma 9.26. Let $U_1 \subset \mathbb{C}$ be the domain bounded by $P_c(\vartheta, \vartheta')$ and $P_c(\tilde{\vartheta}, \tilde{\vartheta}')$. If $p_c^{\circ kn}(c) \in \overline{U}_1$ for all $k \geq 1$, then p_c is simple n-renormalizable.*

In the context of this lemma, we say that *the n-renormalization of p_c is associated to the angles (ϑ, ϑ')*.

Proof. Let $U \subset \mathbb{C}$ be the domain separated from the origin by $P_c(\vartheta, \vartheta')$. Then by Lemma 9.26, the restriction $p_c^{\circ n}\colon U_1 \to U$ is a branched cover of degree 2. This is "almost" a polynomial-like map of degree 2; the problem is that U_1 and U are unbounded and share part of their boundaries, so we do not have $\overline{U}_1 \subset U$.

Pick any $t > 0$ and let V' be the domain U restricted to potentials less than t (that is, using the Böttcher coordinate $\Phi_c\colon \overline{\mathbb{C}} \setminus K_c \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, set $V' := U \setminus \Phi_c^{-1}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_{e^t})$, where $\mathbb{D}_{e^t} = \{z \in \mathbb{C}\colon |z| < e^t\}$). Similarly, let V_1' be the restriction of U_1 to equipotentials below $t/2$. Then V_1' and V' are bounded and the restriction $p_c^{\circ n}\colon V_1' \to V'$ is a branched cover of degree 2, and $\partial V_1' \cap \partial V'$ consists of the ray pair $P(\vartheta, \vartheta')$ at potentials below $t/2$, plus their common landing point, say w. As a landing point of periodic dynamic, rays, w is repelling or parabolic.

All we need to do is "thicken" V_1' and V' near their common boundary; this is a well-known procedure. Let D be a small disk centered at w, small enough so that $\overline{D} \subset p_c^{\circ n}(D)$ if w is repelling, and $\overline{D} \subset p_c^{\circ n}(D) \cup U_1$ if w is parabolic (this will always be satisfied if D is sufficiently small). Choose a small $\varepsilon > 0$ and define Figure!

$$\begin{aligned} V : &= V' \cup p_c^{\circ n}(D) \cup \Phi_c^{-1}\left(\{z \in \mathbb{C}\colon 1 < |z| < e^t; \vartheta - \varepsilon < \arg(z) < \vartheta\}\right) \\ &\qquad \cup \quad \Phi_c^{-1}\left(\{z \in \mathbb{C}\colon 1 < |z| < e^t; \vartheta' < \arg(z) < \vartheta' + \varepsilon\}\right) \end{aligned}$$

and

$$\begin{aligned} V_1 : &= V_1' \cup D \cup \Phi_c^{-1}\left(\{z \in \mathbb{C}\colon 1 < |z| < e^{t/2^n}; \vartheta - \varepsilon/2^n < \arg(z) < \vartheta\}\right) \\ &\qquad \cup \quad \Phi_c^{-1}\left(\{z \in \mathbb{C}\colon 1 < |z| < e^{t/2^n}; \vartheta' < \arg(z) < \vartheta' + \varepsilon/2^n\}\right) \end{aligned}$$

(see Figure 10.1). Then it is easy to check that $p_c^{\circ n}\colon V_1 \to V$ is polynomial-like of degree 2. By the Straightening Theorem 10.2, there is a quadratic polynomial p which has a restriction $p|_{W_1}\colon W_1 \to W$ that is hybrid equivalent to $p_c^{\circ n}\colon V_1 \to V$. Note that any orbit of $p_c^{\circ n}$ on $V_1' \setminus \overline{U}_1$ will eventually leave V_1', so the filled-in Julia set K_{p_c} of $p_c^{\circ n}\colon V_1 \to V$ is contained in $\overline{U}_1$. The unique critical orbit is the orbit of c under iteration of $p_c^{\circ n}$, and it remains in U_1 forever by hypothesis; therefore, $p_c^{\circ n}$ is renormalizable.

Figure 10.1. Illustration of the proof of Lemma 10.5.

By Lemma 9.26, the domains U_1, $p_c(U_1)$, ..., $p_c^{\circ(n-1)}(U_1)$ are disjoint, so the only possible point of intersection of K_{p_c} and $p_c^{\circ k}(K_{p_c})$ (for $k \in \{1, 2, \dots, n-1\}$) could be w; it is the landing point of dynamic rays of period n, so our renormalizations are always of disjoint type or β-type, hence simple. □

10.6. **Lemma (Precharacteristic Ray Pairs)**
Let W by a hyperbolic component of some period $n \geq 2$ with center c_0 and let $P(\vartheta, \vartheta')$ be the parameter ray pair bounding the wake of W. Then for every $k \geq 1$, there are 2^{k-1} preperiodic ray pairs $P_{c_0}(\vartheta_{k,j}, \vartheta'_{k,j})$ (for $j = 1, 2, \dots, 2^{k-1}$) with the following properties (see Figure 10.2):

- $\vartheta < \vartheta_{k,j} < \vartheta'_{k,j} < \vartheta'$;
- $2^n\vartheta_{k,j} = \vartheta_{k-1,j'}$ *and* $2^n\vartheta'_{k,j} = \vartheta'_{k-1,j'}$ *for some* j' *if* $k > 1$;
- $2^n\vartheta_{1,1} = \vartheta'$ *and* $2^n\vartheta'_{1,1} = \vartheta$;
- $|\vartheta'_{k,j} - \vartheta_{k,j}| = 2^{-(k-1)n}|\vartheta'_{1,1} - \vartheta_{1,1}|$;
- $\sum_{k\geq1}\sum_{j=1}^{2^k}|\vartheta'_{k,j} - \vartheta_{k,j}| = |\vartheta' - \vartheta|$;
- *none of these ray pairs separates any of the others from* $P_{c_0}(\vartheta, \vartheta')$;
- *every ray pair other than the* $P_{c_0}(\vartheta_{k,j}, \vartheta'_{k,j})$ *is separated from the critical value by exactly one of the* $P_{c_0}(\vartheta_{k,j}, \vartheta'_{k,j})$.

The ray pairs $P_{c_0}(\vartheta_{k,j}, \vartheta'_{k,j})$ are called precharacteristic ray pairs.

FIGURE 10.2. Illustration of Lemma 10.6.

SKETCH. The ray pair $P_{c_0}(\vartheta_{1,1}, \vartheta'_{1,1})$ is exactly the ray pair $P_{c_0}(\tilde\vartheta, \tilde\vartheta')$ from Lemma 9.26. Let $U_1 \subset \mathbb{C}$ be the domain bounded by $P_{c_0}(\vartheta, \vartheta')$ and $P_{c_0}(\vartheta_{1,1}, \vartheta'_{1,1})$ and U the domain separated from the origin by $P_{c_0}(\vartheta, \vartheta')$; then $p_{c_0}^{\circ n}\colon U_1 \to U$ is a branched cover of degree 2. There are two ray pairs $P_{c_0}(\vartheta_{2,1}, \vartheta'_{2,1})$ and $P_{c_0}(\vartheta_{2,2}, \vartheta'_{2,2})$ within U_1 that map to $P_{c_0}(\vartheta_{1,1}, \vartheta'_{1,1})$ under $p_{c_0}^{\circ n}$, and the construction of the remaining ray pairs follows inductively. The construction shows that none of the $P_{c_0}(\vartheta_{k,j}, \vartheta'_{k,j})$ separates any other from $P(\vartheta, \vartheta')$, and the first four itemized properties in the claim are clearly satisfied.

By Lemma 9.26, we have $|\vartheta'_{1,1} - \vartheta_{1,1}| = (1 - 2^{-n+1})|\vartheta' - \vartheta|$ and thus

$$\begin{aligned}\sum_{k\geq1}\sum_{j=1}^{2^{k-1}}|\vartheta'_{k,j} - \vartheta_{k,j}| &= \sum_{k\geq1} 2^{k-1}2^{-(k-1)n}|\vartheta'_{1,1} - \vartheta_{1,1}| \\ &= 2^{n-1}\sum_{k\geq1} 2^{-k(n-1)}(1 - 2^{-n+1})|\vartheta' - \vartheta| = |\vartheta' - \vartheta| \ .\end{aligned}$$

The last condition will be inserted once we see where it is needed. Idea is that all these ray pairs are on boundary of same Fatou comp. □

10.7. Lemma (Precharacteristic Parameter Ray Pairs)
For all dynamic ray pairs $P_{c_0}(\vartheta_{k,j},\vartheta'_{k,j})$ constructed in Lemma 10.6, there are parameter ray pairs $P(\vartheta_{k,j},\vartheta'_{k,j})$ at the same angles. A polynomial p_c has the property that $p_c^{\circ kn}(c)$ is contained in the closure of the domain bounded by $P_c(\vartheta,\vartheta')$ and $P_c(\vartheta_{1,1},\vartheta'_{1,1})$ for all $k\geq 0$ if and only if $c\in\overline{\mathcal{W}_W}$, but c is not separated from $P(\vartheta,\vartheta')$ by any of the ray pairs $P(\vartheta_{k,j},\vartheta'_{k,j})$.

Proof. We also show that a polynomial p_c has dynamic ray pairs $P_{c_0}(\vartheta_{k,j},\vartheta'_{k,j})$ for all k and j if and only if the parameter c is separated from 0 by $P(\vartheta,\vartheta')$ but by none of the $P(\vartheta_{k,j},\vartheta'_{k,j})$.

The parameter ray pair $P(\vartheta,\vartheta')$ lands at some parabolic parameter $c_W\in\mathcal{M}$ and bounds the wake $\mathcal{W}_W$, and a polynomial p_c has a dynamic ray pair $P_c(\vartheta,\vartheta')$ if and only if $c\in\mathcal{W}_W\cup\{c_W\}$ by Theorem 9.10. Let $U_1(c)$ be the domain bounded by $P_c(\vartheta,\vartheta')$ and $P_c(\vartheta_{1,1},\vartheta'_{1,1})$.

For every p_c with $c\in\mathcal{W}_W$, there exists the dynamic ray pair $P_c(\vartheta_{1,1},\vartheta'_{1,1})=P_c(\tilde\vartheta,\tilde\vartheta')$ by Lemma 9.26. In particular, there are parameters $c\in\mathcal{W}_W$ for which $P_c(\vartheta_{1,1},\vartheta'_{1,1})$ separates the critical value from the origin; since $P_c(\vartheta_{1,1},\vartheta'_{1,1})$ must be characteristic in these cases, Theorem 9.5 implies that there is a parameter ray pair $P(\vartheta_{1,1},\vartheta'_{1,1})$. Let $U_{1,1}$ be the domain in parameter space separated from 0 by $P(\vartheta_{1,1},\vartheta'_{1,1})$.

The critical value $c=p_c^{\circ 0}(c)$ is separated from 0 by the ray pair $P_c(\vartheta,\vartheta')$ whenever this ray pair exists, that is if $c\in\mathcal{W}_W\cup\{c_W\}$, and c is in $\overline{U}_1(c)$ if and only if c is not separated from 0 by $P_c(\vartheta_{1,1},\vartheta'_{1,1})$, i.e., unless the parameter c is separated from 0 by $P(\vartheta_{1,1},\vartheta'_{1,1})$.

The immediate preimages under p_c of $P_c(\vartheta_{1,1},\vartheta'_{1,1})$ are two ray pairs. Their angles coincide with those for p_{c_0} for all parameters in $\mathcal{M}\cap\overline{\mathcal{W}_W}\setminus U_{1,1}$. For parameters in $U_{1,1}$, the preimage rays combine to different ray pairs, while at the landing point $c_{1,1}$ of $P(\vartheta_{1,1},\vartheta'_{1,1})$, the critical value is the landing point of $P_{c_{1,1}}(\vartheta_{1,1},\vartheta'_{1,1})$ and both preimage ray pairs land at the same point. For all $c\in\mathcal{W}_W$, the next $n-1$ preimage steps can be performed unambiguously. Continuing this argument inductively, the claims follow.
Extra explanation needed, or picture? □

For each parameter ray pair $P(\vartheta_{k,j},\vartheta'_{k,j})$, let $U_{k,j}$ be the component of $\mathbb{C}\setminus P(\vartheta_{k,j},\vartheta'_{k,j})$ not containing 0. Define

$$\mathcal{M}_W:=\mathcal{M}\cap\overline{\mathcal{W}_W}\setminus\bigcup_{k,j}U_{k,j}\ . \tag{6}$$

As a nested intersection of compact connected sets, $\mathcal{M}_W$ is compact and connected.

It turns out that each $\mathcal{M}_W$ is homeomorphic to $\mathcal{M}$: this is the explanation for the existence of countably many homeomorphic copies of $\mathcal{M}$ within itself.

10.8. **Theorem (Simple Renormalization and Little Mandelbrot Sets)**
Consider a hyperbolic component W of some period n. Then $\mathcal{M}_W$ is the locus of parameters $c \in \mathcal{M}$ which are simple n-renormalizable so that this renormalization is associated to the angles (ϑ, ϑ'). Moreover, renormalization induces a homeomorphism $\mathcal{M}_W \to \mathcal{M}$ which is conformal on $\mathring{\mathcal{M}}_W$.

Proof. If $c \in \mathcal{M}_W$, then by definition this means that $c \in \overline{\mathcal{W}_W}$, but c is not separated from 0 by any of the parameter rays $P(\vartheta_{k,j}, \vartheta'_{k,j})$. By Lemma 10.7, this means that for every $k \geq 0$, $p_c^{\circ kn}(0) \in \overline{U}_1(c)$, where $U_1(c)$ denotes the domain bounded by the dynamic ray pairs $P_c(\vartheta, \vartheta')$ and $P_c(\vartheta_{1,1}, \vartheta'_{1,1})$. But now Lemma 10.5 says that p_c is n-renormalizable associated to the angles (ϑ, ϑ').

Conversely, if a parameter c is simple n-renormalizable, we need to have $c \in \mathcal{M}$; and if the renormalization is associated to (ϑ, ϑ'), then we need the dynamic ray pairs $P_c(\vartheta, \vartheta')$ and $P_c(\vartheta_{1,1}, \vartheta'_{1,1})$; this implies that $c \in \overline{\mathcal{W}_W} \cap \mathcal{M}$. In order for the critical orbit of $p_c^{\circ n}\colon U_1(c) \to \mathbb{C})$ to remain within $U_1(c)$, the parameter c cannot be separated from 0 by any of the parameter rays $P_c(\vartheta_{k,j}, \vartheta'_{k,j})$, again by Lemma 10.7. This proves the first claim.

For every $c \in \mathcal{M}_W$, the map $p_c^{\circ n}\colon U_1(c) \to U(c)$ is simple n-renormalizable associated to ϑ, ϑ', and since the filled-in Julia set is connected by definition, it follows that $p_c^{\circ n}\colon U_1(c) \to U(c)$ is hybrid equivalent to a quadratic polynomial $p := \chi_W(c)$ with connected Julia set, and so that the polynomial p is unique up to affine conjugation. This defines a map $\chi_W\colon \mathcal{M}_W \to \mathcal{M}$. The fact that this a homeomorphism follows from a deep theorem of Douady and Hubbard [**DHx**]; here we can only sketch a few main points in the argument.

Douady and Hubbard show that $\chi\colon \mathcal{M}_W \to \mathcal{M}$ is continuous in c and "topologically holomorphic", so it "has the same topological properties as a holomorphic map". It can be extended to a neighborhood of $\mathcal{M}_W$ as a branched cover (though not uniquely), and under certain conditions it has a well-defined mapping degree d as a map from $\mathcal{M}_W$ to $\mathcal{M}$. Douady and Hubbard give the following criterion: if c surrounds $\mathcal{M}_W$ once along a simple closed curve $\gamma \subset \mathbb{C}$ so that the straightening map is defined continuously in a neighborhood of $\mathcal{M}_W$ containing γ, then the critical value of $p_c^{\circ n}\colon U_1(c) \to U(c)$ turns exactly d times around the critical point. In our case, it is not hard to see that $d = 1$, so $\chi\colon \mathcal{M}_W \to \mathcal{M}$ is a homeomorphism. To see that $d = 1$, we describe the curve γ in terms of Böttcher coordinates $\Phi(\gamma(t))$; since $\Phi(c) = \Phi_c(c)$, we know the position of the critical value with respect to Böttcher coordinates in the dynamical plane of $p_c^{\circ n}$, and from here one can conclude that $d = 1$. □

10.9. **Corollary (Boundary Points of Renormalization Locus)**
Let W be a hyperbolic component and $c \in \mathcal{W}_W \setminus \mathcal{M}_W$. Then there is a Misiurewicz-Thurston parameter $c_* \in \mathcal{M}_W$ so that two parameter rays landing at c_* separate c from $\mathcal{M}_W \setminus \{c_*\}$, and the image of c_* under the renormalization homeomorphism $\mathcal{M}_W \to \mathcal{M}$

is the landing point of a parameter ray at external angle $m/2^k$ for some $k, m \geq 0$ (i.e., a dyadic Misiurewicz-Thurston parameter).

Conversely, dyadic Misiurewicz-Thurston parameter has the property that its image point under renormalization is the landing point of two preperiodic parameter rays that separate points in $\mathcal{M}$ from $\mathcal{M}_W$.

Roughly speaking, this result says that $\overline{\mathcal{W}_W}$ consists of the "little Mandelbrot set" $\mathcal{M}_W$ together with countably many decorations attached to $\mathcal{M}_W$ at the renormalized images of dyadic Misiurewicz-Thurston parameters.

PROOF. Let n be the period of W. Since $\mathcal{M}_W = \mathcal{M} \cap \mathcal{W}_W \setminus \bigcup_{k,j} U_{k,j}$ by definition in (6), we have $c \in U_{k,j}$ for some k, j. But $U_{k,j}$ is bounded by a parameter ray pair $P(\vartheta_{k,j}, \vartheta'_{k,j})$ with the property that there are two periodic angles ϑ, ϑ' of period n so that $2^{nk}(\vartheta_{k,j}) = \vartheta'$, $2^{nk}(\vartheta'_{k,j}) = \vartheta$. Let c_* be the landing point of the parameter ray pair $P(\vartheta_{k,j}, \vartheta'_{k,j})$.

In the dynamics of p_{c_*}, there is a dynamic ray pair $P_{c_*}(\vartheta_{k,j}, \vartheta'_{k,j})$ that lands at the critical value c_*, and $p_{c_*}^{\circ k}(c_*)$ is a periodic point at which dynamic rays of period n land. Let c^* be the preimage of c_* under the renormalization homeomorphism. Then in the dynamics of c^*, the k-th image of the critical point is a periodic point at which a dynamic ray of period 1 lands. There is only one such ray, the ray $R_{c^*}(0)$, so $p_{c^*}(c^*)$ is the β-fixed point and c^* is the landing point of a ray at angles $m/2^k$ for some integer m.

The converse statement follows by counting: there are 2^{k-1} ray pairs $P(\vartheta_{k,j}, \vartheta'_{k,j})$ for every $k \geq 1$, and equally many dyadic external angles $m/2^k$ (only odd values of m have preperiod k). Since renormalization is a homeomorphism, every dyadic Misiurewicz-Thurston parameter must be reached by the construction described above. □

Note that $p_{c_*}^{\circ k}(c_*)$ is the landing point of two periodic dynamic rays of period n; after straightening, these yield two invariant curves of period 1 that are not necessarily rays. In fact, they are both homotopic to the unique ray of period 0.

10.10. **Theorem (Simple Renormalization Contained in Some $\mathcal{M}_W$)**
Every simple n-renormalizable $c \in \mathcal{M}$ is associated to a pair of angles (ϑ, ϑ') of period n, and thus contained in $\mathcal{M}_W$ for some hyperbolic component W of period n.

PROOF. Suppose p_c is n-renormalizable, i.e., there are two domains $U \subset V \subset \mathbb{C}$ so that $p_c^{\circ n}\colon U \to V$ is a quadratic-like map with connected "little" Julia set K Let K_c be the Julia set of p_c (the "big" Julia set). Then $p_c^{\circ n}(K) = K$ and there is an $m \in \{0, 1, \ldots, n-1\}$ so that $p_c^{\circ m}(K)$ contains the critical point. Set $K_0 := p_c^{\circ m}(K)$ and $K_{i+1} := p_c(K_i)$ for all $i \geq 0$. There are neighborhoods U_0, V_0 of K_0 so that $p_c^{\circ n}\colon U_0 \to V_0$ is a quadratic-like map with connected Julia set (these sets can be chosen so that $p_c^{\circ(n-m}(U_0) = U$ and $p_c^{\circ(n-m}(V_0) = V$, and they can be constructed by appropriate pull-backs of U and V). Set $U_{i+1} := p_c(U_i)$ and $V_{i+1} := p_c(V_1)$ for $i \geq 0$.

We distinguish between renormalizations of disjoint and of β-type. In the former case, the sets K_1, K_2, $\ldots K_n = k_0$ are disjoint. Then we may assume that the sets V_1, V_2,..., V_n are disjoint as well (possibly by pulling back repeatedly under appropriate branches of $p_c^{\circ n}$).

The following argument mirrors the well-known argument "the critical point is an endpoint of the Hubbard tree" (see [**DH1**, ???]). For every $i = 0, 1, 2, \ldots, n$, let s_i be the number of components of $K \setminus V_i$ that contain some K_j. Clearly $s_i \in \{1, 2, \ldots, n-1\}$. Since $n \geq 2$, an easy argument shows that there must be at least 2 numbers s_i and s_j with $i, j \in \{0, 1, \ldots, n-1\}$ and $s_i = s_j = 1$. If K' is such a component of $K \setminus V_i$ and K' does not contain the critical point of p_c, then $p_c(K')$ is a similar component of $K \setminus V_{i+1}$ that does not contain the critical value. It follows that $s_{i+1} \geq s_i$ if $i = 1, 2, \ldots, n-1$ (some component of $K \setminus V_i$ must contain the critical point, and some component of $K \setminus V_{i+1}$ must contain the critical value). This implies that $s_1 = 1$ (it is not hard to show that $s_n \leq 2$).

Let $\gamma_1 \subset \mathbb{C}$ be an injective curve consisting of an arc $\gamma_1' \subset \partial U_1$, together with parts of two dynamic rays that connect the endpoints of γ_1' to ∞. Then γ_1 separates K_1 from all other K_i. The preimage $(p_c^{\circ n})^{-1}(\gamma_1)$ has two components that intersect U_1, and exactly one of them separates K_1 from γ_1. Denote these two components γ_2 and $\tilde{\gamma}_2$ so that γ_2 separates K_1 from γ_1. This construction can be continued and yields a sequence of curves γ_k for $k \geq 1$ so that each γ_{k+1} separates K_1 from γ_k; we also get a similar sequence of curves $\tilde{\gamma}_k$ with $p_c^{\circ n}(\tilde{\gamma}_{k+1}) = \gamma_k$.

Each curve γ_k follows two dynamic rays, say $R_c(\vartheta_k)$ and $R_c(\vartheta_k')$ with $\vartheta_k < \vartheta_k'$. Then we get two sequences of angles $0 < \cdots < \vartheta_k < \vartheta_{k+1} < \cdots < \vartheta_{k+1}' < \vartheta_k' < \cdots < 1$. Both sequences converge, say to limits ϑ, ϑ', and indeed $\vartheta < \vartheta'$ because all γ_k with $k > 1$ separate K_1 from γ_1. Since we have $2^n \vartheta_{k+1} = \vartheta_k$ and $2^n \vartheta_{k+1}' = \vartheta_k'$, it follows that both ϑ and ϑ' are fixed under multiplication by 2^n. A routine hyperbolic contraction argument (within $\mathbb{C} \setminus \bigcup K_i$) shows that the diameters of $\gamma_k \cap K_c$ shrink to zero, and this implies that $R_c(\vartheta)$ and $R_c(\vartheta')$ is a characteristic dynamic ray pair of period n. Let w be its landing point. Then $P_c(\vartheta, \vartheta')$ separates $K_1 \setminus \{w\}$ from all other K_i.

For $k \geq 1$, let $U(k)$ be the domain bounded by γ_{k+1} and $\tilde{\gamma}_{k+1}$ and let $V(k)$ be the domain containing K_1 and bounded by γ_k. Then it is easy to check that $p_c^{\circ n} \colon U(k) \to V(k)$ is quadratic-like with filled-in Julia set K_1, and from this it follows that our renormalization is associated to the angles (ϑ, ϑ').

Now suppose the renormalization is of β-type. Let m be minimal so that K_1 intersects K_{m+1}, and let w be this point of intersection. Then m is a proper divisor of n. The exact period of m cannot be less than m, and in fact it equals m because K_{m+1} intersects K_1 and K_{m+1} in the same point [**Mc**, Theorem 7.13].

By hypothesis, w is the landing point of at least one periodic dynamic ray of period at most n.

This implies that the rays landing at w must separate K_1 from all other K_i, and again it follows that the renormalization is associated to some pair of periodic angles (ϑ, ϑ') of period n so that $P_c(\vartheta, \vartheta')$ lands at w.

The ray pair is $P_c(\vartheta, \vartheta')$ is characteristic, so by Theorem 9.5 there is a parameter ray pair $P(\vartheta, \vartheta')$ of period n; its landing point is the root of a hyperbolic component W of period n, and Theorem 10.8 implies that $c \in \mathcal{M}_W$. □

Renormalization forms a non-abelian semi-group: any two hyperbolic components $W, W' \subset \mathcal{M}$ of respective periods n and n' have their own renormalization homeomorphisms $\chi_W \colon \mathcal{M}_W \to \mathcal{M}$ and $\chi_{W'} \colon \mathcal{M}_{W'} \to \mathcal{M}$, so the parameter $c \in \chi_W^{-1} \circ \chi_{W'}^{-1}(\mathcal{M})$ is n-renormalizable so that $\chi_W(c)$ is still n'-renormalizable. Therefore, c is renormalizable of periods n and nn'. This is not well enough explained.

10.11. Theorem (Renormalization Preserves Trivial Fibers)
*Suppose $\mathcal{M}_W$ is a little Mandelbrot set as in **??** and $\xi_W \colon \mathcal{M}_W \to \mathcal{M}$ is the corresponding straightening homeomorphism. Then the fiber of $c \in \mathcal{M}_W$ is trivial if and only if the fiber of $\chi_W(c)$ is trivial.*

Proof. Suppose a separation line separates $c, c' \in \mathcal{M}$. Then it is easy to find a separation line separating $\chi_W^{-1}(c)$ and $\chi_W^{-1}(c')$ (the map χ_W does not act directly on parameter rays at periodic angles, but it does act on their landing points, which are parabolic parameters). If the fiber of $\chi_W(c)$ is trivial, then the fiber of c intersects $\mathcal{M}_W$ only in c itself. If c is a Misiurewicz-Thurston parameter, then c has trivial fiber by [**Sch3**, where?]. If not, then any $c' \in (\mathcal{M} \setminus \mathcal{M}_W)$ can be separated from c by a parameter ray pair at periodic or preperiodic angles, so the fiber of c is trivial.

Conversely, if $c \in \mathcal{M}_W$ has trivial fiber, then in particular every $c' \in \mathcal{M}_W \setminus \{c\}$ can be separated from c, and we get a separation between $\chi_W(c)$ and $\chi_W(c')$, so $\chi_W(c)$ has trivial fiber. □

10.12. Definition (Infinite Renormalization)
A parameter $c \in \mathcal{M}$ is infinitely renormalizable *if c is n-renormalizable for infinitely many n.*

10.13. Theorem (Yoccoz' Rigidity Result)
Every parameter $c \in \mathcal{M}$ which is not infinitely renormalizable has trivial fiber. □

The proof of this result distinguishes many cases: for parameters on the boundary of hyperbolic components, except primitive roots, this follows from the "No Ghost Limb" Theorem 9.15; primitive roots require a separate argument; see Theorem 9.17. For Misiurewicz-Thurston parameters, this is Theorem 9.21. For the remaining parameters, the proof uses the puzzle technique that was pioneered by Branner, Hubbard, and Yoccoz. These puzzles again distinguish whether the critical orbit is non-recurrent, non-persistently recurrent, or persistently recurrent.

10.14. **Corollary (Non-Infinitely Renormalizable Parameter Rays)**
If an angle $\vartheta \in \mathbb{S}^1$ is not infinitely renormalizable, then $R(\vartheta)$ lands at a parameter $c \in \mathcal{M}$ with trivial fiber.

PROOF. By Lemma **??**, all points in the impression of $R(\vartheta)$ are not infinitely renormalizable and hence have trivial fibers by Theorem 10.13; since every impression is contained in a single fiber, the claim follows. □

There are many results about trivial fibers of infinitely renormalizable parameters $c \in \mathcal{M}$ due to Lyubich, Kahn, Avila and coauthors; see [**?**].

Describe Crossed Renormalization here!

There is more stuff about renormalization and Cremer points in hide.

Some stuff about combinatorial renormalization in hide

Describe (briefly) crossed renormalization

11. Internal Addresses of the Mandelbrot Set

In the early 1990's, Devaney asked the question: *how can you tell where in the Mandelbrot set a given external ray lands, without having Adrien Douady at your side?* In this section, we provide an answer to this question in terms of internal addresses: these are a convenient and efficient way of describing the combinatorial structure of the Mandelbrot set, and of giving geometric meaning to the ubiquitous kneading sequences in human-readable form.

We give the most important results from [**LS**] about the combinatorial geometry of internal addresses. These will be used in Section 12 to derive some algebraic results about permutations of periodic points and Galois groups.

The combinatorial structure of the Mandelbrot set has been studied by many people, notably in terms of *quadratic minor laminations* by Thurston [**Th**], *pinched disks* by Douady [**D3**], or *orbit portraits* by Milnor [**Mi2**] (see also Appendix **??**). All of these results are modeled on parameter rays of the Mandelbrot set at periodic angles, as well as their landing properties. Internal addresses are a natural way to distinguish the components in the complement of parameter rays of bounded periods.

Parameter rays of the Mandelbrot set are defined as follows (see Section 9). By [**DH1**], the Mandelbrot set is compact, connected and full, i.e., there is a conformal isomorphism $\Phi\colon (\mathbb{C}\setminus\mathcal{M}) \to (\mathbb{C}\setminus\overline{\mathbb{D}})$; it can be normalized so that $\Phi(c)/c \to 1$ as $c \to \infty$. The *parameter ray* of $\mathcal{M}$ at angle $\vartheta \in \mathbb{R}/\mathbb{Z}$ is then $\Phi^{-1}(e^{2\pi i\vartheta}(1,\infty))$. If $R(\vartheta)$ and $R(\vartheta')$ land at a common point z, we call $R(\vartheta)\cup R(\vartheta')\cup\{z\}$ a *parameter ray pair* and denote it $P(\vartheta,\vartheta')$. Then $\mathbb{C}\setminus P(\vartheta,\vartheta')$ has two components, and we say that $P(\vartheta,\vartheta')$ *separates* two points or sets if they are in different components. If a polynomial $p_c(z) = z^2 + c$ has connected Julia set, then there are analogous definitions of *dynamic rays* $R_c(\vartheta)$ and *dynamic ray pairs* $P_c(\vartheta,\vartheta')$. General background can be found in Milnor [**Mi1**].

The most important properties of parameter rays at periodic angles are collected in the following well known theorem; see Theorems 9.1 and 9.5 and Lavaurs' Lemma 9.28.

11.1. **Theorem (Properties of Periodic Parameter Ray Pairs)**
For every $n \ge 1$, there are finitely many parameter rays of period up to n. They land in pairs so that any two parameter rays of equal period are separated by a parameter ray of lower period. Every parameter ray pair $P(\vartheta,\vartheta')$ of period n lands at the root of a hyperbolic component W of period n, and the root of every hyperbolic component of period n is the landing point of exactly two parameter rays, both of period n.

The ray pair $P(\vartheta,\vartheta')$ partitions $\mathbb{C}$ into two open components; let $\mathcal{W}_W$ be the component not containing the origin: this is the wake *of W and contains W. This wake is the locus of parameters $c \in \mathbb{C}$ for which the dynamic rays $R_c(\vartheta)$ and $R_c(\vartheta')$ land together at a repelling periodic point; the ray pair $P_c(\vartheta,\vartheta')$ is necessarily characteristic.*

Here a dynamic ray pair is *characteristic* if it separates the critical value from the rays $R_c(2^k\vartheta)$ and $R_c(2^k\vartheta')$ for all $k \ge 1$ (except of course from those on the ray pair $P_c(\vartheta,\vartheta')$ itself).

Every angle $\vartheta \in \mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ has an associated *kneading sequence* $\nu(\vartheta) = \nu_1\nu_2\nu_3\ldots$ (Definition 2.1), defined as the itinerary of ϑ (under angle doubling) on the unit circle $\mathbb{S}^1$ with respect to the partition $\mathbb{S}^1 \setminus \{\vartheta/2, (\vartheta+1)/2\}$, so that

$$\nu_k = \begin{cases} \mathtt{1} & \text{if } 2^{k-1}\vartheta \in (\vartheta/2, (\vartheta+1)/2); \\ \mathtt{0} & \text{if } 2^{k-1}\vartheta \in ((\vartheta+1)/2, \vartheta/2) = \mathbb{S}^1 \setminus [\vartheta/2, (\vartheta+1)/2]; \\ \star & \text{if } 2^{k-1}\vartheta \in \{\vartheta/2, (\vartheta+1)/2\}. \end{cases}$$

If ϑ is non-periodic, then $\nu(\vartheta)$ is a sequence over $\{\mathtt{0}, \mathtt{1}\}$. If ϑ is periodic of exact period n, then $\nu(\vartheta)$ also has period n so that the first $n-1$ entries are in $\{\mathtt{0}, \mathtt{1}\}$ and the n-th entry is $\star$; such $\nu(\vartheta)$ are called $\star$*-periodic*. The partition is such that every $\nu(\vartheta)$ starts with $\mathtt{1}$ (unless $\vartheta = 0$).

The parameter rays at periodic angles of periods up to n partition $\mathbb{C}$ into finitely many components. This partition has interesting symbolic dynamic properties, compare Figure 11.1 and Lemma 11.2: if two parameter rays $R(\vartheta_1)$ and $R(\vartheta_2)$ are in the same component, then the kneading sequences $\nu(\vartheta_1)$ and $\nu(\vartheta_2)$ associated to ϑ_1 and ϑ_2 coincide at least for n entries. The combinatorics of these partitions, and thus the combinatorial structure of the Mandelbrot set, can conveniently be described in terms of *internal addresses*, which are "human-readable" recodings of kneading sequences; see below.

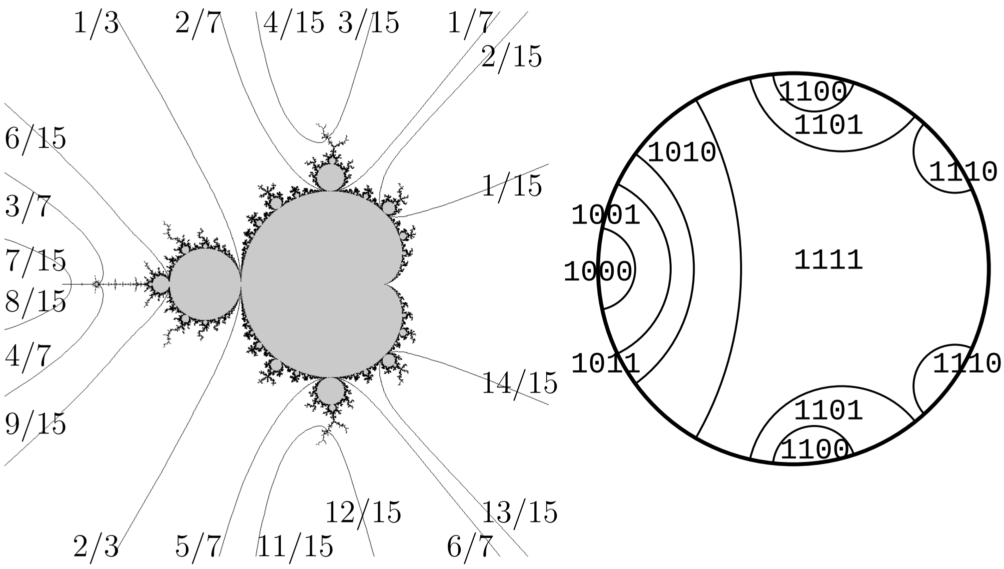

FIGURE 11.1. Left: the 10 parameter rays of period up to 4 in the Mandelbrot set partition $\mathbb{C}$ into 11 components. In each component, the first 4 entries of the kneading sequences $\nu(\vartheta)$ for arbitrary parameter rays $R(\vartheta)$ are constant. Right: the same parameter ray pairs sketched symbolically, and the first 4 entries in the kneading sequences drawn in.

11.2. **Lemma (Parameter Ray Pairs and Kneading Sequences)**
If two parameter rays $R(\vartheta)$ and $R(\vartheta')$ are not separated by a ray pair of period at most n, then ϑ and ϑ' have kneading sequences which coincide for at least n entries (provided neither ϑ nor ϑ' are periodic of period n or less).

If $R(\vartheta)$ and $R(\vartheta')$ with $\vartheta < \vartheta'$ form a ray pair, then $\nu(\vartheta) = \nu(\vartheta') =: \nu_\star$. If in addition both angles are periodic with exact period n, then $\nu_\star$ is $\star$-periodic of period n, we have

$$\lim_{\varphi\nearrow\vartheta} \nu(\varphi) = \lim_{\varphi\searrow\vartheta'} \nu(\varphi) \qquad \textit{and} \qquad \lim_{\varphi\searrow\vartheta} \nu(\varphi) = \lim_{\varphi\nearrow\vartheta'} \nu(\varphi)\ ,$$

and both limits are periodic with period n or dividing n so that their first difference is exactly at the n-th position.

Proof. This result is at the heart of [**LS**], but not explicitly spelled out there (compare [**LS**, Observation 3.3 and Proposition 5.2]). As ϑ varies in $\mathbb{S}^1$, the n-th entry in its kneading sequence changes exactly at angles which are periodic of period dividing n.

Consider two external angles $\vartheta < \vartheta'$ which are not separated by any ray pair of period at most n, and which are not periodic of period up to n (it is quite possible that (ϑ, ϑ') contains a periodic angle of period up to n, as long as the other angle of the same ray pair is also contained in (ϑ, ϑ')). Then for every $k \le n$, the parameter rays of period k with angles $\varphi \in (\vartheta, \vartheta')$ must land in pairs (Theorem 11.1), so the number of such rays is even. Therefore, as the angle varies from ϑ to ϑ', the k-th entry in the kneading sequence of ϑ changes an even number of times, and the kneading sequences of ϑ and ϑ' have identical k-th entries. This settles the first claim, and it shows that $\nu(\vartheta) = \nu(\vartheta')$ if $R(\vartheta)$ and $R(\vartheta')$ land together and neither ϑ nor ϑ' are periodic.

However, if $R(\vartheta)$ and $R(\vartheta')$ land together and one of the two angles is periodic, then both are periodic with the same exact period, say n (Theorem 11.1). In this case, the kneading sequences $\nu(\vartheta)$ and $\nu(\vartheta')$ are $\star$-periodic with exact period n, so they coincide as soon as their first $n-1$ entries coincide; this case is covered by the first claim.

It is quite easy to see that limits such as $\lim_{\varphi\searrow\vartheta} \nu(\varphi)$ exist; moreover, if ϑ is non-periodic, then the limit equals $\nu(\vartheta)$. If ϑ is periodic of exact period n, then $\lim_{\varphi\searrow\vartheta} \nu(\varphi)$ and $\lim_{\varphi\searrow\vartheta} \nu(\varphi)$ are periodic of period n or dividing n, they contain no $\star$, and their first $n-1$ entries coincide with those of $\nu(\vartheta)$. Clearly, $\lim_{\varphi\searrow\vartheta} \nu(\varphi)$ and $\lim_{\varphi\nearrow\vartheta} \nu(\varphi)$ differ exactly at positions which are multiples of n. Finally, $\lim_{\varphi\searrow\vartheta} \nu(\varphi) = \lim_{\varphi\nearrow\vartheta'} \nu(\varphi)$ because for sufficiently small ε, the parameter rays $R(\vartheta+\varepsilon)$ and $R(\vartheta'-\varepsilon)$ are not separated by a ray pair of period at most n, so the limits must be equal. □

The following should be taken as an algorithmic definition of internal addresses in the Mandelbrot set. It is the original definition of internal addresses from [**LS**] (except that in [**LS**], it was applied to hyperbolic components and Misiurewicz-Thurston points, but this makes no difference: see the remark below).

11.3. **Algorithm (Internal Address in the Mandelbrot Set)**
Given a parameter $c \in \mathbb{C}$, the internal address of c in the Mandelbrot set is a strictly increasing finite or infinite sequence of integers. It is defined as follows:

seed: *the internal address starts with $S_0 = 1$ and the ray pair $P(0,1)$;*
inductive step: *if $S_0 \to \ldots \to S_k$ is an initial segment of the internal address of c, where S_k corresponds to a ray pair $P(\vartheta_k, \vartheta'_k)$ of period S_k, then let $P(\vartheta_{k+1}, \vartheta'_{k+1})$ be the ray pair of least period which separates $P(\vartheta_k, \vartheta'_k)$ from c (or for which $c \in P(\vartheta_{k+1}, \vartheta'_{k+1})$); let S_{k+1} be the common period of ϑ_{k+1} and ϑ'_{k+1}. The internal address of c then continues as $S_0 \to \ldots \to S_k \to S_{k+1}$ with the ray pair $P(\vartheta_{k+1}, \vartheta'_{k+1})$.*

This continues for every $k \geq 1$ unless there is a finite k so that $P(\vartheta_k, \vartheta'_k)$ is not separated from c by any periodic ray pair (in particular if $c \in P(\vartheta_k, \vartheta'_k)$).

Remark. The internal address is only the sequence $S_0 \to \ldots \to S_k \to \ldots$ of periods; it does not contain the ray pairs used in the construction. The ray pair $P(\vartheta_{k+1}, \vartheta'_{k+1})$ of lowest period is always unique by Lavaurs' Lemma 9.28.

The internal address of $c \in \mathcal{M}$ can be viewed as a road description for the way from the origin to c in the Mandelbrot set: at any intermediate place, the internal address describes the most important landmark on the remaining way to c; see Figure 11.1. Landmarks are hyperbolic components (or equivalently the periodic parameter rays landing at their roots, see Theorem 9.7), and hyperbolic components are the more important the lower their periods are. The road description starts with the most important landmark: the component of period 1, and inductively continues with the period of the component of lowest period on the remaining way.

Different hyperbolic components (or combinatorial classes) are not distinguished completely by their internal addresses; the remaining ambiguity has a combinatorial interpretation and will be removed by *angled internal addresses*: see Theorem 11.10.

We can give an analogous definition of internal addresses in dynamic planes.

11.4. **Definition (Internal Address in Julia Set)**
Consider a polynomial p_c for which all periodic dynamic rays land and let K_c be the filled-in Julia set. For a point $z \in K_c$, the internal address of z is defined as follows, in analogy to Algorithm 11.3:

seed: *the internal address starts with $S_0 = 1$ and the ray pair $P_c(0,1)$;*
inductive step: *if $S_0 \to \ldots \to S_k$ is an initial segment of the internal address of z, where S_k corresponds to a dynamic ray pair $P_c(\vartheta_k, \vartheta'_k)$ of period S_k, then let $P_c(\vartheta_{k+1}, \vartheta'_{k+1})$ be the dynamic ray pair of least period which separates $P_c(\vartheta_k, \vartheta'_k)$ from z (or for which $z \in P_c(\vartheta_{k+1}, \vartheta'_{k+1})$); let S_{k+1} be the common*

period of ϑ_{k+1} and ϑ'_{k+1}. The internal address of z then continues as $S_0 \to \ldots \to S_k \to S_{k+1}$.[9]

This continues for every $k \geq 1$ unless there is a finite k so that $P_c(\vartheta_k, \vartheta'_k)$ is not separated from z by any periodic ray pair.

Every kneading sequence has an associated internal address as follows (compare Definition 2.1):

11.5. **Definition (Internal Address and Kneading Sequence)**
Given a kneading sequence ν with initial entry $\mathtt{1}$, it has the following associated internal address $1 \to S_1 \to \ldots \to S_k \to \ldots$: we start with $S_0 = 1$ and $\nu_0 = \overline{\mathtt{1}}$. Then define recursively S_{k+1} as the position of the first difference between ν and ν_k, and let $\nu_{S_{k+1}}$ be the periodic continuation of the first S_{k+1} entries in ν (if ν is periodic or period S_k, then the internal address is finite and stops with entry S_k).

Observe that this definition is algorithmic. It can be inverted so as to assign to any finite or infinite strictly increasing sequence starting with 1 (viewed as internal address) a kneading sequence consisting of entries $\mathtt{0}$ and $\mathtt{1}$ and starting with $\mathtt{1}$ (Algorithm 2.3).

For a $\star$-periodic kneading sequence ν of exact period n, we defined $\mathcal{A}(\nu)$ and $\overline{\mathcal{A}}(\nu)$ as the two sequences in which every $\star$ was replaced consistently by $\mathtt{0}$ or consistently by $\mathtt{1}$, chosen so that the internal address of $\mathcal{A}(\nu)$ contains the entry n, while the internal address of $\overline{\mathcal{A}}(\nu)$ does not (Definition 5.14). It turns out that $\mathcal{A}(\nu)$ has exact period n, while the exact period of $\overline{\mathcal{A}}(\nu)$ equals or divides n (Proposition 5.16 and independently Lemma 20.2). The sequences $\mathcal{A}(\nu)$ and $\overline{\mathcal{A}}(\nu)$ are called the *upper* and *lower* periodic kneading sequences associated to ν.

The point of the various algorithmic definitions of internal addresses is of course the following.

11.6. **Proposition (Equal Internal Addresses)**
(1) For every $c \in \mathbb{C}$ so that all periodic dynamic rays of p_c land, the internal address in parameter space equals the internal address of the critical value of c in the dynamical plane of p_c.

(2) Let $P(\vartheta_k, \vartheta'_k)$ be the ray pairs from Algorithm 11.3, let S_k be their periods and let ν^k be the common kneading sequence of ϑ_k and ϑ'_k. Then each ν^k is a $\star$-periodic kneading sequence of period S_k so that

$$\lim_{\varphi \searrow \vartheta_k} \nu(\varphi) = \lim_{\varphi \nearrow \vartheta'_k} \nu(\varphi) = \mathcal{A}(\nu^k) \qquad \text{and} \qquad \lim_{\varphi \nearrow \vartheta_k} \nu(\varphi) = \lim_{\varphi \searrow \vartheta'_k} \nu(\varphi) = \overline{\mathcal{A}}(\nu^k)\ .$$

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[9] If several dynamic ray pairs of equal period S_{k+1} separate $P_c(\vartheta_k, \vartheta'_k)$ from z, take the one which minimizes $|\vartheta'_{k+1} - \vartheta_{k+1}|$. (There is an analog to Lavaurs' Lemma 9.28 in dynamical planes which says that all candidate ray pairs have to land at the same periodic point; this is not hard to prove, but we do not need it here.)

(3) The first S_k entries in $\mathcal{A}(\nu^k)$ coincide with those of ν^{k+1} and, if $c \in R(\vartheta)$, with the first S_k entries of $\nu(\vartheta)$.

(4) For every $\vartheta \in \mathbb{S}^1$, the internal address of any parameter $c \in R(\vartheta)$ in Algorithm 11.3 is the same as the internal address associated to the kneading sequence of ϑ from Definition 11.5.

Proof. The first statement is simply a reformulation of Theorem 11.1 about the correspondence of parameter ray pairs and characteristic dynamic ray pairs: all we need to observe is that the internal address of the critical value in the Julia set uses only characteristic dynamic ray pairs; this follows directly from the definition of "characteristic".

We prove the second statement by induction, starting with the ray pair $P(\vartheta_0, \vartheta'_0)$ with $\vartheta_0 = 0$, $\vartheta'_0 = 1$ and $S_0 = 1$; both angles ϑ_0 and ϑ'_0 have kneading sequence $\nu^0 = \overline{\star}$ and $\mathcal{A}(\nu^0) = \overline{1}$.

For the inductive step, suppose that $P(\vartheta_k, \vartheta'_k)$ is a ray pair of period S_k with $\vartheta_k < \vartheta'_k$, $\nu(\vartheta_k) = \nu(\vartheta'_k) = \nu^k$ and $\lim_{\varphi \searrow \vartheta_k} \nu(\varphi) = \lim_{\varphi \nearrow \vartheta'_k} \nu(\varphi) = \mathcal{A}(\nu^k)$, and c is not separated from $P(\vartheta_k, \vartheta'_k)$ by a ray pair of period S_k or less.

As in Algorithm 11.3, let $P(\vartheta_{k+1}, \vartheta'_{k+1})$ be a ray pair of lowest period, say S_{k+1}, which separates $P(\vartheta_k, \vartheta'_k)$ from c (or which contains c); then $\vartheta_k < \vartheta_{k+1} < \vartheta'_{k+1} < \vartheta'_k$. We have $S_{k+1} > S_k$ by construction, and the ray pair $P(\vartheta_{k+1}, \vartheta'_{k+1})$ is unique by Lavaurs' Lemma 9.28. Since $R(\vartheta_k)$ and $R(\vartheta_{k+1})$ are not separated by a ray pair of period S_{k+1} or less, it follows that the first S_{k+1} entries in $\lim_{\varphi \searrow \vartheta_k} \nu(\varphi) = \mathcal{A}(\nu^k)$ and in $\lim_{\varphi \nearrow \vartheta_{k+1}} \nu(\varphi)$ are equal (the same holds for $\lim_{\varphi \nearrow \vartheta'_k} \nu(\varphi) = \mathcal{A}(\nu^k)$ and $\lim_{\varphi \searrow \vartheta'_{k+1}} \nu(\varphi)$).

Now we show that $\lim_{\varphi \nearrow \vartheta_{k+1}} \nu(\varphi) = \lim_{\varphi \searrow \vartheta'_{k+1}} \nu(\varphi) = \overline{\mathcal{A}}(\nu^{k+1})$; the first equality is Lemma 11.2. The internal address associated to $\nu(\vartheta_k + \varepsilon)$ has no entries in $\{S_k + 1, \ldots, S_{k+1}\}$ provided $\varepsilon > 0$ is sufficiently small (again Lemma 11.2). The internal address associated to $\nu(\vartheta_{k+1} - \varepsilon)$ can then have no entry in $\{S_k + 1, \ldots, S_{k+1}\}$ either, for small ε (or $R(\vartheta_k)$ and $R(\vartheta_{k+1})$ would have to be separated by a ray pair of period at most S_{k+1}). Thus $\lim_{\varphi \nearrow \vartheta_{k+1}} \nu(\varphi) = \overline{\mathcal{A}}(\nu^{k+1})$. The other limiting kneading sequences $\lim_{\varphi \searrow \vartheta_{k+1}} \nu(\varphi)$ and $\lim_{\varphi \nearrow \vartheta'_{k+1}} \nu(\varphi)$ must then be equal to $\mathcal{A}(\nu^{k+1})$.

Claim (3) follows because neither $P(\vartheta_{k+1}, \vartheta'_{k+1})$ nor c are separated from $P(\vartheta_k, \vartheta'_k)$ by a ray pair of period S_k or less.

We prove Claim (4) again by induction. If $c \in R(\vartheta)$, assume by induction that the internal address of $\nu(\vartheta)$ starts with $1 \to \ldots \to S_k$ and $\vartheta_k < \vartheta < \vartheta'_k$; then $\nu(\vartheta)$ coincides with $\mathcal{A}(\nu^k)$ for at least S_k entries. The ray pair of least period separating $R(\vartheta_k)$ and $R(\vartheta)$ is $P(\vartheta_{k+1}, \vartheta'_{k+1})$, so the first difference between $\mathcal{A}(\nu^k)$ and $\nu(\vartheta)$ occurs at position S_{k+1}. Hence the internal address of $\nu(\vartheta)$ as in Definition 11.5 continues as $1 \to \ldots \to S_k \to S_{k+1}$ and we have $\vartheta_{k+1} < \vartheta < \vartheta'_{k+1}$ as required to maintain the induction. □

The previous proof also shows the following useful observation.

11.7. **Corollary (Intermediate Ray Pair of Lowest Period)**
Let $P(\vartheta_1,\vartheta_1')$ and $P(\vartheta_2,\vartheta_2')$ be two parameter ray pairs (not necessarily at periodic angles) and suppose that $P(\vartheta_1,\vartheta_1')$ separates $P(\vartheta_2,\vartheta_2')$ from the origin. If the limits $\lim_{\varphi\nearrow\vartheta_2}\nu(\varphi)$ and $\lim_{\varphi\searrow\vartheta_1}\nu(\varphi)$ do not differ, then the two ray pairs $P(\vartheta_1,\vartheta_1')$ and $P(\vartheta_2,\vartheta_2')$ are not separated by any periodic ray pair. If the limits do differ, say at position n for the first time, then both ray pairs are separated by a unique periodic ray pair $P(\vartheta,\vartheta')$ of period n (but not by ray pairs of lower period); the first $n-1$ entries in $\nu(\vartheta)=\nu(\vartheta')$ coincide with those of $\lim_{\varphi\nearrow\vartheta_2}\nu(\varphi)$ and $\lim_{\varphi\searrow\vartheta_1}\nu(\varphi)$.

Proof. Let n be the position of the first difference in $\lim_{\varphi\searrow\vartheta_1}\nu(\varphi)$ and $\lim_{\varphi\nearrow\vartheta_2}\nu(\varphi)$. Then the number of periodic angles of period dividing n in $(\vartheta_1,\vartheta_2)$ is odd (because only at these angles, the n-th entry in the kneading sequence changes). Since parameter rays at period dividing n land in pairs, there is at least one parameter ray pair at period dividing n which is part of a parameter ray pair separating $P(\vartheta_1,\vartheta_1')$ and $P(\vartheta_2,\vartheta_2')$. If $P(\vartheta_1,\vartheta_1')$ and $P(\vartheta_2,\vartheta_2')$ are separated by a parameter ray pair of period less than n, let n' be the least such period. By Lavaurs' Lemma 9.28, there would be a single such parameter ray pair of period n', and it follows $n'=n$. The remaining claims follow in a similar way. □

Most internal address are infinite; exceptions are related to hyperbolic components as follows.

11.8. **Corollary (Finite Internal Address)**
The internal address of $c\in\mathcal{M}$ is finite if and only if there is a hyperbolic component W with $c\in\overline{W}$. More precisely, if $c\in\overline{W}$ but c is not the root of a hyperbolic component other than W, then the internal address of c terminates with the period of W (if c is the root of a hyperbolic component $W'\neq W$, then the internal address of c terminates with the period of W').

For a non-periodic external angle $\vartheta\in\mathbb{S}^1$, the internal address of $R(\vartheta)$ is finite if and only if $R(\vartheta)$ lands on the boundary of a hyperbolic component W.

Proof. Consider a parameter ray $R(\vartheta)$ with finite internal address and let n be the last entry in this internal address. Then there is a parameter ray pair $P(\vartheta,\vartheta')$ of period n which separates $R(\vartheta)$ from the origin, and no periodic parameter ray pair separates $R(\vartheta)$ from $P(\vartheta,\vartheta')$. The ray pair $P(\vartheta,\vartheta')$ bounds the wake of a hyperbolic component W and lands at the root of W, so $R(\vartheta)$ is in the wake of W (or on its boundary) but not in one of its subwakes. Every such parameter ray lands on the boundary of W; see Corollary 9.16. The converse is clear.

The statements about $c\in\mathcal{M}$ follow in a similar way, using the fact that the limb of W is the union of $\overline{W}$ and its sublimbs at rational internal angles; see Theorem 9.15. □

Remark. The previous result shows (through the topology of the Mandelbrot set) that non-periodic external angles can generate periodic kneading sequences which have finite internal addresses. Of course there are periodic kneading sequences which have infinite internal addresses; but these are never generated by external angles (Lemma 14.6).

For a hyperbolic component W, we have many different ways of associating an internal address to it:

(1) using Algorithm 11.3 within the plane of the Mandelbrot set;
(2) taking a parameter ray $R(\vartheta)$ landing at the root of W and then Algorithm 11.3 for this parameter ray;
(3) taking a parameter ray $R(\vartheta')$ at an irrational angle landing at a boundary point of W (as in Corollary 11.8) and then again Algorithm 11.3 for this parameter ray;
(4) taking a parameter ray $R(\vartheta)$ or $R(\vartheta')$ as in (2) or (3) and then using the internal address associated to the kneading sequence of ϑ or ϑ'; see Definitions 2.1 and 11.5. The periodic angle ϑ generates a $\star$-periodic kneading sequence, while the non-periodic angle ϑ' generates a periodic kneading sequence without $\star$;
(5) for every $c \in W$, we have the internal address of the critical value in the Julia set from Definition 11.4;
(6) specifically if c is the center of W, then the critical orbit is periodic and p_c has a Hubbard tree in the original sense of Douady and Hubbard (Definition 3.4), so we can use all definitions from Proposition 6.8.

All these internal addresses are of course the same: this is obvious for the first three; the next two definitions agree with the first ones by Proposition 11.6 (4) and (1), and the last two agree by Proposition 6.8.

11.9. **Definition (Angled Internal Address)**
For a parameter $c \in \mathbb{C}$, the angled internal address *is the sequence*

$$(S_0)_{p_0/q_0} \to (S_1)_{p_1/q_1} \to \ldots \to (S_k)_{p_k/q_k} \to \ldots$$

where $S_0 \to S_1 \to \ldots \to S_k \to \ldots$ is the internal address of c as in Algorithm 11.3 and the angles p_k/q_k are defined as follows: for $k \geq 0$, let $P(\vartheta_k, \vartheta'_k)$ be the parameter ray pair associated to the entry S_k in the internal address of c; then the landing point of $P(\vartheta_k, \vartheta'_k)$ is the root of a hyperbolic component W_k of period S_k. The angle p_k/q_k is defined such that c is contained in the closure of the p_k/q_k-subwake of W_k. If the internal address of c is finite and terminates with an entry S_k (which happens if and only if c is not contained in the closure of any subwake of W_k), then the angled internal address of c is also finite and terminates with the same entry S_k without angle: $(S_0)_{p_0/q_0} \to (S_1)_{p_1/q_1} \to \ldots \to (S_{k-1})_{p_{k-1}/q_{k-1}} \to (S_k)$.

This definition is illustrated in Figure 11.2. The main point in this definition is that it distinguishes different points in the Mandelbrot set. More precisely, angled internal addresses distinguish combinatorial classes: a *combinatorial class* is a maximal subset of $\mathcal{M} \setminus \{\text{parabolic parameters}\}$ so that no two of its points can be separated by a parameter ray pair at periodic angles; a parabolic parameter c_0 belongs to the combinatorial class of the hyperbolic component of which c_0 is the root. Then a combinatorial class is the equivalence class of parameters in $\mathcal{M}$ so that two parameters c_1 and c_2 are equivalent if and only if for both polynomials, the same periodic and preperiodic parameters land at common points (see Section 9.3 or [**Sch3**, Section 8]).

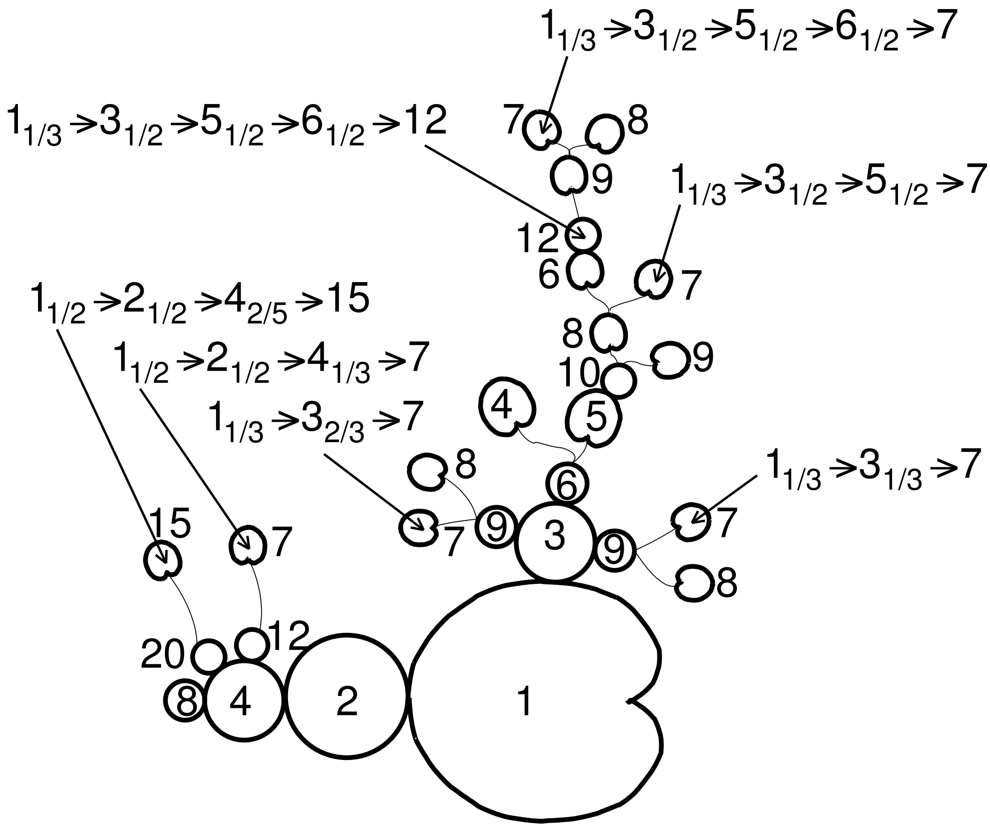


FIGURE 11.2. Angled internal addresses for various hyperbolic components in $\mathcal{M}$.

11.10. **Theorem (Completeness of Angled Internal Addresses)**
If two parameters in $\mathcal{M}$ have the same angled internal address, then they belong to the same combinatorial class. In particular, if two hyperbolic parameters in $\mathcal{M}$ have the same angled internal address, then they belong to the same hyperbolic component.

We postpone the proof of this theorem and of subsequent results and first discuss some interesting consequences.

The Geometry of Internal Addresses.

Now we turn to Devaney's question: *how can you tell where in the Mandelbrot set a given external ray lands, without having Adrien Douady at your side?* Angled internal addresses provide a convenient answer. We already know how to turn an external angle into an internal address; now we turn it into an angled internal address.

It turns out that the internal address without angles completely determines the denominators of any associated angled internal address, but it says nothing about the numerators.

11.11. **Lemma (Estimates on Denominators in Internal Address)**
We have $S_{k+1}/S_k \le q_k < S_{k+1}/S_k + 2$; moreover, $q_k = S_{k+1}/S_k$ whenever the latter is an integer.

If S_{k+1} is a multiple of S_k, then the component of period S_{k+1} is a bifurcation from that of period S_k.

Now we give a precise formula for q_k. It is analogous to Formula (2) in Proposition 5.19. Recall that every internal address has an associated kneading sequence (see Definition 11.5 or Algorithm 2.3). For every kneading sequence $\nu = \nu_1\nu_2\nu_3\ldots$ we define an associated function ρ_ν as follows: for $r \ge 1$, let

$$\rho_\nu(r) := \min\{k > r \colon \nu_k \neq \nu_{k-r}\} \ .$$

The ρ_ν-orbit of r is denoted $\mathrm{orb}_{\rho_\nu}(r)$ (compare Definition 2.2). We usually write ρ for ρ_ν.

11.12. **Lemma (Denominators in Angled Internal Address)**
In an angled internal address $(S_0)_{p_0/q_0} \to \ldots \to (S_k)_{p_k/q_k} \to (S_{k+1})_{p_{k+1}/q_{k+1}}\ldots$, the denominator q_k in the bifurcation angle is uniquely determined by the internal address $S_0 \to \ldots \to S_k \to S_{k+1}\ldots$ as follows: let ν be the kneading sequence associated to the internal address and let ρ be the associated function as just described. Let $r \in \{1, 2, \ldots, S_k\}$ be congruent to S_{k+1} modulo S_k. Then

$$q_k := \begin{cases} \frac{S_{k+1}-r}{S_k} + 1 & \text{if } S_k \in \mathrm{orb}_\rho(r) \ , \\ \frac{S_{k+1}-r}{S_k} + 2 & \text{if } S_k \notin \mathrm{orb}_\rho(r) \ . \end{cases}$$

While the internal address completely specifies the denominators in any corresponding angled internal address, it says says nothing about the numerators: of course, not all internal addresses are realized in the Mandelbrot set (not all are (complex) *admissible*; see Sections 5 and 14), but this is independent of the numerators.

11.13. **Theorem (Independence of Numerators in Angled Int. Address)**
If an angled internal address describes a point in the Mandelbrot set, then the numerators p_k can be changed arbitrarily (coprime to q_k) and the modified angled internal address still describes a point in the Mandelbrot set.

In other words, for every hyperbolic component there is a natural bijection between combinatorial classes of the p/q-sublimb and p'/q-sublimb, for every $q \ge 2$ and all p, p' coprime to q. This bijection would give rise to a homeomorphism between these sublimbs if the Mandelbrot set was locally connected. Unlike other homeomorphisms constructed by surgery as in Branner and Fagella [**BF**] or Riedl [**Ri**], this homeomorphism preserves periods of hyperbolic components.

Here is a way to find the numerators p_k from an external angle.

11.14. **Lemma (Numerators in Angled Internal Address)**
Suppose the external angle ϑ has angled internal address $(S_0)_{p_0/q_0} \to \ldots \to (S_k)_{p_k/q_k} \to (S_{k+1})_{p_{k+1}/q_{k+1}} \ldots$. In order to find the numerator p_k, consider the $q_k - 1$ angles ϑ, $2^{S_k}\vartheta$, $2^{2S_k}\vartheta$, $\ldots$, $2^{(q_k-2)S_k}\vartheta$. Then p_k is the number of these angles that do not exceed ϑ.

If ϑ is periodic, then it is the angle of one of the two parameter rays landing at the root of a hyperbolic component. Here is a way to tell which of the two rays it is.

11.15. **Lemma (Left or Right Ray)**
Let $R(\vartheta)$ and $R(\vartheta')$ be the two parameter rays landing at the root of a hyperbolic component W of period $n \geq 2$, and suppose that $\vartheta < \vartheta'$. Let b and b' be the n-th entries in the binary expansions of ϑ and ϑ'. Then

- *if the kneading sequence of W has n-th entry $\mathtt{0}$, then $b = 1$ and $b' = 0$;*
- *if the kneading sequence of W has n-th entry $\mathtt{1}$, then $b = 0$ and $b' = 1$.*

For example, the hyperbolic component with internal address $1_{1/3} \to 3_{1/2} \to 4$ has kneading sequence $\overline{\mathtt{1100}}$, and the parameter ray $R(4/15)$ lands at its root. The binary expansion of $4/15$ is $0.\overline{0100}$. The 4-th entries in the kneading sequence and in the binary expansion of $4/15$ are $\mathtt{0}$ and 0, so the second ray landing together with $R(4/15)$ has angle $\vartheta < 4/15$ (indeed, the second ray is $R(3/15)$).

The following result complements the interpretation of internal addresses as road descriptions by saying that whenever the path from the origin to a parameter $c \in \mathcal{M}$ branches off from the main road, an entry in the internal address is generated: the way to most parameters $c \in \mathcal{M}$ traverses infinitely many hyperbolic components, but most of them are traversed "straight on" and left into the $1/2$-limb.

11.16. **Proposition (Sublimbs Other Than $1/2$ in Internal Address)**
If a parameter $c \in \mathbb{C}$ is contained in the subwake of a hyperbolic component W at internal angle other than $1/2$, then W occurs in the internal address of c. More precisely, the period of W occurs in the internal address, and the truncation of the angled internal address of c up to this period describes exactly the component W.

Proof. Let n be the period of W. If W does not occur in the internal address of c, then the internal address of c must have an entry $n' < n$ corresponding to a hyperbolic component W' in the wake of W. Denoting the width of W' by $|W'|$, we have $|W'| \geq 1/(2^{n'}-1)$. By Proposition 9.24, if $|W|$ denotes the width of a hyperbolic component W of period n, the width of its p/q-subwake is $|W|(2^n-1)^2/(2^{qn}-1)$, so we must have

$$1/(2^{n'}-1) \le |W|(2^n-1)^2/(2^{qn}-1) < (2^n-1)^2/(2^{qn}-1)$$

or $2^{qn}-1 < (2^{n'}-1)(2^n-1)^2 < (2^n-1)^3 < 2^{3n}-1$, hence $q<3$. □

The internal address of a parameter c also says whether or not this parameter is renormalizable in the sense of Section 10:

11.17. **Proposition (Internal Address and Renormalization)**
Let $c \in \mathcal{M}$ be a parameter with internal address $(S_0)_{p_0/q_0} \to \ldots \to (S_k)_{p_k/q_k} \ldots$.

- *c is* simply renormalizable *of period n if and only if there is a k with $S_k = n$ and $n|S_{k'}$ for $k' \ge k$;*
- *c is* crossed renormalizable *of period n if and only if there is an m strictly dividing n so that $S_k = m$ for an appropriate k and all $S_{k'}$ with $k' > k$ are proper multiples of n (in particular, n does not occur in the internal address). In this case, the crossing point of the little Julia sets has period m.*

Remark. We can also describe combinatorially whether any given parameter rays $R(\vartheta)$ and $R(\vartheta')$ can land at the same point in $\mathcal{M}$: a necessary condition is that they have the same angled internal address. This condition is also sufficient from a combinatorial point of view: suppose $R(\vartheta)$ and $R(\vartheta')$ have the same angled internal address. Then both rays accumulate at the same combinatorial class. If the internal address is finite, then $R(\vartheta)$ and $R(\vartheta')$ land on the boundary of the same hyperbolic component. Otherwise, both rays accumulate at the same combinatorial class. As soon as it is known that this combinatorial class consists of a single point (which would imply local connectivity of $\mathcal{M}$ at that point [**Sch3**]), then both rays land together.

Proofs about Internal Addresses.

Now we give the proofs of the results stated so far; most of them go back to [**LS**]. Many of the proofs are based on the concept of *long internal addresses*, which show that even though internal addresses themselves are a compact road description, they encode refinements to arbitrary detail. Parameter ray pairs have a partial order as follows: $P(\vartheta_1,\vartheta_1') < P(\vartheta_2,\vartheta_2')$ iff $P(\vartheta_1,\vartheta_1')$ separates $P(\vartheta_2,\vartheta_2')$ from the origin $c=0$, or equivalently from $c=1/4$, the landing point of the ray pair $P(0,1)$. It will be convenient to say $P(0,1) < P(\vartheta,\vartheta')$ for every ray pair $P(\vartheta,\vartheta') \ne P(0,1)$. For every parameter $c \in \mathbb{C}$, the set of parameter ray pairs separating c from the origin is totally ordered (by definition, this set always includes the pair $P(0,1)$ as its minimum). A similar partial order can be defined for dynamic ray pairs for every $c \in \mathcal{M}$.

11.18. **Definition (Long Internal Address in the Mandelbrot Set)**
For a parameter ray $c \in \mathbb{C}$, consider the set of periodic parameter ray pairs which separate c from the origin, totally ordered as described above; the long internal address *of c is the collection of periods of these ray pairs with the same total order.*

The long internal address always starts with the entry 1, and it is usually infinite and not well-ordered. For most $c \in \mathbb{C}$, many periods appear several times on the long internal address (which means that several ray pairs of equal period separate c from the origin).

One useful feature of internal addresses is that they completely encode the associated long internal addresses. The following proposition is in fact algorithmic, as its proof shows.

11.19. **Proposition (Long Internal Address Encoded in Internal Address)**
Any internal address completely encodes the associated long internal address.

PROOF. The internal address is a strictly increasing (finite or infinite) sequence of integers, each of which comes with an associated $\star$-periodic kneading sequence. If it is the internal address of some $c \in \mathbb{C}$, then each entry in the internal address represents a parameter ray pair with this period. Corollary 11.7 describes the least period of a periodic parameter ray pair which separates any two given parameter ray pairs, and it also describes the kneading sequence of the ray pair of least period. This allows us to inductively find the periods of all parameter ray pairs of given maximal periods which separate c from the origin, together with the order and kneading sequences of these ray pairs, and in the limit determines the long internal address. $\square$

REMARK. Of course, it makes perfect sense to speak of an *angled long internal address*, which is a long internal address in which all entries (except the last, if there is a last entry) are decorated with the bifurcation angles of the sublimb containing the parameter associated with this address. All entries with angles different from $1/2$ are already contained in the "short" internal address by Proposition 11.16, so the angled internal address completely encodes the angled long internal address.

PROOF OF THEOREM 11.10 (COMPLETENESS OF ANGLED INTERNAL ADDRESSES). Since combinatorial classes are based upon periodic parameter ray pairs, it suffices to prove the claim for periodic ray pairs and thus for hyperbolic components. If the claim is false, then there is a least period S_k for which there are two different hyperbolic components W and W' of period S_k that share the same angled internal address $(S_1)_{p_1/q_1} \to \ldots \to (S_{k-1})_{p_{k-1}/q_{k-1}} \to S_k$. By minimality of S_k, the ray pair of period S_{k-1} is the same in both internal addresses.

By the Branch Theorem 9.13, there are three possibilities: either (1) W is contained in the wake of W' (or vice versa), or (2) there is a hyperbolic component W_* so that W and W' are in two different of its sublimbs, or (3) there is a Misiurewicz-Thurston point c_* so that W and W' are in two different of its sublimbs.

(1) In the first case, there must by a parameter ray pair of period less than S_k separating W and W' by Lavaurs' Lemma 9.28. This ray pair would have to occur in

the internal address of W' (between entries S_{k-1} and S_k) but not of W, and this is a contradiction.

(2) The second possibility is handled by angled internal addresses: at least one of W or W' must be contained in a sublimb of W_* other than the 1/2-sublimb, so by Proposition 11.16, W_* must occur in the internal address, and the angles at W_* in the angled internal address distinguish W and W'.

(3) The case of a Misiurewicz-Thurston c_* point needs more attention. The parameter c_* is the landing point of $k \geq 3$ parameter rays $R(\vartheta_1) \dots R(\vartheta_k)$ at preperiodic angles.

By Proposition 11.19, W and W' have the same long internal address, and by minimality of S_k there is no hyperbolic component W^* of period less than S_k in a sublimb of c^* so that $W \subset \mathcal{W}_{W^*}$.

Let $P(\vartheta, \vartheta')$ be the parameter ray pair landing at the root of W and let c be the center of W. In the dynamics of p_c, there is a characteristic preperiodic point w which is the landing point of the dynamic rays $R_c(\vartheta_1)$, $\dots R_c(\vartheta_K)$ (the definition of characteristic preperiodic points is in analogy to the definition of characteristic periodic points after Theorem 11.1; the existence of the characteristic preperiodic point w is well known ("The Correspondence Theorem"); see e.g., Theorem 9.5 or [**Sch3**, Theorem 2.1]). We use the Hubbard tree T_c of p_c in the original sense of Douady and Hubbard (Definition 3.4). Let $I \subset T_c$ be the arc connecting w to c.

If the restriction $p_c^{\circ S_k}|_I$ is not injective, then let $n \leq S_k$ be maximal so that $p_c^{\circ(n-1)}|_I$ is injective. There is a sub-arc $I' \subset I$ starting at w so that $p_c^{\circ n}|_{I'}$ is injective and $p_c^{\circ n}(I')$ connects $p_c^{\circ n}(w)$ with c. If $n = S_k$ then $I' = I$ because c is an endpoint of $p_c^{\circ n}(I)$, a contradiction; thus $n < S_k$. Since w is characteristic, this implies that $p_c^{\circ n}(I') \supset I'$, so there is a fixed point z of $p_c^{\circ n}$ on I'. If z is not characteristic, then the characteristic point on the orbit of z is between z and c. In any case, we have a hyperbolic component W^* of period $n < S_k$ in a sublimb of c^* so that $W \subset \mathcal{W}_{W^*}$ (again by Theorem 11.1), and this is a contradiction.

It follows that the restriction $p_c^{\circ S_k}|_I$ is injective. There is a unique component of $\mathbb{C} \setminus (R_c(2^{S_k}\vartheta_1) \cup \dots \cup R_c(2^{S_k}\vartheta_k) \cup \{p_c^{\circ S_k}(w)\})$ that contains $p_c^{\circ S_k}(I)$: it is the component containing c and the dynamic ray pair $P_c(\vartheta, \vartheta')$, and thus also the dynamic rays $R_c(\vartheta_1), \dots, R_c(\vartheta_k)$, so this component is uniquely specified by the external angles of c_*. But by injectivity of $p_c^{\circ S_k}|_I$, this also uniquely specifies the component of $\mathbb{C} \setminus (R_c(\vartheta_1) \cup \dots \cup R_c(\vartheta_k) \cup \{z\})$ that must contain I and hence c. In other words, the subwake of c_* containing W (and, by symmetry, W') is uniquely specified. □

Proof of Lemma 11.11 (Estimates on Denominators). The angled internal address $(S_0)_{p_0/q_0} \to (S_1)_{p_1/q_1} \to \dots \to (S_k)_{p_k/q_k} \to (S_{k+1})_{p_{k+1}/q_{k+1}} \dots$, when truncated at periods S_k, describes a sequence of hyperbolic components W_k of periods S_k. If W_{k+1} is contained in the p_k/q_k-sublimb of W_k, then we have $S_{k+1} \leq q_k S_k$ (otherwise an entry $q_k S_k$ would be generated in the internal address); thus $q_k \geq S_{k+1}/S_k$. The other inequality follows from Proposition 9.24: the width of the wake of W_{k+1} cannot exceed the width of the p_k/q_k-subwake of W_k, so $1/(2^{S_{k+1}}-1) \le (2^{S_k}-1)^2/(2^{q_k S_k}-1)$ or

$$2^{q_k S_k} - 1 \le (2^{S_{k+1}}-1)(2^{S_k}-1)^2 < 2^{S_{k+1}+2S_k} - 1 \ ,$$

hence $S_{k+1} > (q_k - 2)S_k$ or $q_k < S_{k+1}/S_k + 2$.

It remains to show that whenever S_{k+1}/S_k is an integer, it equals q_k: there are associated parameter ray pairs $P(\vartheta_k, \vartheta'_k)$ of period S_k and $P(\vartheta_{k+1}, \vartheta'_{k+1})$ of period S_{k+1}, and the limiting kneading sequences $\lim_{\varphi \searrow \vartheta_k} \nu(\varphi)$ and $\lim_{\varphi \nearrow \vartheta_{k+1}} \nu(\varphi)$ are periodic of period S_k and S_{k+1}; both can be viewed as being periodic of period S_{k+1}. Since they correspond to adjacent entries in the internal address, these ray pairs cannot be separated by a ray pair of period up to S_{k+1}, so both limiting kneading sequences are equal. Therefore, Corollary 11.7 implies that the two ray pairs are not separated by any periodic parameter ray pair. If W_{k+1} was not a bifurcation from W_k, then it would have to be in some subwake of W_k whose boundary would be some ray pair separating $P(\vartheta_k, \vartheta'_k)$ from $P(\vartheta_{k+1}, \vartheta'_{k+1})$; this is not the case. So let p_k/q_k be the bifurcation angle from W_k to W_{k+1}; then the corresponding periods satisfy $S_{k+1} = q_k S_k$ as claimed. □

In the following lemma, we use the function ρ_ν as defined before Lemma 11.12.

11.20. **Lemma (Intermediate Ray Pair of Lowest Period)**
Let $P(\vartheta_k, \vartheta'_k)$ and $P(\vartheta_{k+1}, \vartheta'_{k+1})$ be two periodic parameter ray pairs with periods $S_k < S_{k+1}$ and suppose that $P(\vartheta_k, \vartheta'_k)$ separates $P(\vartheta_{k+1}, \vartheta'_{k+1})$ from the origin. Write $S_{k+1} = aS_k + r$ with $r \in \{1, \dots, S_k\}$. Let S be the least period of a ray pair separating $P(\vartheta_k, \vartheta'_k)$ from $P(\vartheta_{k+1}, \vartheta'_{k+1})$. If $S_{k+1} < S \le (a+1)S_k$, then $S = aS_k + \rho_\nu(r)$, where ν is any kneading sequence that has the same initial S_k entries as $\nu(\vartheta_{k+1}) = \nu(\vartheta'_{k+1})$.

Proof. Let B and R denote the initial blocks in $\mathcal{A}(\nu(\vartheta_k)) = \mathcal{A}(\nu(\vartheta'_k))$ consisting of the first S_k or r entries, respectively. Then $\mathcal{A}(\nu(\vartheta_k)) = \overline{B}$. Since $S > S_{k+1}$, Corollary 11.7 implies $\overline{\mathcal{A}}(\nu(\vartheta_{k+1})) = \overline{B^a R}$. Therefore, S is the position of the first difference between $\overline{B}$ and $\overline{B^a R}$, and $S - aS_k$ is the position of the first difference between $\overline{B}$ and $\overline{RB^a}$. Since $S \le (a+1)S_k$, this difference occurs among the first S_k symbols, and these are specified by B and RB. Since R is the initial segment of B of length r, $S - aS_k$ equals $\rho_\nu(r)$ for any sequence ν that starts with B. □

Proof of Lemma 11.12 (Finding denominators). The internal address (without angles) uniquely determines the long internal address by Proposition 11.19. The entry S_k occurs in the internal address and the subsequent entry in the long internal address is $q_k S_k$, so the denominators are uniquely encoded (and depend only on the $q_k S_k$ initial entries in the kneading sequence). Recall the bound $S_{k+1} \le q_k S_k < S_{k+1} + 2S_k$ from Lemma 11.11.

Write again $S_{k+1} = aS_k + r$ with $r \in \{1, \dots, S_k\}$. If $r = S_k$, then S_{k+1} is divisible by S_k and $q_k = S_{k+1}/S_k$ by Lemma 11.11, and this is what our formula predicts.

Otherwise, we have $q_k \in \{a+1, a+2\}$. Below, we will find the lowest period S' between S_k and S_{k+1}, then the lowest period S'' between S_k and S', and so on (of course, the "between" refers to the order of the associated ray pairs). This procedure must eventually reach the bifurcating period $q_k S_k$. If $q_k S_k = (a+1)S_k$, then eventually one of the periods S', S'',... must be equal to $(a+1)S_k$. If not, the sequence S', S'', ...skips $(a+1)S_k$, and then necessarily $q_k S_k = (a+2)S_k$.

We can use Lemma 11.20 for this purpose: we have $S' = aS_k + \rho_\nu(r)$, then $S'' = aS_k + \rho_\nu(\rho_\nu(r))$ etc. until the entries reach or exceed $aS_k + S_k$: if the entries reach $aS_k + S_k$, then $q_k = a+1$; if not, then $q_k > a+1$, and the only choice is $q_k = a+2$. □

Proof of Theorem 11.13 (Numerators arbitrary). We prove a stronger statement: given a hyperbolic component W of period n, then we can combinatorially determine for any $s' > 1$ how many components of period up to s' are contained in the wake $\mathcal{W}_W$, how the wakes of all these components are nested, and what the widths of their wakes are; all we need to know about W are the width of its wake and its internal address (without angles). In particular, these data encode how the internal address of the component W can be continued within $\mathcal{M}$. This proves the theorem (and it also shows that the width of the wake of W is determined by the internal address of W).

For any period s, the number of periodic angles of period s or dividing s within any interval of $\mathbb{S}^1$ of length δ is either $\lfloor \delta/(2^s-1) \rfloor$ or $\lceil \delta/(2^s-1) \rceil$ (the closest integers above and below $\delta/(2^s-1)$). If the interval of length δ is the wake of a hyperbolic component, then the corresponding parameter rays land in pairs, and the correct number of angles is the unique even integer among $\lfloor \delta/(2^s-1) \rfloor$ and $\lceil \delta/(2^s-1) \rceil$. This argument uniquely determines the exact number of hyperbolic components of any period within any wake of given width.

There is a unique component W_s of lowest period s, say, in $\mathcal{W}_W$ (if there were two such components, then this would imply the existence of at least one parameter ray of period less than s and thus of a component of period less than s). The width of $\mathcal{W}_{W_s}$ is exactly $1/(2^s-1)$ (this is the minimal possible width, and greater widths would imply the existence of a component with period less than s).

Now suppose we know the number of components of periods up to s' within the wake $\mathcal{W}_W$, together with the widths of all their wakes and how these wakes are nested. The wake boundaries of period up to s' decompose $\mathcal{W}_W$ into finitely many components. Some of these components are wakes; the others are complements of wakes within other wakes. We can uniquely determine the number of components of period $s'+1$ within each of these wakes (using the widths of these wakes), and then also within each complementary component outside of some of the wakes (simply by calculating differences). Each wake and each complementary component can contain at most one component of period $s'+1$ by Theorem 11.10. The long internal addresses tell us which wakes of period $s'+1$ contain which other wakes, and from this we can determine the widths of the wakes of period $s'+1$. This provides all information for period $s'+1$, and this way we can continue inductively and prove the claimed statement.

Starting with the unique component of period 1, it follows that the width of a wake $\mathcal{W}_W$ is determined uniquely by the internal address of W. $\square$

Proof of Lemma 11.14 (Finding numerators). We only need to find the numerator p_k if $q_k \geq 3$. Let W_k be the unique hyperbolic component with angled internal address $(S_0)_{p_0/q_0} \to \ldots \to (S_k)$ and let $P(\vartheta_k, \vartheta'_k)$ be the ray pair bounding its wake $\mathcal{W}_{W_k}$. Let W'_k be the component with angled internal address $(S_0)_{p_0/q_0} \to \ldots \to (S_k)_{p_k/q_k} \to q_k S_k$; it is an immediate bifurcation from W_k.

By Theorem 11.1, every $c \in \mathcal{W}_{W_k}$ has a repelling periodic point z_c which is the landing point of the characteristic dynamic ray pair $P_c(\vartheta_k, \vartheta'_k)$; we find it convenient to describe our proof in such a dynamic plane, even though the result is purely combinatorial. Let Θ be the set of angles of rays landing at z_c; this is the same for all $c \in \mathcal{W}_{W_k}$. Especially if $c \in W'_k$, it is well known and easy to see that Θ contains exactly q_k elements, the first return map of z_c permutes the corresponding rays transitively and their combinatorial rotation number is p_k/q_k (Lemma 9.3). These rays disconnect $\mathbb{C}$ into q_k sectors which can be labelled $V_0, V_1, \ldots, V_{q_k-1}$ so that the first return map of z_c sends V_j homeomorphically onto V_{j+1} for $j = 1, 2, \ldots, q_k - 2$, and so that V_1 contains the critical value and V_0 contains the critical point and the ray $R_c(0)$. Finally, under the first return map of z_c, points in V_0 near z_c map into V_1, and points in V_{q_k-1} near z_c map into V_0. The number of sectors between V_0 and V_1 in the counterclockwise cyclic order at z_c is then $p_k - 1$, where p_k is the numerator in the combinatorial rotation number.

Now suppose the dynamic ray $R_c(\vartheta)$ contains the critical value or lands at it. Then $R_c(\vartheta) \in V_1$, and $R_c(2^{(j-1)S_k}\vartheta) \in V_j$ for $j = 1, 2, \ldots, q_k-1$. Counting the sectors between V_0 and V_1 means counting the rays $R_c(\vartheta), R_c(2^{S_k}\vartheta), \ldots, R_c(2^{(q_k-2)S_k}\vartheta)$ in these sectors, and this means counting the angles $\vartheta, 2^{S_k}\vartheta, \ldots, 2^{(q_k-2)S_k}$ in $(0, \vartheta)$. The numerator p_k exceeds this number by one, and this equals the number of angles $\vartheta, 2^{S_k}\vartheta, \ldots, 2^{(q_k-2)S_k}$ in $(0, \vartheta]$. $\square$

Proof of Lemma 11.15 (Left or right ray). The n-th entry in the kneading sequence of $\nu(\vartheta)$ is determined by the position of the angle $2^{n-1}\vartheta \in \{\vartheta/2, (\vartheta+1)/2\}$. The n-th entry in the kneading sequence of W equals the n-th entry in the kneading sequence of $\tilde{\vartheta}$ for $\tilde{\vartheta}$ slightly greater than ϑ (for ϑ', we use an angle $\tilde{\vartheta}'$ slightly smaller than $\tilde{\vartheta}$); this is 1 if and only if $2^{n-1}\vartheta = \vartheta/2$ and 0 otherwise. But $2^{n-1}\vartheta = \vartheta/2$ implies $2^{n-1}\vartheta \in (0, 1/2)$, hence $b = 0$, while $2^{n-1}\vartheta = (\vartheta+1)/2$ implies $2^{n-1}\vartheta \in (1/2, 1)$ and $b = 1$. The reasoning for ϑ' is similar. $\square$

Proof of Proposition 11.17 (Internal Address and Renormalization). We only discuss the case of simple renormalization (the case of crossed renormalization is treated in [**RiSch**, Corollary 4.2]).

Fix a hyperbolic component W of period n. Let $\mathcal{M}_W$ be the component of n-renormalizable parameters in $\mathcal{M}$ containing W (all $c \in W$ are n-renormalizable), and let $\Psi_W : \mathcal{M} \to \mathcal{M}_W$ be the tuning homeomorphism; see [**Hai, Mix, Mi2**] or Section 10. Let $1 \to S_1 \to \ldots \to S_k = n$ be the internal address of W. Then the internal address of every $c \in \mathcal{M}_W$ starts with $1 \to S_1 \to \ldots \to S_k = n$ because $\mathcal{M}_W$ contains no hyperbolic component of period less than n, so points in $\mathcal{M}_W$ are not separated from each other by parameter ray pairs of period less than n. All hyperbolic components within $\mathcal{M}_W$, and thus all ray pairs separating points in $\mathcal{M}_W$, have periods that are multiples of n, so all internal addresses of parameters within $\mathcal{M}_W$ have the form $1 \to S_1 \to \ldots \to S_k \to S_{k+1} \ldots$ so that all $S_m \geq n$ are divisible by n. In fact, if $c \in \mathcal{M}$ has internal address $1 \to S'_1 \to \ldots \to S'_{k'} \ldots$, then it is not hard to see that the internal address of $\Psi_W(c)$ is $1 \to S_1 \to \ldots \to n \to nS'_1 \to \ldots \to nS'_k \ldots$ (hyperbolic components of period S' in $\mathcal{M}$ map to hyperbolic components of period nS' in $\mathcal{M}_W$, and all ray pairs separating points in $\mathcal{M}_W$ are associated to hyperbolic components in $\mathcal{M}_W$ that are images under Ψ_W).

For the converse, we need *dyadic Misiurewicz-Thurston parameters*: these are by definition the landing points of parameter rays $R(\vartheta)$ with $\vartheta = m/2^k$; dynamically, these are the parameters for which the singular orbit is strictly preperiodic and terminates at the β fixed point. If $\vartheta = m/2^k$, then the kneading sequence $\nu(\vartheta)$ has only entries $\mathtt{0}$ from position $k+1$, so the internal address of $\nu(\vartheta)$ contains all integers that are at least $2k-1$. But the internal address of $\nu(\vartheta)$ from Algorithm 11.5 equals the internal address of $R(\vartheta)$ in parameter space (Proposition 11.6), and the landing point of $R(\vartheta)$ has the same internal address. Therefore, the internal address of any dyadic Misiurewicz-Thurston parameter contains all sufficiently large positive integers.

Suppose the internal address of some $c \in \mathcal{M}$ has the form $1 \to S_1 \to \ldots \to S_k \to S_{k+1} \ldots$ with $S_k = n$ and all $S_m \geq n$ are divisible by n. There is a component W with address $1 \to S_1 \to \ldots \to S_k$ so that $c \in \mathcal{W}_W$. If $c \notin \mathcal{M}_W$, then c is separated from $\mathcal{M}_W$ by a Misiurewicz-Thurston parameter $c_* \in \mathcal{M}_W$ which is the tuning image of a dyadic Misiurewicz-Thurston parameter (see [**D4**], [**Mi2**, Section 8], or Corollary 10.9). Therefore, the internal address of c_* contains all integers that are divisible by n and sufficiently large, say at least Kn. Let S be the first entry in the internal address of c that corresponds to a component "behind $\mathcal{M}_W$" (so that it is separated from $\mathcal{M}_W$ by c_*). The long internal address of c contains "behind" W only hyperbolic components of periods divisible by n, and this implies that before and after c_* there must be two components of equal period (greater than Kn) that are not separated by a ray pair of lower period. This contradicts Lavaurs' Lemma 9.28. $\square$

Remark. In Definition 11.9, we defined angled internal addresses in parameter space, but they can also be defined dynamically: the angles are combinatorial rotation numbers of rays landing at characteristic periodic points the periods of which occur in the internal address. This allows us to give more dynamic proofs of several theorems that we proved in parameter space. For instance, changing numerators has no impact on whether an angled internal address is realized in the complex plane (Theorem 11.13): for the Hubbard trees at centers of hyperbolic components, this simply changes the embedding of the tree, but not the issue whether such a tree can be embedded (compare the discussion in Corollary 4.11 and Lemma 16.11). (The denominators are determined already by the internal address without angles; see Lemma 11.12). Similarly, the angled internal address completely determines a Hubbard tree with its embedding and thus the dynamics at the center of a hyperbolic component; this is Theorem 11.10: the (finite) angled internal address completely specifies every hyperbolic component.

We can define a partial order of hyperbolic components W, W' in $\mathcal{M}$ as follows: we say that W' is greater than W if $W' \neq W$ and the wake of W contains W' (and thus the wake of W'). Since wakes are by definition bounded by parameter ray pairs at periodic angles, and are thus either nested or disjoint, Theorem 11.1 implies that an equivalent dynamical definition goes as follows: W' is greater than W if any parameter $c' \in W'$ (or even c' in the wake of W') has the property that if $P(\vartheta, \vartheta')$ is the ray pair bounding the wake of W, then $p_{c'}$ has a characteristic dynamic ray pair $P_{c'}(\vartheta, \vartheta')$. (An extension of this partial order to all $c, c' \in \mathcal{M}$ goes as follows: c' is greater than c if there is a parameter ray pair $P(\vartheta, \vartheta')$ at periodic angles that separates c' from c and the origin.)

In Definition 6.1, the space of all $\star$-periodic kneading sequences (realized by a complex polynomial or not) was endowed with a partial order, so that $\nu' > \nu$ if the Hubbard tree for ν' contains a periodic point with itinerary $\mathcal{A}(\nu)$. These two partial orders are related as follows.

11.21. **Proposition (Order for Hyperbolic Components and Kneadings)** *Consider two hyperbolic components $W \neq W'$ with associated kneading sequences ν and ν'. If W' is greater than W, then $\nu' > \nu$. Conversely, if $\nu' > \nu$, then there is a hyperbolic component W'' greater than W so that W'' has kneading sequence ν' and thus the same period and internal address as W'.* □

We omit the proof, which is not difficult: it follows from the dynamical definition of the partial order as given above. Note that it is not true that if $\nu' > \nu$, then W' is greater than W: there can be several hyperbolic components with the same kneading sequence ν' and thus the same internal address, and these components are distinguished by their angled internal addresses. At least one of these components is greater than W, while the other components may not be comparable.

12. Symbolic Dynamics, Permutations, Galois Groups

We investigate which permutations of periodic points of p_c can be achieved by analytic continuation with respect to the parameter c. This makes it possible to determine Galois groups of those polynomials which represent the periodic points. In order to do this, we need some existence theorems of kneading sequences, and these will be derived from results on the structure of the Mandelbrot set using internal addresses.

Every component W of period n has an associated internal address (finite, ending with entry n); compare the remark before Definition 11.9. The component W also has an associated periodic kneading sequence $\nu(W)$ consisting only of entries $\texttt{0}$ and $\texttt{1}$: one way of defining this is to take any parameter ray $R(\varphi)$ with irrational φ landing at ∂W; then $\nu(W) := \nu(\varphi)$. Equivalently, let $P(\vartheta, \vartheta')$ be the parameter ray pair landing at the root of W; then ϑ and ϑ' have period n and $\nu(\vartheta) = \nu(\vartheta')$ are $\star$-periodic of period n, and $\nu(W) = \mathcal{A}(\nu(\vartheta)) = \mathcal{A}(\nu(\vartheta'))$ (see Lemma 11.2 and Proposition 11.6). Of course, the internal address of W is the same as the internal address of $R(\varphi)$, $R(\vartheta)$, $R(\vartheta')$ or of φ, ϑ or ϑ'.

12.1. **Definition (Narrow Component)**
A hyperbolic component of period n is narrow *if its wake contains no component of lower period, or equivalently if the wake has width $1/(2^n-1)$.*

Remark. It follows directly from Proposition 9.24 that if W' is a bifurcation from another component W, then W' is narrow if and only if W has period 1.

If W, W' are two hyperbolic components of periods n and n' so that $W' \subset \mathcal{W}_W$, then we say that W' is *visible* from W if there exists no parameter ray pair of period less than n' that separates W and W'. If $n < n'$, then this is equivalent to the condition that the internal address of W' be formed by the internal address of W, extended by the entry n'.

12.2. **Lemma (Visible Components from Narrow Component)**
For every narrow hyperbolic component of period n, there are visible components of all periods greater than n. More precisely, every p/q sublimb contains exactly n visible components: exactly one component each of period $qn-(n-1)$, $qn-(n-2)$,$\dots$, qn.

Proof. Let W be a narrow hyperbolic component of period n and consider its p/q-subwake. The visible components in this wake have periods at most qn. First we show that any two visible components within the p/q-subwake of W have different internal addresses. By way of contradiction, suppose there are two components W_1 and W_2 of equal period $m \le qn$ with the same internal address. By Theorem 11.10 there must be another hyperbolic component W' in the same subwake of W so that W_1 and W_2 are in different p_1/q'- and p_2/q'-subwakes of W' with $q' \ge 3$ (different hyperbolic components with the same internal address must have different angled internal addresses). Let

$n' > n$ be the period of W'. Then the width of these subwakes is, according to Proposition 9.24,

$$\begin{aligned} |W'|\frac{(2^{n'}-1)^2}{2^{q'n'}-1} &\leq |W|\frac{(2^n-1)^2}{2^{qn}-1}\cdot\frac{(2^{n'}-1)^2}{2^{q'n'}-1} = \frac{2^n-1}{2^{qn}-1}\cdot\frac{(2^{n'}-1)^2}{2^{q'n'}-1} \\ &< \frac{2^n(2^{(2-q')n'})(1-2^{-q'n'})^{-1}}{2^{qn}-1} \leq \frac{2^{(n-n')}\cdot 2}{2^{qn}-1} \leq \frac{1}{2^{qn}-1} \end{aligned}$$

because W' is contained in the p/q-wake of W. But this is not large enough to contain a component of period at most qn, a contradiction.

The next step is to prove that every subwake of W of denominator q contains exactly 2^k external rays with angles $a/(2^{(q-1)n+k}-1)$ for $1 \leq k \leq n$, including the two rays bounding the wake. In fact, Proposition 9.24 says that the width of the wake is $(2^n-1)/(2^{qn}-1)$, so the number of rays one expects by comparing widths of wakes is

$$\frac{(2^n-1)(2^{(q-1)n+k}-1)}{2^{qn}-1} =: 2^k + \alpha,$$

where an easy calculation shows that $-1 \leq \alpha < 1$ and that $\alpha = -1$ can occur only for $k = n$. The actual number of rays can differ from this expected value by no more than one and is even, hence equal to 2^k. Moreover, no such ray of angle $a/(2^{(q-1)n+k}-1)$ can have period smaller than $(q-1)n+k$ because it would land at a hyperbolic component of some period dividing $(q-1)n+k$ — but in the considered wake there would not be room enough to contain a second ray of equal period.

This shows that, for any $k \leq n$, the number of hyperbolic components of period $m = (q-1)n+k$ in any subwake of W of denominator q equals 2^{k-1}. They must all have different internal addresses. The single component of period $(q-1)n+1$ takes care of the case $k = 1$, and its wake subdivides the p/q-subwake of W into two components. There are two components of period $(q-1)n+2$, and since their internal addresses are different, exactly one of them must be in the wake of the component of period $(q-1)n+1$, while the other is not; the latter one is visible from W. (The non-visible component is necessarily narrow, while the visible component may or may not be narrow.)

So far we have taken are of 3 components, and they subdivide the p/q-subwake of W into 4 components. Each component most contain one component of period $(q-1)n+3$, so exactly one of these components is visible from W, and so on. The argument continues as long as we have uniqueness of components for given internal addresses, which is for $k \leq n$. □

12.3. **Lemma (Narrow Visible Components from Narrow Component)**
Suppose W and W' are two hyperbolic components of periods n and $n+s$ with $s > 0$ so that W is narrow and W' is visible from W. Let $k \in \{1, \dots, n-1, n\}$ be so that

$s \equiv k$ modulo n. Then the question whether or not W' is narrow depends only on the first k entries in the kneading sequence of W (but not otherwise on W).

Proof. By Lemma 12.2, every p/q-subwake of W contains exactly one visible hyperbolic component W_m of period $m = (q-1)n + k$ for $k = 1, 2, \ldots, n$. Such a component W_m is narrow unless the wake $\mathcal{W}_{W_m}$ contains a visible component $W_{m'}$ with $m' = (q-1)n + k'$ and $k' \in \{1, 2, \ldots, k-1\}$. If there is such a component, let m' be so that the width $|W_{m'}|$ is maximal (i.e., W_m and $W_{m'}$ are not separated by a parameter ray pair of period less than m').

In order to find out whether the unique visible component $W_{m'}$ is contained in $\mathcal{W}_{W_m}$, we need to compare the kneading sequence ν associated to W with the kneading sequence ν' "just before" $W_{m'}$ and find whether their first difference occurs at position m (according to Corollary 11.7, the position of the first difference is the least period of two ray pairs separating W and $W_{m'}$: if this difference occurs after entry m, then the ray pairs landing at the root of W_m do not separate W from $W_{m'}$; the difference cannot occur before entry m because of visibility of W_m and the choice of m').

By visibility of $W_{m'}$, the kneading sequences ν and ν' coincide for at least $m' > n$ entries. Eliminating these, we need to compare $\sigma^{m'}(\nu)$ with ν' for $m - m' = k - k' < n$ entries. Equivalently, we need to compare the first $k - k'$ entries in $\sigma^{k'}(\nu)$ with ν' or equivalently with ν: these are the entries $\nu_{k'+1} \ldots \nu_k$ and $\nu_1 \ldots \nu_{k-k'}$. But this comparison involves only the first k entries in ν. (The precise criterion is: W' is not narrow if and only if there is a $k' \in \{1, 2, \ldots, k-1\}$ with $\rho(k') = k$; compare (3) in the proof of Corollary 5.20.) $\square$

12.4. **Proposition (Combinatorics of Purely Narrow Components)**
Consider a hyperbolic component W with internal address $1 \to S_1 \to \ldots \to S_k$ and associated kneading sequence ν. Suppose that $\nu_{S_i} = \mathtt{0}$ for $i = 1 = 2, \ldots, k$. Then W is narrow and for every $S_{k+1} > S_k$, there exists a hyperbolic component with internal address $1 \to S_1 \to \ldots \to S_k \to S_{k+1}$; it is narrow if and only if $\nu_{S_{k+1}} = \mathtt{1}$.

Proof. We prove the claim by induction on the length of internal addresses, starting with the address 1 of length 0. The associated component has period 1, it is narrow and has $\nu = \overline{1}$, and there is no condition on ν_{S_i} to check. For every $S_1 > 1$, there exists a hyperbolic component with internal address $1 \to S_1$ (these are components of period S_1 bifurcating immediately from the main cardioid). By the remark after Definition 12.1, these components are narrow, and indeed $\nu_{S_1} = \mathtt{1}$.

Now assume by induction that the claim is true for a narrow component W_{k-1} with internal addresses $1 \to S_1 \to \ldots \to S_{k-1}$ of length $k-1$ and associated kneading sequence μ. We will prove the claim when W is a hyperbolic component with internal address $1 \to S_1 \to \ldots \to S_{k-1} \to S_k$ of length k and with associated internal address ν so that $\nu_{S_k} = \mathtt{0}$. First we show that W is narrow: by the inductive hypothesis, it is narrow if and only if $\mu_{S_k} = \mathtt{1}$, but this is equivalent to $\nu_{S_k} = \mathtt{0}$ because ν and μ first differ at position S_k.

Consider some integer $S_{k+1} > S_k$. By Lemma 12.2, there exists another component W_{k+1} with internal address $1 \to S_1 \to \ldots \to S_k \to S_{k+1}$. We need to show that it is narrow if and only if $\nu_{S_{k+1}} = \mathtt{1}$. If S_{k+1} is a proper multiple of S_k, then the assumption $\nu_{S_k} = \mathtt{0}$ implies $\nu_{S_{k+1}} = \mathtt{0}$, and we have to show that W_{k+1} is not narrow. Indeed, W_{k+1} is a bifurcation from W by Lemma 11.11, and by the remark after Definition 12.1 bifurcations are narrow if and only if they bifurcate from the period 1 component. We can thus write $S_{k+1} = qS_k + S'_{k+1}$ with $S'_{k+1} \in \{S_k + 1, S_k + 2, \ldots, 2S_k - 1\}$.

Again by Lemma 12.2, there exists another component W'_{k+1} with internal address $1 \to S_1 \to \ldots \to S_k \to S'_{k+1}$; by Lemma 12.3, it is narrow if and only if W_{k+1} is. But ν has period S_k, so $\nu_{S_{k+1}} = \nu_{S'_{k+1}}$ and the claim holds for S_{k+1} if and only if it holds for S'_{k+1}. It thus suffices to restrict attention to the case $S_{k+1} < 2S_k$. Whether or not W_{k+1} is narrow is determined by the initial $S_{k+1} - S_k < S_k$ entries in ν.

Now we use the inductive hypothesis for W_{k-1}: there exists a component W' with address $1 \to S_1 \to \ldots \to S_{k-1} \to (S_{k-1} + S_{k+1} - S_k)$, and it is narrow if and only if $\mu_{S_{k-1}+S_{k+1}-S_k} = \mathtt{1}$. The kneading sequences μ and ν first differ at position S_k, so their initial $S_{k+1} - S_k$ entries coincide. Therefore, by Lemma 12.3, the component W' is narrow if and only if W_{k+1} is. Therefore, W_{k+1} is narrow if and only if $\mu_{S_{k-1}+S_{k+1}-S_k} = \mathtt{1}$. Finally, we have

$$\nu_{S_{k+1}} = \nu_{S_{k+1}-S_k} = \mu_{S_{k+1}-S_k} = \mu_{S_{k-1}+S_{k+1}-S_k}$$

by periodicity of ν (period S_k) and of μ (period S_{k-1}) and because the first difference between ν and μ occurs at position $S_k > S_{k+1} - S_k$. This proves the proposition. □

Remark. We call a hyperbolic component W_{k+1} with internal address $1 \to S_1 \to \ldots \to S_k \to S_{k+1}$ and associated kneading sequence ν *purely narrow* if $\nu_{S_i} = \mathtt{0}$ for $i = 1, 2, \ldots, k+1$. Proposition 12.4 implies that this is equivalent to the condition that $1 \to S_1 \to \ldots \to S_{i-1} \to S_i$ describes a narrow component for $i = 1, \ldots, k+1$ (hence the name). The asymmetry in the statement of the proposition (for narrow components, the last entry in ν must be $\mathtt{1}$, rather than $\mathtt{0}$ for all earlier components) is because the condition is with respect to the kneading sequence of period S_k, not with respect to the sequence of period S_{k+1} associated to W_{k+1}.

Remark. For every narrow hyperbolic component, the previous results allow to construct combinatorially the trees of visible components within any sublimb; for purely narrow components, the global tree structure can thus be reconstructed by what we call "growing of trees": see Figure 12.1. These issues have been explored further by Kauko [**Kau1**].

12.5. **Lemma (Maximal Shift of Kneading Sequence)**
For a kneading sequence ν (without $\star$) with associated internal address $1 \to S_1 \to \ldots \to S_k \to \ldots$, the following are equivalent:

(1) *no shift $\sigma^k(\nu)$ exceeds ν with respect to lexicographic ordering;*
(2) *for every $r \geq 1$, $\nu_{\rho(r)} = 0$.*

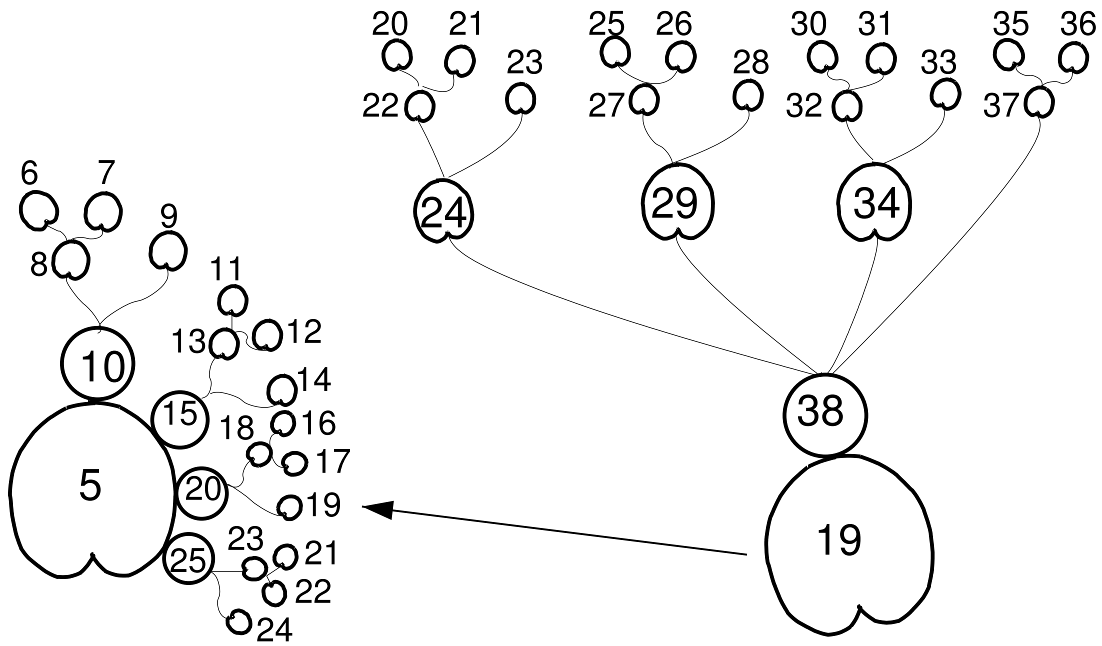


FIGURE 12.1. Left: For a narrow hyperbolic component W (here of period 5), the trees of visible components within any p/q- and p'/q'-subwake are the same when adding $(q'-q)n$ to the periods of all components in the p/q-subwake; this "translation principle" follows directly from Lemma 12.3, using just the combinatorics of internal addresses or kneading sequences (even the embeddings of the trees are the same; this follows from comparisons with dynamical planes). Right: if W' (here of period 19) is visible from W and both are narrow, then the tree of visible components within the 1/2-subwake of W' can be reconstructed from the trees of visible components within various subwakes of W: if n' and n are the periods of W' and W, then the tree formed by the visible components in the 1/2-subwake of W' of periods $n'+1, \ldots, 2n'-1$ (excluding the bifurcating component of period n') equals the tree formed by the visible components of periods $n+1, \ldots, n+n'-1$ in the $1/q$-subwakes of W, adding $n'-n$ to all periods.

Every such kneading sequence is realized by a purely narrow hyperbolic component.

PROOF. Pick some $r \geq 1$. If $\nu_{\rho(r)} = \mathtt{1}$, then the entries $\nu_{r+1}\nu_{r+2}\ldots\nu_{\rho(r)}$ exceed the entries $\nu_1\nu_2\ldots\nu_{\rho(r)-r}$, hence $\sigma^r(\nu) > \nu$ in the lexicographic ordering; conversely, if $\nu_{\rho(r)} = \mathtt{0}$, then $\sigma^r(\nu) < \nu$. Thus both conditions are indeed equivalent.

By Proposition 12.4, it follows inductively that all internal addresses $1 \to S_1 \to \ldots \to S_{k-1} \to S_k$ are realized by narrow components because the associated kneading sequence ν satisfies $\nu_{S_i} = \nu_{\rho(S_{i-1})} = \mathtt{0}$. □

REMARK. The existence of kneading sequences as described in this lemma can also be derived from the general admissibility condition on kneading sequences, see Corollary 5.20 (this is a more abstract and difficult, but also more general result). Schmeling observed that sequences that are maximal shifts with respect to the lexicographic order are exactly the fixed points of the map $\vartheta \mapsto \nu(\vartheta)$.

Symbolic Dynamics and Permutations.

We will now discuss permutations of periodic points of $p_c(z) = z^2 + c$. We make a brief excursion to algebra and describe our theorem first in algebraic terms (for readers that are less familiar with these algebraic formulations, we restate the result in Theorem 12.8). For $n \geq 1$, let $Q_n(z) := p_c^{\circ n}(z) - z$ (consider these as polynomials in z with coefficients in $\mathbb{C}[c]$). The roots of Q_n are periodic points of period dividing n, so we can factor them as

$$Q_n = \prod_{k|n} P_k \ ;$$

this product defines the P_k recursively, starting with $P_1 = Q_1$.

12.6. **Theorem (Galois Groups of Polynomials)**
For every $n \geq 1$, the polynomials P_n are irreducible over $\mathbb{C}[c]$. Their Galois groups G_n consist of all the permutations of the roots of P_n that commute with the dynamics of p_c. There is a short exact sequence

$$0 \longrightarrow (\mathbb{Z}_n)^{N_n} \longrightarrow G_n \longrightarrow S_{N_n} \longrightarrow 0\ ,$$

where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, N_n is the number of periodic orbits of exact period n for p_c with $c \in X_n$, and S_{N_n} is the symmetric group on N_n elements.

In this statement, the injections $(\mathbb{Z}_n)^{N_n} \to G_n$ correspond to independent cyclic permutations of the N_n orbits of period n, while the surjection is the projection from periodic points to periodic orbits and yields arbitrary permutations among the orbits. This theorem was derived independently by algebraic means in [**Bou1, MP**].

A related statement in parameter space is still unsolved: consider the polynomials $\tilde{Q}_n(c) := p_c^{\circ n}(c) - c \in \mathbb{Z}[c]$. Their roots are parameters c for which the critical orbit is periodic of period dividing n (i.e., c is the center of a hyperbolic component of period n), so we can again factor as $\tilde{Q}_n = \prod_{k|n} \tilde{P}_k$ with $\tilde{P}_1 = \tilde{Q}_1$.

12.7. **Conjecture (Galois Groups for Centers of Hyperbolic Components)**
All $\tilde{P}_n$ are irreducible over $\mathbb{Q}$, and their Galois groups are the full symmetric groups.

This would say that the centers of hyperbolic components of fixed period n have the maximal symmetry possible. Manning confirmed this conjecture for low values of n by computer experiments (unpublished).

Our approach for proving Theorem 12.6 will be using analytic continuation, like in the proof of the Ruffini-Abel theorem. This will yield explicit paths along which analytic continuation yields any given permutations. For specific values of $c \in \mathbb{C}$, the P_n are polynomials in $\mathbb{C}[z]$ and factor over $\mathbb{C}$; we write them as $P_n(c)$. Let

$$X_n := \{c \in \mathbb{C} \colon \text{ all roots of } P_n(c) \text{ are simple}\} \ .$$

Then all periodic points of period n can be continued analytically through X_n, so the fundamental group of X_n (with respect to any basepoint) acts on periodic points by analytic continuation. The question is which permutations can be so achieved. Of course, all permutations have to commute with the dynamics: if z is a periodic points of p_c, then any permutation π that is achieved by analytic continuation must have the property that $p_c(\pi(z)) = \pi(p_c(z))$. It turns out that this is the only condition.

12.8. **Theorem (Analytic Continuation of Periodic Points)**
For every period $n \geq 1$, analytic continuation in X_n induces all permutations among periodic points of exact period n that commute with the dynamics.

If z_0 is a double root of $P_n(c)$ for some $c \in \mathbb{C}$, then z_0 is also a double root of Q_n and $(d/dz)Q_n(z_0) = 0$, hence $\mu := (d/dz)p_c^{\circ n}(z_0) = 1$. Here μ is the multiplier of the periodic orbit containing z_0. It is well known that a quadratic polynomial can have at most one non-repelling cycle. If $\mu = 1$, there are two possibilities [**DH1, Mi2, Sch2**]: every point on the non-repelling cycle is either a merger of two points on different orbits of the same exact period n, or it is a merger of one point of period k (so that k strictly divides n) and n/k points from one orbit of period n. In either case, c is the root of a hyperbolic component of period n: it is primitive in the first case and a satellite component in the second. Therefore, c is the landing point of a parameter ray $R(\vartheta)$, where $\vartheta = a/(2^n-1)$ for some $a \in \{0, 1, \ldots, 2^n-2\}$. It follows that $\mathbb{C} \setminus X_n$ is finite, and all periodic points of period n can be continued analytically along curves in X_n (as roots of $P_n(c)$). (However, $\bigcup_n \overline{\mathbb{C} \setminus X_n} = \partial\mathcal{M}$: every $c \in \partial\mathcal{M}$ is a limit point of centers of hyperbolic components [**DH1**], and it follows easily that the same is true for parabolics because for every $\varepsilon > 0$ almost every center has a parabolic parameter at distance less than ε).

12.9. **Lemma (Local Coordinates Near Parabolic Parameters)**
Let c_0 be the root of a hyperbolic component W of period n.

- *If W is primitive, then the parabolic orbit at c_0 is the merger of two periodic orbits of exact period n; when c makes a small loop around c_0, analytic continuation of both orbits interchanges them.*

- *If W is a satellite from a component of period k, then $q := n/k$ is an integer with $q \geq 2$, and the parabolic orbit at c_0 is the merger of one orbit of period n with one orbit of period k; when c makes a small loop around c_0, analytic continuation leaves the period k orbit unchanged and permutes the period n orbit cyclically. Specifically if $k = 1$, the period n orbit is permuted transitively.*

Every other periodic point of p_{c_0} is simple and on a repelling orbit, and it can be continued analytically in a neighborhood of c_0.

Proof. For the the primitive case, see [**Sch2**, Lemma 5.1 and Corollary 5.7] or the proof of [**Mi2**, Lemma 4.2] (the fact that the two orbits are indeed interchanged is equivalent to the fact that no two hyperbolic components of equal period have common boundary points).

In the satellite case, the parabolic orbit of p_{c_0} has exact period k and breaks up under perturbation into one orbit of period n and one of period k (see e.g., [**Mi2**, Theorem 4.1] or [**Sch2**, Lemma 5.1]). The multiplier of the parabolic orbit is $\mu_0 = e^{2\pi i p/q}$ for some p coprime to q (Proposition 9.11). Since $\mu_0 \neq 1$, the period k orbit can be continued analytically in a neighborhood of c_0, so small loops around c_0 can act only on the orbit of period n; any permutation of points on this orbit must commute with the dynamics, so only cyclic permutations are possible.

Perturbing c_0 to nearby parameters c, every parabolic periodic point breaks up into one point w of period k and q points of period n. These q points form, to leading order, a regular q-gon with center w because the first return map of w has the local form $z \mapsto \mu z$ with μ near $e^{2\pi i p/q}$. When c turns once around c_0, analytic continuation induces a cyclic permutation among these q points of period n so that during the loop, the q points of period n continue to lie on (almost) regular q-gons. When the loop is completed, the vertices of the q-gon are restored, so the q-gon will rotate by s/q of a full turn, for some $s \geq 1$. If $s > 1$, then the period n orbit and its multiplier would be restored to leading order after c has completed $1/s$-th of a turn, and since boundaries of hyperbolic components are smooth curves this would imply that c_0 was on the boundary of s hyperbolic components of period n. This is not the case by Proposition 9.11. Therefore $s = 1$. $\square$

The fundamental group of X_n (with respect to any basepoint) acts on the set of periodic points of p_c of period n by analytic continuation. Set $X := \mathbb{C} \setminus (\mathcal{M} \cup \mathbb{R}^+)$: this is a simply connected subset of all X_n and will be used as a "fat basepoint" for the fundamental group of X_n.

For every $c \in X$ we will describe periodic points of p_c using symbolic dynamics: since $c \notin \mathcal{M}$, the critical value c is on the dynamic ray $R_c(\vartheta)$ for some $\vartheta \in \mathbb{S}^1$. Therefore the two dynamic rays $R_c(\vartheta/2)$ and $R_c((\vartheta+1)/2)$ both land at 0 and separate the complex plane into two open parts, say U_0 and U_1 so that $c \in U_1$ (see Figure 12.2). The partition boundary does not intersect the Julia set J_c of p_c, so we have $J_c \subset$

$U_0 \cup U_1$. Every $z \in J_c$ has an associated *itinerary* $\tau_1\tau_2\tau_3\ldots$, where $\tau_k \in \{\mathtt{0},\mathtt{1}\}$ so that $p_c^{\circ(k-1)}(z) \in U_{\tau_k}$.

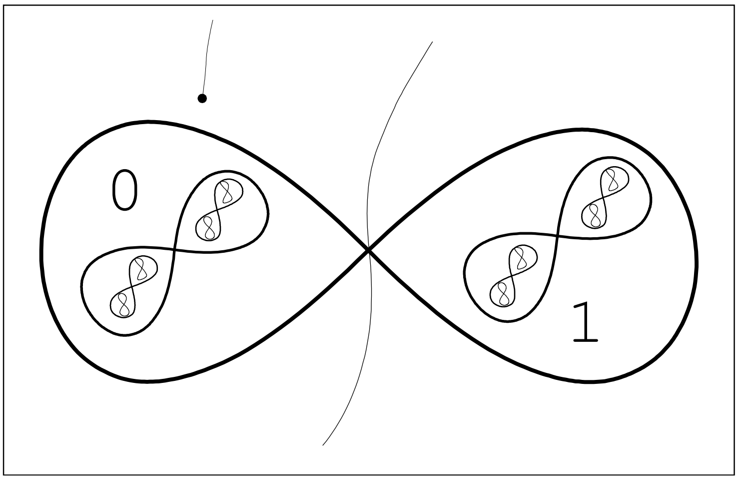


FIGURE 12.2. The partition for disconnected quadratic Julia sets is defined by the two dynamic rays that crash into the critical point; itineraries are defined with respect to this partition.

12.10. **Lemma (Permutations and Symbolic Dynamics)**
Let c_0 be the landing point of the parameter ray $R(\vartheta)$, where $\vartheta = a/(2^n - 1)$. We consider the action of analytic continuation along a small loop around c_0 starting and ending at $R(\vartheta)$. Let W be the component of period n with root c_0.

- *If W is primitive, then analytic continuation along this loop interchanges the two periodic points with itineraries $\mathcal{A}(\nu(\vartheta))$ and $\overline{\mathcal{A}}(\nu(\vartheta))$.*
- *If W is a satellite component, then exactly one of the two itineraries $\mathcal{A}(\nu(\vartheta))$ and $\overline{\mathcal{A}}(\nu(\vartheta))$ has exact period n, and the periodic point with this itinerary is on the orbit that is affected by analytic continuation along the same loop.*

PROOF. Let $U \subset \mathbb{C}$ be a disk neighborhood of c_0 in which c_0 is the only puncture of X_n, and which intersects no parameter ray at n-periodic angles other than $R(\vartheta)$ and the second parameter ray landing at c_0. Let $\gamma \subset X_n$ be the loop under consideration; assume it is small enough so that $\gamma \subset U$, and assume furthermore that γ does not intersect $R(\vartheta)$ except at its endpoints. For external angles $\vartheta_1 < \vartheta < \vartheta_2$ sufficiently close to ϑ, there are two parameters $c_{1,2} \in \gamma$ with $c_i \in R(\vartheta_i)$ for $i = 1, 2$ (the points c_i are near the two ends of γ). For p_{c_1} and p_{c_2}, the periodic dynamic rays $R_{c_1}(\vartheta)$ and $R_{c_2}(\vartheta)$ land at repelling periodic points, say z_1 and z_2.

Consider a curve of parameters within U tending from c_1 to c_0 avoiding $R(\vartheta)$. The analytic continuation of z_1 along this path tends to a periodic point z_0 of p_{c_0} that is either repelling or parabolic. Landing points of periodic rays depend continuously on the parameter whenever the rays land ([**DH1**, Exposé XVII], [**Sch2**, Proposition 5.2]). Therefore, the dynamic ray $R_{c_0}(\vartheta)$ lands at z_0. Since the parabolic parameter c_0 is the landing point of $R(\vartheta)$, the dynamic ray $R_{c_0}(\vartheta)$ lands at the parabolic orbit ([**DH1**]; see also [**Mi2, Sch3**] or Theorem 9.1). Therefore z_1 (and similarly z_2) tend to the parabolic orbit when the parameter tends to c_0. In the primitive case, Lemma 12.9 says that the analytic continuations of z_1 and z_2 are affected by analytic continuation along γ. In the satellite case, exactly one of the points z_1 and z_2 has exact period n, the other one has period dividing n, and the same is true for their itineraries. Again by Lemma 12.9, it is the period n orbit that is affected by analytic continuation. All that remains to be described are the itineraries of z_1 and z_2.

For $i = 1, 2$, the itinerary of z_i within the Julia set of p_{c_i} equals the itinerary of ϑ with respect to ϑ_i, and these are equal to the limiting kneading sequences $\lim_{\varphi\nearrow\vartheta}\nu(\varphi)$ and $\lim_{\varphi\searrow\vartheta}\nu(\varphi)$, hence equal to $\mathcal{A}(\nu(\vartheta))$ and $\overline{\mathcal{A}}(\nu(\vartheta))$. They differ exactly at the n-th position within the period, so they have different numbers of symbols $\mathtt{0}$ within every period. Therefore analytic continuation within X does not move z_1 to z_2, so both are on different orbits throughout X. □

12.11. **Proposition (Symmetric Permutation Group on Orbits)**
Analytic continuation in X_n induces the full symmetric group on the set of periodic orbits of period n.

Proof. The domain X_n is the complement of the finite set of roots of components of period n, and a loop around any of these roots affects at most two periodic orbits of period n; if it does affect two orbits, then both orbits are interchanged. The permutation group among the periodic orbits of period n is thus generated by pair exchanges. As soon as it acts transitively, it is automatically the full symmetric group.

It thus suffices to show that any orbit of period n can be moved to the unique orbit containing the itinerary $\overline{\mathtt{11}\ldots\mathtt{110}}$. In fact, it suffices to show the following: suppose a periodic point has an itinerary containing at least two entries $\mathtt{0}$ during its period; then it can be moved to a periodic point whose itinerary has one entry $\mathtt{0}$ fewer per period. Repeated application will bring any periodic point onto the unique orbit with a single $\mathtt{0}$ per period, i.e., onto the orbit containing the itinerary $\overline{\mathtt{11}\ldots\mathtt{110}}$.

Now consider a periodic point z of period n and assume that its itinerary τ_z contains at least two entries $\mathtt{0}$ per period. Let τ be the maximal shift of τ_z (with respect to the lexicographic order), and let τ' be the same sequence in which the n-th entry (which is necessarily a $\mathtt{0}$ in τ) is replaced by a $\mathtt{1}$, again repeating the first n entries periodically. Then by Lemma 12.5 there is a narrow hyperbolic component W with associated kneading sequence $\nu(W) = \tau$. Let $R(\vartheta)$ be a parameter ray landing at the root of W; then $\mathcal{A}(\nu(\vartheta)) = \nu(W)$, so the $\star$-periodic sequence $\nu(\vartheta)$ coincides with τ and τ' for $n-1$ entries. The component W is primitive: by the remark after Definition 12.1, a narrow component that is not primitive must bifurcate from the period 1 component, and it would then have internal address $1 \to n$ and kneading sequence with a single entry $\mathtt{0}$ in the period. Let c_0 be the root of W. Then by Lemma 12.10, a small loop around c_0 interchanges the periodic points with itineraries τ and τ'. This is exactly the statement we need: we found a loop along which analytic continuation turns z into a periodic point whose itinerary has one entry $\mathtt{0}$ fewer per period. □

Proof of Theorem 12.8. Analytic continuation can achieve only permutations that commute with the dynamics. To see that all of them can actually be achieved, it suffices to specify one loop in X_n that permutes one orbit transitively and leaves all other orbits (of the same period) unchanged: together with transitive permutations of all orbits, this generates all permutations that commute with the dynamics.

Let c_0 be the landing point of the parameter ray $R(1/(2^n-1))$; it is the bifurcation point from the period 1 component to a component of period n. According to Lemma 12.9, a small loop around c_0 induces a transitive permutation on a single orbit of period n and leaves all other orbits unchanged. This is exactly what is needed to prove the claim. □

12.12. **Corollary (Riemann Surface of Periodic Points)**
For every $n \geq 1$, the analytic curve

$$\{(c,z)\colon c \in X_n \textit{ and } z \textit{ is a periodic under } p_c \textit{ of exact period } n\}$$

is connected, i.e., it is a Riemann surface.

These results can be extended to preperiodic points as follows [**Mü**].

12.13. **Corollary (Permutations of Preperiodic Points)**
Consider the set of preperiodic points that take exactly k iterations to become periodic of period n. For all positive integers k and n, analytic continuation along appropriate curves in $\mathbb{C}$ achieves all permutations that commute with the dynamics.

Proof. The parameter ray $R(\vartheta_{k,n})$ with $\vartheta_{k,n} = 1/(2^k 2^{(n-1)})$ lands at a Misiurewicz-Thurston-parameter $c_{k,n}$ for which the dynamic ray $R_{c_{k,n}}(\vartheta)$ lands at the critical value. We have $\nu_{k,n} := \nu(\vartheta_{k,n}) = \mathtt{11\ldots1}\,\overline{\mathtt{11\ldots10}} = \mathtt{1}^k\,\overline{\mathtt{1}^{n-1}\mathtt{0}}$. A small loop around $c_{k,n}$ interchanges the two preperiodic points with itineraries $\mathtt{0}\nu_{k,n}$ and $\mathtt{1}\nu_{k,n}$ (these are the two preimage itineraries of the critical values): every periodic point can be continued analytically in a sufficiently small neighborhood of $c_{k,n}$ and all points on their backwards orbit as long as taking preimages does not involve the critical value. It follows that small loops around $c_{k,n}$ interchange the preperiodic points with itineraries $\tau\mathtt{0}\nu_{k,n}$ and $\tau\mathtt{1}\nu_{k,n}$ for every finite sequence τ over $\{\mathtt{0},\mathtt{1}\}$.

Consider a preperiodic point z with itinerary $\tau\mathtt{1}\,\overline{\mathtt{1}^{n-1}\mathtt{0}}$, where τ is an arbitrary string over $\{\mathtt{0},\mathtt{1}\}$ of length $k-1$ (the entry after τ must be $\mathtt{1}$, or the periodic part in the itinerary would start earlier). If τ has at least one entry $\mathtt{0}$, there is a value k' so that a small loop around $c_{k',n}$ turns the last entry $\mathtt{0}$ within τ into an entry $\mathtt{1}$. Repeating this a finite number of times, z can be continued analytically into the preperiodic point with itinerary $\nu_{k,n}$. Analytic continuation thus acts transitively on the set of preperiodic points with itineraries $\tau\mathtt{1}\,\overline{\mathtt{1}^{n-1}\mathtt{0}}$, for all 2^{k-1} sequences τ of length $k-1$. Since this is achieved by pair exchanges, the full symmetric group on these points is realized.

Two preperiodic points z, z' of p_c are on the same grand orbit if $p_c^{\circ n}(z) = p_c^{\circ n'}(z')$ for some positive integers n, n'. In terms of symbolic dynamics, this is the case if they have the same period, and the periodic parts of their itineraries are cyclic permutations of each other. A permutation of preperiodic points of preperiod k and period n on the same grand orbit commutes with the dynamics if and only if it induces the same cyclic permutations on the periodic parts of the orbit.

For the grand orbit containing the point with itinerary $\overline{\mathtt{1}^{n-1}\mathtt{0}}$, all permutations that commute with the dynamics can thus be achieved by analytic continuation around Misiurewicz-Thurston parameters $c_{k,n}$ and the root of the hyperbolic component $1_{1/n} \to n$ (a loop around the latter induces a transitive cyclic permutation of the periodic orbit containing the periodic point with itinerary $\overline{\mathtt{1}^{n-1}\mathtt{0}}$).

Since analytic continuation induces the full symmetric group on the set of grand orbits, the claim follows. □

Note that any permutation of preperiodic points of preperiod k and period n induces a permutation of preperiodic points of preperiod in $\{k-1, k-2, \dots, 1, 0\}$ and period n. Analytic continuation takes place in X_n from which finitely many Misiurewicz-Thurston points are removed (and there is an ambiguity at centers of hyperbolic components of period n: the critical point should count both as a periodic point and a preperiodic point of preperiod 1).

Analogous results can also be derived for the families of *unicritical polynomials*, parametrized in the form $z \mapsto z^d + c$ for $d \geq 2$. Again, analytic continuation allows us to achieve all permutations of periodic points that commute with the dynamics; for details, see [**LS**, Section 12]. Note that the analogous statement for general degree d polynomials is much weaker: the bigger the space of maps, the easier it is to achieve permutations by analytic continuation.

A related study was done by Blanchard, Devaney, Keen [**BDK**]: analytic continuation in the shift locus of degree d polynomials realizes all automorphisms of the shift over d symbols (in the special case of $d = 2$, this corresponds to a loop around $\mathcal{M}$, and this interchanges all entries $\mathtt{0}$ and $\mathtt{1}$ in itineraries; indeed, this is the only non-trivial automorphism of the 2-shift).

A simple space where not all permutations can be achieved is the space of quadratic polynomials, parametrized as $z \mapsto \lambda z(1-z)$ with $\lambda \in \mathbb{C}$: the two fixed points are $z = 0$ and $z = 1 - 1/\lambda$ and they cannot be permuted by analytic continuation. This is related to the fact that the λ-space is not a true parameter space; every affine conjugacy class of quadratic polynomials is represented twice: the λ-space is the double cover over the true parameter space (written as $z \mapsto z^2 + c$) that distinguishes the two fixed points. Another example is the space $f_c\colon z \mapsto (z^2 + c)^2 + c$ of second iterates of quadratic polynomials. Fixed points of such maps may have period 1 or 2 for $z \mapsto z^2 + c$; this yields obstructions for permutations of fixed points of f_c.

Include in Sec 5 a quotable theorem that kneading sequences are admissible iff they satisfy the condition!

13. More Algorithms

In this section, we collect old and new algorithms to turn the various combinatorial data into each other. In the first part of this section, we give two ways how to determine conjugate external angles (the second angle for a ray pair in the Mandelbrot set, given a periodic angle). In the second part, we describe algorithms related to Hubbard trees (find the associated external angle and vice versa, find its kneading sequence). The third part presents some more combinatorial algorithms. In the fourth part, we discuss spider theory find a complex parameter for a postcritically finite quadratic polynomial with given combinatorics, and conversely how to read off from the complex polymnomial its external angles or kneading sequence.

Conjugate Periodic Angles

It has been known since the work of Douady and Hubbard [**DH1**] (see also [**Mi2, Sch2**]) that for the Mandelbrot set, parameter rays at periodic angles land in pairs (see Theorem 9.1). Given a periodic external angle, its *conjugate angle* will be the angle of the second ray landing at the same point (Proposition 13.2). We will give two algorithms how to find conjugate angles: in Algorithm 13.3, we directly construct the conjugate angle, and in Algorithm 13.4, we construct all pairs of conjugate angles in the order of increasing periods. Conjugate angles will also help to construct the Hubbard tree associated to any periodic angle (Algorithm 13.5).

We say that two pairs of different angles (α, α') and (β, β') are *unlinked* if $\{\beta, \beta'\}$ is in a single connected component of $\mathbb{S}^1 \setminus \{\alpha, \alpha'\}$ (or equivalently, if $\{\alpha, \alpha'\}$ is in a single connected component of $\mathbb{S}^1 \setminus \{\beta, \beta'\}$).

The *length* of a pair (α, β) will be the shortest unsigned distance along $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, written $\ell(\alpha, \beta)$.

13.1. **Lemma (Conjugate External Angle)**
Let $\vartheta \in \mathbb{S}^1$ be an external angle of exact period $n \geq 2$. Then there is a conjugate external angle $\vartheta' \in \mathbb{S}^1$ *with the following properties:*

(1) *The exact period of ϑ' is n as well;*
(2) *for all $k \in \{1, 2, \ldots, n-1\}$, $\ell(2^k\vartheta, 2^k\vartheta') > \ell(\vartheta, \vartheta')$;*
(3) *all pairs of angles $(2^k\vartheta, 2^k\vartheta')$ are unlinked, for $k = 0, 1, \ldots, n-1$;*
(4) *the interval (ϑ, ϑ') (or (ϑ', ϑ), whichever is shorter) does not intersect the forward orbit of ϑ or ϑ';*
(5) *if ϑ_1 and ϑ_1' are the preperiodic inverse images of ϑ and ϑ', then $(\vartheta_1, -\vartheta_1')$ is unlinked to all pairs $(2^k\vartheta, 2^k\vartheta')$;*
(6) *the angles ϑ and ϑ' have the same kneading sequence;*
(7) *the first $n-1$ entries in the following four itineraries coincide: $I_\vartheta(\vartheta)$, $I_\vartheta(\vartheta')$, $I_{\vartheta'}(\vartheta)$, and $I_{\vartheta'}(\vartheta')$.*

Among all n-periodic angles in $\mathbb{S}^1$, ϑ' is already uniquely described by the first three conditions.

PROOF. This result is folklore and is proved along the way to the proof that parameter rays of the Mandelbrot set at periodic angles land in pairs; we give a constructive proof in Algorithm 13.3 below. □

13.2. **Proposition (Second Ray in the Mandelbrot Set)**
Let ϑ' be the conjugate angle of a periodic angle ϑ. Then the two parameter rays $R(\vartheta)$ and $R(\vartheta')$ of the Mandelbrot set land together.

PROOF. It is shown in all three references cited above that when two parameter rays $R(\vartheta)$ and $R(\vartheta')$ at periodic angles land together, then the landing point is a parameter $c \in \mathbb{C}$ so that the polynomial $p_c(z) = z^2 + c$ has a parabolic periodic orbit, and the dynamic rays $R_c(\vartheta)$, $R_c(\vartheta')$ land at a common point on this orbit so that these rays form a *characteristic ray pair*, which means that the first three properties in Lemma 13.1 are satisfied. By the uniqueness statement in that lemma, the claim follows. □

13.3. **Algorithm (Conjugate External Angle)**
Given an external angle ϑ of exact period $n \geq 2$, the conjugate angle ϑ' can be found as follows: let ϑ_1 be the preperiodic preimage of ϑ, and for $k = 2, 3, \dots, n$, let ϑ_k be the unique preimage of ϑ_{k-1} such that $(2^{n-k}\vartheta, \vartheta_k)$ and $(2^{n-1}\vartheta, \vartheta_1)$ are unlinked; then

$$\vartheta' = \vartheta + \frac{\vartheta_n - \vartheta}{1 - 2^{-n}} = \vartheta_n + 2^{-n}\frac{\vartheta_n - \vartheta}{1 - 2^{-n}} \ . \tag{7}$$

See Figure 13.1 for the angle $\vartheta = 6/15$ of period 4.

PROOF. (0) This proof uses ideas from Thurston's theory of quadratic minor laminations [**Th**], but it is self-contained. It was first pointed out to us by Mary Rees.

First observe that ϑ' is indeed periodic of period n: we have

$$2^n\vartheta' = 2^n\vartheta_n + \frac{\vartheta_n - \vartheta}{1 - 2^{-n}} = \vartheta + \frac{\vartheta_n - \vartheta}{1 - 2^{-n}} = \vartheta' \ ;$$

at this point, we do not claim that n is the exact period of ϑ'. In this proof, we will suppose that $\vartheta_n > \vartheta$ in $(0, 1)$, hence $\vartheta' > \vartheta_n$; the other case is analogous.

The two preimages of ϑ are $\{\vartheta_1, 2^{n-1}\vartheta\} = \{\vartheta/2, (\vartheta+1)/2\}$: they form a diameter in $\mathbb{S}^1$ which is the boundary for the itinerary with respect to ϑ (note that $2^{n-1}\vartheta$ is always the periodic preimage of ϑ, while $\vartheta/2$ may be the periodic or preperiodic preimage, depending on ϑ).

By construction, the first $n - 1$ entries in the itinerary of ϑ_n (with respect to ϑ) are the same as the itinerary of ϑ, i.e. the kneading sequence of ϑ (indeed, since $2^n\vartheta_n = 2^n\vartheta = \vartheta$, the entire itineraries of ϑ_n and ϑ coincide, although the first angle is preperiodic and the second periodic; this is because both $\vartheta/2$ and $(\vartheta + 1)/2$ are coded

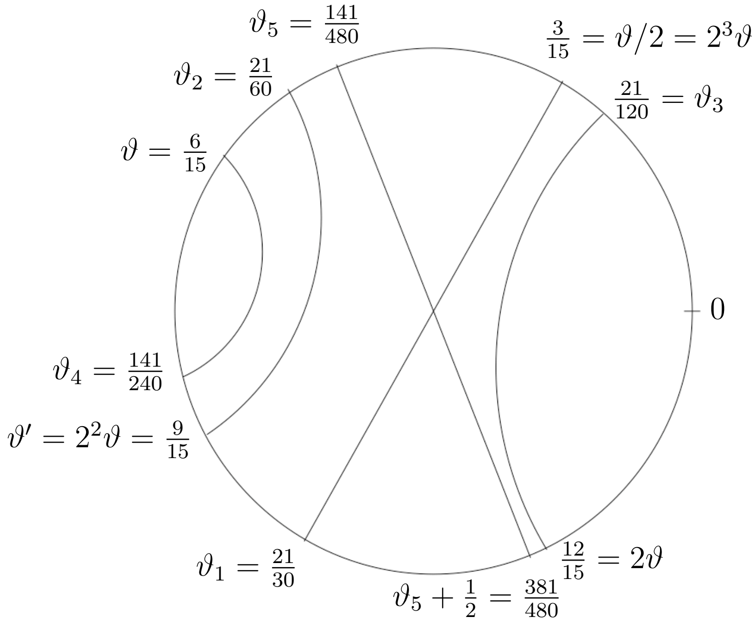


FIGURE 13.1. The conjugate angle of $\vartheta = 6/15$ is $9/15$; both have period $n = 4$.

by the entry $\star$). In order to show that the first $n-1$ entries in the itineraries of ϑ_n and ϑ' coincide, it suffices to show that for $k = 1, 2, \ldots, n$, the interval $(2^k\vartheta_n, 2^k\vartheta')$ does not contain ϑ.

(1) As a first step, note that all pairs $(2^{n-k}\vartheta, \vartheta_k)$ are unlinked: otherwise, if $(2^{n-k}\vartheta, \vartheta_k)$ and $(2^{n-m}\vartheta, \vartheta_m)$ were linked and $k > m$ say, all four angles would be contained in the closure of a connected component of $\mathbb{S}^1 \setminus \{\vartheta/2, (\vartheta+1)/2\}$. If $m > 1$, then one could iterate one step further so that $(2^{n-(k-1)}\vartheta, \vartheta_{k-1})$ and $(2^{n-(m-1)}\vartheta, \vartheta_{m-1})$ were also linked, and they would still be contained in a half circle because of their itineraries. However, when finally $m = 2$, then $(2^{n-(k-m+1)}\vartheta, \vartheta_{k-m+1})$ and $(2^{n-1}\vartheta, \vartheta_1)$ would be linked as well, but this would violate the way ϑ_{k-m+1} was constructed.

Let ϑ_{n+1} and $\vartheta_{n+1} + \frac{1}{2}$ be the two preimages of ϑ_n so that ϑ_{n+1} is closest to $2^{n-1}\vartheta$ (and $\vartheta_{n+1} + \frac{1}{2}$ is closest to ϑ_1). Then the argument just given extends to show that all $(2^{n-k}\vartheta, \vartheta_k)$ are unlinked with $(2^{n-1}\vartheta, \vartheta_{n+1})$ and with $(\vartheta_1, \vartheta_{n+1} + \frac{1}{2})$. Since they are also unlinked with $(2^{n-1}\vartheta, \vartheta_1)$, it follows that they are unlinked with $(2^{n-1}\vartheta, \vartheta_{n+1} + \frac{1}{2})$ and with $(\vartheta_1, \vartheta_{n+1})$.

The unlinking property of the $(2^{n-k}\vartheta, \vartheta_k)$ implies that no $(2^{n-k}\vartheta, \vartheta_k)$ can have greater length than $(\vartheta_1, \vartheta_{n+1})$ (which has the same length as the opposite pair $(2^{n-1}\vartheta, \vartheta_{n+1} + \frac{1}{2})$). This implies that no $(2^{n-k}\vartheta, \vartheta_k)$ (with $k \in \{1, 2, \ldots, n-1\}$) can have shorter length than (ϑ_n, ϑ): note that if (α, β) has length x, then $(2\alpha, 2\beta)$ has length $\min(2x, 1-2x)$;

therefore, if $(2^{n-k}\vartheta, \vartheta_k)$ is the pair with shortest length with $k < n$, then $(2^{n-(k-1)}\vartheta, \vartheta_{k-1})$ must be the longest pair and longer than $(\vartheta_1, \vartheta_{n+1})$, which is impossible.

Since $(2^k\vartheta, 2^k\vartheta') = (2^k\vartheta, \vartheta_{n-k}] \cup [\vartheta_{n-k}, 2^k\vartheta')$ and the length of $[\vartheta_{n-k}, 2^k\vartheta')$ doubles every time k is increased by one, while $(\vartheta, \vartheta_n]$ is as most as long as any $(2^k\vartheta, \vartheta_{n-k}]$, it follows that $\ell(\vartheta, \vartheta') < \ell(2^k\vartheta, 2^k\vartheta')$ for all $k \in \{1, 2, \dots, n-1\}$.

Since no $(\vartheta_{k-1}, 2^{n-(k-1)}\vartheta)$ is shorter than (ϑ_n, ϑ) and both are unlinked, this implies that the interval (ϑ, ϑ_n) does not contain any $2^{n-k}\vartheta$, i.e. no angle on the forward orbit of ϑ.

(2) Now suppose that the first $n-1$ entries in the itineraries of ϑ_n and ϑ' (with respect to ϑ) do not coincide; say that the first difference is at position k. Then the interval $[2^{k-1}\vartheta_n, 2^{k-1}\vartheta']$ contains one of $\vartheta/2$ and $(\vartheta+1)/2$, so $[2^k\vartheta_n, 2^k\vartheta']$ contains ϑ and $[2^n\vartheta_n, 2^n\vartheta'] = [\vartheta, \vartheta']$ contains $2^{n-k}\vartheta$. But $[\vartheta, \vartheta'] = [\vartheta, \vartheta_n] \cup [\vartheta_n \cup \vartheta']$ and $\ell(\vartheta', \vartheta_n) < 1/(2^n - 1)$ by (7). Since adjacent n-periodic angles differ by $1/(2^n-1)$ and ϑ' is n-periodic, it follows that $2^{n-k}\vartheta \in [\vartheta, \vartheta_n]$, which is a contradiction. Therefore, the first $n-1$ entries in the itineraries $I_\vartheta(\vartheta')$ and $I_\vartheta(\vartheta_n)$ coincide (and those of $I_\vartheta(\vartheta_n)$ and $I_\vartheta(\vartheta)$ do by construction).

(3) It also follows that the exact period of ϑ' is n: if the exact period was $k < n$, then $(2^k\vartheta_n, 2^k\vartheta') = (\vartheta_{n-k}, \vartheta')$ would imply that $\vartheta_{n-k} \in (\vartheta, \vartheta_n)$, while $2^k\vartheta \notin (\vartheta, \vartheta_n)$ in contradiction to the unlinking properties.

(4) Now we show that all pairs $(2^k\vartheta, 2^k\vartheta')$ are unlinked. We already know from Step (1) that no $2^k\vartheta \in (\vartheta, \vartheta_n)$, and no $2^k\vartheta \in (\vartheta_n, \vartheta')$ because ϑ' is n-periodic and the distance between any two n-periodic angles is a multiple of $1/(2^n-1)$. The same argument shows that no $2^k\vartheta' \in (\vartheta_n, \vartheta')$.

The k-th iterate of $(\vartheta_n, \vartheta')$ is $(\vartheta_{n-k}, 2^k\vartheta')$ with length less than 1. Therefore, if $2^k\vartheta' \in (\vartheta, \vartheta')$, then either $\vartheta_{n-k} \in (\vartheta, \vartheta')$, or $\vartheta \in (\vartheta_{n-k}, 2^k\vartheta')$. But in the first case, the unlinking property is violated because $2^k\vartheta \notin (\vartheta, \vartheta')$. In the second case, $n-k$ further iterations would yield $2^{n-k}\vartheta \in (\vartheta, \vartheta')$, a contradiction. Therefore, the interval (ϑ, ϑ') does not intersect the forward orbits of ϑ or of ϑ'.

The periodic preimage of (ϑ, ϑ') is $(2^{n-1}\vartheta, 2^{n-1}\vartheta')$ and the preperiodic preimage is $(2^{n-1}\vartheta + \frac{1}{2}, 2^{n-1}\vartheta' + \frac{1}{2})$. The "short" intervals $(2^{n-1}\vartheta, 2^{n-1}\vartheta' + \frac{1}{2})$ and $(2^{n-1}\vartheta + \frac{1}{2}, 2^{n-1}\vartheta')$ map under doubling onto (ϑ, ϑ'), so they cannot intersect the forward orbits of ϑ and ϑ'. Since no $(2^k\vartheta, 2^k\vartheta')$ intersects the diagonal $(2^{n-1}\vartheta, 2^{n-1}\vartheta + \frac{1}{2})$, it follows that all $(2^k\vartheta, 2^k\vartheta')$ are unlinked with $(2^{n-1}\vartheta, 2^{n-1}\vartheta')$; as above, it follows indeed that all pairs $(2^k\vartheta, 2^k\vartheta')$ are unlinked (any linked pair could be mapped forward until one of the affected pairs was $(2^{n-1}\vartheta, 2^{n-1}\vartheta')$).

We have now justified the first five properties in Lemma 13.1.

(5) Since the first $n-1$ entries in the kneading sequences of ϑ and ϑ' are the same, both have a $\star$ at position n, and both have period at most n, it follows that the entire kneading sequences coincide. Since neither ϑ nor ϑ' have an element of their forward orbits in (ϑ, ϑ'), it follows that itineraries of both ϑ and ϑ' do not change when the

partition $(2^{n-1}\vartheta, 2^{n-1}\vartheta+\frac{1}{2})$ is replaced by $(2^{n-1}\vartheta', 2^{n-1}\vartheta'+\frac{1}{2})$ (except that a $\star$ may turn into `0` or `1`, or conversely, which happens at the earliest in the n-th entry); in other words, the first $n-1$ entries in $I_\vartheta(\vartheta)$ and in $I_{\vartheta'}(\vartheta)$ coincide, and so do the first $n-1$ entries in $I_\vartheta(\vartheta')$ and in $I_{\vartheta'}(\vartheta')$. But we had seen in (2) that the first $n-1$ entries in $I_\vartheta(\vartheta')$ and in $I_\vartheta(\vartheta)$ coincide, so all these sequences have identical initial $n-1$ entries and Property 7 is shown.

This holds in particular for the kneading sequences of ϑ and of ϑ', which are both n-periodic with a $\star$ at position n, so both kneading sequences coincide as soon as their initial $n-1$ entries coincide. This settles the remaining Property 6, and ϑ' is indeed the conjugate angle to ϑ.

(6) It remains to show uniqueness of ϑ' using the first three properties in Lemma 13.1. Along the way, we will show directly that most of the properties in the lemma follow from the first three. Suppose again that $\vartheta < \vartheta'$.

Note that the fourth property in Lemma 13.1 follows directly from the second and third (if (ϑ, ϑ') contains $2^k\vartheta$, then because of the unlinking condition it must also contain $2^k\vartheta'$ and vice versa, but this contradicts the length condition).

To show that Property 7 follows from the first three, suppose that for $k \le n-1$, the k-th entry in $I_{\vartheta'}(\vartheta)$ differs from $I_\vartheta(\vartheta) = \nu(\vartheta)$. That implies either $2^{k-1}\vartheta \in (\vartheta/2, \vartheta'/2)$ or $2^{k-1}\vartheta \in ((\vartheta+1)/2, (\vartheta'+1)/2)$, hence in both cases $2^k\vartheta \in (\vartheta, \vartheta')$, a contradiction. Similarly, the first $n-1$ entries in $I_\vartheta(\vartheta')$ and in $I_{\vartheta'}(\vartheta')$ coincide.

Now suppose that the k-th entries in $I_\vartheta(\vartheta)$ and in $I_\vartheta(\vartheta')$ differ, for $k \le n-1$. This implies that $\vartheta/2 \in (2^{k-1}\vartheta, 2^{k-1}\vartheta')$ or $(\vartheta+1)/2 \in (2^{k-1}\vartheta, 2^{k-1}\vartheta')$. But neither $2^{k-1}\vartheta$ nor $2^{k-1}\vartheta'$ can be contained in $(\vartheta/2, \vartheta'/2)$ or in $((\vartheta+1)/2, (\vartheta'+1)/2)$ (otherwise, $2^k\vartheta$ or $2^k\vartheta'$ would be in (ϑ, ϑ')), hence it follows that $(2^{k-1}\vartheta, 2^{k-1}\vartheta')$ is linked with both $(\vartheta'/2, (\vartheta+1)/2)$ and $((\vartheta'+1)/2, \vartheta/2)$; but one of them equals $(2^n\vartheta, 2^n\vartheta')$, and this is a contradiction. This proves Property 7, of which Property 6 is a special case.

Now for uniqueness of ϑ'. There is a unique $\vartheta_n \in (\vartheta, \vartheta')$ such that multiplication by 2^n maps $(\vartheta_n, \vartheta')$ homeomorphically onto (ϑ, ϑ'). By Property 4, (ϑ, ϑ') does not intersect the forward orbits of ϑ or ϑ'. Now $\vartheta \notin (2^k\vartheta_n, 2^k\vartheta')$ for $k = 0, 1, \dots, n$ (or $n-k$ iterations further, we would have $2^{n-k}\vartheta \in (\vartheta, \vartheta')$), so the first $n-1$ entries in the itineraries $I_\vartheta(\vartheta_n)$ and $I_\vartheta(\vartheta')$ coincide, and by Property 7 they coincide also with the corresponding entries in the kneading sequence of ϑ. But this uniquely determines ϑ_n among $\{\alpha \in \mathbb{S}^1 \colon 2^n\alpha = \vartheta\}$, so ϑ_n is exactly the angle constructed in the claim of the algorithm and ϑ' is the least n-periodic angle greater than ϑ_n. Therefore, the construction of ϑ' at the beginning of the proof is the only one possible. □

13.4. **Algorithm (The Lavaurs' Algorithm for Conjugate External Angles)**
The ray pairs at periodic angles can be determined recursively as follows: suppose all ray pairs for periods less than n are known. Let $\vartheta_1, \vartheta_2, \dots$ *be the finitely many external angles of exact period n, ordered within* $(0,1)$*. Inspect these angles in increasing order; for every* ϑ_i *which has not yet been determined as the conjugate angle for a smaller* ϑ_j*,*

the conjugate angle of ϑ_i is the least angle $\vartheta_k > \vartheta_i$ which is not separated from ϑ_i by a ray pair of lower period.

PROOF. From Theorem 9.1, we know that all periodic parameter rays land in pairs at equal periods, and ray pairs at different periods do not cross. The only ambiguity can thus occur if some angle ϑ_i can be connected to more than one angle of equal period without crossing ray pairs of lower periods. If ϑ_i was not conjugate to the least angle available, then there would be two ray pairs $(\vartheta_i, \vartheta_j)$ and $(\vartheta_k, \vartheta_l)$ of the same period with $\vartheta_i < \vartheta_k < \vartheta_l < \vartheta_j$ which were not separated by a ray pair of lower period, and this contradicts Lavaurs' Lemma 9.28. □

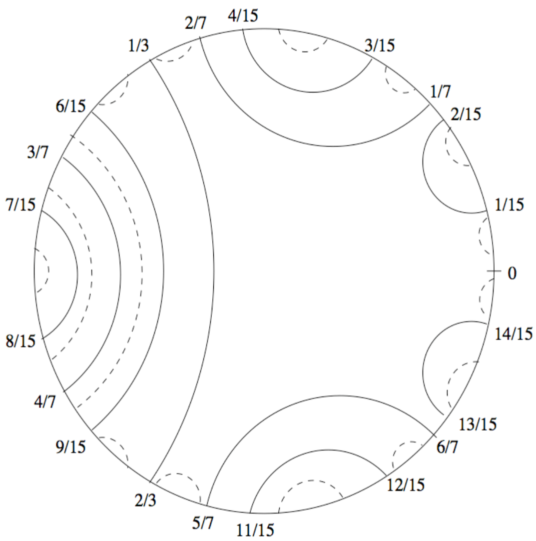


FIGURE 13.2. Ray pairs up to period 4 by Lavaurs' algorithm.

REMARK. Here we show how this algorithm works for low periods, see Figure 13.2. The only two angles of period 2 are $1/3$ and $2/3$, and they form a ray pair. For period 3, there are the six angles $1/7, \ldots, 6/7$ forming the pairs $(1/7, 2/7)$, $(3/7, 4/7)$, $(5/7, 6/7)$. Period 4 is a little more interesting: we have the 12 angles $a/15$ for $a = 1, 2, \ldots, 14$ (except $5/15 = 1/3$ and $10/15 = 2/3$), and we connect $(1/15, 2/15)$, $(3/15, 4/15)$, $(6/15, 9/15)$ (note that $6/15$ and $7/15$ are separated by $(3/7, 4/7)$!), $(7/15, 8/15)$, $(11/15, 12/15)$ and $(13/15, 14/15)$. So far, for every angle there was only one angle available that it could be connected to without crossing ray pairs of lower periods, and there was no choice. Things are different for period 5: the rays at angles $1/31$, $2/31$, $9/31$, $10/31$, as well as the symmetric ones $30/31$, $29/31$, $22/31$, $21/31$, are all unseparated by ray pairs of lower periods (they have identical internal addresses $1 \to 5$, and the corresponding parameter rays all land on the boundary of the main cardioid of the Mandelbrot set). These angles form the ray pairs $(1/31, 2/31)$ and $(9/31, 10/31)$ etc. (and not $(1/31, 10/31)$ and $(2/31, 9/31)$ etc., or $(1/31, 30/31)$ and $(2/31, 29/31)$ etc.).

The Hubbard Tree

13.5. **Algorithm (From the External Angle to the Hubbard Tree)**
Given a periodic external angle ϑ, the associated Hubbard tree can be constructed as follows: construct the conjugate angle ϑ', draw the n leafs $(2^k\vartheta, 2^k\vartheta')$ (for $k = 0, 1, \ldots, n-1$) in $\overline{\mathbb{D}}$ and pick a point c_{k+1} on each leaf $(2^k\vartheta, 2^k\vartheta')$. The action of f on these points is by cyclic permutation. Note that $c_0 = c_n$ will be the critical point; let us abbreviate the leaf $(2^k\vartheta, 2^k\vartheta')$ that contains it by L_0.

Also draw the leaf $(2^{n-1}\vartheta + 1/2, 2^{n-1}\vartheta' + 1/2)$ and call it L_0'. The images of L_0 and L_0' are the same, namely (ϑ, ϑ').

Next draw the preimages of L_0 and L_0', except when they are separated by some $(2^k\vartheta, 2^k\vartheta')$ from all other angles $2^i\vartheta$. Draw the preimages of the preimage leafs of the previous step, except when they are separated by $(2^k\vartheta, 2^k\vartheta')$ from all other angles $2^i\vartheta$. Continue this way until the preimages of order n are found.

Pick a point on each of the new (i.e., different from $(2^k\vartheta, 2^k\vartheta')$) preimage leaf. The arguments of the proof of Algorithm 13.3 show that all these leafs are unlinked.

Now for each component U of $\overline{\mathbb{D}} \setminus \{$drawn leafs$\}$, connect the points in its boundary leafs as follows:

- *If U has only one boundary leaf (which by construction is $(2^k\vartheta, 2^k\vartheta')$ for some $k < n$), then ignore U. The point c_k will be an endpoint of the Hubbard tree.*
- *If U has two boundary leafs, connect the chosen points on these leafs by an arc within U.*
- *If U has $m \geq 3$ boundary leafs, connect the chosen points on these leafs by an m-star within U.*

In the resulting tree, take the minimal subtree connecting all the points c_k, $k = 1, \ldots, n$. Call this tree T; it is the Hubbard tree.

Proof. Since the action of f is known on the critical orbit, the action on the other marked points (the branch points) follows uniquely.

The squaring map acts homeomorphically on non-central components of $\overline{\mathbb{D}} \setminus (L_0 \cup L_0')$, there is no critical point in these components. Since there is no leaf between L_0 and L_0', $0 \in L_0$ is indeed the only marked point that can serve as critical point in T.

It remains to show that the configuration of the branch points of T is correct. Suppose by contradiction that instead of an m-star in component U, the boundary leafs of U should have been connected in a more complicated way. Then U would contain at least two branch points, say x and y. By expansivity of Hubbard trees, these branch points will be separated by 0 under iteration of f. Since there are no more than n branch points, this separation occurs within n iterates. But then x and y should have been separated by a preimage leaf (ray pair) of L_0 or L_0' of order $\leq n$. This contradiction establishes the right configuration. $\square$

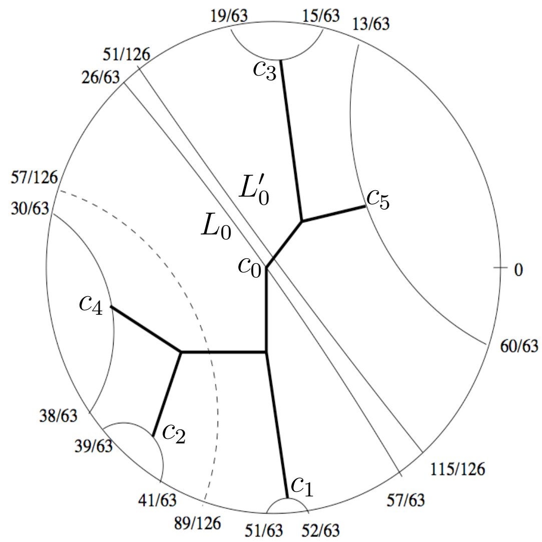


FIGURE 13.3. Construction of the Hubbard tree for external angle 52/63.

Figure 13.3 gives an example for external angle 52/63 with conjugate angle 51/63. The orbits of these angles are $52/63 \to 41/63 \to 19/63 \to 38/63 \to 13/63 \to 26/63 \to 52/63$ and $51/63 \to 39/63 \to 15/63 \to 30/63 \to 60/63 \to 57/63 \to 51/63$. One component of $\overline{\mathbb{D}} \setminus \cup_{k=0}^{3}(2^k\vartheta, 2^k\vartheta')$ contains the four leafs in its boundary. Taking the preimages of L_0 (the dashed line) up to order 1 already reveals that this component should contain a subtree homeomorphic to the letter H to connect the points c_0, c_1, c_2, c_4 on these boundary leafs.

13.6. **Algorithm (From the Hubbard Tree to the External Angle)**
Let T be a Hubbard tree without evilorbits and finite critical orbit $\{0, c_1, \dots, c_n\}$ (where $c_n = 0$ or not). Suppose T comes with an embedding in the plane $\mathbb{C}$. The remark following Proposition 3.10 states that we can extend T as to contain the β fixed point (with itinerary $\overline{\mathsf{0}}$) and its preimage $-\beta$ (with itinerary $\mathsf{1}\overline{\mathsf{0}}$). In practice, this can be done as follows. Let T_0 and T_1 be the components of $T \setminus \{0\}$ such that $c_1 \in T_1$. Find the subgraph $Y \subset T_0$ of points whose itineraries starts with the most $\mathsf{0}$'s. There are three possibilities:

- *$T_0 = \emptyset$. In this case, $0 = c_n$. Extend T with an arc $[0, \beta]$ and let f map it homeomorphically onto $[c_1, \beta]$. Attach another arc $[c_{n-1}, -\beta]$ to c_{n-1} and map it homeomorphically onto $[0, \beta]$.*

- *There is a subgraph $Y \subset T_0$ of points with itinerary $\overline{\mathtt{0}}$. Then f maps Y homeomorphically onto itself, and for each component Y' of Y, f fixes the local arm of Y' towards* 0*. This implies that the other arms of Y' are mapped homeomorphically among themselves, and so they all have itinerary $\overline{\mathtt{0}}$. In other words, Y' has only one arm, and contains at least one endpoint of T. By expansivity, Y has only one component and contains precisely one endpoint. This endpoint is fixed, and lies of the critical orbit, hence it equals c_n. Call it β and let $-\beta := c_{n-1} \in T_1$.*
- *There is a maximal $m \geq 2$ and a subgraph $Y \subset T_0$ of point whose itinerary starts with exactly $m-1$ symbols $\mathtt{0}$ followed by a $\mathtt{1}$. Again Y is connected, because any arc connecting different components of Y must belong to Y itself. Let $y \in \overline{Y}$ be such that $f^{\circ m-1}(y)$ (which lies in $T_1 \cup \{0\}$) is closest to* 0*. Attach an arc $[y, \beta]$ to y and let f map it homeomorphically onto $[f(y), \beta]$. The embedding of $[y, \beta]$ in $\mathbb{C}$ can be determined (up to homotopy) by observing that $f^{\circ m-1}$ should map $Y \cup [y, \beta]$ onto $f^{\circ m-1}(Y) \cup [f^{\circ m-1}(y), \beta]$ in a homeomorphic, orientation preserving fashion. Finally, find $y' \in T_1$ such that $f(y') = y$, attach an arc $[y', -\beta]$ embedded in such a way that f maps $T \cup [y', -\beta]$ in an orientation preserving fashion onto $T \cup [y, \beta]$.*

Let R_β and $R_{-\beta}$ be disjoint rays coming from ∞ and landing at β resp. $-\beta$. Together with the arc $[-\beta, \beta]$, they separate the plane, say into a region $\mathbb{C}_0$ and a region $\mathbb{C}_1$. Now define $\vartheta_k = j$ if $c_k \in \mathbb{C}_j$. If $c_k \notin [-\beta, \beta]$ for all $k \geq 1$, then (ϑ_k) gives the binary expansion of the external angle ϑ: $\vartheta = \sum_{k \geq 1} \vartheta_k 2^{-k}$.

Note that ϑ depends on the embedding, and the choice of the regions $\mathbb{C}_0$ and $\mathbb{C}_1$. However, all different results ϑ are external angles generating ν.

If $c_k \in [-\beta, \beta]$ for some maximal $k \leq n$, then we argue as follows. First extend f continuously to a neighborhood V of $T \cup [-\beta, \beta]$. Let U be a small neighborhood of c_k and choose a component U' of $U \setminus (T \cup R_\beta \cup [-\beta, \beta] \cup R_{-\beta})$. (The choice is arbitrary, but each choice gives rise to a different external angle.) Let $\vartheta_k = 0$ or 1 according to whether $U' \subset \mathbb{C}_0$ or $\mathbb{C}_1$,

For $i < k$, determine the digit ϑ_i by pulling back U' along the backward orbit $c_k, c_{k-1}, \dots, c_i$. Because $T \cup R_\beta \cup [-\beta, \beta] \cup R_{-\beta}$ is forward invariant, this pullback of U' is indeed contained in $\mathbb{C}_0$ or $\mathbb{C}_1$.

To determine the digits ϑ_i for $i > k$, we distinguish two case:

- *If 0 is periodic (and hence $k = n$), then $f|U$ is not homeomorphic, but we can extend (ϑ_i) periodically in this case.*
- *If $0 \neq c_i$ for all $i > k$, then $f^{\circ i-k}|U'$ is well-defined for U' sufficiently small, and disjoint from $T \cup R_\beta \cup [-\beta, \beta] \cup R_{-\beta}$. In fact, since* $\mathrm{orb}(0)$ *is preperiodic, (ϑ_i) will be preperiodic too, although its period may be a proper multiple of the period of* $\mathrm{orb}(0)$.

Now $\vartheta = \sum_{k \geq 1} \vartheta_k 2^{-k}$ (for each choice of U') is an external angle generating ν.

Remark. This algorithm shows that if c_1 is periodic, then it has exactly two external angles. These are of course conjugate.

13.7. **Corollary (Admissible Kneading Sequences and External Angles)**
A $\star$-periodic kneading sequence ν is admissible if and only if there is an external angle $\vartheta \in \mathbb{S}^1$ such that ν is the kneading sequence generated by ϑ.

Proof. Every $\star$-periodic kneading sequence ν has an associated Hubbard tree by Theorem 3.10, and by definition of admissibility of the sequence, the tree can be embedded into the plane so that Algorithm 13.6 finds an external angle which generates ν.

Conversely, given a periodic external angle ϑ, Algorithm 13.5 constructs a Hubbard tree embedded in the plane with the kneading sequence of ϑ, which is thus admissible. □

Remark. It is easy to conclude that the corollary also holds for non-periodic kneading sequences; see Corollary 14.2. A periodic kneading sequence ν without $\star$ (with exact period n) is realized by an external angle (necessarily irrational) if and only if there is a $\star$-periodic kneading sequence ν_* so that $\nu = \mathcal{A}(\nu_*)$; see Theorem 14.10.

More Combinatorial Algorithms

13.8. **Algorithm (Cutting Times Algorithm for the Internal Address)**
In a Hubbard tree (T, f), the internal address can be read off as follows: let $\gamma_1 := [c_1, x_0] := [c_1, c_0]$ and given $\gamma_n = [c_n, x_n]$ for $n \geq 1$, let

$$\gamma_{n+1} := \begin{cases} [c_{n+1}, c_0] & \text{if } c_0 \in [c_{n+1}, f(x_n)]; \\ [c_{n+1}, f(x_n)] & \text{otherwise .} \end{cases}$$

Then the internal address associated to (T, f) is the sequence of indices n for which γ_n ends at c_0.

Remark. The name of the algorithm comes from the following observation: start with the arc γ_1 and map it forward under the dynamics; each time it covers c_0, cut the arc at c_0 and retain only the part near the orbit point of c_1. This assures that every time the arc is mapped forward homeomorphically, and the internal address records the times at which the arc was cut.

Proof. By construction, for every n there is a point $\zeta_n \in [c_1, c_0]$ such that the map $f^{\circ(n-1)} \colon [c_1, \zeta_n] \to \gamma_n$ is a homeomorphism. Now n is a cutting time if and only if ζ_n is a precritical point, and clearly $(c_1, \zeta_n]$ cannot contain a precritical point which maps to 0 before ζ_n does: hence the sequence of cutting times is exactly the sequence of closest precritical points on $(c_1, c_0]$, and these give the internal address by Proposition 6.8. □

13.9. **Algorithm (From Ordered Kneading Sequence to External Angle)** *Let $\vartheta_1 \in \mathbb{S}^1$ be a periodic angle of exact period $n \geq 2$, let $\vartheta_k := 2^{k-1}\vartheta_1$ for $k = 2, 3, \ldots$ (with $\vartheta_{n+1} = \vartheta_1$) and $\vartheta_0 := \vartheta_n + 1/2$ (the preperiodic preimage of ϑ_1). Knowing only the cyclic order of $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$, the external angle of ϑ_1 can be found as follows:*

(1) *Among the two oriented intervals $[\vartheta_0, \vartheta_n]$ and $[\vartheta_n, \vartheta_0]$, let I_1 be the one containing ϑ_1 and let I_0 be the other interval.*
(2) *Find two consecutive points $\vartheta_i, \vartheta_j \in I_0$ such that $(\vartheta_{i+1}, \vartheta_{j+1}) \supset (\vartheta_i, \vartheta_j)$. Mark an arbitrary point $z \in (\vartheta_i, \vartheta_j)$ as a fixed point for angle doubling.*
(3) *Find two consecutive points $\vartheta_{i'}, \vartheta_{j'} \in I_1$ such that $z \in (\vartheta_{i'+1}, \vartheta_{j'+1})$, and mark an arbitrary point $z' \in (\vartheta_{i'}, \vartheta_{j'})$.*
(4) *Define two intervals $J_0 := (z, z')$ and $J_1 := (z', z)$ (with respect to the order inherited from $\mathbb{S}^1$). Define numbers $x_1, x_2, \ldots, x_n \in \{0, 1\}$ such that $x_i = 0$ if $\vartheta_i \in J_0$ and $x_i = 1$ if $\vartheta_i \in J_1$.*
(5) *The binary representation of $\vartheta_1 \in (0, 1)$ is $0.\overline{x_1 x_2 \ldots x_n}$.*

Proof. We know that $\{\vartheta_0, \vartheta_n\}$ is a diameter of $\mathbb{S}^1$ formed by the two preimages of ϑ_1. Angle doubling on the circle has exactly one fixed point (at angle $0 = 1$), which is separated from ϑ_1 by $\{\vartheta_0, \vartheta_n\}$. Given the choice of I_1, it follows that I_0 contains the fixed point. Since ϑ_1 is periodic by assumption with period $n \geq 2$, all $\vartheta_i \neq 0$, so there is exactly one pair of consecutive points $\vartheta_i, \vartheta_j \in I_0$ which contains the fixed point 0. Clearly, for consecutive points $\vartheta_i, \vartheta_j \in \mathbb{S}^1$, the interval $(\vartheta_i, \vartheta_j)$ (of length less than $1/2$) contains a fixed point if and only if $(\vartheta_{i+1}, \vartheta_{j+1}) \supset (\vartheta_i, \vartheta_j)$. Therefore, ϑ_i and ϑ_j are uniquely determined.

Angle doubling sends the interior of I_1 homeomorphically onto $\mathbb{S}^1 \setminus \{\vartheta_1\}$, so there is a unique $z' \in I_1$ as claimed, which is the unique pre-fixed point other than the fixed point itself, i.e. $z' = 1/2$.

Since the orbit of ϑ_1 never visits 0 or $1/2$, we have $\vartheta_i \in J_0$ if and only if $\vartheta_i \in (0, 1/2)$ and $\vartheta_i \in J_1$ if and only if $\vartheta_i \in (1/2, 1)$, and the itinerary of ϑ_1 with respect to the partition $(0, 1/2)$ and $(1/2, 1)$ is exactly the binary expansion of ϑ_1. □

Spider Theory

Apart from its combinatorial data, a quadratic polynomial can of course also be specified by its complex parameter c in the parametrization $z \mapsto z^2 + c$. We show now how to turn external angles into the parameter c and vice versa. In the case that the critical orbit is periodic, let the *characteristic Fatou component* be the Fatou component containing the critical value. It has a unique boundary point which is fixed under the first return map of the component; it is called the *root* of this component. At least two dynamic rays land at this root (we exclude the trivial case of period 1 in the rest of this section). If there are exactly two rays, they are called the *characteristic external ray* (or supporting rays) of the polynomial; their external angles angles are the *characteristic angles.* If there are more than two rays, then the two rays which surround the critical value closest possible are the *characteristic rays.* An *extended characteristic ray* is a characteristic ray into the root of the characteristic Fatou component, extended to the critical value by an arbitrary simple closed curve within the characteristic Fatou component.

In the preperiodic case, there is a positive finite number of rays landing at the critical value; we call each of them a characteristic ray.

A *spider* for a postcritically finite quadratic polynomial with critical orbit $c_0 = 0, c_1, c_2, \ldots, c_{n-1}$ (and $c_n = c_l$ for some $l \in \{0, 1, \ldots, n-1\}$ is a collection of injective curves $\gamma_i \colon [0,1] \to \overline{\mathbb{C}}$ (for $i = 1, 2, \ldots, n$) with $\gamma_i(0) = \infty$ and $\gamma_i(1) = c_i$ such that these curves are disjoint, except that they all begin at ∞. These curves are known as *spider legs.* We might as well suppose that they all approach ∞ tangentially, which makes it easier to see that the legs have a well-defined cyclic order at ∞. The notion of spiders is taken from [**HS**], where the *spider algorithm* was introduced (Algorithm 13.11).

We start with an algorithm to find the characteristic angles of a critically periodic parameter c. In a sense, this is the spider algorithm run backwards. It should be noted that it can be implemented effectively on a computer, much as the spider algorithm itself; this is described in [**HS**].

13.10. **Algorithm (From the Parameter to External Angles)**
Let p be a quadratic polynomial in which the critical orbit is periodic of period $n > 1$ and let $c_0, c_1, c_2, \ldots, c_{n-1}, c_n = c_0$ be the critical orbit, starting with the critical point c_0.

(1) *For $i = 1, 2, \ldots, n$, connect all the postcritical points c_i to ∞ by curves $\gamma_i \colon [0,1] \to \overline{\mathbb{C}}$ which are injective and disjoint, except that they all meet at ∞. These form the initial spider Γ^0.*
(2) *From a spider Γ^{k-1}, construct a new spider Γ^k as follows: for $i \in \{1, 2, \ldots, n-1\}$, take the leg $\gamma_{i+1} \in \Gamma^{k-1}$ and let $\tilde{\gamma}_i$ be the image of γ_{i+1} under the branch of p^{-1} which sends c_{i+1} to c_i. For $i = n$, let $\tilde{\gamma}_0$ and $\tilde{\gamma}_n$ be the two branches of $p^{-1}(\gamma_1)$, always chosen so that the cyclic order near ∞ of γ_0, γ_1 and γ_n is the same in every iteration of the algorithm (but both possibilities are permitted). Then Γ^k is the spider with legs $\tilde{\gamma}_i$ for $i = 1, 2, \ldots, n$.*
(3) *Iterate this sequence of spiders until Γ^k and Γ^{k+1} have legs at all c_i which are homotopic to each other relative to the postcritical set (which will be shown to happen after finitely many steps). Let γ_i be the leg at c_i in this homotopy class, for all i. Then the γ_i are homotopic to the orbit of an extended characteristic dynamic ray of the polynomial.*
(4) *Given the cyclic order near ∞ of $\gamma_0, \gamma_1, \ldots, \gamma_n$, use Algorithm 13.9 to find the external angle associated to the leg γ_1.*

PROOF. Suppose that two spiders Γ^{k-1} and $(\Gamma')^{k-1}$ are homotopic (in the sense that for every $i \in \{1, 2, \ldots, n\}$, the legs γ_i and γ_i' are homotopic relative to the postcritical set). Then Γ^k and $(\Gamma')^k$ are also homotopic. Hence the map from Γ^{k-1} to Γ^k descends to the quotient space obtained from the space of all spiders modulo the equivalence relation given by homotopy.

The first claim which needs justification is the fact that after finitely many steps, the spiders Γ^{k-1} and Γ^k have homotopic legs at the critical value. To see this, we use the existence of a Hubbard tree T within the filled-in Julia set of p: this is a Hubbard tree in the sense of Definition 3.2 with respect to the dynamics of the polynomial p; its existence was shown by Douady and Hubbard [**DH1**], using local connectivity and hence pathwise connectivity of the filled-in Julia set. Do we want to include the proof in this paper? We may replace Γ^0 by a homotopic spider for which every leg γ_i intersects the Hubbard tree only finitely often. For $k \geq 0$, let N_k be the combined number of times the legs of Γ^k intersect T (where every Γ^k is replaced by a homotopic spider with minimal intersections).

Note that for every $N \geq 1$ the number of homotopy classes of spiders with $N_k \leq N$ is finite. Since the critical value c_1 is an endpoint of T, all legs at c_1 which do not intersect T (except at c_1) are homotopic to each other. The orbit of any extended characteristic dynamic ray supplies an invariant spider with no intersections; this is the spider we want to find.

We will show that $N_k = 0$ for k large enough. The construction assures that $N_{k+1} \leq N_k$: under the pull-back, new intersections cannot be created; they can be destroyed because T is not fully backwards invariant: there are subsets of T which have only one inverse image. Moreover, even if Γ^k is so that the intersection number is minimal within its homotopy class, then Γ^{k+1} can possibly be homotopic to a spiders with fewer intersections.

To see that $N_k = 0$ for sufficiently large k, let γ_1^k be the leg at c_1 for the spider Γ^k and suppose that γ_1^k is not homotopic to a leg which does not intersect T except at c_1. Let $z_1 \in T \cap \gamma_1^k$ be a point of intersection so that there is a single postcritical point $c_k \in [c_1, z_1]$, where $[c_1, z_1] \subset T$ denotes the unique subarc in T connecting c_1 and z_1 (if there is no such point, then z_1 is an inessential point of intersection). Pulling back along the orbit of c_i, we obtain a sequence of arcs $[c_0, z_0] \subset p^{-1}(T)$, $[c_{n-1}, z_{n-1}] \subset p^{-2}(T)$,.... Exactly one of c_{k-1} and $-c_{k-1}$ is in $[c_0, z_0]$; if it is $-c_{k-1}$ and not c_{k-1}, then the point z_0 is an inessential point of intersection of $T \cap \gamma_n^{k+1}$ and $N_{k+1} < N_k$. Otherwise, we can continue this argument: after $k-1$ iterations, the pull-back of c_k is c_1 which is an endpoint of the Hubbard tree, so it can no longer be an interior point of some arc in T, and one point of intersection has disappeared.

In any case, every intersection point of Γ^0 with T eventually either disappears or can be homotoped away, so that there is a finite k_0 such that for $k \geq k_0$, the spider Γ^k is homotopic to one without intersections. But then the legs of these Γ^k at c_1 must all be homotopic to each other: since T is connected and c_1 is an endpoint of T, there is only one homotopy class of curves γ_1 from c_1 to ∞ which avoids T except at the endpoint. There are two homotopy classes of legs at c_0: one is γ_0 and the other one is γ_n; they are distinguished by the cyclic order at ∞ of γ_0, γ_1 and γ_n. Given the homotopy class of γ_n, the homotopy classes of γ_{n-1}, γ_{n-2}, $\dots$, γ_2 are determined uniquely by pull-backs. These are the homotopy classes of the extended dynamic rays into the critical orbit.

We have now justified the claims in Step 3: for a finite k, the spiders Γ^k and Γ^{k+1} are homotopic (it is clear that from then on, all further spiders will be homotopic too). Moreover, all γ_i are homotopic to the orbit of an extended characteristic dynamic ray.

Therefore, we know the cyclic order at ∞ of a characteristic dynamic ray, which is the same as the cyclic order on $\mathbb{S}^1$ of the associated external angle. Now Algorithm 13.9 allows us to determine the characteristic external angle. (Note that the two choices for γ_0 and γ_n give two different cyclic orders, which exactly specify the two choices for the characteristic external angle.) □

It is not difficult to give an analogous algorithm in the preperiodic case.

The following algorithm is known as the Spider Algorithm. It was introduced in [**HS**]. It turns combinatorial data (an external angle) into analytical data (the complex parameter of the polynomial), so it can yield only an approximate solution with arbitrary accuracy (of course, since the parameter is the root of a known polynomial with integer coefficients, it is in principle determined by the knowledge of the polynomial and an approximate value with known accuracy).

13.11. **Algorithm (From the External Angle to the Parameter)**
Let $\vartheta_1 \in \mathbb{S}^1$ be an external angle which is periodic of period n, and let $\vartheta_j := 2^{j-1}\vartheta_1$ for $j \geq 1$.

(1) *The initial spider Γ^0 (also known as the standard spider) has endpoints $w_j^0 = \exp(2\pi i\vartheta_j)$, and its legs γ_j^0 are straight radial lines from w_j^0 to ∞.*
(2) *Take the spider Γ^k with endpoints w_j^k and legs γ_j^k. As an inductive hypothesis, we may assume that for each j, its leg γ_j^k tends to ∞ tangentially to the leg γ_j^0. Fix the quadratic polynomial $p^k(z) = z^2 + w_1^k$. For $j = 0, 1, \dots, n-1$, the leg γ_{j+1}^k has exactly one inverse image under p^k which is tangent to γ_j^0; let $\tilde\gamma_j^{k+1}$ be this inverse leg, and let w_j^{k+1} be its endpoint. Then the union over j of the legs $\tilde\gamma_j^{k+1}$ form the spider Γ^{k+1}; it satisfies the inductive claim about the approach of its legs to ∞.*
(3) *In this sequence (Γ^k) of spiders, the sequence of endpoints w_j^k converges to a limit point w_j for every j. The point w_1 is the parameter of the unique normalized quadratic polynomial $z^2 + w_1$ which has a period n critical orbit and for which the dynamic ray at external angle ϑ_1 is a characteristic ray.*

□

The proof of this algorithm uses analytic machinery completely different from the contents of this paper. It can be found in [**HS**]. We should note that the description of the spider algorithm is slightly different: in [**HS**] the choice of branch of γ_{j+1}^k under the inverse of p^k does not use external angles of rays, but rather their kneading sequence. The resulting algorithm is the same.

Also want angles in angled internal address.

New result: any segment of tree, when iterated, either covers every interior point of the tree as an interior point of the image, or there is a renormalizable subtree.

14. External Angles and Admissibility

In this section, we give a complete description of those kneading sequences that are generated by external angles. For most kneading sequences, the existence of such an external angle is equivalent to admissibility of the kneading sequence. Exceptions are periodic kneading sequences without $\star$: these require a deeper investigation.

14.1. **Proposition (Admissibility in Terms of External Angle ($\star$-periodic))**
A $\star$-periodic kneading sequence ν is admissible (in the sense of Definition 5.1) if and only if there is an angle $\vartheta \in \mathbb{S}^1$ which generates ν (via Definition 2.1, i.e., $\nu = \nu(\vartheta)$).

In this case the angle ϑ has the same exact period as ν, and (if this period is at least 2) ν has a conjugate angle ϑ' of the same exact period with the following properties: ϑ' also has kneading sequence ν, the interval (ϑ, ϑ') contains no angle $2^k\vartheta$ or $2^k\vartheta'$ with $k \geq 1$, and among the two angles $\vartheta/2$ and $\vartheta'/2$, exactly one is periodic and the other one is preperiodic.

Proof. This is a purely combinatorial statement which can be shown algorithmically, see Corollary 13.7. Here we sketch a shorter proof using analytic methods and some well-known background from complex dynamics. Is this really easier than in 13.7? Perhaps delete this entire proposition, or write only what's better than in 13.7.

Let n be the period of ν. If $n = 1$, then $\nu = \overline{\star}$ is trivially admissible, and it is generated by $\vartheta = 0$. From now on, we will suppose that $n \geq 2$.

Suppose that ν is generated by an angle $\vartheta \in \mathbb{S}^1$, which is necessarily periodic of exact period n. Then there exists a unique angle ϑ', necessarily of the same exact period, so that the two parameter rays $R(\vartheta)$ and $R(\vartheta')$ land together (Theorem 9.1). The landing point of these two parameter rays is the root of a hyperbolic component W of period n (Theorem 9.7), and the ray pair $P(\vartheta, \vartheta')$ separates W from the origin. Let c be the center of W. Then by Theorem 9.5, the dynamics rays $R_c(\vartheta)$ and $R_c(\vartheta')$ land together, and the ray pair $P_c(\vartheta, \vartheta')$ separates the critical value from the remaining critical orbit, as well as from all rays $R_c(2^k\vartheta)$ and $R_c(2^k\vartheta')$ that are not part of the ray pair $P_c(\vartheta, \vartheta')$.

The polynomial p_c has a Hubbard tree in the original sense of Douady and Hubbard (as in Definition 3.4) for which the kneading sequence of the critical orbit equals $\nu(\vartheta) = \nu(\vartheta')$. To see this, note that the rays $R_c(\vartheta/2)$, $R_c(\vartheta'/2)$, $R_c((\vartheta + 1)/2)$ and $R_c((\vartheta' + 1)/2)$ all land at the boundary of the Fatou component containing the critical point. These four rays form two ray pairs, and these subdivide $\mathbb{C}$ into three parts, say U_0, U_1 and $U_\star$, so that $U_\star$ contains the critical point, U_1 contains the critical value and $U_0 = -U_1$. The only point on the critical orbit within $U_\star$ is 0. Similarly, all rays $R_c(2^k\vartheta)$ and $R_c(2^k\vartheta')$ for $k \geq 1$ are in U_0 or U_1, except those on $\partial U_\star$. It is now easy to verify the claim that the kneading sequence of the critical orbit (with respect to the Hubbard tree) equals the kneading sequence of the angle ϑ or equivalently of ϑ': they can all be read off from the visits to the sets U_0 and U_1. This shows that every kneading sequence ν which comes from an external angle is realized by a Hubbard tree that can be embedded, so by Theorem 5.2, ν is admissible in the sense of Definition 5.1.

For the final statement, assume by contradiction that $\vartheta/2$ and $\vartheta'/2$ are both periodic. Then the rays $R_c(\vartheta/2)$ and $R_c(\vartheta'/2)$ land together and bound a domain $V_0 \subset \mathbb{C}$ consisting of rays $R(\varphi)$ with $\varphi \in (\vartheta/2, \vartheta'/2)$, and the domain $-V_0$ is disjoint from V_0. The polynomial $z \mapsto z^2 + c$, maps both V_0 and $-V_0$ conformally onto the domain bounded by the rays $R(\vartheta)$ and $R(\vartheta')$ and containing the rays $R_c(\varphi)$ with $\varphi \in (\vartheta, \vartheta')$; but this domain contains the critical value. HB: I don't quite understand this previous bit. The same argument works if $\vartheta/2$ and $\vartheta'/2$ are both preperiodic. Therefore, exactly one of $\vartheta/2$ and $\vartheta'/2$ must be periodic as claimed.

Conversely, let an admissible $\star$-periodic ν with period n be given. Its Hubbard tree can be embedded into the complex plane by definition. Therefore, it satisfies Poirier's conditions [**Poi**] (see Theorem 23.2) which guarantee that it occurs indeed as the Hubbard tree of a complex quadratic polynomial, say $z \mapsto z^2 + c$, with a periodic critical orbit. Then there are two angles ϑ, ϑ' as above which generate ν. $\square$

14.2. **Corollary (Admissibility in Terms of External Angle)**
A $\star$-periodic or non-periodic kneading sequence ν is admissible if and only if there is an angle $\vartheta \in \mathbb{S}^1$ which generates ν.

Proof. The $\star$-periodic case has been shown in Proposition 14.1. Therefore, we suppose that ν is non-periodic.

Let ϑ be an external angle which generates ν. Suppose ν is not admissible, *i.e.*, it fails the admissibility condition for some period m. This depends only on the $\rho_\nu(m)$ initial entries in ν, so all $\star$-periodic kneading sequences with the same initial entries are also non-admissible. But a periodic angle $\vartheta' \in \mathbb{S}^1$ sufficiently close to ϑ will produce a kneading sequence which coincides with ν for more than $\rho_\nu(m)$ entries and which is admissible by Proposition 14.1. So we have a contradiction.

Conversely, suppose that ν is admissible. Let ν^k be the sequence of approximating $\star$-periodic kneading sequences from Definition 7.1. Then $\nu^k < \nu$ by Proposition 6.8 (2), and all ν^k are admissible by Lemma 7.3. Find angles ϑ^k generating ν^k. By compactness, the ϑ^k have an accumulation point $\vartheta \in \mathbb{S}^1$. We claim that ϑ generates ν. This is clear if ϑ is non-periodic. We show that ϑ cannot be periodic. Indeed, if ϑ is periodic, then $\nu(\vartheta)$ is $\star$-periodic, and we can choose a subsequence of the ϑ^k which converges monotonically to ϑ. It follows easily that this subsequence of ν^k must converge to $\mathcal{A}(\nu(\vartheta))$ or to $\overline{\mathcal{A}}(\nu(\vartheta))$, but these are periodic kneading sequences without $\star$ and hence different from ν. $\square$

It follows from Theorem 18.4 that the set of kneading sequences which are generated by angle doubling has positive measure within the symbol space Σ (using the $\frac{1}{2}$-$\frac{1}{2}$-product measure).

It remains to consider the case of periodic kneading sequences without $\star$. This is a rather complicated story; the final result will be given in Theorem 14.10. We begin by a sequence of lemmas which prove that periodic sequences without $\star$ are admissible only if they are upper sequences.

Recall that if φ is said to have some *itinerary with respect to* ϑ, we mean the itinerary with respect to the partition $\{(\vartheta/2,(\vartheta+1)/2),((\vartheta+1)/2,\vartheta/2),\{\vartheta/2,(\vartheta+1)/2\}\}$ with symbols $\mathtt{1},\mathtt{0},\star$ respectively.

14.3. **Lemma (Periodic Itineraries in Markov Case)**
Suppose that the angle ϑ is periodic or preperiodic. Then φ has a periodic itinerary with respect to ϑ if and only if φ is periodic.

Proof. Let $\Theta := \{\vartheta/2,(\vartheta+1)/2\}\cup\{\vartheta,2\vartheta,4\vartheta,\dots\}$ be the finite set consisting of the partition boundary and its forward orbit, and let $I_1, I_2,\dots,I_k$ be the components of $\mathbb{S}^1\setminus\Theta$. These intervals form a Markov partition of $\mathbb{S}^1$ with respect to angle doubling because Θ is forward invariant, and the itineraries of all points within any I_j start with the same symbol. Since angle doubling is expanding, it follows that every periodic itinerary can only be realized by a single angle within each I_j, and all such angles are periodic. (The period of φ may be a multiple of the period its itinerary. In this case, several angles have the same itinerary.) $\square$

Remark. In a similar way, it follows that every periodic itinerary is actually realized by a periodic angle, except if ϑ is periodic: in this case, the kneading sequence ν of ϑ is $\star$-periodic, and the periodic itineraries $\mathcal{A}(\nu)$ and $\overline{\mathcal{A}}(\nu)$ are not realized (and neither are their shifts).

14.4. **Lemma (Periodic Gaps)**
Suppose there are closed infinite sets $T_0,T_1,T_2,\dots,T_n\subset\mathbb{S}^1$ with $2T_k=T_{k+1}$ for all k and $T_n=T_0$. Suppose also that each set T_k is contained in a semicircle. Then there is a $\vartheta\in\mathbb{S}^1$ so that one of the sets T_k contains $\{\vartheta/2,(\vartheta+1)/2\}$.

The sets T_k in this lemma combinatorially describe external angles of dynamic rays landing at Siegel disk boundaries (at least when the corresponding Julia sets are locally connected).

Proof. If $(\vartheta_1,\vartheta_2)$ is a component of $\mathbb{S}^1\setminus T_k$, then we call $\{\vartheta_1,\vartheta_2\}$ a *boundary leaf* of T_k; the length of this leaf is defined as the length of the shortest interval containing the leaf $\{\vartheta_1,\vartheta_2\}$. Every T_k has a unique boundary leaf of maximal length, say ℓ_k. Renumber the T_k cyclically so that $\ell_0=\max\{\ell_k\}$. For a leaf $\{\vartheta_1,\vartheta_2\}$, we call $\{2\vartheta_1,2\vartheta_2\}$ the image leaf provided $2\vartheta_1\neq 2\vartheta_2$. I f the interval $(\vartheta_1,\vartheta_2)$ has length less than $1/2$, then under angle doubling it maps to one of the two components of $\mathbb{S}^1\setminus\{2\vartheta_1,2\vartheta_2\}$.

If $\ell_k\le 1/4$, then $\ell_{k+1}=2\ell_k$. Thus $\ell_0\in(1/4,1/2]$, and the image of the leaf with length ℓ_0 is a leaf with length $\varepsilon_1:=1-2\ell_0$ (if $\ell_0=1/2$, then the image is a single point). Note also that no T_k can contain an interval, hence every T_k has infinitely many boundary leaves.

Suppose that $\varepsilon_1 > 0$. Every T_k has only finitely many boundary leaves of length ε_1 or greater. The only chance for the orbit of any leaf of length greater than ε_1 to ever acquire length less than ε_1 is to have a leaf of length greater than ℓ_0 at the previous iteration, and this is excluded by hypothesis. As a result, every leaf with length greater than ε_1 must be (eventually) periodic. The angle doubling map from T_k to T_{k+1} is a bijection on boundary leaves for every k. But short boundary leaves have their lengths doubled, so they eventually acquire lengths greater than ε_1, and this is impossible injectively. This contradiction shows the claim $\varepsilon_1 = 0$, and hence $\ell_0 = 1/2$. □

In the statement and proof of the following result, we use the standard notation $\omega(\vartheta)$ for the set of accumulation points of the orbit of ϑ (the ω-limit set).

14.5. **Lemma (Periodic Sequences and Non-Periodic Angles)**
Let $\varphi, \vartheta \in \mathbb{S}^1$ and let τ be the itinerary of φ with respect to ϑ. If τ is periodic without $\star$, but φ is non-periodic, then the orbit of ϑ is infinite and there is an $m \geq 0$ so that ϑ has periodic itinerary $\sigma^{\circ m}(\tau)$ (a shift of τ). Moreover, $\vartheta \in \omega(\vartheta)$.

Proof. By Lemma 14.3, ϑ cannot be periodic or preperiodic, so its orbit must be infinite. Let n be the exact period of τ.

Our first claim is that $\varphi \in \omega(\varphi)$ or $\vartheta \in \omega(\varphi)$. Since φ has an infinite orbit on $\mathbb{S}^1$, there is an accumulation point $x \in \mathbb{S}^1$ of the set $U := \{\varphi, 2^n\varphi, 2^{2n}\varphi, 2^{3n}\varphi, \dots, \}$. Note that each angle in U has the same itinerary. For a fixed $\varepsilon > 0$, there is a $k > 0$ so that $|x - 2^k\varphi| < \varepsilon$. The interval $[x, 2^k\varphi]$ (or $[2^k\varphi, x]$, depending on orientation) can be pulled back $k-1$ times along the orbit of φ unless some pull-back interval contains ϑ. If none of them do, then we obtain an interval $[x', \varphi]$ or $[\varphi, x']$ of length $\varepsilon/2^{k-1}$ where $2^{k-1}x' = x$ and x' is another accumulation point of the set U. If this works for arbitrarily small $\varepsilon > 0$, then $\varphi \in \omega(\varphi)$. However, if there are arbitrarily small $\varepsilon > 0$ for which we cannot pull back $k-1$ times, then it follows that $\vartheta \in \omega(\varphi)$. This proves the claim.

If $\vartheta \in \omega(\varphi)$, then the itinerary of ϑ is a shift of τ. To see this, recall that ϑ is not periodic, so for every $i > 0$ there is an $\varepsilon > 0$ so that if $|x'' - \vartheta| < \varepsilon$, then the first i entries in the itineraries of ϑ and x'' coincide. By applying this to $x'' = 2^k\varphi$, we see that ϑ also has periodic itinerary without being a periodic angle.

We may thus use ϑ for φ and repeat the argument. So regardless of whether $\omega(\varphi)$ contains φ or ϑ, we find a non-periodic angle φ (possibly equal to ϑ) with periodic itinerary, and $\varphi \in \omega(\varphi)$.

Let $T = \overline{U}$. Since φ is an accumulation point of itself, it follows that $2^nT = T$. Moreover, T is infinite and each of its points is an accumulation point of T. Let $T_k := 2^{k-1}T$ for $k \geq 1$. Then $T_{k+n} = T_k$ for all k. Each T_k is the closure of a set of points with identical itineraries, so T_k is contained in the closure of one of the two components of $\mathbb{S}^1 \setminus \{\vartheta/2, (\vartheta+1)/2\}$. By Lemma 14.4, there is an angle ϑ' and a set T_k with $\{\vartheta'/2, (\vartheta'+1)/2\} \subset T_k$; more precisely, we must have $\vartheta' = \vartheta$, hence $\vartheta \in T_{k+1}$ (otherwise, the itineraries of points in T_k would have different initial entries). Since the angle ϑ is in the closure of the set of points with identical itineraries and ϑ is non-periodic, we conclude that ϑ has the same periodic itinerary. As above, it also follows that ϑ is an accumulation point of its own orbit. □

14.6. **Lemma (Periodic Kneading Sequence is Upper Sequence)**
If a non-periodic angle ϑ has periodic kneading sequence ν, then ν has no $\star$ and is an upper sequence (i.e., the exact period is contained in the associated internal address).

Proof. As ϑ is non-periodic, ν cannot be $\star$-periodic, and all we need to show is that ν is not a lower sequence. Let n be the exact period of ν, let $U = \{\vartheta, 2^n\vartheta, 2^{2n}\vartheta, 2^{3n}\vartheta, \dots\}$ and $U_k := 2^{k-1}U$ for $k \geq 0$. Let also $T := \overline{U}$ and $T_k := \overline{U}_k$ for each k. By Lemma 14.5, ϑ is an accumulation point of its own orbit, which implies that $2^nT = T$. Since ν is periodic, each T_k is contained in a semicircle.

As before, we say that $\{\vartheta_1, \vartheta_2\}$ is a boundary leaf of T_k if $(\vartheta_1, \vartheta_2)$ is a component of $\mathbb{S}^1 \setminus T_k$, and the length of this leaf is the length of the shortest interval containing the leaf $\{\vartheta_1, \vartheta_2\}$. The sets T_k are *unlinked* in the sense that if $\{\vartheta_1, \vartheta_2\}$ is a boundary leaf of T_k, then $T_{k'}$ is contained in the closure of one of the components of $\mathbb{S}^1 \setminus \{\vartheta_1, \vartheta_2\}$). To show why this is true, let us call $\{\vartheta_1, \vartheta_2\} \subset \mathbb{S}^1$ is a *precritical leaf* of Step k if $2^k\vartheta_1 = 2^k\vartheta_2 = \vartheta$. For every $k \geq 1$, there are exactly 2^{k-1} precritical leaves of Step k, so that all precritical leaves of all Steps are disjoint from each other and unlinked. Indeed, the critical leaf is the unique precritical leaf of Step 1, and there are exactly 2 unlinked precritical leaves of Step 2 (preimages of the critical leaf, one on each side of the critical leaf). Continue by induction (using the fact that ϑ is non-periodic). The construction implies that if two points on $\mathbb{S}^1$ have itineraries which differ within the first k entries, then they are separated by a precritical leaf of at most Step k. Since the itineraries of the points in U_k and $U_{k'}$ differ within the first n entries, the precritical leaves of at most Step n separate the sets U_k, and the closures T_k are indeed unlinked.

Let $\ell_k \geq \ell'_k$ be the lengths of the longest two boundary leaves of T_k. We claim:

$$\text{The longest boundary leave in } T_1 \text{ is precritical with Step } n. \tag{8}$$

First note that $0 < \ell'_k < \ell_k \leq 1/2$ for all k because each T_k is infinite and contained in a semicircle. If $\ell_k \leq 1/4$, then $\ell_{k+1} = 2\ell_k$ and $\ell'_{k+1} = 2\ell'_k$. If $\ell_k > 1/4$, then also $\ell'_k \geq 1/4$ (otherwise, T_{k+1} would not be contained in a semicircle). In this case, all boundary leaves of T_k except the two longest have combined length $\ell_k - \ell'_k \leq 1/4$, so each other individual boundary leaf of T_k has length less than 1/4.

The two longest leaves in $T_0 = T_n$ have lengths $1/4 \leq \ell'_0 < \ell_0 = 1/2$, and their images are one leaf with length $1 - 2\ell'_0$ and a single point ϑ, see Figure 14.1. Let b_1 be the longest leaf of T_1; it has length $\ell_1 = 1 - 2\ell'_0$. The boundary of T_0 is contained in

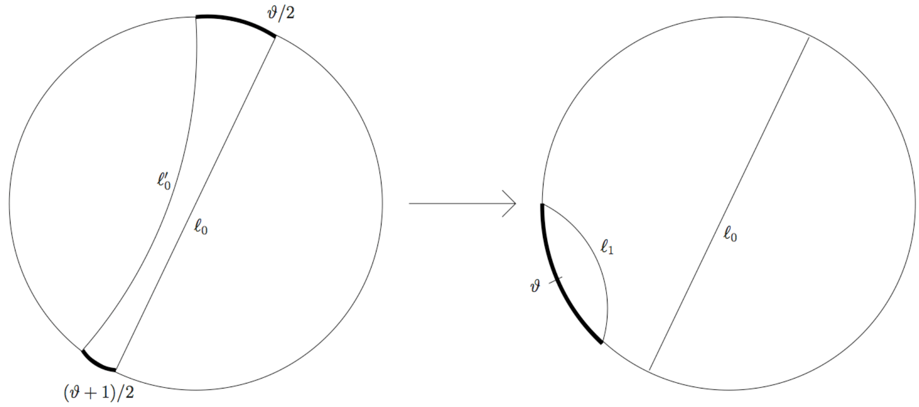


FIGURE 14.1. The action of the angle doubling on the leaves ℓ_0 and ℓ_0'.

two short intervals of combined length $\ell_0 - \ell_0'$, and their images under doubling meet at ϑ with combined length $1 - 2\ell_0' = \ell_1$. Therefore, b_1 is the image of the second longest leaf of T_0.

Let m_k be the combined lengths of all boundary leaves in T_k other than the longest two, so $m_k \leq \ell_k - \ell_k'$. If $\ell_k \leq 1/4$, then all leaves just double in length, so $m_{k+1} = 2m_k$. If for some $k < n$, $\ell_k > 1/4$ and hence $\ell_k' \geq 1/4$, then the second longest leaf becomes the longest leaf of T_{k+1}, with length $\ell_{k+1} = 1 - 2\ell_k'$, whereas the longest leaf of T_k is mapped to a leaf of T_{k+1} with length $1 - 2\ell_k > 1 - 2\ell_0' \geq m_0$. If this leaf is not the second longest of T_{k+1}, then $m_{k+1} > m_0$, and the same is true for all m_i, $i \geq k+1$. This is a contraction, because $m_n = m_0$. Therefore the longest two leaves of T_k remain the longest two until T_n is reached.

It follows that if b_1 is not precritical, then after $n-1$ iterations, b_1 maps to the second longest boundary leaf of T_n, and hence b_1 is periodic of period n. Let $J_\pm \subset \mathbb{S}^1$ be the two short arcs of combined length $\ell_1 - \ell_1'$ that cover the boundary of T_1. Multiplication by 2^n preserves orientation and is injective on T_1, except that the two points of its second longest leaf (which is then precritical) are mapped to ϑ. It follows that one of $J_\pm$ is mapped to a short arc. But this contradicts that 2^n expands distances on $J_\pm$. This contradiction completes the proof of Claim (8).

We define the internal address of ϑ in this context as the sequence $S_0, S_1, \ldots, S_m$ of strictly increasing integers as follows. We start with the critical leaf, which is precritical of STEP $S_0 = 1$. If the entry S_{k-1} stands for a precritical leaf a_{k-1} of STEP S_{k-1}, then let a_k be the leaf of least STEP-number which separates a_{k-1} from U_1, and let S_k be the STEP-number of a_k. It is easy to see that a_k is uniquely defined and $S_k > S_{k-1}$ for all k; this sequence terminates with the leaf $a_m = b_1$ of STEP $S_m = n$.

Finally, we claim that the internal address of ϑ in this context equals the internal address of the kneading sequence ν associated to ϑ (compare Proposition 6.8: our construction can be viewed as an internal address by precritical points). The limiting itinerary of points $\vartheta/2+\varepsilon$ and $(\vartheta+1)/2-\varepsilon$ (as $\varepsilon \searrow 0$) is the concatenation 1ν. Therefore the precritical leaf a_2 of least Step separating ϑ from a_1 has Step $S_1 = \rho_\nu(1)$ steps, using ρ_ν from Definition 2.2. Similarly, if $a_2 = \{\vartheta_2, \vartheta_2'\}$ with $\vartheta_2 < \vartheta_2'$, then the limiting itinerary of points $\vartheta_2 + \varepsilon$ and $\vartheta_2' - \varepsilon$ is the concatenation of the first S_0 entries of ν, followed by ν, and thus $S_2 = \rho_\nu(S_1)$, and so on. Thus $S_k = \rho_\nu^{\circ k}(1)$, which proves the claim.

Since the internal address of ν terminates with the entry $S_m = n$, which is the exact period of ν, the sequence ν is an upper sequence in the sense of Definition 5.14. □

Remark. Lower periodic sequences without $\star$ can be the itineraries of non-periodic angles φ with respect of angles $\vartheta \neq \varphi$. For example, the upper sequence $\overline{1100}$ occurs as the kneading sequence of certain angles ϑ by Corollary 14.8, so 2ϑ generates the itinerary $\overline{1001}$ which is a lower sequence. Proposition 14.6 means that $\vartheta \neq \varphi$.

14.7. **Lemma (Kneading Sequence $\overline{1}$ Generated)**
There are uncountably many external angles which generate the kneading sequence $\overline{1}$ (one such angle for every irrational rotation number).

Proof. We follow the arguments of Bullett and Sentenac [**BuS**] (see also Theorem 26.2). For $\vartheta \in [0,1]$, define the function $g_\vartheta\colon [0,1] \to [0,1]$ by

$$g_\vartheta(x) = \begin{cases} \vartheta & \text{if } 0 \leq x \leq \vartheta/2 \\ 2x & \text{if } \vartheta/2 < x < (\vartheta+1)/2 \\ \vartheta & \text{if } (\vartheta+1)/2 \leq x \ . \end{cases}$$

Then g_ϑ descends to a continuous self-map of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ of degree one. The map g_ϑ has a well-defined rotation number r_ϑ, depending continuously on ϑ, see *e.g.* [**ALM1**]. Moreover, r_ϑ is rational if and only if g_ϑ has a periodic point, and $r_\vartheta = 0$ if and only if g_ϑ has a fixed point. (In fact, the map $\vartheta \to r_\vartheta$ is a monotone surjection from $\mathbb{S}^1$ to itself and has topological degree one.)

For $\vartheta = 0$, g_ϑ has a fixed point at 0, so $r_0 = 0$. For $\vartheta \neq 0$, there is no fixed point, so $r_\vartheta \neq 0$. Since r_ϑ depends continuously on ϑ, there are uncountably many values of ϑ for which r_ϑ is irrational, so g_ϑ has no periodic point. For such values of ϑ, the orbit of ϑ must always remain in the interval $(\vartheta/2, (\vartheta+1)/2)$, and so the kneading sequence of ϑ is $\overline{1}$. □

Remark. Bullett and Sentenac show more; compare Section **??**: the set of angles ϑ with kneading sequence $\overline{1}$ has Hausdorff dimension 0, and for each such angle ϑ the set of all points with itinerary $\overline{1}$ also has Hausdorff dimension 0. They also show interesting number theoretic relations between ϑ and r_ϑ; compare Section **??**. Check Goldberg and Milnor, and also discuss the Farey algorithm somewhere.

14.8. **Corollary (Upper Periodic Kneading Sequences Generated)**
For every admissible $\star$-periodic kneading sequence ν_*, the upper sequence $\mathcal{A}(\nu_*)$ is generated by a non-periodic external angle.

PROOF. Let n be the period of ν_*. The case $n = 1$ has been treated in Lemma 14.7; now we discuss the general case and reduce it to the case $n = 1$ by a version of tuning (compare Section 10).

Since ν_* is admissible, Proposition 14.1 (and Theorem 9.1) shows that there are two external angles $0 < \vartheta < \vartheta' < 1$ of exact period n which generate the kneading sequence ν_* so that the interval (ϑ, ϑ') contains no angle $2^k\vartheta$ or $2^k\vartheta'$. Moreover, among the two angles $\vartheta/2$ and $\vartheta'/2$, exactly one is periodic and the other one is preperiodic.

Let $I_1 = (\vartheta'/2, (\vartheta + 1)/2)$ and $I_0 = ((\vartheta' + 1)/2, \vartheta/2)$ (in the positive orientation, so that $0 \in I_0$ and $2I_1 = 2I_0 = \mathbb{S}^1 \setminus [\vartheta, \vartheta'])$. See Figure 14.2

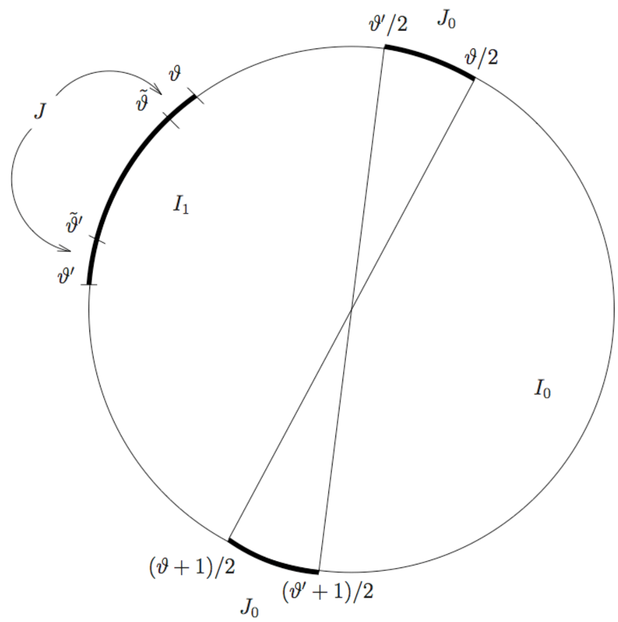


FIGURE 14.2. The intervals I_0, I_1, J_0 and J_0' and boundary angles $\vartheta < \tilde\vartheta < \tilde\vartheta' < \vartheta'$.

There are two strings $A = A_1 \dots A_n$, $B = B_1 \dots B_n \in \{0, 1\}^n$ so that

$$\vartheta = \frac{1}{2^n - 1} \sum_{k=1}^{n} 2^{n-k} A_k \text{ and } \vartheta' = \frac{1}{2^n - 1} \sum_{k=1}^{n} 2^{n-k} B_k,$$

cf. the Staircase Algorithm 26.4. By Lemma 14.7, there are uncountably many external angles with kneading sequence $\overline{1}$. Let h be one of them and write $h = \sum h_k 2^{-k}$ for

$k \geq 1$ in the binary expansion with $h_k \in \{0, 1\}$. Replace every 0 by the string A and every 1 by the string B; this yields the binary expansion of an angle $\vartheta_h \in (\vartheta, \vartheta')$.

There are two angles $0 < \vartheta < \tilde{\vartheta} < \tilde{\vartheta}' < \vartheta'$ so that $2^n : (\vartheta, \tilde{\vartheta}) \to (\vartheta, \vartheta')$ and $2^n : (\tilde{\vartheta}', \vartheta') \to (\vartheta, \vartheta')$ are homeomorphisms. The binary expansions of $\tilde{\vartheta}$ and $\tilde{\vartheta}'$ are written as $ABBBBB\ldots$ and $BAAAAA\ldots$, cf. the rotation sequences of Definition 26.7. Then for every $k \geq 0$, the angle $2^{kn}\vartheta_h$ is contained in $J := (\vartheta, \tilde{\vartheta}) \cup (\tilde{\vartheta}', \vartheta')$. Let also $J_0 := (\vartheta/2, \vartheta'/2) \cup ((\vartheta + 1)/2, (\vartheta' + 1)/2)$.

We claim that the interval (ϑ, ϑ') contains no angle $2^k\tilde{\vartheta}$ or $2^k\tilde{\vartheta}'$ for $k \geq 1$. Indeed, since $2^n\tilde{\vartheta} = \vartheta'$ and $2^n\tilde{\vartheta}' = \vartheta$, this could happen only for $k \in \{1, 2, \ldots, n-1\}$, and since $2^n : (\vartheta, \tilde{\vartheta}) \to (\vartheta, \vartheta')$ and $2^n : (\tilde{\vartheta}', \vartheta') \to (\vartheta, \vartheta')$ are homeomorphisms, $2^k\tilde{\vartheta} \in (\vartheta, \vartheta')$ or $2^k\tilde{\vartheta}' \in (\vartheta, \vartheta')$ would imply that $2^{n-k}\vartheta \in (\vartheta, \vartheta')$ or $2^{n-k}\vartheta' \in (\vartheta, \vartheta')$, a contradiction.

Our next claim is if $\varphi \in (\vartheta, \tilde{\vartheta})$, then $2^k\varphi \in J_0$ implies that $k \geq n-1$ (and an analogous statement holds for $\varphi \in (\tilde{\vartheta}', \vartheta')$). If $2^k\varphi \in J_0$, then $2^k(\vartheta, \tilde{\vartheta})$ must contain the component of J_0 that contains $2^k\varphi$. (Otherwise, J_0 contains $2^k\vartheta$ or $2^k\tilde{\vartheta}$, and one of $2^{k+1}\vartheta$ or $2^{k+1}\tilde{\vartheta}$ would be contained in (ϑ, ϑ'), a contradiction). But if $k \leq n-1$, then $n-k$ iterations later the forward image of a boundary angle of J_0 would be contained in (ϑ, ϑ'), and this is a contradiction again.

The previous claim implies that for every $\varphi \in J$ and every fixed $k \in \{0, 1, \ldots, n-2\}$, all three angles $2^k\vartheta$, $2^k\vartheta'$ and $2^k\varphi$ are in $I_\mathtt{0}$ or all three are in $I_\mathtt{1}$. None of these angles can be in $\overline{J}_0 = \mathbb{S}^1 \setminus (I_\mathtt{0} \cup I_\mathtt{1})$ and the angles $2^k\vartheta$ and $2^k\vartheta'$ are both in $I_\mathtt{0}$ or both in $I_\mathtt{1}$ because the kneading sequences of ϑ and ϑ' coincide. Finally, if $\varphi \in (\vartheta, \tilde{\vartheta})$, then $2^k\varphi$ and $2^k\vartheta$ must both be in $I_\mathtt{0}$ or both be in $I_\mathtt{1}$ because of the previous claim. If $\varphi \in (\tilde{\vartheta}', \vartheta)$, then the same is true for $2^k\varphi$ and $2^k\vartheta'$.

This implies that the first $n-1$ entries in the itinerary of φ coincides with the first $n-1$ entries in ν_*, and in particular that the kneading sequence of ϑ_h coincides with ν_* except possibly at entries that are divisible by n. (Keep in mind that the kneading sequence of ϑ_h is defined with respect to a different partition than the kneading sequences of ϑ or ϑ', but points in the interval $I_\mathtt{1}$ always generate an entry `1` and points in the interval $I_\mathtt{0}$ always generate an entry `0`, and this suffices.)

So far, the argument works for all angles ϑ_h whose binary expansion is an infinite concatenation of strings A and B. To treat the entries in the kneading sequence of ϑ_h that are divisible by n, we need the specific form of ϑ_h derived from Lemma 14.7. First observe that the last entries $A_n \neq B_n$ are different. This is a rephrasing of the statement made earlier that exactly one of $\vartheta/2$ and $\vartheta'/2$ is periodic.

Suppose that the last entry in A is 0 and the last entry in B is 1: this means that $\vartheta/2$ and $(\vartheta' + 1)/2$ are periodic. Then the sequence of n-th, $2n$-th, $3n$-th etc. entries in the kneading sequence of ϑ_h are exactly the kneading sequence of h, *i.e.,* they are all equal to `1` (this requires considering a few cases but is not difficult). Similarly, if the last entry in A is 1 and the last entry in B is 0, then the n-th, $2n$-th, $3n$-th etc. entries in the kneading sequence of ϑ_h are all $\mathtt{0}$.

This shows that the kneading sequence of ϑ_h is periodic of period n without $\star$, so it equals either $\mathcal{A}(\nu_*)$ or $\overline{\mathcal{A}}(\nu_*)$. As the latter case is excluded by Lemma 14.6, this completes the proof of the corollary. □

14.9. **Lemma (Periodic Sequences in Admissible Cylinders)**
Let $\nu_1 \ldots \nu_n \in \{\mathtt{0}, \mathtt{1}\}^n$ be an admissible word (with $\nu_1 = \mathtt{1}$) such that an entry n is generated in the internal address. Then the kneading sequence $\nu := \overline{\nu_1 \ldots \nu_n}$ is admissible.

Proof. By hypothesis, the internal address of ν is finite, say $1 \to S_1 \to S_2 \to \ldots \to S_k$ with $S_k = n$. If ν fails the admissibility condition for m, then $\rho(m) \in \mathrm{orb}_\nu(1)$, hence $\rho(m) \le n$. But then the entire word $\nu_1 \ldots \nu_n$ is non-admissible. □

Remark. The assumption about the entry n in the internal address is necessary, as the example $\nu = \overline{\mathtt{1011\,1010\,11}\star}$ with internal address $1 \to 2 \to 4 \to 8 \to 10 \to 11$ shows: it fails the admissibility condition for $m = 6$ (with $\rho(6) = 11$), and so does $\overline{\mathcal{A}}(\nu) = \overline{\mathtt{1011\,1010\,111}}$ (with $\rho(6) = 12$ and $r = m = 6$). However, the word $\mathtt{1011\,1010\,111}$ is admissible and realized for example by the internal address $1 \to 2 \to 4 \to 8 \to 10 \to 14$.

14.10. **Theorem (Admissibility of Periodic Sequences)**
Let ν_* be a $\star$-periodic kneading sequence of some exact period $n \ge 2$. Define $\nu = \mathcal{A}(\nu_*)$ and $\nu' = \overline{\mathcal{A}}(\nu_*)$ and let n' be the exact period of ν'. Then the exact period of ν is n, and the following are equivalent:

1. *the sequence ν is generated by an angle $\vartheta \in \mathbb{S}^1$ (necessarily non-periodic);*
2. *ν_* is generated by an angle $\vartheta_* \in \mathbb{S}^1$ (necessarily periodic);*
3. *ν is admissible in the sense of Definition 5.1;*
4. *ν_* is admissible in the sense of Definition 5.1;*
5. *the sequence ν' is admissible, and either $n' = n$ or $\nu' = \mathcal{A}(\nu'_*)$ for some $\star$-periodic sequence ν'_*.*

Moreover, the following are equivalent:

6. *the sequence ν' is generated by an angle $\vartheta \in \mathbb{S}^1$ (necessarily non-periodic);*
7. *ν_* is admissible in the sense of Definition 5.1, $n' < n$, and $\nu' = \mathcal{A}(\nu'_*)$ for some admissible $\star$-periodic sequence ν'_* (necessarily of period n').*

Existence of an external angle ϑ generating ν' implies, but is not implied by, admissibility of ν'. Admissibility of $\mathcal{A}(\nu_*)$ implies, but is not implied by, admissibility of $\overline{\mathcal{A}}(\nu_*)$.

Remark. This theorem can be interpreted in the context of the Mandelbrot set as follows: for any $\star$-periodic kneading sequence ν_*, the sequences $\mathcal{A}(\nu_*)$ and $\overline{\mathcal{A}}(\nu_*)$ are generated by external angles only if ν_* is generated by an external angle (or equivalently admissible), so there is a parameter ray pair $P(\varphi, \varphi')$ at periodic angles with $\nu_* = \nu(\varphi) = \nu(\varphi')$, and the landing point of $P(\varphi, \varphi')$ is the root of a hyperbolic component W. There are uncountably many parameter rays at irrational external angles ϑ landing at ∂W, and their associated kneading sequences are always $\mathcal{A}(\nu_*)$. If W is a satellite component, *i.e.,* there is a component W' from which W bifurcates, then the parameter rays at irrational angles ϑ' landing on the boundary of W' all generate $\overline{\mathcal{A}}(\nu_*)$. In this case, the period of W' equals the exact period n' of $\overline{\mathcal{A}}(\nu_*)$, and n' properly divides n. However, if W' is a primitive component, then the kneading sequence $\overline{\mathcal{A}}(\nu_*)$ has period $n' = n$ and it is not generated by any external angles.

Since admissibility of a kneading sequence ν_* means that ν_* occurs in the Mandelbrot set, we can rephrase this theorem in the following way: every $\star$-periodic kneading sequence ν_* satisfies exactly one of the following four cases (where again n and n' denote the exact periods of $\mathcal{A}(\nu_*)$ and $\overline{\mathcal{A}}(\nu_*)$):

primitive root in $\mathcal{M}$: $n' = n$ and the sequences $\mathcal{A}(\nu_*)$, ν_* and $\overline{\mathcal{A}}(\nu_*)$ are admissible; $\mathcal{A}(\nu_*)$ and ν_* are generated by external angles, but $\overline{\mathcal{A}}(\nu_*)$ is not. Example: $\nu_* = \overline{\mathtt{10}\star}$;

bifurcation in $\mathcal{M}$: n' strictly divides n and $\overline{\mathcal{A}}(\nu_*)$ is an upper sequence for some ν'_*. The sequences $\mathcal{A}(\nu_*)$, ν_* and $\overline{\mathcal{A}}(\nu_*)$ are admissible as well as generated by external angles. Example: $\nu_* = \overline{\mathtt{101}\star}$;

minimal non-existing sequence near primitive root: n' strictly divides n and $\overline{\mathcal{A}}(\nu_*)$ is a lower sequence for some ν'_*. The sequence $\overline{\mathcal{A}}(\nu_*)$ is admissible, but ν_* and $\mathcal{A}(\nu_*)$ are not; none of $\mathcal{A}(\nu_*)$, ν_* and $\overline{\mathcal{A}}\nu_*$ is generated by an external angle. Example: $\nu_* = \overline{\mathtt{10110}\star}$;

non-existing sequence: n' divides n, possibly strictly; $\mathcal{A}(\nu_*)$, ν_* and $\overline{\mathcal{A}}(\nu_*)$ are non-admissible (within non-admissible subtrees, all three previous cases occur; the third one describes the creation of a new evil orbit; none of $\mathcal{A}(\nu_*)$, ν_*, $\overline{\mathcal{A}}(\nu_*)$ are generated by external angles. Example: $\nu_* = \overline{\mathtt{1011001101100}\star}$.

HB: Do I remember correctly that Karsten Keller wrote about this too in is monograph. Not that this should affect our writing about this, but if true, it might appropriate be mentioning this when we get to Karsten in the Other/Further

Proof. By Proposition 5.16, the exact periods of ν_* and $\mathcal{A}(\nu_*)$ coincide.
(4) $\Leftrightarrow$ (3): This is Lemma 6.11.
(4) $\Leftrightarrow$ (2): This is Proposition 14.1.
(4) $\Rightarrow$ (1): This is Corollary 14.8.
(1) $\Rightarrow$ (4): Suppose the periodic sequence ν is generated by an angle $\vartheta \in \mathbb{S}^1$; since ν is not $\star$-periodic, ϑ cannot be periodic. Arbitrarily close to ϑ there are periodic angles which generate $\star$-periodic angles of period greater than n, so there are admissible kneading sequences that start with the same initial n entries as ν. Since ν is an upper sequence, Lemma 14.9 shows that ν is admissible.

Now we focus on the equivalent (3) ⇔ (5). Let $\nu = \overline{\nu_1\nu_2\dots\nu_n}$ be the upper sequence and $\nu' = \overline{\nu_1\nu_2\dots\nu_n'}$ (with opposite symbols ν_n and ν_n') lower sequence. Let ρ and ρ' be their respective ρ-functions as in Definition 2.2.

(3) ⇒ (5): Suppose lower sequence $\nu' = \overline{\nu_1\nu_2\dots\nu_n'}$ fails Admissibility Condition 5.1 at entry m. Then $n \in \mathrm{orb}_\rho(1)$ and $n \notin \mathrm{orb}_{\rho'}(1)$. Also, taking $s := \rho'(m) - m \pmod m$, we have $m \in \mathrm{orb}_{\rho'}(s)$ by part (3) of Admissibility Condition 5.1. We divide the argument into the following cases:

(i) $n < m$. Take $m' = m - n$. By periodicity of ν' we have $\rho'(m') - m' = \rho'(m) - m = s$, so by Lemma 20.1 there is $t \le \rho'(m') < \rho'(m)$ such that $t \in \mathrm{orb}_{\rho'}(s) \cap \mathrm{orb}_{\rho'}(1)$. But this contradicts that $\rho'(m) = \min\{\mathrm{orb}_{\rho'}(s) \cap \mathrm{orb}_{\rho'}(1)\}$.

(ii) $\rho'(m) < n$. Then ν was already inadmissible, because admissibility depends only on the first $\rho'(m) = \rho(m)$ entries of ν' (or ν).

(iii) $\rho'(m) = n$. This is impossible because $\rho'(m) \in \mathrm{orb}_{\rho'}(1)$ by Lemma 20.1.

(iv) $m < n < \rho'(m)$. First we show that $\rho'(m) < n + m$. Indeed, if $\rho'(m) \ge n + m$, then for $\tilde\nu = \overline{\nu_1\dots\nu_m}$ we have $\tilde\rho(n) > n + m$. Therefore $\tilde\nu$ is periodic of exact period l dividing $\gcd(m, n)$. Now $l < m$ contradicts part (2) of Admissibility Condition 5.1 because $\rho'(l) = \rho'(m)$. If on the other hand $l = m$, then n is a multiple of m, and it is easy to check that ν already fails the Admissibility condtion at entry m.

Thus we can assume that $t := \rho'(m) - m < n$. Then

$$\begin{aligned} \nu_{m+1}\dots\nu_n'\nu_1\dots\nu_{\rho'(m)-n} &= \nu_{m+1}\dots\nu_n'\nu_{n+1}\dots\nu_{\rho'(m)} \\ &= \nu_1\dots\nu_{n-m}\nu_{n-m+1}\dots\nu_t'. \end{aligned}$$

This shows that for the ρ-function of the upper sequence ν we have $\rho(m) = n$ and $\rho(n-m) = \rho(m) - m = t < n$. Since $m \in \mathrm{orb}_\rho(s)$ for $s = t \pmod m$, this implies that $m \in \mathrm{orb}_\rho(n - m \pmod m) = \mathrm{orb}_\rho(\rho(m) - m \pmod m)$. Hence m fails the admissibility condition already for ν.

Therefore the cases (ii) and (iv) both lead to ν being inadmissible. Finally, if the exact period n' of ν' is a proper divisor of n, and $n' \notin \mathrm{orb}_{\rho'}(1)$, then ν fails the Admissibility Condition at entry n'.

(5) ⇒ (3): Assume that ν' is the upper sequence of an admissible $\star$-periodic kneading sequence ν_*' of period n', so n' is the last entry of $\mathrm{orb}_{\rho'}(1)$. Suppose now that ν fails the admissibility condition at entry m. Let ρ be the ρ-function of ν. If $\rho(m) < n$, then m also failed the admissibility condition for ν', so we are left with two cases:

(i) $m < n < \rho(m)$. Write $t = \rho(m) - m$, and $s = t \pmod m$, so $m \in \mathrm{orb}_\rho(s)$ by part (3) of Definition 5.1. We have for ν

$$\begin{aligned} \nu_{m+1}\dots\nu_n\nu_{n+1}\dots\nu_{\rho(m)} &= \nu_{m+1}\dots\nu_n\nu_1\dots\nu_{\rho(m)-n} \\ &= \nu_1\dots\nu_{n-m}\nu_1\dots\nu_t'. \end{aligned}$$

Therefore $\rho(n-m) = t$. But this means that for ν', we have $\rho'(m) = n$ and $\rho'(\rho'(m) - m) = \rho'(n-m) = t$ and $m \in \mathrm{orb}_{\rho'}(\rho'(m) - m \pmod m)$. This means again that ν'

failed the Admissibility Condition already at entry m.

(ii) $\rho(m) = n$. Take again $s = \rho(m) - m \pmod m$. If also $n - n' < m$, then $s < n'$ and for ν' neither n' nor $2n'$ belong to $\mathrm{orb}_{\rho'}(s)$. This contradicts Lemma 20.2. If on the other hand $m < n - n'$, then we can take $k = m \pmod{n'}$ and find $\rho(k) = \rho(m) = n$. But then $k < n'$ and for ν' itself we have $\rho'(k) > 2n'$, again contradicting Lemma 20.2. This completes the proof of this implication.

REMARK. Our standard non-admissible example $\nu_* = \overline{\mathtt{101\,10}\star}$ of period 6 with internal address $1 \to 2 \to 4 \to 5 \to 6$ has an admissibile lower sequence $\overline{\mathcal{A}}(\nu_*) = \overline{\mathtt{101}} = \overline{\mathcal{A}}(\overline{\mathtt{10}\star})$. This shows that the condition $\overline{\mathcal{A}}(\nu_*) = \mathcal{A}(\nu'_*)$ is essential.

Now we discuss the second set of equivalences.

(6) ⇒ (7): If the kneading sequence ν' is generated by a non-periodic angle ϑ, then ν' is an upper sequence by Lemma 14.6, so we can write $\nu' = \mathcal{A}(\nu'_*)$ for some $\star$-periodic sequence ν'_*, which is admissible by the first part of the theorem. The sequences ν' and ν'_* have identical internal addresses which terminate with the exact period n' (Proposition 5.16). Since $\nu' = \overline{\mathcal{A}}(\nu_*)$, it follows that n' properly divides n, because if $n' = n$ then $\nu'_* = \nu_*$, a contradiction. The internal address of ν_* is then equal to the internal address of ν'_* (or equivalently of ν'), extended by an entry n. By Lemma 6.12, ν_* is admissible.

(7) ⇒ (6): If $\nu' = \mathcal{A}(\nu'_*)$, then the first part of the theorem proves that ν' is generated by an external angle. □

14.11. **Corollary (Preperiodic Kneading Sequences)**

If an angle $\vartheta \in \mathbb{S}^1$ generates a preperiodic kneading sequence ν, then ϑ is preperiodic itself so that ϑ and ν have identical preperiod, while the period of ν divides the period of ϑ (possibly properly).

PROOF. If ϑ is irrational and has a preperiodic kneading sequence ν, then $2^k\vartheta$ is still irrational and has a periodic itinerary, and this contradicts Lemma 14.5. Preperiodic kneading sequences are thus generated rational angles, and only preperiodic angles are possible. The statements about preperiod and period are easy to check. □

REMARK. It does occur that the period of ν strictly divides the period of ϑ. For example, the angles $\vartheta_1 = 9/56 = 0.001\,\overline{001}$, $\vartheta_2 = 11/56 = 0.001\,\overline{010}$ and $\vartheta_3 = 15/56 = 0.010\overline{001}$ which all generate the same kneading sequence $\nu = \mathtt{110}\,\overline{\mathtt{1}}$. This has some interesting consequences, for example on the number of preperiodic rays associated to Misiurewicz-Thurston points. All three parameter rays $R(9/56)$, $R(11/56)$ and $R(15/56)$ land together at the same parameter. Moreover, in the context of spiders, this gives rise to a Thurston Obstruction (see [**HS**, Section 6]).

HB: I don't really follow this paragraph below, or why it should be in. A separate proof of Corollary 14.11 uses the fact that the preperiodic kneading sequence has an associated Hubbard tree with finite critical orbit, and only finitely many external angles are associated to the critical orbit (which terminates on a repelling periodic orbit). In contrast, a periodic kneading sequence also has a finite Hubbard tree, but infinitely many external angles may be associated critical orbit: under continued pull-back of the Hubbard tree, the points on the critical orbit acquire infinitely many branches and thus allow for infinitely many angles.

15. Construction of Hubbard Trees

We give an explicit construction of the Hubbard tree of a $\star$-periodic kneading sequence.

Let us start with a lemma to justify Step 2 in the construction of Hubbard trees.

15.1. **Lemma (Interaction of Tame and Evil Orbits)**
Let (T, f) be the Hubbard tree with $\star$-periodic kneading sequence ν and internal address $1 \to S_1 \to \ldots \to S_k \to S_{k+1}$ and let p_1 be the S_k-periodic characteristic point whose itinerary coincides with ν for S_k entries. If T contains an m-periodic evil orbit with $\rho(m) = S_{k+1}$, then its characteristic point x_1 lies on (p_1, c_1), and p_1 is not a branch point.

Proof. Since x_1 is the characteristic point of its orbit, $x_1 \in [c_1, c_0]$. If $x_1 \in [p_1, 0]$, then $\mathrm{orb}(x_1)$ is contained in the connected hull of $\mathrm{orb}(p_1)$. By "orbit forcing" this would imply that the evil orbit $\mathrm{orb}(x_1)$ already existed in the Hubbard tree with internal address $1 \to \ldots \to S_k$, and then $\rho(m) \le S_k$, contradicting that $\rho(m) = S_{k+1}$. Hence x_1 lies behind p_1.

For the second assertion, suppose that p_1 has $q > 2$ arms and x_1 has $q' > 2$ arms. Denote the global arms of p_1 by $G_0, \ldots, G_{q-1}$ and the global arms of x_1 by $G'_0, \ldots, G'_{q'-1}$, as in Lemma 4.6. Recall that $S_{k+1} = aS_k + r = a'm + r'$ where $a' = q' - 2$ and $a = q - 2$ or $q - 1$ depending on whether $S_k \in \mathrm{orb}_\rho(r)$ or not, see Propositions 5.19 and 5.13.

We distinguish two cases, both leading to a contradiction:

- $a'm \ge aS_k$. In this case $r' \le r \le S_k$. Since m violates the admissibility condition, $m \in \mathrm{orb}_\rho(r')$ which implies that $\zeta_m \in (x_1, \zeta_{r'})$ and $\zeta_{r'} \in (x_1, c_{1+a'm}) \subset G'_{q'-1} \subset G_1$. Therefore $0 \in f^{\circ r'-1}(G_1)$, contradicting that $f^{\circ S_k}$ maps G_1 homeomorphically onto G_2.
- $a'm < aS_k$ (and therefore $m < aS_k$). Since m violates the admissibility condition, S_k does not divide m, so $p_{1+m} \ne p_1$ and by Lemma 4.1, $p_{1+m} \in G_0$. Since $c_{1+m} \in G_1$, the local arm L at p_{1+m} towards c_{1+m} points to some other point in $\mathrm{orb}(p_1)$. Continuing the iteration, the images of L will point to some point in $\mathrm{orb}(p_1)$ or to c_1 until (and including the iterate where) p_{1+m} reaches p_1. This is the argument of Lemma 4.1. Let j be the smallest integer such that $jS_k \ge m$. Clearly $1 \le j \le a$. We have $f^{\circ jS_k - m}(p_{1+m}) = p_1$. Now $f^{\circ jS_k - m}(L)$ points towards $f^{\circ jS_k}(c_1) \in G_{j+1}$. Clearly $G_{j+1} \ne G_1$ and if $G_{j+1} \ne G_0$ as well, then G_{j+1} contains no point in $\mathrm{orb}(p_1)$ nor the critical value, contradicting the above property of the local arm L. The final possibility is that $G_{j+1} = G_1$; this can only occur if $j = a = q-1 \ge 2$. Choose $t \ge 1$ such that $m_0 := a'm - tS_k \in \{1, \ldots, S_k - 1\}$. Since $\rho(a'm) = S_{k+1} > aS_k$, we have by Lemma 5.10 that also $\rho(m_0) = S_{k+1} > 2S_k$. This, however, contradicts Lemma 20.2 (2), finishing the proof. □

15.2. **Algorithm (Construction of Hubbard Tree)**

Let ν be a $\star$-periodic kneading sequence and $1 \to S_1 \to \ldots \to S_l$ be its internal address. Write $\nu = \overline{\nu_1 \ldots \nu_{S_l - 1}\star}$. Our construction of the Hubbard tree of ν will be inductive.

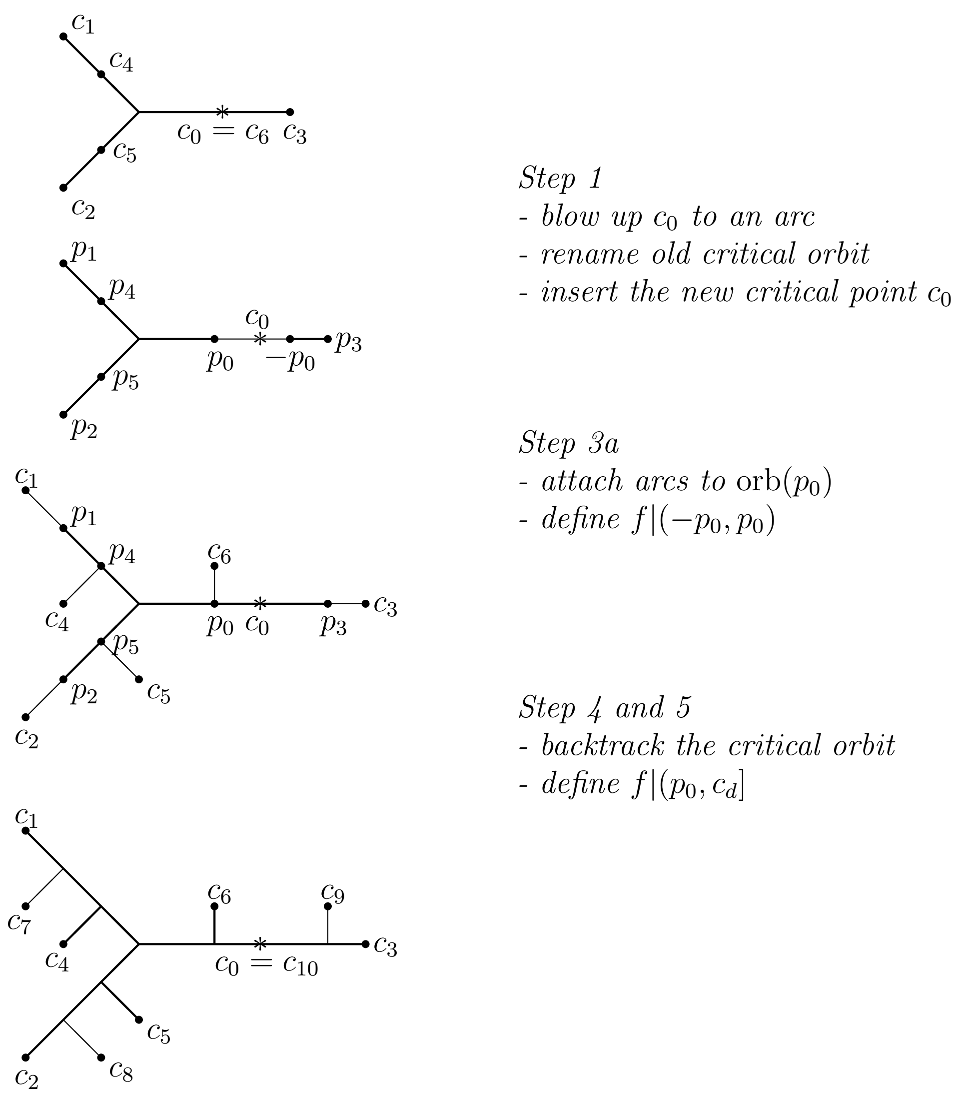


FIGURE 15.1. Construction of $1 \to 3 \to 6 \to 10$ from $1 \to 3 \to 6$.

If the internal address is $1 \to S_1$, then the Hubbard tree is an S_1-star with a fixed center vertex (the α-fixed point) and the S_1 arms are permuted cyclically. The critical point is the endpoint of one of the arms and has period S_1.

Suppose that we have the Hubbard tree (T, f) for internal address $1 \to \ldots \to S_k$. The induction step consists in constructing the Hubbard tree $(\tilde{T}, \tilde{f})$ with internal address $1 \to \ldots \to S_k \to S_{k+1}$. This is done in five steps (illustrated in Figures 15.1 and 15.2).

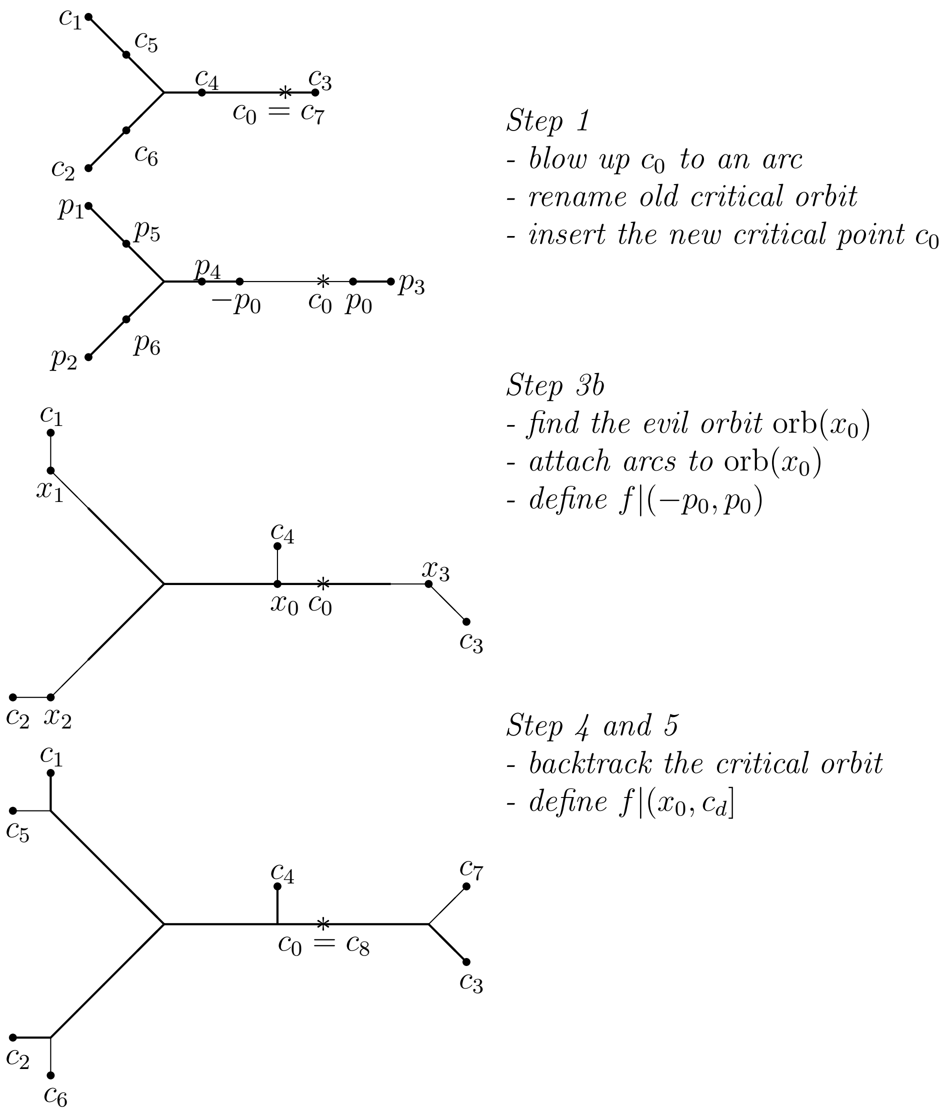


FIGURE 15.2. Construction of $1 \to 3 \to 6 \to 7 \to 8$ from $1 \to 3 \to 6 \to 7$.

Step 1. *Replace the critical point c_0 by a closed arc $[-p_0, p_0]$ where p_0 is adjacent to the ν_{S_k}-side of T. Rename the points $c_1, \dots, c_{S_{k-1}}$ to $p_1, \dots, p_{S_k-1}$. Let $f(p_{S_k-1}) = p_0$, $f(p_0) = f(-p_0) = p_1$ and leave the action on $p_1, \dots, p_{S_k-2}$ as well as on the other marked points (the branch points) unchanged.*

As a result, p_1 has the itinerary $\overline{\nu_1 \dots \nu_{S_k}}$ and the action f is defined on all the branch points and endpoints of $T \cup (p_0, -p_0)$ and their orbits. Extend f homeomorphically to the arcs connecting these points, except for $(-p_0, p_0)$.

Take a new critical point $c_0 = c_{S_{k+1}}$ on $(-p_0, p_0)$.

Step 2. *Write $S_{k+1} = aS_k + r$ $(1 \le r \le S_k)$ and let* [10]

$$q := \begin{cases} a+1 & \text{if } S_k \in \mathrm{orb}_\rho(r), \\ a+2 & \text{if } S_k \notin \mathrm{orb}_\rho(r). \end{cases}$$

Next check if the $\star$-periodic kneading sequence $\overline{\nu_1 \dots \nu_{S_{k+1}-1}\star}$ violates the admissibility condition for period m such that $\rho(m) = S_{k+1}$, (see Proposition 5.12) [11]. *Write $S_{k+1} = a'm + r'$ $(1 \le r' \le m)$. We have three possibilities* [12]*:*

- *If $q > 2$, put $d = (q-2)S_k$ and proceed with Step 3a.*
- *If the integer m exists, put $d = a'm$ and proceed with Step 3b.*
- *If neither of the above, put $d = 0$ and proceed with Step 4.*

Step 3a. *Attach $q-2$ closed arcs to each of the points $p_0, \dots, p_{S_k-1}$. Map $(-p_0, p_0)$ is a 2-to-1 fashion onto (one of) the arc(s) attached to p_1 such that its end point is $c_1 := f(c_0)$. Extend f as follows: Let f map $[p_1, c_1]$ homeomorphically onto one of the new arcs at p_2. (Thus $c_2 = f(c_1)$ is the endpoint of this arc.) Next let f map $[p_2, c_2]$ homeomorphically onto one of the (new) arcs at p_3. (Thus $c_3 = f(c_2)$ is the endpoint of this arc.)*

Continue until the point $c_d = c_{(q-2)S_k}$ is defined, as endpoint of the last remaining arc at p_0. Now f is defined on T with all its attached arcs, except for the arc $(p_0, c_d]$. Continue with Step 4.

Step 3b. *Take points $x_0 = x_m$ and $-x_0$ on $(-p_0, p_0)$ on the ν_m-side resp. opposite side of c_0. We track the orbit x_m backwards such that the itinerary of x_1 will be $\overline{\nu_1 \dots \nu_m}$. Find y_{m-1} on the ν_{m-1}-side of the tree created so far such that $f(y_{m-1})$ is closest to x_m. If $f(y_{m-1}) = x_m$, then take $x_{m-1} = y_{m-1}$. Otherwise attach an arc $[y_{m-1}, x_{m-1}]$ to y_{m-1} and let f map it homeomorphically onto $[f(y_{m-1}), x_m]$.*

Continue this way, i.e., if x_i is defined, then find y_{i-1} on the ν_{i-1}-side of tree created so far such that $f(y_{i-1})$ is closest to x_i. If $f(y_{i-1}) = x_i$, then take $x_{i-1} = y_{i-1}$. Otherwise attach an arc $[y_{i-1}, x_{i-1}]$ to y_{i-1} and let f map it homeomorphically onto $[f(y_{i-1}), x_i]$.

---

[10] This choice is of course motivated by Proposition 5.19

[11] If there is such m we expect the creation of an evil orbit

[12] It follows from Lemma 15.1 that these cases are mutually exclusive.

If however x_i lies beyond p_1 (i.e., $p_1 \in [c_0, x_1]$) and $y_{i-1} = \pm p_0$ (or more generally, y_{i-1} is a boundary point of the arc containing c_0 on which f is not yet defined), then take a point $w \in (y_{i-1}, c_0) \setminus [x_0, -x_0]$. Attach an arc $[x_{i-1}, w]$ to w and let f map $(y_{i-1}, x_{i-1}]$ homeomorphically onto $(f(y_{i-1}), x_i]$.

We stop when y_1 and x_1 are found. By construction (we give the proof in Proposition 15.3 below), $p_1 \in (x_1, c_0)$.

Now attach a' closed arcs to each of the points $x_1, \dots, x_m$. Extend f as follows: Map $(-p_0, p_0)$ in a 2-to-1 fashion onto the union of $(p_1, x_1]$ and one of the new arcs at x_1 (possibly extending $f|(\pm p_0, w]$), in such a way that $f(x_0) = f(-x_0) = x_1$ and the endpoint of the attached arc is $c_1 := f(c_0)$. Next let f map $[x_1, c_1]$ homeomorphically onto one of the (new) arcs at x_2. (Thus $c_2 = f(c_1)$ is the endpoint of this arc.) Next let f map $[x_2, c_2]$ homeomorphically onto one of the (new) arcs at x_3. (Thus $c_3 = f(c_2)$ is the endpoint of this arc.)

Continue until the point $c_d = c_{a'm}$ is defined, as endpoint of the last remaining arc at x_m. Now f is defined on T with all its attached arcs, except for the arc $(x_m, c_d]$. Continue with Step 4.

Step 4. *In this step we create the points c_i for $i = S_{k+1} - 1$ down to $d + 1$. This is done as follows: Find $y_{S_{k+1}-1}$ on the $\nu_{S_{k+1}-1}$-side of the tree created so far such that $f(y_{S_{k+1}-1})$ is closest to $c_0 = c_{S_{k+1}}$. If $f(y_{S_{k+1}-1}) = c_0$, then take $c_{S_{k+1}-1} = y_{S_{k+1}-1}$. Otherwise attach an arc $[y_{S_{k+1}-1}, c_{S_{k+1}-1}]$ to y_{m-1} and let f map it homeomorphically onto $[f(y_{S_{k+1}-1}), c_0]$.*

Continue this way, i.e., if c_i is defined, then find y_{i-1} on the ν_{i-1}-side of tree created so far such that $f(y_{i-1})$ is closest to c_i. If $f(y_{i-1}) = c_i$, then take $c_{i-1} = y_{i-1}$. Otherwise attach an arc $[y_{i-1}, c_{i-1}]$ to y_{i-1} and let f map it homeomorphically onto $[f(y_{i-1}), c_i]$.

However, if c_i lies beyond p_1 (i.e., $p_1 \in [c_0, c_1]$) and y_{i-1} is a boundary point of the arc which f is not yet defined, then take an interior point w on this arc. Attach an arc $[c_{i-1}, w]$ to w and let f map $(y_{i-1}, c_{i-1}]$ homeomorphically onto $(f(y_{i-1}), c_i]$.

Step 4 is finished when y_{d+1} and c_{d+1} are found.

Step 5. *We distinguish three cases according to whether Step 3 is skipped, Step 3a or Step 3b is carried out.*

1. *Step 3. is skipped, $d = 0$. By construction (proof in Proposition 15.3 below), $p_1 \in (c_1, c_0)$. Map $(-p_0, p_0)$ in a 2-to-1 fashion onto $(p_1, c_1]$ such that $f(c_0) = c_1$. (This possibly extends $f|(\pm p_0, w]$.)*
2. *Step 3a is carried out, $d = (q-2)S_k$. By construction (Proposition 15.3), $p_1 \in (c_{d+1}, c_0)$. Map $(p_0, c_d]$ homeomorphically onto $(p_1, c_{d+1}]$.*
3. *Step 3b is carried out, $d = a'm$. By construction (Proposition 15.3), $x_1 \in (c_{d+1}, c_0)$. Map $(x_0, c_d]$ homeomorphically onto $(x_1, c_{d+1}]$.*

This finishes the induction step.

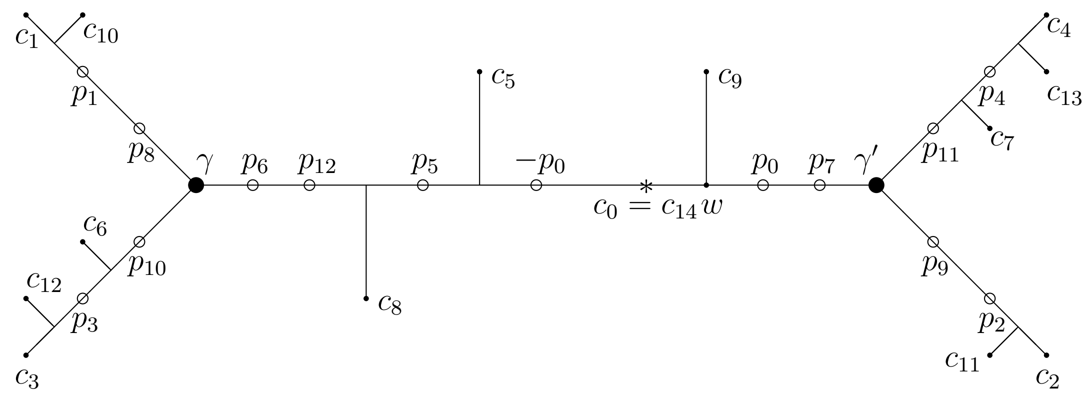


FIGURE 15.3. The Hubbard tree for $1 \to 2 \to 6 \to 7 \to 13 \to 14$ with orb(p_0) drawn in and preperiodic branch point $w \in [p_0, c_9]$; it maps to the 2-periodic branch point γ in six iterates.

REMARK. The introduction of points w in Step 3b and Step 4 corresponds to strictly preperiodic branch points beyond p_1 in $\tilde{T}$. An example of this is realized by the internal address $1 \to 2 \to 6 \to 7 \to 13 \to 14$, see Figure 15.3, where the point $w \in [c_0, p_0]$ is strictly perperiodic to the period 2 branch point.

Also, if p_1 is a branch point with q arms, or x_1 is an evil branch point with $a' + 2$ arms, then the global arm G_{q-1} at p_1, or $G_{a'+1}$ at x_1, labelled as in Lemma 4.6, can contain preperiodic branch points. One example is the tree with internal address $1 \to 2 \to 4 \to 5 \to 11$, see Figure 5.2. Another example is the tree with internal address $1 \to 2 \to 4 \to 5 \to 7 \to 13$, see Figure 15.4. In this case, the characteristic periodic point p_1 of period 6 has three arms, and Step 3a is carried out to find them. The arm $[p_1, c_8]$ contains a preperiodic branch point. An example with an evil orbit is the tree with internal address $1 \to 2 \to 4 \to 6 \to 10 \to 20 \to 40 \to 49$. One can check that this tree contains an evil characteristic point x_1 of period 30 with three arms. The arm G_2 contains a preperiodic branch point with arms leading to x_1, to c_{31} and to c_{47}.

15.3. **Proposition (Construction of Hubbard Tree is Valid)**
Algorithm 15.2 is correct. In particular, if (T, f) is a Hubbard Tree with internal address $1 \to S_1 \to \ldots \to S_k$, then the above induction step yields a valid Hubbard tree with internal address $1 \to S_1 \to \ldots \to S_{k+1}$.

PROOF. By induction, we can assume that the constructed tree (T, f) is a valid Hubbard tree with internal address $1 \to S_1 \to \ldots \to S_k$. Section 3 gives the existence of trees $(\tilde{T}, \tilde{f})$ with internal address $1 \to S_1 \to \ldots \to S_{k+1}$. We need to show that the constructed tree is equivalent to $(\tilde{T}, \tilde{f})$.

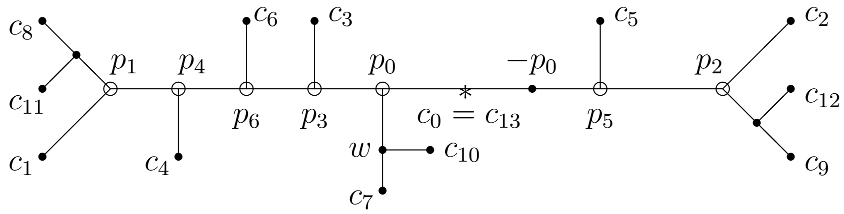


FIGURE 15.4. The Hubbard tree for $1 \to 2 \to 4 \to 5 \to 7 \to 13$ with $\mathrm{orb}(p_0)$ drawn in and preperiodic branch point $w \in [p_0, c_7]$; it maps to p_1 in three iterates.

The forcing relations discussed in Section 6 show that (T, f) appears as a subtree of $(\tilde{T}, \tilde{f})$; more precisely, T is homeomorphic to the connected hull $[p_0, \dots, p_{S_k-1}]$ of the characteristic periodic point with itinerary $\overline{\nu_1 \dots \nu_{S_k}}$. The action f corresponds to the restriction of $\tilde{f}$ to $Z := \{z \in \tilde{T}; \tilde{f}^{\circ i} \in [p_0, \dots, p_{S_k-1}] \text{ for all } i \geq 0\}$. In other words, there is a map $\pi : Z \to T$ preserving itineraries ($\mod \star$) such that $\pi \circ \tilde{f} = f \circ \pi$.

The algorithm supplies the points $\{p_0, \dots, p_{S_k-1}\} = \pi^{-1}(\mathrm{orb}_T(c_0))$ with the right number of arms in $\tilde{T}$ (cf. Proposition 5.19). If the admissibility condition fails, then the algorithm backtracks the evil orbit $\mathrm{orb}(x_1)$ in $\tilde{T}$ and supplies the right number of arms (cf. Proposition 5.13). Also it backtracks the critical orbit in $\tilde{T}$. In particular, the various claims in the algorithm follow from the properties of $(\tilde{T}, \tilde{f})$. To be precise:

Claim of Step 3b: $p_1 \in (x_1, c_0)$. This is contained in Lemma 15.1.

Claim of Step 5(1): $p_1 \in (c_1, c_0)$. This is contained in *e.g.* Proposition 6.8, part (4).

Claim of Step 5(2): $p_1 \in (c_{d+1}, c_0)$. This follows from Lemma 4.6.

Claim of Step 5(3): $x_1 \in (c_{d+1}, c_0)$. This follows from Lemma 4.6 as well.

Since the construction gives a tree which is the connected hull of $\{c_1, \dots, c_{S_{k+1}}\}$, this tree must indeed be $\tilde{T}$. □

V. Measure, Dimension, and Biaccessibility

16. Combinatorial Biaccessibility

In this section and the next, we will discuss *biaccessible* points in Julia sets and in the Mandelbrot set: these are landing points of at least two external rays. We will estimate the Hausdorff dimension of the set of biaccessible points from above and below.

In this section, we work with a combinatorial definition of biaccessibility for itineraries and kneading sequences and estimate their Hausdorff dimension. We also estimate the Hausdorff dimension for the set of corresponding external angles. In Section 17, we will transfer these results to the complex planes containing Julia sets and the Mandelbrot set, using the usual topological definition of biaccessibility.

Definition of Combinatorial Biaccessibility. Let $\Sigma = \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$ be equipped with the metric $d(x,\tilde{x}) = \sum_{i\geq 1}|x_i - \tilde{x}_i|2^{-i}$. The *binary expansion map* $b\colon \Sigma \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is given by $b(x_1x_2\ldots) = \sum x_i 2^{-i}$; it is injective except for the countably many dyadic rationals. We denote the Hausdorff dimension of subsets of Σ and of $\mathbb{S}^1$ by $\dim_H$, defining the Hausdorff dimension of $Y \subset \Sigma$ by $\dim_H(Y) := \dim_H(b(Y))$.

For any kneading sequence $\nu \in \Sigma^\star$ and $x \in \Sigma$, let

$$\rho_{\nu,x}(n) := \min\{k > n \ : \ x_k \neq \nu_{k-n}\} \ . \tag{9}$$

Obviously, $\rho_{\nu,\nu} = \rho$ for ρ as in Definition 2.2.

16.1. **Definition (Biaccessible Itinerary)**
An itinerary $x \in \Sigma$ will be called biaccessible with respect to ν *if there is a $k \geq 2$ such that*

$$\operatorname{orb}_{\rho_{\nu,x}}(1) \cap \operatorname{orb}_{\rho_{\nu,x}}(k) = \emptyset \ . \tag{10}$$

A kneading sequence ν will be called biaccessible *if there is a $k \geq 2$ such that*

$$\operatorname{orb}_{\rho}(1) \cap \operatorname{orb}_{\rho}(k) = \emptyset \ . \tag{11}$$

In particular, periodic and $\star$-periodic kneading sequences are biaccessible.

Remark. The lemma below explains why this rather abstract definition indeed corresponds to the geometric concept of biaccessibility. To prove it, we need an abstract model of the Julia set (see Section 21), rather than just the Hubbard tree, because c_1 is always an end point of the Hubbard tree, but it can have two arms in the Julia set, for example for $\nu = \mathtt{10}\overline{\mathtt{11}}$ where $\operatorname{orb}_\rho(1) \cap \operatorname{orb}_\rho(3) = \emptyset$.

16.2. **Lemma (Correspondence of Biaccessibility in Julia sets)**
Let J be an abstract Julia set with kneading sequence ν. A point $y \in J$ separates J if and only if the itinerary x of y is biaccessible with respect to ν. More precisely, the number of arms (also called the valency, [**Pen1**]*) of y is equal to the number of disjoint $\rho_{\nu,x}$-orbits.*

Proof. We defined *closest precritical points to y* analogous to Definition 5.5 as the points η_k such that $f^{\circ k}(\eta_k) = c_1$, and there is no precritical point of lower Step on the arc $[y, \eta_k]$. Their uniqueness follows as in Lemma 5.6, and the Step of the next precritical point to y on the arc $[y, \eta_k]$ equals $\rho_{\nu,x}(k)$.

Therefore every disjoint $\rho_{\nu,x}$-orbit belongs to a disjoint arc ending in y, and hence y has at least as many arms in J as there are disjoint $\rho_{\nu,x}$-orbits. Conversely, if $G \subset J$ is an arm of y, then it contains a precritical point. The precritical point $\eta \in G$ of lowest Step must be η_k for some $k \geq 1$, and have its $\rho_{\nu,x}$-orbit disjoint from $\rho_{\nu,x}$-orbits stemming from other arms of y. This proves the lemma. □

The following observation follows directly from $\rho_{\nu,\nu} = \rho$ and from Proposition 7.9 on endpoints of the parameter tree.

16.3. **Lemma (Correspondence of Biaccessibility in the Parameter Tree)**
A kneading sequence is biaccessible if and only if it is not an endpoint of the parameter tree. □

Remark. For locally connected Julia sets and for points in the Mandelbrot set near which $\mathcal{M}$ is locally connected (see the discussion in Section 17), geometric biaccessibility is equivalent to the symbolic version of Definition 16.1.

Abstract Lemmas on Hausdorff Dimension. The motor for the dimension estimates will be the following elementary lemma.

16.4. **Lemma (Hausdorff Dimension of Sample Sets)**
Given $v, u \geq 1$ with $v < u$, construct nested compact sets $A_s \subset [0,1]$ (for $s \geq 0$) as follows:

- $A_0 = [0,1]$;
- *Divide each of the $(u-v)^s$ intervals of A_s of length u^{-s} into u equal intervals and remove the closures of v of them, chosen arbitrarily. Then A_{s+1} is the union of the closures of the remaining $(u-v)^{s+1}$ intervals of length $u^{-(s+1)}$ each.*

Let $A = \cap_s A_s$. Then $\dim_H(A) = \dfrac{\log(u-v)}{\log u}$.

Proof. Since A_s consists of $(u-v)^s$ intervals of length u^{-s}, the box dimension of A is $\frac{\log(u-v)}{\log u}$. Therefore $\dim_H(A) \leq \frac{\log(u-v)}{\log u}$.

For the lower bound we use the measure μ_s on A_s which assigns mass $(u-v)^{-s}$ to each of the $(u-v)^s$ intervals of A_s and refine it to a measure μ on A using Kolmogorov's extension theorem, [**Ch**]. For a point $x \in A$, let $I_s(x)$ be the interval of A_s containing x. If $B(x;\varepsilon)$ is the ε-ball around x, then the interval $I_s(x)$ is contained in the ball

$B(x;\varepsilon)$ for $u^{-s} < \varepsilon \leq u^{-s+1}$. Therefore

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{\log \mu(B(x;\varepsilon))}{\log 2\varepsilon} &\geq \liminf_{s\to\infty} \frac{\log \mu(I_s(x))}{\log 2u^{-s+1}} \\
&= \liminf_{s\to\infty} \frac{-s\log(u-v)}{(1-s)\log u + \log 2} \\
&= \frac{\log(u-v)}{\log u}.
\end{aligned}$$

This is true for every $x \in A$. The theory of pointwise dimensions (see *e.g.* [**Pes**, Theorem 3.1]) gives $\dim_H(A) \geq \dim_H(\mu) \geq \frac{\log u - v}{\log u}$. (This result can also be derived from the Frostman lemma.) □

16.5. **Corollary (Hausdorff Dimension of Concatenations of Blocks)**
For distinct blocks $X_1, \dots, X_k$ of 0*s and* 1*s, none of which is a suffix of another, let*

$$B = \Big\{ x = W_1 W_2 \dots \ : \ W_i \in \{X_1, \dots, X_k\} \Big\} \subset \Sigma.$$

Then $\dim_H(B) \geq \dfrac{\log k}{m \log 2}$ *for* $m = \max_i |X_i|$.

Proof. Extend each block X_i to a block $\tilde{X}_i$ of length m. Since no X_i is a suffix of any other X_j, the resulting blocks $\tilde{X}_i$ are distinct. Then Lemma 16.4 immediately gives that $\dim_H(\tilde{B}) = \log k/(m \log 2)$ for $\tilde{B} = \{x = W_1 W_2 \dots \ : \ W_i \in \{\tilde{X}_1, \dots, \tilde{X}_k\}\}$. Indeed, $\tilde{B}$ can be transformed into a subset of $\mathbb{S}^1$ using the binary extension map $b : \Sigma \to \mathbb{S}^1$ that makes the shift on Σ commute with the angle doubling map on $\mathbb{S}^1$.

Now define $h : \Sigma \to \Sigma$ by replacing every 'non-overlapping' occurrence of a block X_i in $x = x_1 x_2 x_3 \dots \in \Sigma$ by the block $\tilde{X}_i$ and leaving the other coordinates x_j untouched. More precisely, we work from left to right: whenever we encounter a block X_j not overlapping with an occurrence of some block X_i replaced previously, then we replace it with $\tilde{X}_i$. Then h maps B bijectively and continuously onto $\tilde{B}$, and the Lipschitz constant of h is at most 1. Therefore $\dim_H(B) \geq \dim_H(\tilde{B}) = \log k/(m \log 2)$, as required. □

16.6. **Lemma (No Increase of Hausdorff Dimension)**
Let p be a polynomial and $K > 0$ and suppose that $I : \mathbb{S}^1 \to \Sigma$ is a map such that the preimage $I^{-1}(C)$ of any n-cylinder consists of at most $p(n)$ intervals of length $\leq K2^{-n}$. Then $\dim_H(I^{-1}(Q)) \leq \dim_H(Q)$ for any set $Q \subset \Sigma$.

Proof. Let $\varepsilon > 0$ be arbitrary and take any $\delta'' > \delta' > \delta = \dim_H(Q)$. Let N be so large that $K^{\delta''} p(n) < 2^{n(\delta''-\delta')}$ for all $n \geq N$. Let $\{U_i\}$ be a cover of Q such that $\mathrm{diam}(U_i) < 2^{-N}$ for each i and $\sum_i \mathrm{diam}(U_i)^{\delta'} < \varepsilon$. (Note that in the standard metric on Σ^1, $\mathrm{diam}(U) = 2^{-|U|}$ where $|U|$ denotes the length of the cylinder.) Without loss of generality we can assume that each U_i is a cylinder set of length $n_i \geq N$. Then $\{I^{-1}(U_i)\}_i$ defines a countable cover $\{V_j\}_j$ of $I^{-1}Q$, each interval V_j has length at most $K2^{-n_i}$, and

$$\begin{aligned}
\sum_i \mathrm{diam}(V_i)^{\delta''} &= \sum_{n\geq N}\sum_{|U_i|=n}\sum_{V_i\subset I^{-1}(U_i)} \mathrm{diam}(V_i)^{\delta''} \leq \sum_{n\geq N}\sum_{|U_i|=n} K^{\delta''}p(n)2^{-n\delta''} \\
&\leq \sum_{n\geq N}\sum_{|U_i|=n} K^{\delta''}p(n)2^{-n(\delta''-\delta')}\mathrm{diam}(U_i)^{\delta'} \\
&\leq \sum_{n\geq N}\sum_{|U_i|=n} \mathrm{diam}(U_i)^{\delta'} < \varepsilon.
\end{aligned}$$

Since this is true for every $\varepsilon > 0$ and $\delta' > \delta$, it follows that $\dim_H(I^{-1}(Q)) \leq \delta$. □

Dimension Estimates for Symbol Spaces. We will produce two pairs of bounds for the Hausdorff dimension of biaccessible itineraries (and kneading sequences). The first pair is very accurate if ν starts as `10000...`, so for parameters near the tip of the left antenna of the Mandelbrot set, where the Hausdorff dimension of biaccessible angles is close to 1, as we will prove eventually in Corollary 17.11. At the tip of the antenna (*i.e.,* $z \mapsto z^2 - 2$) the Julia set is the interval $[-2, 2]$ and all external angles are biaccessible, except for $\vartheta = 0$ and $\vartheta = 1/2$.

The second pair works accurately for kneading sequences that are renormalizable several time, and each next renormalization emerges from a direct bifurcation from the previous hyperbolic component. It can be shown that the biaccessible angles of certain infinitely renormalizable Julia sets (such as the Feigenbaum map) have Hausdorff dimension 0.

(1) For a kneading sequence $\nu = \mathtt{1}\cdots \in \Sigma^1$, but $\nu \neq \mathtt{1}\overline{\mathtt{0}}$, let

$$N := 1 + \min\{i > 1 \ : \ \nu_i = \mathtt{1}\}, \tag{12}$$

so that $N - 1$ is the position of the second `1` in ν. Set

$$L_1(N) := \begin{cases} \frac{\log\left(2^{\lfloor N/2\rfloor -1}-1\right)}{\log 2^{\lfloor N/2\rfloor -1}} & \text{if } N \geq 6, \\ \frac{1}{2} & \text{if } N = 5, \end{cases} \qquad \text{and} \qquad U_1(N) := \frac{\log(2^N - 1)}{\log 2^N}.$$

(2) For a kneading sequence $\nu \in \Sigma^1$ with internal address $1 \to S_1 \to S_2 \to \dots$, let

$$\kappa := \sup\{k \geq 1 : S_j \text{ is a multiple of } S_{j-1} \text{ for all } 1 \leq j \leq k\}. \tag{13}$$

This is well-defined except for certain infinitely renormalizable kneading sequences, and for such we can take κ to be an arbitrarily large integer. Define

$$L_2(\kappa) := \frac{1}{S_{\kappa+1}} \quad \text{and} \quad U_2(\kappa) := \sqrt{\frac{7}{2S_\kappa}} \quad \text{for } S_\kappa \geq 4. \tag{14}$$

We will argue that the Hausdorff dimensions of sets of biaccessible itineraries, kneading sequences and eventually biaccessible external angles fall in the interval

$$\mathcal{I}(N,\kappa) := \Big[\ \max\{L_1(N), L_2(\kappa)\} \ , \ \min\{U_1(N), U_2(\kappa)\} \ \Big] \ . \tag{15}$$

16.7. **Lemma (Dimension of Biaccessible Itineraries and Kneadings)**
(1) For any kneading sequence ν, the Hausdorff dimension of biaccessible itineraries with respect to ν is in $\mathcal{I}(N(\nu),\kappa(\nu))$.
(2) The Hausdorff dimension of biaccessible kneading sequences ν is in $\mathcal{I}(N(\nu),\kappa(\nu))$.

Proof. Let

$$\Sigma_k = \Big\{ x \in \Sigma \ : \ k = \min\{i \ : \ \mathrm{orb}_{\rho_{\nu,x}}(1) \cap \mathrm{orb}_{\rho_{\nu,x}}(i) = \emptyset\} \Big\} .$$

Upper bound $U_1(N)$: We will prove that $\dim_H(\Sigma_k) \le \frac{\log(2^N-1)}{\log 2^N}$ with $N = N(\nu)$ for any ν by showing that every n-cylinder contains at least one $n+N$-cylinder which is disjoint from Σ_k.

Choose an integer $n > k$. We write cylinder sets as $C_{e_1\dots e_n} = \{x \in \Sigma : x_1 \dots x_n = e_1 \dots e_n\}$. By increasing n if necessary, we can assume that $C_{e_1\dots e_n}$ is such that $\rho(i) \le n$ for $i = 1, 2, \dots, N$. Take $a := \max\{i \le n \colon i \in \mathrm{orb}_{\rho_{\nu,x}}(1)\}$ and $b := \max\{i \le n \colon i \in \mathrm{orb}_{\rho_{\nu,x}}(k)\}$. Then clearly $\rho_{\nu,x}(a) > n$ and $\rho_{\nu,x}(b) > n$, and also $a, b \ge N+1$. Suppose that $C_{e_1\dots e_n} \cap \Sigma_k \ne \emptyset$; then $a \ne b$. Let $w_a = \nu_{n-a+1} \dots \nu_{n-a+N}$ and $w_b = \nu_{n-b+1} \dots \nu_{n-b+N}$. Recall that $\rho_{\nu,x}(a)$ finds the first difference between $x_{a+1}x_{a+2}\dots$ and $\nu_1\nu_2\dots$. Then $\rho_{\nu,x}(a) \le n+N$ unless x starts with $e_1 \dots e_n w_a$, and similarly for b.

Our task is the following: given ν, n and $e_1 \dots e_n$, we want to find at least one $n+N$-cylinder in $C_{e_1\dots e_n}$ which is disjoint from Σ_k.

Let $\mathtt{0}\dots\mathtt{0}$ and $\mathtt{01}\dots\mathtt{0}$ be the two words of length N which contain no $\mathtt{1}$, except possibly at the second position. The following three cases are easy to check:

- **Case 1:** $w_a \ne \mathtt{0}\dots\mathtt{0}$ and $w_b \ne \mathtt{0}\dots\mathtt{0}$. We claim that the cylinder $C_{e_1\dots e_n\mathtt{0}\dots\mathtt{0}}$ is disjoint from Σ_k. Indeed, for $x \in C_{e_1\dots e_n\mathtt{0}\dots\mathtt{0}}$, we get $\rho_{\nu,x}(a) \in \{n+1, \dots, n+N\}$, and after that $\mathrm{orb}_{\rho_{\nu,x}}(a)$ increases in steps of 1 until it reaches $n+N$, hence $n+N \in \mathrm{orb}_{\rho_{\nu,x}}(a) \subset \mathrm{orb}_{\rho_{\nu,x}}(1)$. Similarly, $n+N \in \mathrm{orb}_{\rho_{\nu,x}}(k)$, which proves the claim.
- **Case 2:** $w_a = \mathtt{0}\dots\mathtt{0}$ and $w_b = \mathtt{1}\dots$. This time, we claim that $C_{e_1\dots e_n\mathtt{01}\dots\mathtt{0}}$ is disjoint from Σ_k: for $x \in C_{e_1\dots e_n\mathtt{01}\dots\mathtt{0}}$, we have $\rho_{\nu,x}(a) = n+2$, and after that, $\mathrm{orb}_{\rho_{\nu,x}}(a)$ increases in steps of 1 up to $n+N$, so again $n+N \in \mathrm{orb}_{\rho_{\nu,x}}(a)$. This time, $\rho_{\nu,x}(b) = n+1$ and $\rho_{\nu,x}(\rho_{\nu,x}(b)) = n+N$, so $n+N \in \mathrm{orb}_{\rho_{\nu,x}}(1) \cap \mathrm{orb}_{\rho_{\nu,x}}(k)$.
- **Case 3:** $w_a = \mathtt{0}\dots\mathtt{0}$ and $w_b = \mathtt{0}\dots$. Now the entire $N+1$-cylinder $C_{e_1\dots e_n\mathtt{1}}$ is disjoint from Σ_k: for $x \in C_{e_1\dots e_n\mathtt{1}}$, we have $\rho(a) = n+1 = \rho(b)$.

By interchanging the roles of w_a and w_b one can see that these three cases cover all possibilities. Hence each n-cylinder contains at least one $n+N$-cylinder that is disjoint from Σ_k. By Lemma 16.4, Σ_k is contained in a Cantor set of Hausdorff dimension

$\frac{\log(2^N-1)}{\log 2^N}$. The upper bound in the lemma follows by taking the union over all Σ_k.

Upper bound $U_2(\kappa)$: By the definition of κ, we can write $S_j = p_j S_{j-1}$ for $1 \le j \le \kappa$, and

$$\nu_1 \dots \nu_{S_j} = (\nu_1 \dots \nu_{S_{j-1}})^{p_j-1}(\nu_1 \dots \nu'_{S_{j-1}}),$$

where $\nu'_i = \mathtt{1}$ if $\nu_i = \mathtt{0}$ and vice versa. Every $n < S_\kappa$ can be written uniquely as

$$n = \sum_{j=0}^{\kappa-1} a_j S_j, \qquad 0 \le a_j < p_j.$$

If $\rho(n) < S_{\kappa+1}$, then

$$\rho(n) = n + aS_h \text{ for } h \ge \min\{j : a_j \ne 0\} \text{ and some } 0 < a \le p_h. \tag{16}$$

Now if $x \in \Sigma_k$, then we can enumerate the entries of these orbits as $1 = u_0 < u_1 < \dots$ and $k = v_0 < v_1 < \dots$. If $u_s < v_t < u_{s+1}$ and $v_t - u_s < S_\kappa$, then we can define $h = h(s,t)$ as in (16) and conclude that $\rho(v_t) \ge u_s + S_{\kappa+1}$, or $\rho_{\nu,x}(v_t) = v_t + a_t S_h$ for some $a_t \ge 1$. This means in the former case, that $x_{u_s+1} \dots x_{u_s+S_{\kappa+1}-1} = \nu_1 \dots \nu_{S_{\kappa+1}-1}$, and reduces the number of possible biaccessible $u_s + S_{\kappa+1} - 1$-subcylinders of the cylinder starting with $x_1 \dots x_{u_s}$ to only one. In the latter case, we have two cases:

- $v_{t+1} > u_{s+1}$, but then $\rho_{\nu,x}(u_s) - u_s = \min\{S_j : S_j + u_s > v_t\}$. In this case $h(t, s+1) = h(s,t)$, and we say that the *roles* of u and v *switch*.
- $v_{t+1} < u_{s+1}$, and then $\rho_{\nu,x}(v_t) - u_s = \min\{S_j : S_j + u_s > v_t\}$. In this case $h(s, t+1) > h(s,t)$.

The analogous reasoning applies when $v_s < u_t < v_{s+1}$. This means that the function $h(s,t)$ is non-decreasing, until it reaches the value κ. It can stay constant for a while, but there is only one way of doing this, namely by switching the role of u and v at every step. Again, this reduces the number of possible subcylinders of the cylinder starting with $x_1 \dots x_{u_s}$ to only one.

To illustrate this, let us give an example:

$$\nu = \mathtt{101011101010100}\dots$$

with $\kappa = 3$ and internal address $1 \to 2 \to 6 \to 12 \to 15 \to \dots$, and

$$x = \underbrace{1}_{u_0=1}\ \overbrace{1}^{v_0,h=0}\ \overbrace{0}^{v_1,h=1}\ 1\ 0\ 1\ \overbrace{1}^{v_2,h=2}\ 1\ 0\ 1\ 0\ 1\ \underbrace{1}_{u_1,h=2}\ 1\ 0\ 1\ 0\ 1\ \overbrace{1}^{v_3,h=2}\ \dots$$

We see that $h(u,v)$ stays constant if the roles of u and v switch, and increases otherwise. Furthermore, two consecutive switches of roles takes $S_{h(u,v)+1}$ digits.

We can code the consecutive switches and non-switches by integers $l_j \ge 0$: each l_j indicates the the number of switches where $h(u,v)$ remains constant at j. If $l_j = 0$, it means that $h(u,v)$ increases from below j to above j. There are $r \ge 1$ occurrences of $h(u,v) \geq \kappa$ before $h(u,v)$ drops below κ again. Let $m_j \geq S_\kappa$, $1 \leq j \leq r$, denote the distances between the remaining switches before $h(u,v)$ drops below κ again. (If $r = 0$, then there are no such m_js.) Thus the whole loop from $h(u,v) = 0$ to the last $h(u,v) \geq \kappa$ takes at least $\sum_{j=0}^{\kappa} l_j S_j + \sum_{j=1}^{r} m_j$ digits. Let us introduce a second index i to indicate the loopnumber. Then the pair $(l_{i,j})_{i=1,j=0}^{n,\ \kappa}, (m_{i,j})_{i=1,j=1}^{n,\ r_i}$ encodes a cylinder set in Σ_k going through n loops, and the cylinder length is at least $k + \sum_{i=1,j=0}^{n,\kappa} l_{i,j} S_j + \sum_{i=1,j=1}^{n,r_i} l_{i,j} m_{i,j}$.

Let $\delta = U_2(\kappa)$. The cylinders encoded by $(l_{i,j})_{i=1,j=0}^{n,\ \kappa}, (m_{i,j})_{i=1,j=1}^{n,\ r_i}$ form a cover of Σ_k with diameter $< 2^{-(k+nS_\kappa)}$. Its δ-dimensional Hausdorff measure is bounded by

$$\sum 2^{-\delta\ \sum_{i=1,j=0}^{n,\kappa} l_{i,j} S_j} \cdot \sum 2^{-\delta\ \sum_{i=1,r_i=1,j=1}^{n,\infty,r_i} m_{i,j}},$$

where the first main sum runs over all combinations of $n(\kappa+1)$ positive integers $l_{i,j}$ and the second main sum over all combinations of integers $m_{i,j} \geq S_\kappa$. Using geometric series, the estimate $\sum_{l=0}^{\infty} 2^{-l\alpha} \leq 1 + \int_0^\infty 2^{-x\alpha} dx = 1 + \frac{1}{\alpha \log 2}$, and changing the order of product and sum, we can rewrite this quantity as

$$\prod_{i=1}^{n} \left(\prod_{j=0}^{\kappa} \sum_{l_{i,j}=0}^{\infty} 2^{-\delta l_{i,j} S_j} \cdot \sum_{r_i=1}^{\infty} \prod_{j=1}^{r_i} \sum_{m_{i,j}=S_\kappa}^{\infty} 2^{-\delta m_{i,j}} \right)$$
$$\leq \prod_{i=1}^{n} \left(\prod_{j=0}^{\kappa} \left(1 + \frac{1}{\delta S_j \log 2} \right) \cdot \sum_{r_i=1}^{\infty} \left(\frac{2^{-\delta S_\kappa}}{1 - 2^{-\delta}} \right)^{r_i} \right). \tag{17}$$

Next observe that $S_j \geq 2^j$ for $0 \leq j \leq \kappa$, and hence $\prod_{j=0}^{\kappa}(1 + \frac{1}{\delta S_j \log 2}) \leq 2^{2/\delta}$. The second factor is another geometric series, and can be computed as

$$\sum_{r_i=1}^{\infty} \left(\frac{2^{-\delta S_\kappa}}{1 - 2^{-\delta}} \right)^{r_i} = 2^{-\delta S_\kappa - \log(1 - 2^{-\delta} - 2^{-\delta S_\kappa})/\log 2}.$$

For $S_\kappa \geq 4$ we can estimate $-\log(1 - 2^{-\delta} - 2^{-\delta S_\kappa})/\log 2 \leq 3/(2\delta) - \varepsilon$ for some $\varepsilon > 0$. Therefore expression (17) is bounded by $2^{n(7/(2\delta) - \delta S_\kappa - \varepsilon)}$, and since $\delta = \sqrt{7/2S_\kappa}$, the exponent $n(7/(2\delta) - S_\kappa \delta - \varepsilon) = -\varepsilon n < 0$. Therefore the limit as $n \to \infty$ is zero, and $\dim_H(\Sigma) \leq U_2(\kappa)$.

Remark. The estimate $U_2(\kappa)$ is not sharp. For example, if $S_\kappa = 4$, we can take and $\delta = 0.91 < 0.935 \dots = \sqrt{7/8}$, and still find that $2/\delta + \delta S_\kappa - \log(1 - 2^{-\delta} - 2^{-\delta S_\kappa})/\log 2 \approx -0.0748$ is negative. If $S_\kappa = 3$, then $\nu = \texttt{110}\dots$ and $U_1(N) = U_1(3) = \log 7/\log 8 \approx 0.936$ gives a better upper bound. On the other hand, given any $\varepsilon > 0$, we can take $\delta = \sqrt{(2+\varepsilon)/S_\kappa}$ as upper bound provided S_κ is sufficiently large.

Lower bound $L_1(N)$: First assume that $N = N(\nu) \geq 6$. Observe that the head of

$\nu = \mathtt{100\dots01}\dots$ contains $N-3$ zeroes in a row. Take $M = \lfloor N/2 \rfloor - 1$, and let

$$B = \{x = W_1 W_2 \dots \in \Sigma \ : \ W_i \text{ is an } M\text{-block } \neq \mathtt{00\dots0}\}.$$

In particular, no $x \in B$ contains $N-3$ consecutive symbols $\mathtt{0}$. It follows that $\rho_{\nu,x}(i) - i < N$ for all $i \geq 1$, and that every $x \in B$ allows at least two disjoint $\rho_{\nu,x}$-orbits. The Hausdorff dimension of B is $\log(2^{\lfloor N/2 \rfloor -1} - 1)/\log 2^{\lfloor N/2 \rfloor -1}$ according to Corollary 16.5.

Now let us treat the case $N = 5$, so $\nu = \mathtt{1001}\dots$. In this case, we take

$$B = \{x = W_1 W_2 \dots \in \Sigma \ : \ W_i = \mathtt{11} \text{ of } \mathtt{10} \text{ for } i \geq 2\}.$$

The every $x \in B$ allows two disjoint $\rho_{\nu,x}$-orbits. The Hausdorff dimension of B is $\log(2^n)/\log 4^n = \frac{1}{2}$ according to Corollary 16.5. This proves lower bound $L_1(N)$.

Remark. The same idea gives lower bounds for other heads of kneading sequences:

$$\nu = \mathtt{10110}\cdots: \quad \text{Taking } W_i = \mathtt{1111} \text{ or } \mathtt{1010} \text{ gives } \dim_H(B) = \frac{1}{4}.$$
$$\nu = \mathtt{10100}\cdots: \quad \text{Taking } W_i = \mathtt{11111} \text{ or } \mathtt{11010} \text{ gives } \dim_H(B) = \frac{1}{5}.$$
$$\nu = \mathtt{101111}\cdots: \quad \text{Taking } W_i = \mathtt{101110} \text{ or } \mathtt{111010} \text{ gives } \dim_H(B) = \frac{1}{6}.$$

Incidentally, for the latter two examples, these bounds equal the respective bounds $L_2(\kappa)$ below. The bound $\frac{1}{4}$ for $\nu = \mathtt{10110}\dots$ is better than $L_2(\kappa) = \frac{1}{5}$.

Lower bound $L_2(\kappa)$: Write $\nu = \nu_1\nu_2\dots$ and define $V = \nu_1\nu_2\dots\nu_{S_\kappa - 1}\nu'_{S_\kappa}$ and $\hat{V} = \nu_1\nu_2\dots\nu_{S_{\kappa+1}-1}\nu'_{S_{\kappa+1}}$, where $\nu'_i = \mathtt{1}$ if $\nu_i = \mathtt{0}$ and vice versa. Let

$$B = \{x = W_1 W_2 \dots \in \Sigma \ : \ W_i = V \text{ or } \hat{V}\}.$$

Corollary 16.5 gives that $\dim_H(B) \geq \frac{1}{S_{\kappa+1}}$ as claimed, so it suffices to show that each $x \in B$ allows two disjoint $\rho_{\nu,x}$-orbits. By construction of $x \in B$, $\rho_{\nu,x}^{\circ i}(S_\kappa) = |W_1 W_2 \dots W_i|$ for all $i \geq 0$. We will show that $\rho_{\nu,x}$-orbit of $S_{\kappa-1}$ is disjoint from this. Note that V is the concatenation of $S_\kappa/S_{\kappa-1}$ blocks $\nu_1\nu_2\dots\nu_{S_{k-1}}$. Therefore, for any integer a, $1 \leq a < S_\kappa/S_{\kappa-1}$,

$$\rho_{\nu,VV}(aS_{\kappa-1}) = \rho_{\nu,V\hat{V}}(aS_{\kappa-1}) = S_\kappa + aS_{\kappa-1}$$

where we extended the definition of $\rho_{\nu,x}$ to the case where x is a finite block. Also

$$\rho_{\nu,\hat{V}V}(aS_{\kappa-1}) = \rho_{\nu,\hat{V}\hat{V}}(aS_{\kappa-1}) = S_\kappa.$$

Let $n = S_{\kappa+1} - S_\kappa$, so we can write $\hat{V} = VW$ for $W = \nu_1\dots\nu_n$. Furthermore $W = V^i X$ for some $i \geq 0$ and $m := |X| < S_\kappa$. We can use (16) to compute $\rho_{XV,XV}(m) = aS_h$ for some $h \leq \kappa - 1$ and $1 \leq a \leq S_{h+1}/S_h$. If $h = \kappa - 1$, then X is the concatenation of at most $p_\kappa - 1$ blocks $\nu_1\dots\nu_{S_{\kappa-1}}$ and in this case we readily find

$$\rho_{\nu,\hat{V}V}(S_\kappa) = \rho_{\nu,\hat{V}\hat{V}}(S_\kappa) = S_\kappa + aS_{\kappa-1}.$$

If $h \le \kappa - 2$, then

$$\rho_{\nu,\hat{V}V}(S_\kappa) = \rho_{\nu,\hat{V}\hat{V}}(S_\kappa) = S_\kappa + \rho(n) = S_{\kappa+1} + (\rho(n) - n) = S_{\kappa+1} + (\rho(m) - m).$$

Since $\rho(m) \le S_{\kappa-1}$, Lemma 20.1 gives that $S_{\kappa-1} \in \mathrm{orb}_\rho(\rho(m) - m)$, and therefore $S_{\kappa+1} + S_{\kappa-1}$ belongs to the $\rho_{\nu,\hat{V}V}$-orbit (and to the $\rho_{\nu,\hat{V}V}$-orbit) of S_κ.

Combining these facts, we derive that the $\rho_{\nu,x}$-orbit of $S_{\kappa-1}$ contains $|W_1 \dots W_i| + a_i S_{\kappa-1}$ for each i and some $1 \le a_i < p_\kappa$, and hence is disjoint from the $\rho_{\nu,x}$-orbit of S_κ.

A remark to make: The proof for both lower bounds involves the construction of sets of itineraries x such that $\{\sigma^{\circ n}(x) : n \in \mathbb{N}\}$ is bounded away from ν, and hence these sequences belong to points in a hyperbolic set X in the Julia set. It can be expected that (similar to repelling periodic points) all dynamical rays with impressions intersecting X actually land at points in X. DIERK, DO YOU THINK THIS IS INDEED PROVED SOMEWHERE??? Therefore the lower bounds are valid irrespective of whether the Julia set is locally connected or not.

Estimates for kneading sequences: Now for the second statement, we repeat the proof with

$$\Sigma_{N,\kappa,k} = \left\{ \nu \in \Sigma \ : \ \begin{array}{l} N(\nu) = N, \ \kappa(\nu) = \kappa, \\ k = \min\{i \ : \ \mathrm{orb}_\rho(1) \cap \mathrm{orb}_\rho(i) = \emptyset\} \end{array} \right\}. \tag{18}$$

For $\nu \in \Sigma_{N,\kappa,k}$, instead of comparing subwords of ν with a fixed itinerary, we compare subwords of ν with ν itself. In the above arguments, only a comparison with $\nu_1 \dots \nu_M$ is of importance for $M = \max\{N, S_{\kappa+1}\}$.

If $M = N$, then the sequences $\nu \in \Sigma_{N,\kappa,k}$ are contained in two N-cylinders defined by $\mathtt{10\dots010}$ and $\mathtt{10\dots011}$. We can extend ν arbitrarily with blocks as in the proof of $L_1(N)$ without violating Admissibility Condition 5.1. Thus for each cylinder the estimate by $L_1(N)$ and $U_1(N)$ goes through without further changes.

If $M = S_{\kappa+1}$ and then the initial block $\nu_1 \dots \nu_{S_{k+1}}$ satisfies the admissibility condition. We can extend ν arbitrarily with blocks $\nu_1 \dots \nu'_{S_\kappa}$ and $\nu_1 \dots \nu'_{S_{\kappa+1}}$ without compromising admissibility. Therefore the lower bound $L_2(\kappa)$ follows. The proof for the upper bound $U_2(\kappa)$ goes through as well. □

Dimension Estimates for External Angles. Next we investigate how these dimension estimates transfer to subsets of the circle with the angle doubling map $g(x) = 2x \bmod 1$ acting on it.

16.8. **Lemma (Itinerary Map does not Increase Hausdorff Dimension)**
For every external angle $\vartheta \in \mathbb{S}^1$, the map $I_\vartheta \colon \mathbb{S}^1 \to \Sigma$ from Definition 2.1 assigning to external angles their itineraries with respect to ϑ has the property that $\dim_H(I_\vartheta^{-1}(Q)) \le \dim_H(Q)$ for any set $Q \subset \Sigma$.

PROOF. The arcs $A_0 = [\frac{\vartheta}{2}, \frac{\vartheta+1}{2})$ and $A_1 = [\frac{\vartheta+1}{2}, \frac{\vartheta}{2})$ form the partition of $\mathbb{S}^1$ which yields itineraries, except that A_0 and A_1 are half-open, while they are supposed to be open in Definition 2.1. Only countably many angles $\varphi \in \mathbb{S}^1$ are affected by the differing definition, so this has no effect on the Hausdorff dimension. The two sets A_0 and A_1 correspond to the two 1-cylinders C_0 and C_1 of Σ.

Let $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the angle doubling map. We claim that for every n-cylinder C (with $n \geq 1$), $I_\vartheta^{-1}(C)$ consists of at most n half-open intervals of total length 2^{-n}, and $g^{\circ(n-1)}\colon I_\vartheta^{-1}(C)$ is a bijection onto either A_0 or A_1. Indeed, for $n = 1$ this is clear. For the inductive step observe that for every half-open interval A, $\vartheta \notin A \subset \mathbb{S}^1$, containing its lower boundary point, $g^{-1}(A) \cap A_i$ is exactly one half-open interval with half the length. If $\vartheta \in A$, then $g^{-1}(A) \cap A_i$ is the union of two half-open intervals which together have half the length of A. Now the result follows from Lemma 16.6. □

REMARK. Note that the above proof shows that for any ϑ, I_ϑ transforms Lebesgue measure into the $(\frac{1}{2}, \frac{1}{2})$-product measure μ on Σ.

16.9. **Theorem (Hausdorff Dimension of Biaccessible Angles)**
For every external angle $\vartheta \in \mathbb{S}^1 \setminus \{1/2\}$, the set of external angles $\varphi \in \mathbb{S}^1$ with biaccessible itineraries has Hausdorff dimension in $\mathcal{I}(N, \kappa)$, where $N = N(\nu(\vartheta))$ and $\kappa = \kappa(\nu(\vartheta))$ are as in (12) *and* (13) *respectively. In particular, the set of external angles with biaccessible itineraries has Hausdorff dimension less than* 1. □

PROOF. The upper bound of the dimension follows immediately from Lemmas 16.7 and 16.8.

For the lower bound, observe that if C is any cylinder set of length n, then by the proof of Lemma 16.8, $I_\vartheta^{-1}(C)$ consists of at most n intervals of total length 2^{-n}. At least one of these intervals must have length $2^{-n}/n$.

Let $B \subset \Sigma$ be the subset of biaccessible itineraries indicated in the proof of Lemma 16.7 and let δ be its Hausdorff dimension. Take $0 < \delta'' < \delta' < \delta$ and $K > 0$ arbitrary. Then there exists n so large that

- $(\frac{1}{n} 2^{-n})^{\delta''} > 2^{-\delta' n}$;
- $\sum_i (2^{-|C_i|})^{\delta'} > 2K$, where $\{C_i\}$ is a cover of B with cylinder sets each of length n. Since the upper box dimension of B equals its Hausdorff dimension (see Lemma 16.4), it is no restriction to give all cylinders in this cover the same length.

For each C_i, let A_i be an interval in $I_\vartheta^{-1}(C_i)$ of length $\geq 2^{-n}/n$. Let $\{V_j\}_j$ be any open cover of $I_\vartheta^{-1}(B)$ with intervals of length $< 2^{-n}/(2n)$. For each i, let $\mathcal{V}_i = \{V_j : V_j \subset A_i\}$, so $\sum_{V_j\in\mathcal{V}_i}|V_j|^{\delta''} > \frac{1}{2}|A_i|^{\delta''}$. Therefore

$$\begin{aligned}\sum_j |V_j|^{\delta''} &\geq \sum_i \sum_{V_j\in\mathcal{V}_i} |V_j|^{\delta''} \geq \frac{1}{2}\sum_i |A_i|^{\delta''} \\ &\geq \frac{1}{2}\sum_i \Big(\frac{1}{|C_i|}2^{-|C_i|}\Big)^{\delta''} \geq \frac{1}{2}\sum_i 2^{-\delta'|C_i|} > K.\end{aligned}$$

Since K and $\delta'' < \delta' < \delta$ are arbitrary, we get $\dim_H(I_\vartheta^{-1}(B)) \geq \delta$. This proves the lemma. □

The results given so far are on a purely combinatorial level. In Section 17 we translate them into statements about topological biaccessibility of points in Julia sets; see especially Corollary 17.8: for every $c \in \mathcal{M}\setminus\{-2\}$, the set of biaccessible points has harmonic measure zero and the corresponding angles have Hausdorff dimension less than 1.

We now turn to the discussion of (combinatorial) parameter space.

16.10. **Lemma (Hausdorff Dim. of Admissible Biaccessible Kneadings)**
The set of admissible biaccessible kneading sequences ν with $N(\nu) = N$ and $\kappa(\nu) = \kappa$ has Hausdorff dimension in $\mathcal{I}(N, \kappa)$.

Proof. Lemma 16.7 estimates the Hausdorff dimension of the set of biaccessible kneading sequences ν with $N(\nu) = N$, admissible or not. This gives the upper bound in the claim.

For the lower bound, take any n-cylinder C intersecting $\Sigma_{N,\kappa,k}$ (as in (18)) with $n > \rho(k) > \max\{N, S_{\kappa+1}\}$. Assume also that no $\nu \in C$ gives rise to an evil m-periodic point for any $m \leq k$.

We claim that all kneading sequences of $C \cap \Sigma_{N,\kappa,k}$ are admissible. Indeed, if this is false and $C \cap \Sigma_{N,\kappa,k}$ contains a kneading sequence ν' which fails the admissibility condition for some period m, then we may replace ν' by a $\star$-periodic kneading sequence ν which differs from ν after position $\rho(m)$, and ν fails the admissibility condition as well. Let T be the Hubbard tree for ν and let z be the characteristic periodic point of the evil orbit of period m. Clearly, $m > k$. By the admissibility condition and Lemma 20.1, $\rho(m) \in \mathrm{orb}_\rho(1)$, hence $\rho(m) \notin \mathrm{orb}_\rho(k)$. (Indeed, if $\rho(m) = qm + r$ for $r \in \{1, \dots, m\}$ then the admissibility condition implies that $m \in \mathrm{orb}_\rho(r) \setminus \mathrm{orb}_\rho(1)$. Then Lemma 20.1 implies that $\mathrm{orb}_\rho(r)$ and $\mathrm{orb}_\rho(1)$ intersect at the latest at $\rho(m)$.) By Lemma 5.9, there is a closest precritical point $\zeta_{\rho(m)} \in [z, c_1]$.

If there is a closest precritical point $\zeta_k \in T$, then $\zeta_{\rho(m)} \notin [\zeta_k, c_1]$ by Lemma 5.7 because $\rho(m) \notin \mathrm{orb}_\rho(k)$, hence ζ_k is in the same global arm of z as c_1; call this arm G. By Lemma 4.6, $f^{\circ i}(G) \not\ni 0$ for $0 \leq i < m$, which is a contradiction to the fact that $k < m$. If there is no ζ_k in T, one can extend the Hubbard tree as in Theorem 20.12 as to contain ζ_k. Then ζ_k must be in an extension of G, but Lemma 4.6 shows that for the local arm in G, its first m forward images are local arms pointing away from the critical point, and this is again a contradiction. Therefore, all of $C \cap \Sigma_{N,\kappa,k}$ is admissible. This shows that the Hausdorff dimension of the admissible kneading sequences is at least $\dim_H(\Sigma_{N,\kappa,k} \cap C) \in \mathcal{I}(N,\kappa)$. □

Let $I : \mathbb{S}^1 \to \Sigma$ be the map that assigns the kneading sequence to an external parameter angle. In order to investigate how Hausdorff dimension behaves under I^{-1}, we need the following result.

16.11. **Lemma (Upper Bound for Number of Embeddings)**
A Hubbard tree in which the critical orbit is periodic with period n has less than n embeddings into the plane which respect the circular order of the local arms at every branch point.

Proof. Let $1 \to S_1 \to \ldots \to S_k$ be the internal address of the tree with $S_k = n$. We may suppose that all branch points are tame (or there would be no embedding at all, see Proposition 4.10). By Proposition 5.19, the periods of all branch points appear on the internal address. Let $p_0, \ldots, p_{k-1}$ be the tame characteristic periodic points of periods $S_0, \ldots, S_{k-1}$ (compare Proposition 6.8). Let their numbers of arms be $q_0, \ldots, q_{k-1}$; according to Proposition 5.19 they satisfy

$$S_{i+1} = \begin{cases} (q_i - 1)S_i + r_i & \text{if } S_i \in \operatorname{orb}_\rho(r_i) \\ (q_i - 2)S_i + r_i & \text{if } S_i \notin \operatorname{orb}_\rho(r_i) \end{cases}$$

where the r_i are uniquely defined by the condition $1 \le r_i \le S_i$.

Since only branch points contribute to the number of embeddings, let us write $i(0), i(1), \ldots, i(l)$ for the indices of p_i that are branch points. Obviously $k > i(l)$.

By Corollary 4.11, there are precisely $a := \prod_{s=0}^{l} \varphi(q_{i(s)})$ dynamically compatible embeddings of the Hubbard tree into the plane. Here $\varphi(q)$ is the Euler function counting the integers $1 \le i < q$ that are coprime to q; it gives the number of transitive order preserving permutations on q points on the circle. Clearly $a \le (q_{i(0)} - 1) \prod_{s=1}^{l} (q_{i(s)} - 1)$. We will show that $a < S_k$.

The arc $[p_{i(t)}, c_1]$ contains the closest precritical point $\zeta_{S_{i(t)+1}}$, and $f^{\circ(q_{i(t)}-2)S_{i(t)}}$ maps it to a precritical point ζ_t of $\textsc{Step}(\zeta_t) = S_{i(t)+1} - (q_{i(t)} - 2)S_{i(t)}$. Lemma 4.6 implies that the arm G_1 of $p_{i(t)}$ containing c_1 homeomorphically survives $f^{\circ(q_{i(t)}-2)S_{i(t)}}$ and ζ_t lies in a different arm of $p_{i(t)}$ as the critical point. However, ζ_t and $\zeta_{S_{i(t-1)+1}}$ lie in the same global arm of $p_{i(t-1)}$, which homeomorphically survives another $(q_{i(t-1)} - 2)S_{i(t-1)}$ iterates. Inductively repeating this argument gives

$$S_{i(t)+1} > (q_{i(t)} - 2)S_{i(t)} + (q_{i(t-1)} - 2)S_{i(t-1)} + \cdots + (q_{i(0)} - 2)S_{i(0)}.$$

Choose $u_1 = S_{i(1)}$, $u_0 = u_1/(q_{i(0)} - 2)$ and

$$u_{t+1} := (q_{i(t)} - 2)u_t + (q_{i(t-1)} - 2)u_{t-1} + \cdots + (q_{i(0)} - 2)u_0\ .$$

Then by induction $u_{t+1} = (q_{i(t)} - 1)u_t$, and therefore

$$u_{t+1} = (q_{i(t)} - 1)u_t = u_1 \prod_{s=1}^{t} (q_{i(s)} - 1).$$

Hence $S_k \geq S_{i(l)+1} > u_{l+1} = S_{i(1)} \prod_{s=1}^{l} (q_{i(s)} - 1)$. It is easily checked that $S_{i(1)} \geq q_{i(0)} - 1$. Therefore $S_k > a$ as asserted. □

16.12. **Lemma (Kneading Map does not Increase Hausdorff Dimension)** *Let $I : \mathbb{S}^1 \to \Sigma$ be the map assigning the kneading sequence to an external angle in parameter space. Then $\dim_H(I^{-1}(Q)) \leq \dim_H(Q)$ for any set $Q \subset \Sigma$.*

Proof. We claim that for every n-cylinder $C \subset \Sigma$, $I^{-1}(C)$ consists of at most $\frac{1}{2}n(n+1)$ arcs of length $\leq \frac{1}{2^n - 1}$. Indeed, if α is such that the n-th entry $I_n(\alpha) = \star$, say that $2^{n-1}\alpha = \frac{m+\alpha}{2}$ for some $m \geq 1$, then for $\alpha' = \alpha + \frac{1}{2^n-1}$ we have $2^{n-1}\alpha' = \frac{m+1+\alpha'}{2}$. Therefore every component of $I^{-1}(C)$ must be contained in an arc $(\alpha, \alpha + \frac{1}{2^n-1})$ for some $\alpha \in \mathbb{S}^1$. This shows that I^{-1} is Lipschitz on each branch.

Let T be a Hubbard tree with a periodic critical point; say the period is $m = S_k$. The external angles of T depend on the specific embedding of T in the plane; by Lemma 16.11 there are at most m different embeddings. Each embedding of T comes with two external angles, see Algorithm 13.6. (Note that only trees with strictly preperiodic critical points can have more than two external angles.) So there are at most $2m$ external angles realizing the kneading sequence $\overline{\nu_1 \dots \nu_{m-1}\star}$. Each arc in $I^{-1}(\nu_1 \dots \nu_n)$ has two boundary points having kneading sequences $\overline{\nu_1 \dots \nu_{m-1}\star}$ for some $m \leq n$. Therefore the total number of arcs is bounded by $\sum_{m=1}^{n} m = \frac{1}{2}n(n+1)$. This proves the claim. Now use Lemma 16.6 to finish the proof. □

16.13. **Theorem (Hausdorff Dimension of Biaccessible Parameter Angles)** *The set of external angles α for which $I(\alpha)$ is a biaccessible kneading sequence with $N(I(\alpha)) = N$ and $\kappa(I(\alpha)) = \kappa$ has Hausdorff dimension in $\mathcal{I}(N, \kappa)$.*

In particular, this implies that the harmonic measure of the biaccessible points in the boundary of the Mandelbrot set is 0: in Corollary 17.11, we transfer our dimension estimates to the set of external angles of biaccessible parameters in the Mandelbrot set; this gives a much stronger result than just zero harmonic measure.

Proof. The upper bound follows immediately from Lemmas 16.7 and 16.12. For the lower bound, take $k > M := \max\{N, S_{\kappa+1}$ and a cylinder set $C = C_{e_1 \dots e_n}$ intersecting $\Sigma_{N,\kappa,k}$. Let us pick C so that $n \in \mathrm{orb}_\rho(i)$ for each $i \leq M$, and that no $\nu \in C$ gives rise to an evil period m with $\rho(m) \leq n$.

Using Lemma 16.7, we are able to find a subset $B \subset C$ of Hausdorff dimension $\delta = \max\{L_1(N), L_2(\kappa)\}$. Moreover, for all $\nu \in B$, $r_i := \rho(i) - i \leq M$ for all $i \geq 1$.

Therefore $n \in \mathrm{orb}_\rho(r_i)$, so it follows (see Propositions 5.13 and 5.19) that every $\nu \in C$ corresponds to an admissible Hubbard tree T, whose periodic branch points have period $\leq M$. By Corollary 4.11, T has a bounded number of embeddings, hence the map $I : I^{-1}(B) \to B$ is bounded-to-one.

A second property of $B \subset C$ is that if $\tilde{C} = C_{e_1 \dots e_m}$ is any subcylinder intersecting B, then all four subcylinders $C_{e_1 \dots e_m e_{m+1} e_{m+2}}$ satisfy Admissibility Condition 5.1. Therefore the single arc $A := I^{-1}(\tilde{C})$ is divided into four pieces by points of the form $\frac{k}{2^{m+1}-1}$ or $\frac{k}{2^{m+2}-1}$ (where k is an integer), and $|A| > \frac{1}{2^{n+2}-1}$. It follows that the map I restricted to $I^{-1}(B)$ is Lipschitz on each of its branches. Therefore, the set of biaccessible external angles contains a Cantor set $I^{-1}(B)$ of Hausdorff dimension δ. □

We call a kneading sequence *n-renormalizable* if the internal address associated to it contains the entry n, and every further entry is divisible by n. Such internal addresses correspond to n-renormalizable dynamics; compare Section 10. We call an external angle n-renormalizable if its kneading sequence is n-renormalizable. A kneading sequence or an external angle is *infinitely renormalizable* if it is n-renormalizable for infinitely many values of n. The following lemma is well known, see [**Ma**].

16.14. **Lemma (Hausdorff Dimension of Renormalizable Angles)**
For any $n \geq 2$, the Hausdorff dimension of the n-renormalizable kneading sequences and of n-renormalizable external angles is at most $1/n$. The Hausdorff dimension of infinitely renormalizable kneading sequences and external angles is 0.

Proof. If a kneading sequence is n-renormalizable, then the associated internal address contains after entry n only entries which are divisible by n. If the kneading sequence is divided in blocks of length n, then any block can differ from the first one only at the last position. Since ν_1 is always `1` and the n-th entry must be such that n occurs in the internal address, there are at most $2^{n-2} \cdot 2^{k-1}$ possibilities for the first kn entries of n-renormalizable kneading sequences. Therefore the set of n-renormalizable kneading sequences has Hausdorff dimension at most $(\log 2)/\log(2^n) = 1/n$. Infinitely renormalizable kneading sequences are n-renormalizable for arbitrarily large n, and their Hausdorff dimension is 0.

The statement about external angles follows from Lemma 16.12. □

17. Topological Biaccessibility

In this section, we will transfer the combinatorial results about (combinatorial) biaccessibility from Section 16 to actual results about the topology of Mandelbrot and Julia sets, providing estimates about the Hausdorff dimension of external angles of rays landing at biaccessible points. This involves discussions about whether certain external rays land and if so, if they land together whenever the combinatorics predicts that. The only possible obstacle is lack of local connectivity.

We will distinguish "combinatorial biaccessibility" of an itinerary or kneading sequence (in the sense of Definition 16.1) from "topological biaccessibility" of a point in the sense below.

17.1. **Definition (Accessible and Biaccessible)**
Let $K \subset \mathbb{C}$ be compact, connected and full (i.e. $\mathbb{C} \setminus K$ is connected). A point $z \in K$ is accessible *if there is a curve $\gamma : [0,1] \to \overline{\mathbb{C}}$ such that $\gamma(0) = z$, $\gamma(1) = \infty$ and $\gamma(t) \notin K$ for $t > 0$. The point z is* (topologically) biaccessible *if there are two such curves which are not homotopic within $\overline{\mathbb{C}} \setminus K$.*

Remark. A point $z \in \partial K$ is accessible if and only if an external ray lands at z, and it is biaccessible if and only if at least two external rays land at z. This is Lindelöf's Theorem; see Ahlfors [**Ah1**, Theorem 3.5].

The *impression* of an external ray $R(\vartheta)$ is the set

$$I(\vartheta) := K \cap \bigcap_{\varepsilon>0} \overline{\bigcup_{\vartheta' \in [\vartheta-\varepsilon, \vartheta+\varepsilon]} R(\vartheta')} \ . \tag{19}$$

Equivalently, the impression of the ray $R(\vartheta)$ is the set of all possible subsequential limits $\lim \Phi^{-1}(r_n e^{2\pi i \vartheta_n})$ with $r_n \searrow 1$ and $\vartheta_n \to \vartheta$.

If the impression of a ray consists of a single point, then this implies that the ray lands. If a ray lands, then its impression contains the landing point (but may be larger). Compare the discussion in [**Pom, Pet, Sch1, Sch3**].

17.2. **Proposition (Non-Biaccessibility in Julia Sets)**
Suppose that for a quadratic polynomial p_c, the dynamic ray $R_c(\vartheta)$ lands at the critical value. Let ν be the kneading sequence of ϑ. For an angle $\varphi \in \mathbb{S}^1$ not on the backward orbit of ϑ, let x be the itinerary of φ with respect to ϑ. If x is not combinatorially biaccessible with respect to ν, then for every $\varphi' \in \mathbb{S}^1$ with $\varphi' \neq \varphi$ the impressions $I(\varphi)$ and $I(\varphi')$ are disjoint.

17.3. **Lemma (Dynamic Rays Accumulates at Critical Value)**
Every parameter $c \in \mathcal{M} \setminus \{-2\}$ without irrationally indifferent periodic orbit has an angle $\vartheta \in \mathbb{S}^1 \setminus \{1/2\}$ which satisfies at least one of the following:

- *the dynamic ray $R_c(\vartheta)$ lands at the critical value;*
- *the critical value is in the impression of the dynamic ray $R_c(\vartheta)$, and there is a sequence of dynamic ray pairs $P_c(\vartheta_n, \vartheta'_n)$ with $\vartheta_n \to \vartheta$, $\vartheta'_n \to \vartheta$;*
- *the critical value is in the impression of the dynamic ray $R_c(\vartheta)$, and there are an angle ϑ' and two sequences of dynamic ray pairs $P_c(\vartheta_n, \vartheta'_n)$, $P_c(\tilde{\vartheta}_n, \tilde{\vartheta}'_n)$ with $\vartheta_n < \vartheta < \tilde{\vartheta}_n < \tilde{\vartheta}'_n < \vartheta' < \vartheta'_n$ for all n, so that $\vartheta_n, \tilde{\vartheta}_n \to \vartheta$ and $\vartheta'_n, \tilde{\vartheta}'_n \to \vartheta'$ (in other words, both sequences of approximating ray pairs converge to a possible ray pair $P_c(\vartheta, \vartheta')$, even though we do not claim that this latter ray pair exists);*
- *there is an attracting or parabolic periodic orbit and $R_c(\vartheta)$ lands at the boundary of the Fatou component containing the critical value.*

PROOF. If p_c has an attracting or parabolic orbit, then infinitely many dynamic rays land at the Fatou component containing the critical value, and the result is clear. Since the case of irrationally indifferent orbits is excluded, we may assume that all periodic orbits are repelling and there are no bounded Fatou components. In particular, the critical value is in the Julia set.

Let the internal address of c be $1 \to S_1 \to S_2 \ldots$. Then there is a sequence of parameter rays $P(\vartheta_k, \vartheta'_k)$ with period S_k separating c from $P(\vartheta_{k-1}, \vartheta'_{k-1})$. If we suppose $\vartheta_k < \vartheta'_k$), then there are limiting angles α, α' with $\vartheta_k \nearrow \alpha$ and $\vartheta'_k \searrow \alpha'$. By the Correspondence Theorem 9.5, there are dynamic ray pairs $P_c(\vartheta_k, \vartheta'_k)$ separating the critical value from P

To prove that $\vartheta \neq 1/2$ unless $c = -2$, suppose first that both fixed points of p_c are repelling. One of them, called β, is the landing point of $R_c(0)$; the other one is called α and is the landing point of at $q \geq 2$ periodic dynamic rays; the characteristic ray pair separates the critical value from β. Then q preperiodic rays land at $-\alpha$, and continued pull-backs of these rays along the branch of p_c^{-1} fixing β shows that the impression of $R_c(0)$ is the point β alone. Similarly, the impression of $R_c(1/2)$ is the point $-\beta$ alone. Therefore, the only case when the critical value is in the impression $R_c(1/2)$ is when $c = -\beta$, so $p_c(c)$ is the β fixed point, and this happens only for $c = -2$. In the case that p_c has an attracting or parabolic fixed point, the Fatou component containing the critical value is the landing point of infinitely many dynamic rays. The last case is that p_c has an irrationally indifferent fixed point, and in this case we have nothing to prove. $\square$

17.4. **Lemma (No Wandering Triangles in Julia Sets)**
Suppose a polynomial p_c has three sequences of ray pairs $P_c(\alpha_n, \alpha'_n)$, $P_c(\beta_n, \beta'_n)$, $P_c(\gamma_n, \gamma'_n)$ so that each $P_c(\alpha_n, \alpha'_n)$ separates $P_c(\alpha_{n-1}, \alpha'_{n-1})$ from $P_c(\alpha_{n+1}, \alpha'_{n+1})$ and from all $P_c(\beta_k, \beta'_k)$, $P_c(\gamma_k, \gamma'_k)$, and similarly for the other two sequences. ????

17.5. **Proposition (Non-Biaccessibility in Julia Sets)**
Consider a quadratic polynomial p_c without irrationally indifferent periodic orbits so that $c \in \mathcal{M}$. Let ϑ be an angle as in Lemma 17.3. Let ν be the kneading sequence associated to ϑ (as in Definition 2.1). For an angle $\varphi \in \mathbb{S}^1$ not on the backward orbit of ϑ, let x be the itinerary of φ with respect to ϑ. If x is not combinatorially biaccessible with respect to ν, then for every $\varphi' \in \mathbb{S}^1$ with $\varphi' \neq \varphi$ the impressions $I(\varphi)$ and $I(\varphi')$ are disjoint.

Proof. Suppose first that the dynamic ray $R_c(\vartheta)$ lands at the critical value. Then $p_c^{-1}(R_c(\vartheta))$ consists of two rays which land at the critical point 0. Let P_1 be the ray pair formed by these two rays and their landing point; it cuts $\mathbb{C}$ into two parts. Then $p_c^{-1}(P_1)$ consists of two ray pairs, $p_c^{-1}(p_c^{-1}(P_1))$ consists of four ray pairs etc., and all these ray pairs are disjoint. Let $\mathcal{P}_n$ be the union of all the 2^{n-1} ray pairs consisting of rays which map to $R_c(\vartheta)$ after exactly n steps. For a ray pair $P \subset \mathcal{P}_n$, we set $\textsc{Step}(P) := n$.

Two external angles φ, φ' have the same initial N entries in their itineraries with respect to ν if and only if the rays $R_c(\varphi)$ and $R_c(\varphi')$ are in the same connected component of $\mathbb{C} \setminus \bigcup_{n=1}^{N} \mathcal{P}_n$; if they are not, then the position of the first difference in their itineraries equals the lowest n such that a ray pair in $\mathcal{P}_n$ separates $R_c(\varphi)$ and $R_c(\varphi')$.

We construct a sequence of ray pairs P_k as follows: as above P_1 is the ray pair landing at the critical point, and P_{k+1} is the unique ray pair which separates P_k from $R_c(\varphi)$ such that $\textsc{Step}(P_{k+1})$ is least possible. We claim that the sequence $\textsc{Step}(P_k)$ equals $\operatorname{orb}_{\rho_{\nu,x}}(1)$. Indeed, both sequences start with $\textsc{Step}(P_1) = 1$. For the inductive step, note that $\textsc{Step}(P_{k+1})$ records the least number of iterations when P_k and $R_c(\varphi)$ are mapped to different sides of P_1; by construction, this happens after as many iterations as the itineraries of $R_c(\varphi)$ (which is x) and P_k coincide, and this is exactly $\rho_{\nu,x}(\textsc{Step}(P_k))$.

For $k \geq 1$, let $w_k \in (0, 1/2]$ be the difference of external angles of the two rays in P_k; then clearly $w_k > w_{k+1} > 0$ for all k, so $w_k \searrow w_\infty$ for some $w_\infty \geq 0$.

If $w_\infty = 0$, then every ray $R_c(\varphi')$ with $\varphi' \neq 0$ is separated from $R_c(\varphi)$ by some ray pair P_k, and the impressions of $R_c(\varphi)$ and $R_c(\varphi')$ can intersect at most at the landing point of P_k; however, this landing point is separated from the impression of $R_c(\varphi)$ by P_{k+1}, so $R_c(\varphi)$ and $R_c(\varphi')$ have disjoint impressions and the claim follows.

Therefore, all we need to show is that $w_\infty > 0$ implies that x is combinatorially biaccessible with respect to ν. Indeed, if $w_\infty > 0$, then there is a lowest integer $s > 0$ for which there is a ray pair $P_1' \subset \mathcal{P}_s$ that is separated from 0 by all P_k. As above, construct a sequence (P_k') of ray pairs such that P_{k+1}' is the unique ray pair which separates P_k' from $R_c(\varphi)$ such that $\textsc{Step}(P_{k+1}')$ is least possible. All P_k' are on the same side of all P_l as $R_c(\varphi)$. If $s := \textsc{Step}(P_1')$, then as above the sequence $\textsc{Step}(P_k')$ equals $\operatorname{orb}_{\rho_{\nu,x}}(s)$, and indeed $\operatorname{orb}_{\rho_{\nu,x}}(s) \cap \operatorname{orb}_{\rho_{\nu,x}}(1) = \emptyset$, so x is combinatorially biaccessible as claimed. This finishes the proof if $R_c(\vartheta)$ lands at c.

Now suppose the critical value is in the accumulation set of $R_c(\vartheta)$ and there is a sequence of periodic dynamic ray pairs $P_c(\vartheta_n, \vartheta'_n)$ so that $\vartheta_n \to \vartheta$ and $\vartheta'_n \to \vartheta$. Then all $P_c(\vartheta_n, \vartheta'_n)$ separate the critical value from the origin and there are two sequences of periodic or preperiodic ray pairs $P_c(\vartheta_n/2, (\vartheta'_n + 1)/2)$ and $P_c((\vartheta_n + 1)/2, \vartheta'_n/2)$ whose angles converge to $\vartheta/2$ and $(\vartheta+1)/2$. If any two angles φ, φ' not on the backwards orbit of ϑ have different initial entries in their itineraries with respect to ν, then they are in different components of $\mathbb{S}^1 \setminus \{\vartheta/2, (\vartheta+1)/2\}$, and then $R_c(\varphi)$ and $R_c(\varphi')$ are separated by a dynamic ray pair $P_c(\vartheta_n/2, (\vartheta'_n + 1)/2)$ (as well as by $P_c((\vartheta_n + 1)/2, \vartheta'_n/2)$), for sufficiently large n. It follows again that $R_c(\varphi)$ and $R_c(\varphi')$ have disjoint impressions. If the itineraries of φ and φ' differ in the n-th position, then an inductive argument shows that $R_c(\varphi)$ and $R_c(\varphi')$ are separated by a ray pair on the backwards orbit of $P_c(\vartheta_n/2, (\vartheta'_n+1)/2)$ (and of $P_c((\vartheta_n+1)/2, \vartheta'_n/2)$) for some n, and the same conclusion holds.

The next case we consider is that the critical value is in the accumulation set of $R_c(\vartheta)$ and there are two sequences of periodic dynamic ray pairs $P_c(\vartheta_n, \vartheta'_n)$ and $P_c(\tilde\vartheta_n, \tilde\vartheta'_n)$ with $\vartheta_n < \vartheta < \tilde\vartheta_n < \tilde\vartheta'_n < \vartheta' < \vartheta'_n$ for all n so that $\vartheta_n, \tilde\vartheta_n \to \vartheta$ and $\vartheta'_n, \tilde\vartheta'_n \to \vartheta$. Then there are four sequences of periodic or preperiodic ray pairs $P_c(\vartheta_n/2, (\vartheta'_n + 1)/2)$, $P_c((\vartheta_n+1)/2, \vartheta'_n/2)$, $P_c(\tilde\vartheta_n/2, (\tilde\vartheta'_n+1)/2)$ and $P_c((\tilde\vartheta_n+1)/2, \tilde\vartheta'_n/2)$ whose angles converge to $\vartheta/2$ and $(\vartheta + 1)/2$. As before, we claim that if φ and φ' have different itineraries with respect to ϑ, then $R_c(\varphi)$ and $R_c(\varphi')$ have disjoint impressions. If the itineraries of two angles φ, φ' differ in the first position, then the rays $R_c(\varphi)$ and $R_c(\varphi')$ are separated by infinitely many ray pairs from our sequences, and the impressions are again disjoint. If the itineraries differ at a later position, then the inductive argument proceeds as above.

If the dynamic ray $R_c(\vartheta)$ lands at the boundary of the Fatou component containing the critical value, then the two dynamic rays $R_c(\vartheta/2)$ and $R_c((\vartheta + 1)/2$ land at the Fatou component containing the critical point, and the landing points of both rays can be connected within this Fatou component by a curve; the reasoning is similar as in the first case.

This finishes the proof of the proposition. □

We now prove that biaccessible points of all quadratic Julia sets J_c with $c \neq -2$ have harmonic measure zero. In fact, we prove the stronger statement that the associated external angles have Hausdorff dimension less than 1.

For the statement of the following theorem, recall from Section 11 that every $c \in \mathcal{M}$ has an associated internal address, which is a (finite or infinite) strictly increasing sequence of integers starting with 1. The parameter $c = -2$ is a Misiurewicz-Thurston parameter (the leftmost parameter of $\mathcal{M} \cap \mathbb{R}$) with the unique internal address $1 \to 2 \to 3 \to 4 \dots$ that contains all positive integers, and every $c \neq -2$ has a different internal address.

17.6. **Theorem (Biaccessibility in Julia Sets and Hausdorff Dimension)** *For every $c \in \mathcal{M} \setminus \{-2\}$, there is a set $A_c \subset \mathbb{S}^1$ of Hausdorff dimension less than 1 so that for all $\varphi \in \mathbb{S}^1 \setminus A_c$, the impression of $R_c(\varphi)$ is disjoint from any other impression $R_c(\varphi')$ with $\varphi' \neq \varphi$. In particular, biaccessible points must have external angles in A_c.*

More precisely, let N be such that $N-1$ is the first positive integer that does not occur in the internal address of $c \in \mathcal{M} \setminus \{-2\}$. Then the external angles of rays landing at biaccessible points of J_c have Hausdorff dimension in $\mathcal{I}(N)$, where $\mathcal{I}(N)$ is defined as in Equation (15).

PROOF. Suppose that p_c does not have an irrationally indifferent periodic orbit. By Proposition 17.5, there is an angle $\vartheta \in \mathbb{S}^1 \setminus \{1/2\}$ so that every angle $\varphi \in \mathbb{S}^1$ not on the backward orbit of ϑ has the property that the impression $I(\varphi)$ is disjoint from $I(\varphi')$ for all $\varphi' \neq \varphi$ provided that the itinerary of φ with respect to ϑ is not combinatorially biaccessible. If there is a ray pair $P_c(\varphi, \varphi')$, then either φ must be one of the countably many angles on the backwards orbit of ϑ, or φ is combinatorially biaccessible. By Theorem 16.9, the set of angles φ which are combinatorially biaccessible has Hausdorff dimension in $\mathcal{I}(N)$. This proves the upper bound on the Hausdorff dimension if there is no irrationally indifferent orbit.

If p_c has an irrationally indifferent orbit, then there is a hyperbolic component W with $c \in \partial\mathcal{M}$. Let c_0 be the center of W. Then the claim is true (with the same value of N) for p_{c_0}. If there is a ray pair $P_c(\varphi, \varphi;')$ for p_c but not for c_0, then the landing point of $P_c(\varphi, \varphi')$ cannot be repelling (or the ray pair would persist in a parameter space neighborhood of c by Lemma 9.4, hence it would have to exist throughout W), so the landing point must be on the irrationally indifferent orbit. This orbit must be in the Julia sets, so it must be a Cremer point. It follows from a result of Bullett and Sentenac [**BuS**] see Theorem 26.2 that the Hausdorff dimension of rays landing at a Cremer point (if any) is zero.

For the lower bound, suppose first that the critical value of p_c is the landing point of a dynamic ray $R_c(\vartheta)$ with $\vartheta \neq 1/2$. Then by Theorem 16.9 there is a set of angles A_c of Hausdorff dimension in $\mathcal{I}(N)$ so that all angles in A_c are combinatorially biaccessible. We will show that enough of these angles have the property that the corresponding rays are part of ray pairs. We continue notation and ideas from the proof of Proposition 17.5. If an angle $\varphi \in \mathbb{S}^1$ not on the backwards orbit of ϑ is biaccessible, then there are two sequences of ray pairs P_k and P'_k with angles on the backwards orbit of ϑ and with the following properties: every P_k separates P_{k-1} from all P'_l and from $R_c(\varphi)$, and every P'_l separates P'_{l-1} from all P_k and from $R_c(\varphi)$. It follows that the angles in P_k and in P'_l converge: if $P_k = P_c(\alpha_k, \alpha'_k)$ and $P'_l = (\beta_l, \beta'_l)$, then $\alpha_k \to \alpha$, $\alpha'_k \to \alpha'$ and $\alpha < \beta_l < \beta'_l < \alpha'$, hence $\alpha' \neq \alpha$. Similarly $\beta_l \to \beta$, $\beta'_l \to \beta' \neq \beta$.

We claim that $\alpha = \beta$ and $\alpha' = \beta'$.

again that p_c does not have an irrationally indifferent periodic orbit and let $\vartheta \in \mathbb{S}^1 \setminus \{1/2\}$ be an angle as in Proposition 17.5.

There is a converse at least for locally connected Julia sets.

17.7. **Corollary (Biaccessibility in Locally Connected Julia Sets)**
Suppose the Julia set of a quadratic polynomial p_c is locally connected and the dynamic ray $R_c(\vartheta)$ lands at the critical value. Let ν be the kneading sequence of ϑ, let $\varphi \in \mathbb{S}^1$ be an angle not on the backwards orbit of ϑ, and let x be the itinerary of φ with respect to ϑ. Then x is combinatorially biaccessible with respect to ν if and only if the landing point of $R_c(\varphi)$ is topologically biaccessible.

Sketch. If the critical value is in the Julia set, then local connectivity implies that there is a dynamic ray that lands at the critical value. If not, the critical point is within a bounded Fatou component, and this component must be part of an attracting or parabolic cycle and there are dynamic rays landing on the boundary of every Fatou component.

If the landing point of $R_c(\varphi)$ is biaccessible, then biaccessibility of x is the content of Proposition 17.2. For the converse, the reasoning is similar as for Lemma 17.10 below and is thus omitted. Write more about this? □

For polynomials f_c, one of several equivalent ways of defining *harmonic measure* (Brolin measure) ω_c of a subset $B \subset J_c$ is by putting [**Pom**]

$$\omega_c(B) = \mathrm{Leb}\{\vartheta \in \mathbb{S}^1 \colon R_c(\vartheta) \text{ lands at a point in } B\} .$$

Here Leb is Lebesgue measure on the circle. It follows from results of Fatou [**Hm**] or Riesz & Riesz [**RR**] (compare also [**Mi1**]) that the rays at Lebesgue-a.e. angles land.

17.8. **Corollary (Biaccessibility and Dimension for Julia Sets)**
For every $c \in \mathcal{M}\setminus\{-2\}$ so that the Julia set J_c is locally connected and the critical value is the landing point of a dynamic ray, the biaccessible points in J_c have zero harmonic measure and the associated external angles have Hausdorff dimension less than 1.

More precisely, the internal address of c within $\mathcal{M}$ has an associated kneading sequence $\nu \neq 1\overline{0}$, and there is an $N \geq 3$ so that the second 1 in ν occurs at position $N-1$. Then the external angles of rays landing at biaccessible points have Hausdorff dimension in $\mathcal{I}(N)$, where $\mathcal{I}(N)$ is defined as in Equation (15).

Proof. The internal address of c is defined in Algorithm 11.3 using periodic parameter ray pairs. If this address is $1 \to 2 \to 3 \to 4 \to \dots$, then no periodic parameter ray pair separates c from the landing point c' of the parameter ray $R(1/2)$ (this is the leftmost "antenna tip" of $\mathcal{M}$ at $c' = -2$). But c' is a Misiurewicz-Thurston parameter, so by Theorem 9.21 we must have $c' = c$.

In all other cases, the kneading sequence associated to the internal address of c has at least two entries 1, and N is well-defined. The dynamic ray at angle $1/2$ always lands at the preimage of a fixed point, and for $c \neq -2$ this is not the critical value. Therefore, for $c \neq -2$, the critical value is the landing point of a ray other than $R_c(1/2)$.

By Theorem 16.9, the set of external angles with biaccessible itineraries has Hausdorff dimension in $\mathcal{I}(N)$. By Corollary 17.7, an external angle φ has biaccessible itinerary if and only if the landing point of $R_c(\varphi)$ is topologically biaccessible, and this proves the claim. □

Related results have been proven by Smirnov, by Zakeri and by Zdunik [**Sm1, Za3, Zd**]. Zakeri shows that the set of biaccessible points has zero Brolin measure for polynomial Julia sets of arbitrary degrees provided they are locally connected. Smirnov and Zdunik independently prove the same statement with the assumption of "locally conneced" weakened to "connected". It is plausible that the Hausdorff dimension of external angles with biaccessible landing points is always positive for all parameters in $\mathcal{M}$ in the wake of a primitive component of period greater than 1 (for parameters in hyperbolic components that arise by a finite chain of bifurcations from the main cardioid, there are only countably many ray pairs altogether).

What about loc conn Siegel disks in the previous corollary? We are also losing some cases in the step from 15.2 to 15.4. And we should include a proof of 15.3.

Now we turn to parameter space and relate combinatorial biaccessibility to topological biaccessibility for boundary points of the Mandelbrot set. As for Julia sets, we will use ray pairs, i.e., pairs of external rays which land at the same point. For Julia sets, such ray pairs were easy to obtain by pulling back a single ray pair which lands at the critical point. For the Mandelbrot set, we use the Douady-Hubbard theory of parameter rays at periodic external angles: all such parameter rays for any given period land in pairs (Theorem 9.1). If the kneading sequences of two angles ϑ and ϑ' (not necessarily periodic) first differ at their n-th entries and neither ϑ nor ϑ' have period n, then the parameter rays $R(\vartheta)$ and $R(\vartheta')$ are separated by a parameter ray pair of period n (Lemma 11.2). In particular, if $R(\vartheta)$ and $R(\vartheta')$ land at a common point, then ϑ and ϑ' have identical kneading sequences.

17.9. **Proposition (Non-Biaccessibility in the Mandelbrot Set)**
Suppose an angle $\vartheta \in \mathbb{S}^1$ is such that its associated kneading sequence ν is an endpoint of the parameter tree (in the sense of Definition 7.8). Then for every $\vartheta' \in \mathbb{S}^1$ with $\vartheta' \neq \vartheta$, the impressions of the parameter rays $R(\vartheta)$ and $R(\vartheta')$ are disjoint; in particular, these two parameter rays do not land at the same point. It follows that no point in the impression of $R(\vartheta)$ is biaccessible.

Proof. Only non-periodic kneading sequences can be endpoints of the parameter tree, so the internal address of ν is infinite, say $1 \to S_1 \to \ldots \to S_k \to \ldots$. For $s \geq 1$, let again ν^s be the $\star$-periodic kneading sequences of period s which coincide with ν for $s-1$ entries. By Corollary 7.7, $\nu^s < \nu$ if and only if s occurs on the internal address of ν. Note that ν is admissible, hence all $\nu^s < \nu$ are admissible too (Lemma 7.3).

In Algorithm 11.3, the internal address of $R(\vartheta)$ within the Mandelbrot set is recursively defined as the sequence of periods S_k such that there is a parameter ray pair

$P(\vartheta_k, \vartheta'_k)$ of period S_k which separates $R(\vartheta)$ from $P(\vartheta_{k-1}, \vartheta'_{k-1})$ so that S_k is least possible; the recursion starts with the ray pair $P(\vartheta_0, \vartheta'_0)$ with $\vartheta_0 = 0$, $\vartheta'_0 = 1$ and $S_0 = 1$.

By Proposition 11.6, the internal address of $R(\vartheta)$ thus defined coincides with the internal address of the kneading sequence associated to ϑ. By construction, $\{\vartheta_k, \vartheta'_k\}$ separates 0 from $\{\vartheta_{k+1}, \vartheta'_{k+1}\}$ (within $\mathbb{S}^1$), hence $\vartheta_k < \vartheta_{k+1} < \vartheta < \vartheta_{k+1} < \vartheta'_k$ for all k. It follows that the two sequences (ϑ_k) and (ϑ'_k) converge monotonically to two limits $\vartheta_\infty, \vartheta'_\infty$ with $\vartheta_\infty \le \vartheta \le \vartheta'_\infty$.

If $\vartheta_\infty = \vartheta = \vartheta'_\infty$, then we are through: given $\vartheta' \ne \vartheta$, then there is an index k such that $\vartheta' \notin (\vartheta_k, \vartheta'_k)$, so the impressions of $R(\vartheta)$ and $R(\vartheta')$ can intersect at most at the landing point of $P(\vartheta_k, \vartheta'_k)$, but this landing point is separated from $R(\vartheta)$ by the ray pair $P(\vartheta_{k+1}, \vartheta'_{k+1})$. Therefore, the impressions of $R(\vartheta)$ and $R(\vartheta')$ are disjoint.

It remains to show that $\vartheta_\infty = \vartheta'_\infty$ whenever ν is an endpoint of the parameter tree (that hypothesis has not been used yet). For every k, the kneading sequences associated to ϑ_k and ϑ'_k are both equal to ν^{S_k} (Lemma 11.2).

If $\vartheta_\infty < \vartheta'_\infty$, then there is a ray pair $P(\varphi, \varphi')$ at periodic angles $\varphi < \varphi'$ with $\{\varphi, \varphi'\} \subset (\vartheta_\infty, \vartheta'_\infty) \subset (\vartheta_k, \vartheta'_k)$. Let ν_φ be the common kneading sequence of φ and φ'. By Proposition 11.6, $\nu_\varphi > \nu^{S_k}$ for all k, hence $\nu_\varphi > \nu$ by Definition 7.1. But then ν violates Definition 7.8 of endpoints of the parameter tree. □

We have a converse if the ray $R(\vartheta)$ lands and the topology of the Mandelbrot set is sufficiently well-behaved near the landing point. The concept of fibers from Section 9.4 helps to give a precise condition.

17.10. **Lemma (Biaccessibility and Fiber in the Mandelbrot set)**
For a parameter ray $R(\vartheta)$, suppose that the fiber containing its impression consists of a single parameter c and suppose that c is not on the boundary of a hyperbolic component. Then the kneading sequence of ϑ is combinatorially biaccessible if and only if c is the landing point of at least two parameter rays (which is equivalent to biaccessibility of c).

PROOF. Let ν be the kneading sequence of ϑ. If c is the landing point of two parameter rays, then c is clearly topologically biaccessible. If c is topologically biaccessible, then by Proposition 17.9, ν cannot be an endpoint of the parameter tree, so it must be combinatorially biaccessible.

We only need to prove that if ν is combinatorially biaccessible, then c is the landing point of at least two parameter rays. Let $1 \to \ldots \to S_k \to \ldots$ be the internal address of c; it is infinite by Corollary 11.8. Let $P(\vartheta_k, \vartheta'_k)$ be the corresponding sequence of parameter ray pairs of period S_k from Algorithm 11.3. Then for all k, we have $\vartheta_k < \vartheta_{k+1} < \vartheta'_{k+1} < \vartheta'_k$ as usual, so there are two monotone limits $\vartheta_k \nearrow \vartheta_\infty$ and $\vartheta'_k \searrow \vartheta'_\infty$. We claim that $\vartheta'_\infty \ne \vartheta_\infty$.

For every k, let ν^{S_k} be the kneading sequence of ϑ_k and ϑ'_k. Then every ν^{S_k} coincides with ν for $S_k - 1$ entries. Since ν is not an endpoint of the parameter tree, there is an admissible $\star$-periodic kneading sequence $\nu_* > \nu$. By Definition 7.1, this means $\nu_* > \nu^{S_k}$ for all k. Among all periodic external angles with kneading sequence ν_*, there is one, say φ, with $\vartheta_k^- < \varphi < \vartheta_k^+$ for every k (Proposition 11.21). Since the ray pairs $P(\vartheta_k, \vartheta'_k)$ are nested, and φ is part of a ray pair (φ, φ') (possibly after switching φ and φ'), it follows that $\vartheta_\infty \le \varphi < \varphi' \le \vartheta'_\infty$ and hence $\vartheta' \neq \vartheta$.

The construction of internal addresses assures that the parameter rays $R(\vartheta_\infty)$ or $R(\vartheta')$ cannot be separated from $R(\vartheta)$ by a periodic parameter ray pair (possibly, $\vartheta \in \{\vartheta_\infty, \vartheta'_\infty\}$); therefore, all three rays have their impressions in the same fiber. Since this fiber equals $\{c\}$ by hypothesis, the parameter rays $R(\vartheta_\infty)$ and $R(\vartheta'_\infty)$ land at c. $\square$

17.11. **Corollary (Biaccessibility and Dimension for the Mandelbrot Set)**
The set of biaccessible parameters in the Mandelbrot set has harmonic measure zero. Stronger yet, for every closed interval $I \subset \mathbb{S}^1 \setminus \{1/2\}$, the set of external angles $\vartheta \in I$ so that the parameter ray $R(\vartheta)$ lands at a biaccessible parameter has Hausdorff dimension strictly between 0 and 1; if I contains $1/2$, then the corresponding set of angles has Hausdorff dimension 1.

Proof. For $N \ge 3$, let $I_N \subset \mathbb{S}^1$ be the set of external angles whose kneading sequences start with $\mathtt{1}$, followed by $N-3$ entries $\mathtt{0}$ and then either $\mathtt{1}$ or $\star$. Then $\bigcup_N I_N = \mathbb{S}^1 \setminus \{0, 1/2\}$ (we remarked after Equation (15) that the only angle generating $\nu = \mathtt{1}\overline{\mathtt{0}}$ is $\vartheta = 1/2$).

By Proposition 17.9, the parameter ray $R(\vartheta)$ can land at a biaccessible point $c \in \mathcal{M}$ only if the kneading sequence ν of ϑ is not an endpoint of the parameter tree. By Lemma **??**, this means that ν is not biaccessible. By Theorem 16.13, the set of such angles in I_N has Hausdorff dimension less than 1, and the upper bound less than 1 is uniform for bounded N. Therefore, the Hausdorff dimension of the set of biaccessible angles in $\bigcup_{N \le M} I_N$ is still less than 1 for every finite M, and the Lebesgue measure of these angles in $\bigcup_{N \ge 3} I_N$ is still zero. Now $I_3 = (0, 1/3] \cup [2/3, 1)$ and the I_N exhaust $\mathbb{S}^1 \setminus \{0, 1/2\}$ so that for every $\varepsilon > 0$, the set $\mathbb{S}^1 \setminus (\{0\} \cup (1/2 - \varepsilon, 1/2 + \varepsilon))$ is contained in finitely many I_N. This shows that every compact interval $I \subset \mathbb{S}^1 \setminus \{1/2\}$ is contained in finitely many I_N union $\{0\}$, so the angles $\vartheta \in I$ for which $R(\vartheta)$ lands at a biaccessible parameter have Hausdorff dimension less than 1. This finishes the claim about the upper bound.

For the lower bound, consider any particular set I_N. Let $J_N \subset I_N$ be the set of angles with biaccessible kneading sequences. By Theorem 16.13, J_N has Hausdorff dimension in $\mathcal{I}_N$ where the lower bound tends to 1 as $N \to \infty$. The set of external angles which generate periodic kneading sequences has Hausdorff dimension 0 by either Bullett and Sentenac or Sec 14 . The set of external angles which are infinitely renormalizable also has Hausdorff dimension 0 by Lemma 16.14. The remaining angles in J_N satisfy the same estimates on Hausdorff dimension, and the corresponding rays land at parameters with trivial fibers by Corollary **??**; by Corollary 11.8, the landing points are not on the boundaries of hyperbolic components. Therefore, Lemma 17.10 shows that the landing points are biaccessible. This takes care of the lower bound. □

The previous discussion specificallly excluded periodic kneading sequences. Their relation to biaccessibility is particularly simple.

17.12. **Lemma (Biaccessibility for Periodic Sequences)**
A parameter ray $R(\vartheta)$ lands on the boundary of a hyperbolic component if and only if the kneading sequence of ϑ is periodic (with or without $\star$).

If the kneading sequence ν associated to ϑ is $\star$-periodic, then there is always another angle ϑ' so that $R(\vartheta)$ and $R(\vartheta')$ form a parameter ray pair. If ν is periodic but not $\star$-periodic, then $R(\vartheta)$ lands, but is never part of a ray pair.

Proof. If a parameter ray lands on the boundary of a hyperbolic component, then its kneading sequence is periodic by Corollary 11.8. Conversely, if ϑ generates a periodic kneading sequence ν, then the internal address associated to ν is finite by Lemma 14.6, and $R(\vartheta)$ lands on the boundary of a hyperbolic component again by Corollary 11.8.

If ν is $\star$-periodic, then we are simply saying that parameter rays at periodic angles land in pairs; see Theorem 9.1. If ν is periodic but not $\star$-periodic, then the ray $R(\vartheta)$ lands at the boundary of a hyperbolic component of $\mathcal{M}$. It follows from Corollary 9.16 that no other ray lands together with $R(\vartheta)$. □

18. Measure of Admissible Kneading Sequences

The main result of this section is that the admissible kneading sequences in Σ^1 form a Cantor set for which the $\frac{1}{2}$-$\frac{1}{2}$ product measure is positive.

We defined admissibility of $\star$-periodic and preperiodic kneading sequences in terms of non-existence of evil orbits in the associated Hubbard trees (Definition 4.9). In Theorem 5.2 we showed that this is equivalent to the combinatorial admissibility condition from Definition 5.1, and we used this condition to define admissibility for kneading sequences which are neither $\star$-periodic nor preperiodic. In Corollary 14.2 we showed that this admissibility condition for a kneading sequence ν is equivalent to the existence of an external angle which generates ν in the sense of Definition 2.1 (except for certain periodic kneading sequences without $\star$). Recall that Σ^1 is the space of all $\mathtt{0}$-$\mathtt{1}$-sequences starting with $\mathtt{1}$. We equip Σ^1 with the product topology and the $\frac{1}{2}$-$\frac{1}{2}$-product measure μ, normalized so that $\mu(\Sigma^1) = 1$. The main result of this section is to show that admissible kneading sequence have positive measure in Σ^1.

18.1. **Definition (n-Admissible Cylinders and Kneading Sequences)**
Let $E \subset \Sigma^1$ be the set of all sequences which satisfy the Admissibility Condition 5.1 for every m (the set of admissible kneading sequences*). For $n \geq 1$, a finite word $\nu_1 \dots \nu_n$ with $\nu_i \in \{\mathtt{0}, \mathtt{1}\}$ and $\nu_1 = \mathtt{1}$ is called an* admissible n-word *if there is a $\nu \in E$ which begins with $\nu_1 \dots \nu_n$. An* admissible n-cylinder *is the collection of sequences in Σ^1 which share an given admissible n-word. Finally, let E_n be the union of the admissible n-cylinders; this is the* set of n-admissible kneading sequences*.*

The following facts will be used repeatedly: whether or not $\nu \in \Sigma^1$ fails the admissibility condition for period m depends only on the first $\rho_\nu(m)$ entries in ν, so if ν fails the condition for period m, then an entire $\rho_\nu(m)$-cylinder is non-admissible. Thus $E = \bigcap_{n \geq 1} E_n$ is a decreasing intersection of sets which are simultaneously open and compact.

18.2. **Lemma (Admissible Kneading Sequences Form Cantor Set)**
The set $E \subset \Sigma^1$ of admissible kneading sequences forms a Cantor set. Hence E is measurable with respect to μ. Moreover, $\Sigma^1 \setminus E$ is dense in Σ^1.

Proof. Any violation of the admissibility condition for period m discards an entire cylinder subset of Σ^1. The set of non-admissible kneading sequences is a union of such cylinder sets and hence open. As a closed subset of the compact set Σ^1, the set E is compact. Clearly, E is totally disconnected because $\Sigma^1 \supset E$ is.

Now we show that E contains no isolated points. Fix an admissible $\nu \in E$. If the internal address of ν is infinite, then we approximate ν from below: $\nu > \nu^k$ for all $\star$-periodic sequences ν^k corresponding to finite truncations of the internal address as in Definition 7.1. All ν^k and $\mathcal{A}(\nu^k)$ are admissible by Lemmas 7.3 and 6.11, and $\mathcal{A}(\nu^k) \to \nu$ in the topology of E as $k \to \infty$.

The other case is that ν has finite internal address; suppose that n is the last entry. Then we approximate ν from above. By Proposition 5.16, ν is periodic with exact period n. Let ν_* be the $\star$-periodic kneading sequence of period n with the same internal address as ν; then $\nu = \mathcal{A}(\nu_*)$ and ν_* is admissible by Lemma 6.11. Choose an integer $s \geq 2$ and let ν'_* be the $\star$-periodic kneading sequence of period sn which coincides with ν for $sn-1$ entries. The internal address of ν'_* is that of ν_*, extended by the last entry sn. Hence by Lemma 6.10, ν'_* is admissible. Then $\mathcal{A}(\nu'_*)$ is admissible, again by Lemma 6.11. As $s \to \infty$, the sequence $\mathcal{A}(\nu'_*)$ converges to ν, and ν is non-isolated.

Finally, non-admissible sequences are dense in Σ^1 because any finite word can be continued into a non-admissible one (compare Example 5.3). □

18.3. **Corollary (Admissible Periodic Sequences)**
Let R_k be the number of admissible periodic kneading sequences of period k in Σ^1. Then

$$\lim_{k\to\infty} \frac{R_k}{2^{k-1}} = \mu(E) \ .$$

Proof. We have $\mu(E_k) = R_k 2^{-(k-1)}$ (Lemma 14.9) and $\mu(E) = \lim_k \mu(E_k)$. □

18.4. **Theorem (Admissible Kneading Sequences Have Positive Measure)**
The set of admissible kneading sequences has positive measure: $\mu(E) > 0$.

Proof. Fix $n_1 := 100$, and define a sequence of integers $\{n_i\}_{i\geq 1}$ so that n_{i+1} is the largest integer less than $\frac{3}{2}n_i$ for which $n_{i+1} - n_i$ is divisible by $2i$ (this gives 100, 148, 220, 324, 484,...). Let $m_i := (n_{i+1} - n_i)/2$. Then one can check that $(1.45)^i < m_i$. Clearly, $n_{i+1} < \frac{3}{2}n_i < n_i + \frac{3}{4}n_{i-1} < n_i + n_{i-1}$ and $m_i < n_i/4$.

Let F_1 be some cylinder of length n_1 containing an admissible sequence so that no block of 40 consecutive $\mathtt{0}'s$ appears among the first n_1 entries. For $i \geq 1$, define

$$F_{i+1} := \left\{ \begin{array}{ll} \nu \in F_i : & \text{every } k \leq n_i + m_i \text{ has an } s \in \operatorname{orb}_{\rho_\nu}(k) \cap \operatorname{orb}_{\rho_\nu}(1) \\ & \text{with } n_i + m_i \leq s \leq n_{i+1}, \\ & \text{and the entries } n_i,\ n_i+1,\ \ldots,\ n_{i+1}-1 \text{ in } \nu \text{ do} \\ & \text{not contain a block of } \lfloor n_i/8 \rfloor \text{ zeroes} \end{array} \right\}$$

For every $\nu \in F_{i+1}$, every $k \leq n_i + m_i$ satisfies $\rho_\nu(k) \leq n_{i+1}$, so every F_{i+1} is a union of cylinder sets of length n_{i+1}. Note that the second condition for F_{i+1} implies that the first i entries in any $\nu \in F_i$ do not contain a contiguous block of $\lfloor i/4 \rfloor$ zeroes. Indeed, in order to find the longest block of consecutive zeroes among the first i entries in ν, find the largest j such that $\frac{5}{4}n_j < i$. There may be consecutive zeroes at positions $n_{j+1} - \lfloor n_j/8 \rfloor + 1, \ldots, n_{j+1} - 1$, followed immediately by zeroes at $n_{j+1}, \ldots,$ $n_{j+1} + \lfloor n_{j+1}/8 \rfloor - 2$, yielding a total of $\lfloor n_j/8 \rfloor + \lfloor n_{j+1}/8 \rfloor - 2 \leq \lfloor 5n_j/16 \rfloor - 1 \leq \lfloor i/4 \rfloor - 1$ consecutive zeroes (and they may not even be finished by position i).

The two main steps in the proof are $E \supset \cap_i F_i$ and $\mu(F_{i+1}) \geq c_i \mu(F_i)$ for numbers $c_i > 0$ with $\prod_i c_i > 0$, from which the conclusion will follow.

(1) By induction over $i \geq 1$, we show that all $\nu \in \cap_j F_j$ are n_i-admissible for all i. They are n_1-admissible by hypothesis. Suppose $\nu \in \cap_j F_j$ is n_i-admissible but not n_{i+1}-admissible; then ν fails the Admissibility Condition 5.1 for some period m with $n_i < \rho(m) \leq n_{i+1}$. Let $r \in \{1, 2, \ldots, m\}$ be congruent to $\rho(m)$ modulo m.

If $m \leq n_{i-1} + m_{i-1}$, then by definition of F_{i-1}, there is an $s \in \mathrm{orb}_\rho(m) \cap \mathrm{orb}_\rho(1)$ with $s \leq n_i$; since $m \notin \mathrm{orb}_\rho(1)$, we have $\rho(m) \leq s \leq n_i$, which is a contradiction. Hence $m > n_{i-1} + m_{i-1}$.

If $r \geq n_{i-1} + m_{i-1}$, then $\rho(m) \geq m + r > 2n_{i-1} + 2m_{i-1} = n_i + n_{i-1} > n_{i+1}$, again a contradiction. Thus $r < n_{i-1} + m_{i-1}$, and there is an $s \in \mathrm{orb}_\rho(r) \cap \mathrm{orb}_\rho(1)$ with $s \leq n_i$. By the admissibility condition, $m \in \mathrm{orb}_\rho(r)$ and $m \notin \mathrm{orb}_\rho(1)$, hence $s > m$. Thus $s \in \mathrm{orb}_\rho(m)$ and $\rho(m) \leq s \leq n_i$ in contradiction to n_i-admissibility of ν. This final contradiction shows that $E \supset \cap_i F_i$ as claimed.

(2) Given ν and an integer r, let $J_\nu(r) = \{k \leq r : \rho(k) > r\}$. Obviously, $J_\nu(r)$ depends only on the first r entries of ν. We claim that for all $i \geq 1$,

$$\text{if } \nu \in F_i \text{ and } n_i + m_i \leq r < n_{i+1} \text{ then } \#J_\nu(r) < i/2 + 150\ . \tag{20}$$

Indeed, if $k \in J_\nu(r)$, then $\nu_{k+1} \ldots \nu_r = \nu_1 \ldots \nu_{r-k}$. Suppose that $r - k > n_2$. Then there is a unique $j \geq 1$ with $n_{j+2} \geq r - k > n_{j+1}$, and $r < n_{i+1}$ implies $j < i$. Since all $k' \in \{1, 2, \ldots, n_j + m_j\}$ have $\rho(k') \leq n_{j+1} < r - k$ by definition of F_{i+1}, it follows that all $k' \in \{k+1, k+2, \ldots, k + n_j + m_j\}$ have $\rho(k') \leq k + n_{j+1} < r$, hence $k' \notin J_\nu(r)$.

We turn this into an inductive argument: write $J_\nu(r) = \{k_1, k_2, k_3, \ldots, k_S\}$ with $k_s < k_{s+1}$ for all s. Let j_s be so that $n_{j_s+2} \geq r - k_s > n_{j_s+1}$. Then $k_{(s+1)} > k_s + n_{j_s} + m_{j_s}$, hence

$$\begin{aligned} n_{j_{(s+1)}+1} < r - k_{(s+1)} &< r - k_s - n_{j_s} - m_{j_s} \leq n_{j_s+2} - n_{j_s} - m_{j_s} \\ &= n_{j_s} + 2m_{j_s} + 2m_{j_s+1} - n_{j_s} - m_{j_s} = m_{j_s} + 2m_{j_s+1} \\ &< n_{j_s}/4 + n_{j_s+1}/2 < n_{j_s}\ , \end{aligned}$$

so $j_{s+1} + 1 < j_s$ or $j_{s+1} \leq j_s - 2$. Therefore, after at most $i/2$ turns of this argument we are left with elements $k_s \in J_\nu(r)$ with $r - k_s \leq n_2$ or $k_s \geq r - n_2$, and there are at most $n_2 + 1$ of those. It follows $\#J_\nu(r) \leq i/2 + n_2 + 1 < i/2 + 150$ as claimed.

(3) Now we show that $\mu(F_{i+1})$ is not too small compared to $\mu(F_i)$. Suppose $i \geq 40$, which implies $\lfloor \log_2(i/2 + 150) \rfloor + 1 \leq \lfloor i/4 \rfloor$. Let C be any $n_i + m_i$-cylinder in F_i and pick $\nu \in C$. Divide the integer interval $[n_i + m_i + 1, n_{i+1}]$ into m_i/i blocks $B_1, B_2, \ldots$ of length i. By Equation (20), $\#J_\nu(n_i + m_i) < i/2 + 150$, and there are 2^i different possibilities for B_1 which extend C. We claim that at least $2^{i/2}$ of them are *barriers* in the sense that for all $k \in J_\nu(n_i + m_i)$, we have $n_i + m_i + 2\lfloor i/4 \rfloor \in \mathrm{orb}_\rho(k)$. This is useful in view of the first condition of the definition of F_i. To construct these barriers, divide B_1 into three subblocks: the first two of length $\lfloor i/4 \rfloor$ each, the last of length $i - 2\lfloor i/4 \rfloor \geq i/2$.

In the second subblock, every entry is `0`. The third subblock is filled arbitrarily with `0` and `1`, for which there are at least $2^{i/2}$ possibilities. The first subblock needs more care.

In the first subblock of B_1, choose the first entry (at position $n_i + m_i + 1$) so that $n_i + m_i + 1 \in \mathrm{orb}_\rho(k)$ for at least half of the elements $k \in J_\nu(n_i + m_i)$; choose the second entry so that $n_i + m_i + 2 \in \mathrm{orb}_\rho(k)$ for at least half of the remaining elements in $J_\nu(n_i + m_i)$, and so on. The first $\lfloor \log_2(i/2 + 150) \rfloor + 1$ entries in B_1 suffice so that for every $k \in J_\nu(n_i + m_i)$, $\mathrm{orb}_\rho(k)$ contains at least one of these positions. Since $\lfloor \log_2(i/2 + 150) \rfloor + 1 \le \lfloor i/4 \rfloor$, we have used only positions within the first subblock of B_1. Fill the remaining entries within the first subblock of B_1 arbitrarily.

Since the first i entries of ν do not contain a contiguous block of $\lfloor i/4 \rfloor$ zeroes, it follows that the orbit of every $k \in J_\nu(n_i + m_i)$ visits an entry in the second subblock of B_1, and $\mathrm{orb}_\rho(k)$ contains the last position in the second subblock of B_1, which is $n_i + m_i + 2\lfloor i/4 \rfloor \in \mathrm{orb}_\rho(k)$.

An analogous construction can be repeated for all the other blocks B_2, B_3,.... Suppose that at least a single block B_j has a barrier as above. Then the construction yields n_{i+1}-cylinders which satisfy the first condition for F_{i+1}: for $k \le n_i + m_i$, we either have $\rho(k) \le n_i + m_i$ and we consider $\rho(k)$ instead of k, or $k \in J_\nu(n_i + m_i)$ and $\mathrm{orb}_\nu(k)$ visits $n_i + m_i + 2\lfloor i/4 \rfloor + (j-1)i$; therefore, $n_i + m_i + 2\lfloor i/4 \rfloor + (j-1)i \in \mathrm{orb}_\nu(k)$ for every $k \le n_i + m_i + 2\lfloor i/4 \rfloor$, in particular for $k = 1$.

Given the $n_i + m_i$-cylinder C, there are 2^i continuations into $n_i + m_i + i$-cylinders in F_{i+1}, and at least $2^{i/2}$ of them have a barrier in B_1 as constructed above, so at most a relative proportion of $1 - 2^{-i/2}$ has no barrier within B_1. The same bound for the relative proportions holds for B_2, B_3, ... We find that the proportion of n_{i+1}-cylinders in C with no barrier at any block B_j for $j = 1, \ldots, m_i/i$ is $(1 - 2^{-i/2})^{m_i/i}$. We have $m_i > (1.45)^i > 2^{i/2}$, so $m_i 2^{-i/2} > \alpha^i$ with $\alpha > 1$. Therefore

$$(1 - 2^{-i/2})^{m_i/i} < \left(\exp(-2^{-i/2})\right)^{m_i/i} = \exp\left(-2^{-i/2}\frac{m_i}{i}\right) < \exp(-\alpha^i/i) < i/\alpha^i,$$

and the relative proportion of n_{i+1}-cylinders in F_{i+1} satisfying the first condition is at least $1 - i/\alpha^i$.

We still have to take care of the second condition for F_{i+1}. Among the 2^{2m_i} continuations from any n_i-cylinder into an n_{i+1}-cylinder, there are $2^{2m_i - \lfloor n_i/8 \rfloor}$ which have $\lfloor n_i/8 \rfloor$ contiguous entries `0` beginning at any given position $n_i, n_i + 1, \ldots, n_{i+1} - \lfloor n_i/8 \rfloor$, and less than $2m_i 2^{2m_i - \lfloor n_i/8 \rfloor}$ which have $\lfloor n_i/8 \rfloor$ contiguous `0`'s between entries n_i and $n_{i+1} - 1$ (inclusively). The proportion of n_{i+1}-cylinders in F_i which do not contain $\lfloor n_i/8 \rfloor$ consecutive `0`'s is thus at least $1 - 2m_i 2^{-\lfloor n_i/8 \rfloor} < 1 - \frac{1}{2} n_i 2^{-\lfloor n_i/8 \rfloor}$.

Therefore

$$\begin{aligned}\mu(F_{i+1}) &\geq \left[1-\left(1-2^{-i/2}\right)^{m_i/i}\right]\left[1-\frac{1}{2}n_i 2^{-\lfloor n_i/8\rfloor}\right]\mu(F_i)\\ &> \left[1-\frac{i}{\alpha^i}\right]\left[1-\frac{n_i}{2\cdot 2^{\lfloor n_i/8\rfloor}}\right]\mu(F_i)\end{aligned}$$

for $i \geq 40$. Since $\sum_i i/\alpha^i < \infty$ and $\sum_i n_i/2^{\lfloor n_i/8\rfloor} < \infty$, it follows

$$\mu(E) \geq \mu\left(\bigcap_i F_i\right) \geq \mu(F_{40})\prod_{i\geq 40}\left[1-\frac{i}{\alpha^i}\right]\left[1-\frac{n_i}{2\cdot 2^{\lfloor n_i/8\rfloor}}\right] > 0$$

because F_{40} contains at least one cylinder set. This proves the theorem. $\square$

Numerical computations (shown in Table 1) suggest that about 80 percent of Σ^1 is admissible. Unfortunately, the convergence is too slow to make precise estimates for $\mu(E)$.

n	non-admis.	total	fraction
6	1	32	0.031250
7	2	64	0.031250
8	7	128	0.054688
9	17	256	0.066406
10	44	512	0.085938
11	96	1 024	0.093750
12	221	2 048	0.107910
13	473	4 096	0.115479
14	1 028	8 192	0.125488
15	2 160	16 384	0.131836
16	4 544	32 768	0.138672
17	9 408	65 536	0.143555
18	19 488	131 072	0.148682
19	39 984	262 144	0.152527
20	81 963	524 288	0.156332
21	167 138	1 048 576	0.159395
22	340 393	2 097 152	0.162312
23	691 104	4 194 304	0.164772
24	1 401 610	8 388 608	0.167085
25	2 836 989	16 777 216	0.169098
26	5 737 023	33 554 432	0.170977
27	11 586 451	67 108 864	0.172652
28	23 382 085	134 217 728	0.174210
29	47 143 911	268 43 5456	0.175625
30	94 995 724	536 870 912	0.176943
31	191 292 387	1 073 741 824	0.178155
32	385 016 428	2 147 483 648	0.179287

TABLE 1. The number of non-admissible cylinders of length n, among all the 2^{n-1} cylinders in Σ^1 of this length, for $n = 6, 7, \dots, 32$. Examples: for $n = 6$, the 2^5 total cylinders of length 6 are the binary words of length 6 starting with `1`, and the only non-admissible cylinder is `101100` (the only non-admissible $\star$-periodic sequence is $\overline{\texttt{10110}\star}$, but the cylinder `101 101` is admissible). For $n = 7$, the two non-admissible cylinders are the two continuations of `101100`; for $n = 8$, we have four continuations, as well as the new cylinders `1001 1000`, `1101 1100` and `1011 1100`: in the sense of Proposition 6.7, these sequences branch off from the admissible subtree at the kneading sequences $\overline{\texttt{100}\star}$, $\overline{\texttt{110}\star}$ and $\overline{\texttt{1011}\star}$ of periods 4, 4 and 5(!).

19. Typical Critical Points

A point $z \in J$ is typical with respect to harmonic measure ω and a continuous function φ, if it satisfies Birkhoff's Ergodic Theorem. In this section we show that for almost every external parameter angle, the critical value is typical. As a corollary we derive that for almost every external parameter angle f_c is a Collet-Eckmann map, and the critical value has the typical Lyapunov exponent.

Given an ergodic measure preserving transformation $(X, f, \mathcal{B}, \mu)$ a point $x \in X$ is called *typical* if it satisfies Birkhoff's Ergodic Theorem, *i.e.,* given an $L^1(\mu)$ function φ,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \varphi(f^{\circ i}(x)) = \int \varphi \, d\mu.$$

In this section we will discuss whether the critical value of f_c is a typical point with respect to harmonic measure ω_c on the Julia set J_c. By definition, ω_c is isomorphic to Lebesgue measure on $\mathbb{S}^1$. However, for our discussion we need the connection between ω_c and $\frac{1}{2}$-$\frac{1}{2}$-product measure μ on Σ. Denote the harmonic measure on $\partial\mathcal{M}$ by ω.

19.1. **Theorem (Typical Critical Values)**
Let $\varphi : \mathbb{C} \to \mathbb{R}$ be continuous. For ω-a.e. $c \in \partial\mathcal{M}$, the critical value c is a φ-typical point with respect to harmonic measure ω_c on the Julia set J_c.

Proof. Let $G = \{c \in \partial\mathcal{M} : J_c \text{ is locally connected}\}$. This excludes some external angles for which f has an indifferent periodic point, or is infinitely renormalizable [**HY, Ly2**], but at most a set of Hausdorff dimension 0 (see Lemma 16.14). Thus $\omega(G) = 1$. For each $c \in G$, we can define $\varphi_c : \Sigma \to \mathbb{R}$ by $\varphi_c(x) = \varphi(z)$ where z is the unique point in J_c such that the itinerary of z equals x. It is clear that φ_c is continuous. Take any n-cylinder set $C \subset \Sigma$. The set of external angles with itinerary in C is the union of at most n intervals of total length $2^{-n} = \mu(C)$, see the proof of Lemma 16.8. Since also all external angles with the same itinerary land at the same point, $\int_\Sigma \varphi_c \, d\mu = \int_{J_c} \varphi \, d\omega_c$. Suppose by contradiction that the theorem is false. Then there exists a compact set $G_0 \subset G$ with $\omega(G_0) > 0$ and $\varepsilon_0 > 0$ such that for every $c \in G_0$

$$\limsup_n \left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi_c \circ \sigma^{\circ i}(\nu) - \int_\Sigma \varphi_c \, d\mu \right| > \varepsilon_0. \tag{21}$$

Let $M = \sup |\varphi_c|$ and find a partition P of Σ into cylinder sets C such that

$$|\sup\{\varphi_c(x) : x \in C\} - \inf\{\varphi_c(x) : x \in C\}| \le \frac{\varepsilon_0}{2}$$

for all $C \in P$ and $c \in G_0$. Denote the visit frequency of ν to C by

$$V_n(C) = \frac{1}{n} \#\{0 \le i < n : \sigma^{\circ i}(\nu) \in C\}\ .$$

Obviously, for each C and $n \geq 1$, there exists

$$u_C(n) \in [\inf\{\varphi_c(x) : x \in C\}, \sup\{\varphi_c(x) : x \in C\}]$$

such that $\frac{1}{n}\sum_{i=0}^{n-1}(\mathbb{1}_C \cdot \varphi_c) \circ \sigma^{\circ i}(\nu) = V_C(n)u_C(n)$. We claim that for at least one of the cylinders C,

$$\limsup_n |V_C(n) - \mu(C)| > \eta_C := \frac{\varepsilon_0}{2M}\mu(C).$$

Indeed, if this were not the case, then for all sufficiently large n,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=0}^{n-1}\varphi_c \circ \sigma^{\circ i}(\nu) - \int_\Sigma \varphi_c \, d\mu\right| &= \sum_{C\in P}\left|\frac{1}{n}\sum_{i=0}^{n-1}(\mathbb{1}_C \cdot \varphi_c) \circ \sigma^{\circ i}(\nu) - \int_C \varphi_c \, d\mu\right| \\
&\leq \sum_{C\in P} |V_C(n) - \mu(C)|\, u_C(n) + \sum_{C\in P}\frac{\varepsilon_0}{2}\mu(C) \\
&\leq \sum_{C\in P}\eta_C u_C(n) + \frac{\varepsilon_0}{2} \leq \varepsilon_0.
\end{aligned}$$

This contradicts (21) proving the claim. So let us take an integer t so large that $\eta_C > \frac{1}{t}$. Without loss of generality we can assume that

$$\frac{1}{n}\#\left\{0 \leq i < n : \sigma^{\circ i}(\nu) \in C\right\} < 2^{-|C|} - \frac{1}{t} \tag{22}$$

infinitely often. Here $|C|$ denotes the length of the cylinder, so $2^{-|C|} = \mu(C)$. It is more convenient to iterate $\sigma^{\circ|C|}$; (22) implies that there exists $0 \leq k < |C|$ such that

$$\frac{1}{n}\#\left\{0 \leq i < n : \sigma^{\circ i|C|+k}(\nu) \in C\right\} < 2^{-|C|} - \frac{1}{t}$$

infinitely often. Write

$$U_{C,n,m,k} = \left\{\nu \in \Sigma : \#\left\{0 \leq i < n : \sigma^{\circ i|C|+k}(\nu) \in C\right\} = m\right\}.$$

Then $U_{C,n,m,k}$ consists of $\binom{n}{m}$ intervals of length $(2^{-|C|})^m(1-2^{-|C|})^{n-m}$. Writing $\varepsilon = \frac{m}{n}$ and using Stirling's formula, we obtain

$$\binom{n}{m}\left[2^{-|C|m}(1-2^{-|C|})^{n-m}\right]^r \leq \frac{(2^{-|C|})^{\varepsilon nr}(1-2^{-|C|})^{(1-\varepsilon)nr}}{\varepsilon^{\varepsilon n}(1-\varepsilon)^{(1-\varepsilon)n}}.$$

This tends to 0 exponentially as $n \to \infty$ for all

$$r > r_\star(\varepsilon, |C|) := \frac{\varepsilon\log\varepsilon + (1-\varepsilon)\log(1-\varepsilon)}{\varepsilon\log 2^{-|C|} + (1-\varepsilon)\log(1-2^{-|C|})}.$$

The function $r_\star$ is increasing in ε for $0 \le \varepsilon < 2^{-|C|}$ and therefore

$$\sum_{m=0}^{n(2^{-|C|}-\frac{1}{t})} \binom{n}{m} \left[2^{-|C|m}(1-2^{-|C|})^{n-m}\right]^r \\ \le m\binom{n}{m} \left[2^{-|C|m}(1-2^{-|C|})^{n-m}\right]^r \Big|_{m=[n(2^{-|C|}-\frac{1}{t})]}$$

also tends to 0 exponentially for $r > r_\star(2^{-|C|}-\frac{1}{t},|C|)$. In particular

$$\sum_{n\ge N} \sum_{m=0}^{n(2^{-|C|}-\frac{1}{t})} \binom{n}{m} \left[2^{-|C|m}(1-2^{-|C|})^{n-m}\right]^r$$

tends to 0 as $N \to \infty$ for these values of r. Hence the set

$$U_{C,t,k} = \left\{\nu \in \Sigma : \frac{1}{n}\#\{0 \le i < n : \sigma^{\circ i|C|+k}\nu \in C\} < 2^{-|C|} - \frac{1}{t} \text{ infinitely often}\right\},$$

being contained in

$$\bigcap_N \bigcup_{n\ge N} \bigcup_{0\le m<n(2^{-|C|}-\frac{1}{t})} U_{C,n,m,k} \ ,$$

has Hausdorff dimension at most $r_\star(2^{-|C|}-\frac{1}{t},|C|) < 1$ for any cylinder C, $0 \le k < |C|$ and $t > 0$. By Lemma 16.12, the non-typical external angles are contained in the countable union $\cup_C \cup_{k=0}^{|C|} \cup_{t=1}^{\infty} I^{-1}(U_{C,t,k})$ of sets of Hausdorff dimension < 1. □

Remark. Part of the argument in this proof are related to results proven by Barreira and Schmeling [**BaS**]. The next result was proven earlier by Graczyk and Świątek [**GSw**]. Smirnov [**Sm2**] also showed, using a combinatorial characterization of the Collet-Eckmann property, that the non-Collet-Eckmann external angles have Hausdorff dimension 0. We apply Theorem 19.1 to an $L^1(\omega_c)$-function φ instead of a continuous function, and make sure that φ is integrable for all measures ω_c under consideration.

19.2. **Corollary (Collet-Eckmann Maps)**
For ω-a.e. $c \in \partial\mathcal{M}$, f_c satisfies the Collet-Eckmann condition, i.e., there exist $K > 0$ and $\lambda > 1$ (depending on c) such that

$$|(f_c^{\circ n})'(c)| \ge K\lambda^n \textit{ for all } n.$$

In fact, we show slightly more, namely that for ω-a.e. $c \in \partial\mathcal{M}$, the Lyapunov exponent of c equals $\int \log|f_c'|\, d\omega_c > 0$.

Proof. Let $\varphi = \log|f_c'| = \log 2|z|$ and h_{ω_c} be the measure theoretic entropy. Since harmonic measure maximizes entropy, $h_{\omega_c} = \log 2$. The variational principle (see e.g. [**Wa**]) implies that the pressure $P_t = P(-t\log|f_c'|)$, taken at $t = 1$, is an upper bound for $-\int \varphi\, d\omega_c \le h_{\omega_c} - \int \log|f_c'|\, d\omega_c$. Since P_t is decreasing (and strictly decreasing in a neighborhood of 0), we obtain $P_1 < P_0 = h_{top}(f_c|_{J_c}) = \log 2$, the topological entropy of the map. Take $0 < \varepsilon < \log 2 - P_1$. Note that $|z| \le 2$ for all $z \in J_c$ and $c \in \partial\mathcal{M}$. For each f_c-invariant measure μ and $c \in \partial\mathcal{M}$, we find

$$\int |\varphi|\, d\mu \le -\int \log 2|z|\, d\mu + 2\int_{|z|>\frac{1}{2}} \log 2|z|\, d\mu \le P_1 + 4\log 2 \le 5\log 2.$$

Hence $\varphi \in L^1(\mu)$ uniformly over all invariant measures μ and $c \in \partial\mathcal{M}$. Let G_n be the set of all $c \in \partial\mathcal{M}$ such that for all f_c-invariant probability measures μ,

$$\int_{|z|<\frac{1}{n}} \varphi\, d\mu > -\varepsilon. \tag{23}$$

Obviously $\partial\mathcal{M} = \cup_n G_n$. For any $n \ge 1$ and $c \in G_n$,

$$\liminf_k \frac{1}{k} \sum_{i\in\{1,\dots k\},|f^{\circ i}(0)|<\frac{1}{n}} \varphi(f_c^{\circ i}(0)) > -\varepsilon. \tag{24}$$

Indeed, otherwise there exists a subsequence $\{k_j\}$ along which (24) fails. Take a weak limit μ of $\frac{1}{k_j}\sum_{i=1}^{k_j} \delta_{f_c^{\circ i}(0)}$, where δ_z is the Dirac measure at z. Then μ is f_c-invariant and $\int_{|z|<\frac{1}{n}} \varphi\, d\mu \le -\varepsilon$. This contradicts (23). Let $\varphi_L = \max\{\varphi, -L\}$ and take L so large that $\varphi(z) = \varphi_L(z)$ for $|z| > \frac{1}{n}$. Then Theorem 19.1 shows that for ω-a.e. $c \in G_n$,

$$\begin{aligned}
\liminf_k \frac{1}{k}\log|(f_c^k)'(c)| &= \lim_k \frac{1}{k}\sum_{i=1}^k \varphi(f_c^{\circ i}(c)) \\
&\ge \lim_k \frac{1}{k}\sum_{i=1}^k \varphi_L(f_c^{\circ i}(c)) + \liminf_k \frac{1}{k} \sum_{i\in\{1,\dots k\},|f^{\circ i}(0)|<\frac{1}{n}} \varphi(f_c^{\circ i}(0)) \\
&\ge \int \varphi_L\, d\omega_c - \varepsilon \\
&\ge \int \varphi\, d\omega_c - \varepsilon \ge h_{\omega_c} - P_1 - \varepsilon > 0.
\end{aligned}$$

By taking the union over all G_n, we see that f_c is Collet-Eckmann for ω-a.e. c. Since ε is arbitrary, we actually find $\liminf_k \frac{1}{k}\log|(f_c^k)'(c)| \ge \int \varphi\, d\omega_c$. The other inequality $\limsup_k \frac{1}{k}\log|(f_c^k)'(c)| \le \int \varphi\, d\omega_c$ is proven similarly. $\square$

A. Appendix

20. Existence of Hubbard Trees

In this section we prove the existence of Hubbard trees for $\star$-periodic and preperiodic kneading sequences. First we give two technical lemmas on the structure of ρ-orbits which will be used to predict the position of certain marked points in the tree. The proof of the Existence Theorem uses the triod map from Definition 3.14 extensively. Starting with the set $\{\star\nu, \nu, \sigma(\nu), \dots\}$ we use the triod map to find the branch points and therefore all marked points (represented by itineraries) of the tree. Further triod arguments then determine how to connect marked points by edges, and allow for verifying the properties of the dynamics on the tree.

The purpose of this section is to prove the following result which was announced in Section 3.

3.10. **Theorem (Existence and Uniqueness of Hubbard Trees)**
Every $\star$-periodic or preperiodic kneading sequence is realized by a Hubbard tree; this tree is unique (up to equivalence).

The following technical lemma on the behavior of ρ-orbits for arbitrary kneading sequences establishes a combinatorial result which will imply that the critical value is an endpoint of the Hubbard tree[1].

20.1. **Lemma (Critical Value is Endpoint (Combinatorial Version))**
Let $\nu \in \Sigma^\star$ be arbitrary. Then for each $k \in \mathbb{N}^*$ such that $\rho(k) - k < \infty$, there exists an i with $\rho^{\circ i}(\rho(k) - k) \le \rho(k)$ such that $\rho^{\circ i}(\rho(k) - k) \in \operatorname{orb}_\rho(1)$. In particular, if ν fails Admissibility Condition 5.1 at k, then $\rho(k) \in \operatorname{orb}_\rho(1)$.

Proof. If ν is $\star$-periodic, say of period N, then $m := \rho(k) - k$ is either infinite or at most N (depending on whether or not k is divisible by N). In the finite case, $N \in \operatorname{orb}_\rho(m) \cap \operatorname{orb}_\rho(1)$ inevitably. Therefore we only need to consider $\nu \in \Sigma^1$. We argue by induction on m, using the induction hypothesis IH$[m]$:

> **IH$[m]$:** For every $\nu \in \Sigma^1$ and corresponding ρ-function for which there exists a k such that $\rho(k) - k = m$, the orbits $\operatorname{orb}_\rho(1)$ and $\operatorname{orb}_\rho(m)$ intersect at the latest at $\rho(k)$.

Remark. IH$[m]$ does not imply that $\operatorname{orb}_\rho(1) \cap \operatorname{orb}_\rho(m)$ contains $\rho(k)$, not even if k is minimal such that $\rho(k) - k = m$. For example, if

$$\nu = \mathtt{1011001101101}\dots$$

[1] Using the interpretation of the ρ-function as cutting times in the Hubbard tree, $\operatorname{orb}_\rho(1)$ finds the sequence of closest precritical points in $[0, c_1]$ while $\operatorname{orb}_\rho(\rho(k) - k)$ finds the sequence of closest precritical points between the critical value c_1 and the postcritical point c_k. The lemma can be interpreted as saying that within the Hubbard tree, the arcs from c_1 to the critical point and to any other postcritical point c_k intersect, so c_1 is an endpoint of the tree.

with $m = 6$ and $k = 7$, then $m \in \mathrm{orb}_\rho(1)$, but $\rho(m) > k > m$.

The induction hypothesis is trivially true for $m = 1$. So assume that IH$[m']$ holds for all $m' < m$. Take $\nu \in \Sigma^1$ arbitrary and k minimal such that $\rho(k) - k = m$. If no such k exists, then IH$[m]$ is true for this ν by default. Let $m_0 \in \mathrm{orb}_\rho(m)$ be maximal such that $m_0 \le \rho(k)$; thus $\rho(m_0) > \rho(k)$. We distinguish two cases:

Case I: $m_0 < \rho(k)$. If $m_0 \le k$, then $\rho(m_0) > \rho(k)$ implies $m_0 < k$ and

$$\nu_1 \dots \nu_{k-m_0+1} \dots \nu_{\rho(k)-m_0} = \nu_{m_0+1} \dots \nu_{k+1} \dots \nu_{\rho(k)} \ ,$$

hence $\rho(k - m_0) - (k - m_0) = \rho(k) - k = m$, contradicting minimality of k. Therefore $k < m_0 < \rho(k)$. Since $\rho(m_0) > \rho(k)$ and

$$\nu_{k+1} \dots \nu_{m_0+1} \dots \nu_{\rho(k)} = \nu_1 \dots \nu_{m_0-k+1} \dots \nu'_m$$

(where ν'_m is the opposite symbol of ν_m), we have $\rho(m_0 - k) = m$. Consider $\tilde\nu := \nu_1 \dots \nu_{m_0-1}\nu'_{m_0}\nu_{m_0+1} \dots$ (with arbitrary continuation) with associated function $\tilde\rho$. Then $\tilde\rho(k) = m_0$.

(i) If $m_0 = m$, then the fact that $\rho(m_0 - k) = m$ implies $\tilde\rho(m_0 - k) > m_0$, so $\tilde\rho(k) \notin \mathrm{orb}_{\tilde\rho}(m_0 - k)$.

(ii) If $m_0 > m$, then $\tilde\rho(k) = m_0 \notin \mathrm{orb}_{\tilde\rho}(m)$, and $\tilde\rho(m_0 - k) = \rho(m_0 - k) = m < m_0$ again implies $\tilde\rho(k) \notin \mathrm{orb}_{\tilde\rho}(m_0 - k)$.

So in both cases $\tilde\rho(k) \notin \mathrm{orb}_{\tilde\rho}(m_0 - k)$. Now $\tilde\rho(k) - k = m_0 - k < \rho(k) - k = m$, so by the induction hypothesis IH$[m_0 - k]$, $\mathrm{orb}_{\tilde\rho}(1)$ and $\mathrm{orb}_{\tilde\rho}(m_0 - k)$ meet at or before $\tilde\rho(k)$; since $\tilde\rho(k) \notin \mathrm{orb}_{\tilde\rho}(m_0 - k)$, they meet before $\tilde\rho(k) = m_0$.

As a result, also $\mathrm{orb}_\rho(1)$ and $\mathrm{orb}_\rho(m_0 - k)$ meet before $m_0 < \rho(k)$, and since $\rho(m_0 - k) = m$, it follows that $\mathrm{orb}_\rho(1)$ and $\mathrm{orb}_\rho(m)$ meet before $\rho(k)$.

Case II: $m_0 = \rho(k)$. In this case $\rho(k) \in \mathrm{orb}_\rho(m)$. Let $n_0 \in \mathrm{orb}_\rho(1)$ be maximal such that $n_0 \le \rho(k)$, hence $\rho(n_0) > \rho(k)$. If $n_0 = \rho(k)$ there is nothing to prove, so assume that $n_0 < \rho(k) < \rho(n_0)$. As in Case I (by minimality of k), we only need to consider the case that $k < n_0 < \rho(k) < \rho(n_0)$. Since $\nu_{k+1} \dots \nu_{\rho(k)} = \nu_1 \dots \nu'_m$, we have $\rho(n_0 - k) = m$ (similarly as above). Set $\tilde\nu := \nu_1 \dots \nu'_{n_0} \dots$ with associated function $\tilde\rho$. Then $\tilde\rho(k) = n_0 < \rho(k)$ and by IH$[n_0 - k]$, $\mathrm{orb}_{\tilde\rho}(1)$ and $\mathrm{orb}_{\tilde\rho}(n_0 - k)$ meet at the latest at $\tilde\rho(k) = n_0$.

(i) If $m < n_0$, then $\tilde\rho(n_0 - k) = \rho(n_0 - k) = m$, so $\mathrm{orb}_{\tilde\rho}(1)$ and $\mathrm{orb}_{\tilde\rho}(m)$ meet at the latest at n_0. But $n_0 \notin \mathrm{orb}_{\tilde\rho}(1)$, so in fact $\mathrm{orb}_{\tilde\rho}(1)$ and $\mathrm{orb}_{\tilde\rho}(m)$ meet before n_0. But then $\mathrm{orb}_\rho(1)$ and $\mathrm{orb}_\rho(m)$ also meet before $n_0 < \rho(k)$.

(ii) If $m = n_0$, then $\mathrm{orb}_\rho(1)$ and $\mathrm{orb}_\rho(m)$ obviously meet at $n_0 < \rho(k)$.

(iii) The case $m > n_0$ is impossible: We have $\rho(k) - k = m > n_0 > k$, so $\rho(k) > 2k$. Since $\rho(n_0) > \rho(k) = m + k > n_0 + k$, we find that $\nu_{k+1} \dots \nu_{n_0+1} \dots \nu_{\rho(k)-1} = \nu_1 \dots \nu_{n_0-k+1} \dots \nu_{\rho(k)-k-1}$, hence $\rho(n_0 - k) \ge \rho(k) - k > n_0$. For the sequence $\tilde\nu$ this means that $\tilde\rho(n_0 - k) = n_0$, while $n_0 \notin \mathrm{orb}_{\tilde\rho}(1)$. Therefore $\mathrm{orb}_{\tilde\rho}(1)$ and $\mathrm{orb}_{\tilde\rho}(n_0 - k)$ do not meet at or before n_0; since $\tilde\rho(k) - k = n_0 - k$, this contradicts IH$[n_0 - k]$.

This completes Case II and proves that $\mathrm{orb}_\rho(1)$ and $\mathrm{orb}_\rho(m)$ intersect at the latest at $\rho(k)$, where k is minimal with the property that $\rho(k)-k=m$. For an arbitrary k with $\rho(k)-k=m$, let k' minimal with this property. Then the two orbits meet at the latest at $\rho(k')=m+k'\le m+k=\rho(k)$, so the statement holds for arbitrary k. This proves IH$[m]$.

To prove the lemma for $\nu\in\Sigma^1$ and arbitrary k, set $m:=\rho(k)-k$. Then $\mathrm{orb}_\rho(m)$ and $\mathrm{orb}_\rho(1)$ meet at the latest at $\rho(k)$. This proves the first statement.

Now for the second statement, assume that k' is the largest multiple of k less than $\rho(k)$, so $\rho(k')=\rho(k)$. Since k failing the admissibility means that $k\in\mathrm{orb}_\rho(\rho(k')-k')$ but $k\notin\mathrm{orb}_\rho(1)$, the second statement is an immediate consequence. □

The following combinatorial lemma will be used to locate the images of certain closest precritical points in Hubbard trees.

20.2. **Lemma (Combinatorics of ρ-Orbits)**
Let $\nu\in\Sigma^1$ (not containing a $\star$) and let m belong to the internal address of ν.

(1) *If s is such that $s<m<\rho(s)$, then* $\mathrm{orb}_\rho(\rho(m-s)-(m-s))\ni m$.
(2) *If $\rho(m)>2m$, then for every $s\in\{1,\dots,m\}$, either m or $2m$ belongs to $\mathrm{orb}_\rho(s)$.*
(3) *If $\rho(m)=\infty$, then m is the exact period of ν.*

Proof. Let $\nu=\nu_1\nu_2\dots$ be the kneading sequence. Using Lemma 20.1 we prove two claims.

Claim 1: *If $k\ge 1$ and $l\in\mathrm{orb}_\rho(k)\cap\mathrm{orb}_\rho(1)\setminus\{k\}$, then $l\in\mathrm{orb}_\rho(\rho(k)-k)$.*

Assume by contradiction that $l\notin\mathrm{orb}_\rho(\rho(k)-k)$. Consider $\tilde\nu=\overline{\nu_1\dots\nu_l}$ and let $\tilde\rho$ be the corresponding ρ-function. Note that $\rho(k)=\tilde\rho(k)$ because $l\ge\rho(k)$. We have $l=\max\mathrm{orb}_{\tilde\rho}(1)$, but $l\notin\mathrm{orb}_{\tilde\rho}(\rho(k)-k)$. This contradicts Lemma 20.1.

Claim 2: *If $k<m<\rho(k)$ and $m'\in\mathrm{orb}_\rho(1)$ is such that $\rho(m')=m$, then $m'\in\mathrm{orb}_\rho(m-k)$.*

Consider $\tilde\nu=\overline{\nu_1\dots\nu_{m'}}$. Then $m'=\max\mathrm{orb}_{\tilde\rho}(1)$ and $\tilde\rho(k)=m$ (this follows directly from $\rho(k)>m$ and $\rho(m')=m$). Therefore Lemma 20.1 implies that $m'\in\mathrm{orb}_{\tilde\rho}(m-k)$, and Claim 2 is proved.

To prove the first assertion, let m' be as in Claim 2. We have $m'\in\mathrm{orb}_\rho(m-s)$. Hence $m=\rho(m')\ge\rho(m-s)$. By Claim 1, $m\in\mathrm{orb}_\rho(\rho(m-s)-(m-s))$.

Now we prove the second assertion. If $m\in\mathrm{orb}_\rho(s)$ there is nothing to prove. Hence we may assume without loss of generality that $s<m<\rho(s)$.

Assume first that $\rho(s)<\infty$. Assertion (1) implies $\rho(m-s)-(m-s)\le m$, hence $\rho(m-s)\le 2m-s<\rho(m)-s$. If $\rho(s)\ge\rho(m)$, then

$$\begin{aligned}\nu_1\dots\nu_{m-s}\nu_{m-s+1}\dots\nu_{\rho(m)-s-1} &= \nu_{s+1}\dots\nu_m\nu_{m+1}\dots\nu_{\rho(m)-1}\\ &= \nu_{s+1}\dots\nu_m\nu_1\dots\nu_{\rho(m)-m-1}\ ,\end{aligned}$$

hence $\rho(m-s) \geq \rho(m) - s$, a contradiction. Hence $\rho(m) > \rho(s)$ and, similarly as above,

$$\begin{aligned} \nu_{s+1} \dots \nu_m \nu_1 \dots \nu_{\rho(s)-m} &= \nu_{s+1} \dots \nu_m \nu_{m+1} \dots \nu_{\rho(s)} \\ &= \nu_1 \dots \nu_{m-s} \nu_{m-s+1} \dots \nu'_{\rho(s)-s} \ , \end{aligned}$$

where ν'_j denotes the opposite symbol of ν_j. Therefore $\rho(m-s) = \rho(s) - s$. By Assertion (1), $m \in \mathrm{orb}_\rho(\rho(m-s)-(m-s)) = \mathrm{orb}_\rho(\rho(s)-m)$. (Note that this implies $\rho(s) \leq 2m$.) Since $\nu = \nu_1 \dots \nu_m \nu_1 \dots \nu_m \dots$, we then have $2m \in \mathrm{orb}_\rho(\rho(s))$ as claimed (we simply start m entries later).

If $\rho(s) = \infty$, we first change the entry ν_k for some $k > 2m$. Then still $\rho(m) > 2m$ and $\rho(s) = k > 2m$, and we can use the above argument.

For the third assertion, consider $\tilde{\nu} = \nu_1 \dots \nu_m \nu_1 \dots \nu_m \nu'_1 \dots$ with ρ-function $\tilde{\rho}$. Then $\tilde{\rho}(m) = 2m + 1 > 2m$. If $s < m$ is the exact period of ν, then $\rho(s) = \infty$ and $\tilde{\rho}(s) = 2m+1$. Therefore $m, 2m \notin \mathrm{orb}_{\tilde{\rho}}(s)$, contradicting the second assertion. □

20.3. **Lemma (ν is Always Endpoint)**
Fix a sequence $\nu \in \Sigma^\star$. If the triod $[\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \nu]$ (with $k, l \geq 1$) can be iterated infinitely often, then the sequence ν will be chopped off eventually. If the `stop` *case is reached, the initial $\star$ does not occur in the last sequence. Hence ν cannot be an interior point of such a triod.*

Proof. Suppose the triod can be iterated infinitely often but ν is never chopped off. By the definition of the ρ-function, $\sigma^{\circ k}(\nu)$ differs from ν differ at entry $\rho(k)$, so one of the two is chopped off at step $\rho(k) - k$. If the third sequence is chopped off, then we are done, so we may assume that it is the first sequence which is chopped off. Therefore,

$$\varphi^{\circ(\rho(k)-k)}\left([\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \nu]\right) = [\nu, \cdot\cdot, \sigma^{\circ(\rho(k)-k)}(\nu)]$$

(the second entry depends on whether or not $\sigma^{\circ l}(\nu)$ has been chopped off in the meantime). The next time that the first and third sequences differ is at iterate $\rho(\rho(k)-k)$, and again we may assume the first sequence is chopped off, yielding

$$\varphi^{\circ\rho(\rho(k)-k)}\left([\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \nu]\right) = [\nu, \cdot\cdot, \sigma^{\circ\rho(\rho(k)-k)}(\nu)] \ ,$$

and in general the first sequence is chopped off at steps in $\mathrm{orb}_\rho(\rho(k)-k)$.

Similarly, the second sequence is chopped off at iteration steps in $\mathrm{orb}_\rho(\rho(l)-l)$. By Lemma 20.1, there is an iteration step $s \in \mathrm{orb}_\rho(\rho(k)-k) \cap \mathrm{orb}_\rho(\rho(l)-l)$. Take s minimal with this property. At this step, the first and second sequences both differ from the third, so in this step the sequence ν must be chopped off (or the `stop` case is reached, but this is excluded by hypothesis). This settles the first claim.

For the second claim, the `stop` case can be reached only if ν is $\star$-periodic, say of period n. We can assume by Lemma 5.10 that $1 \leq k < l < n$. By Lemma 20.1, $\mathrm{orb}_\rho(\rho(k)-k)$ and $\mathrm{orb}_\rho(\rho(l)-l)$ intersect $\mathrm{orb}_\rho(1)$ at an entry $s \leq \max\{\rho(k), \rho(l)\}$, and since ν is $\star$-periodic, $\max\{\rho(k), \rho(l)\} \leq n$. The above argument implies that ν is chopped off at some iterate $s' \leq s$. If $s' = n$, then $\rho(k) = n$ or $\rho(l) = n$. If $\rho(l) \geq n$, then the `stop` case is reached after $n - l$ iterates of φ, and the $\star$ is in the second sequence. If $\rho(k) \geq n$, then the `stop` case is reached after $n - k$ iterates of φ, and the $\star$ is in the first sequence. Finally, if $s < n$, then the triod after s iterates will be $\varphi^{\circ s}([\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \nu]) = [\sigma^{\circ k'}(\nu), \sigma^{\circ l'}(\nu), \nu]$ for some $k', l' \leq n$, and we can repeat the argument. □

Constructing a Hubbard Tree. We will now begin to construct a Hubbard tree: we define a finite set of vertices and connect them by edges, and then we endow this tree with a continuous map which satisfies all the hypotheses of a Hubbard tree from Definition 3.2. For the rest of this section, we fix a $\star$-periodic or preperiodic sequence ν and set

$$S(\nu) = \{\star\nu, \nu, \sigma(\nu), \sigma^{\circ 2}(\nu), \dots\}.$$

Clearly, $\sigma(S(\nu)) \subset S(\nu)$ and $S(\nu) \cap \{\mathtt{0}\nu, \mathtt{1}\nu\} = \emptyset$.

In order to introduce the other marked points of the Hubbard tree, we need to analyze the behavior of the triod map of Definition 3.13 in more detail.

Branch Points $\mathfrak{b}(s, t, u)$. Fix a $\star$-periodic or preperiodic ν. Take $s, t, u \in S(\nu) \cup \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$ and consider the triod $[s, t, u]$. If each of these sequences is chopped off infinitely often under iteration of φ, then $[s, t, u]$ is called a *branched* triod, and it has a "branch point" called $\mathfrak{b}(s, t, u)$: that is the sequence which is defined by *majority vote* of the three sequences in each iteration step. More precisely, the n-th entry of this sequence is defined by majority vote among the first entries in $\varphi^{\circ(n-1)}([s, t, u])$. Clearly, $\mathfrak{b}(s, t, u) \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$. If s, t, u are (pre)periodic, then $\mathfrak{b}(s, t, u)$ is necessarily (pre)periodic too. Moreover, $\mathfrak{b}(s, t, u)$ differs from all sequences s, t, and u: otherwise, if say $\mathfrak{b}(s, t, u) = s$, then because of the majority construction, the first sequence in the triod $[s, t, u]$ always coincides with at least one other sequence, so it is never chopped off, contrary to our assumption.

Types of Triods. Based on Equation (1) in Definition 3.13, we can distinguish five types of triods:

(1) The triod can be iterated indefinitely so that all three sequences are chopped off infinitely often (this implies all three sequences remain distinct under iteration and the `stop` case is never reached); in this case we call the triod *branched*. The sequence $\mathfrak{b}(s, t, u) \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}^*}$ obtained by majority vote is the *branch point* of the triod.

(2) The triod can either be iterated indefinitely and precisely two sequences are chopped off infinitely often whereas the remaining sequence is never chopped, or the iteration reaches the `stop` case so that the sequence that lands on $\star$ at the `stop` case has never been chopped off before; in this case we call the

triod *flat*. The sequence which reaches the $\star$ in the `stop` case or which is never chopped off is called the *middle point* of the flat triod.

(3) Not all three sequences remain distinct during the iteration, *i.e.,* two or three sequences become identical at some iterate of φ; in this case we say that the triod has *collapsing* sequences. Note that collapsing sequences means that one of these sequences must have been equal to $\mathtt{0}\nu$ or $\mathtt{1}\nu$ the iterate before collapsing. Note also that if φ can be iterated indefinitely, but only one sequence ever gets chopped off infinitely often, then the remaining sequences must collapse.

(4) The iteration of φ reaches the `stop` case so that the $\star$ is in a sequence that was chopped off before.

(5) The iteration of φ can be carried on forever without collapsing, and some sequence is chopped off at least once, but not infinitely often.

For us, the most important cases are the first two because we will show in Lemma 20.4 that (3)–(5) do not occur when s, t, u are distinct and are equal to forward shifts of the kneading sequence ν.

Note that all five cases are mutually distinct and cover all possible cases: if two or three sequences collapse, then the `stop` case cannot be reached and at most one sequence can be chopped off infinitely often, and we are exactly in case (3). Otherwise, there is no collapsing. If we can iterate forever without collapsing, then either some sequence is chopped off a positive finite number of times and we are exactly in case (5); or every sequence is chopped off either infinitely many times or not at all, and the number of chopped sequences must be 2 or 3, so we are in cases (2) or (1). The last possibility is that the `stop` case is reached; depending on whether the sequence landing on $\star$ had or had not been chopped off before, we are in case (4) or (2).

20.4. **Lemma (If All Three Sequences are Chopped)**
Let $\nu \in \Sigma^\star$ be $\star$-periodic or preperiodic. If $[\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \sigma^{\circ m}(\nu)]$ is such that each of the three sequences is chopped off at least once, then each sequence is chopped off infinitely often, and the triod can be iterated forever without reaching the `stop` *case.*

Proof. First note that $\sigma^{\circ s}(\nu) \neq \mathtt{0}\nu$ or $\mathtt{1}\nu$ for all s. Therefore the triod can be iterated until the `stop` case is reached, if ever.

Suppose the first chopping occurs after s iterations, and it is the third sequence which is chopped off. The resulting triod is $[\sigma^{\circ(k+s)}(\nu), \sigma^{\circ(l+s)}(\nu), \nu]$. By Lemma 20.3, the `stop` case is never reached with the $\star$ at the third position.

By assumption, there are iteration times when the first and second sequences are chopped off, and it follows similarly that the `stop` case can never be reached with the $\star$ at any position. The triod can thus be iterated infinitely often.

If there is a last iterate $\varphi^{\circ s}([\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \sigma^{\circ m}(\nu)])$ in which the third sequence is chopped off, then this iterate has the form $[\sigma^{\circ k'}(\nu), \sigma^{\circ l'}(\nu), \nu]$, and this contradicts

Lemma 20.3. Therefore, during the iteration of the triod, all sequences are chopped off infinitely often. □

The Structure of Triods in $S(\nu)$. Every triod in $S(\nu)$ is either flat or branched: no triod can collapse, and it follows from Lemma 20.3 that every sequence which gets chopped off once will never be the center of a flat triod when the `stop` case is reached. Therefore if a triod cannot be iterated forever then it reaches the `stop` case and is flat. If a triod can be iterated forever, then at least two sequences get chopped off infinitely often, and if all three sequences get chopped off at least once, then by Lemma 20.4 the triod is branched.

20.5. **Lemma (Branch Points of Branched Triods)**
Fix a $\star$-periodic or preperiodic ν. Suppose that $s,t,u \in S(\nu) \cup \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$ are such that $[s,t,u]$ is a branched triod; set $v := \mathfrak{b}(s,t,u)$. If $\sigma^{\circ k}(w) \notin \{\mathtt{0}\nu, \mathtt{1}\nu\}$ for all $w \in \{s,t,u,v\}$ and all $k \geq 0$, then $[s,t,v]$, $[s,u,v]$ and $[t,u,v]$ are flat with v in the middle.

Proof. By assumption, the triod map can be iterated forever on $[s,t,u]$. If the iteration stops for $[s,t,v]$ after finitely many steps, then either the iteration reaches one of the sequences $\mathtt{0}\nu$ and $\mathtt{1}\nu$ (which is excluded by hypothesis), or the `stop` case in (1) is reached; but since $v \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$ is constructed by majority vote among the sequences s,t,u, this can never occur. The majority vote also assures that v can never be chopped off along the iteration, so $[s,t,v]$ is flat with v in the middle. The reasoning for the other two sequences $[s,u,v]$ and $[t,u,v]$ is the same. □

Vertices (Marked Points) of the Tree. Set

$$V := S(\nu) \cup \bigcup_{[s,t,u]} \{\mathfrak{b}(s,t,u)\},$$

where the union runs over all branched triods $[s,t,u]$ with vertices in $S(\nu)$, so their branch points $\mathfrak{b}(s,t,u)$ are well-defined. The set V is σ-invariant because $S(\nu)$ is and $\sigma(\mathfrak{b}(s,t,u))$ equals $\mathfrak{b}(\varphi([s,t,u]))$.

The Triod Map Can Be Iterated. Triods consisting of vertices in V can be branched and flat (types (1) and (2)). In order for φ to be well-defined, we need to ensure that types (3)–(5) do not occur. In addition, we want to know that for $s,t,u \in V$, the triod map can only result in elements of V. This is the contents of the following lemma.

20.6. **Lemma (V is Closed under Taking Triods)**
For each triple of different sequences $s,t,u \in V$, $\mathfrak{b}(s,t,v) \in V$. In particular, no triod $[s,t,u]$ is of type (3)–(5).

Proof. Take $s,t,u \in V$ arbitrary but distinct. Let us first show that $V \cap \{\mathtt{0}\nu, \mathtt{1}\nu\} = \emptyset$, so $[s,t,u]$ is not of type (3). To see why this is true, suppose that $0\nu \in V$ (the case $\mathtt{1}\nu \in V$ is analogous). Since $\mathtt{0}\nu \notin S(\nu)$, this implies that $\mathtt{0}\nu = \mathfrak{b}(s,t,u)$ for a branched triod $[s,t,u]$ with $s,t,u \in S(\nu)$. We may suppose that $s_1 = t_1 = \mathtt{0}$ and $u_1 \in \{\mathtt{0}, \star, \mathtt{1}\}$. If $u_1 = \mathtt{0}$, then $\varphi([s,t,u]) = [\sigma(s), \sigma(t), \sigma(u)]$ is a triod with vertices in $S(\nu)$ and $\sigma(\mathfrak{b}(s,t,u)) = \nu$ in the middle. This contradicts the fact that ν is an endpoint (Lemma 20.3). If $u_1 \in \{\star, \mathtt{1}\}$, then $\varphi([s,t,u]) = [\sigma(s), \sigma(t), \nu]$ is a triod with middle point ν. This contradicts that ν is an endpoint (Lemma 20.3) once more. As a result, every triod in V can be iterated forever without sequences collapsing, unless the `stop` case is reached.

The next step is to find $s', t', u' \in S(\nu)$ such that $\mathfrak{b}(s',t',u') = \mathfrak{b}(s,t,u)$, showing that $\mathfrak{b}(s,t,u) \in V$. As s,t,u are taken distinct, at least two of them are chopped off under iteration of φ. Assume that s is chopped off first and t second. If $s \in S(\nu)$, then put $s' = s$; otherwise, $s = \mathfrak{b}(s^1, s^2, s^3)$ for some $s^1, s^2, s^3 \in S(\nu)$. We iterate the triod map φ on $[s,t,u]$, and keep track what happens to $[s^1, s^2, s^3]$.

- As long as $[s,t,u]$ has unanimous vote (*i.e.,* $s_1 = t_1 = u_1$), we refrain from making a selection among s^1, s^2, s^3 and take $\varphi([s^1, s^2, s^3])$ for the next iterate.
- If s has minority vote (*i.e.,* $s_1 \neq t_1 = u_1$), then select $s' = s^k$ for any $k = 1,2,3$ such that s^k shares vote with s. We have $\varphi([s,t,u]) = \varphi([s',t,u])$.

We iterate this algorithm until s has reached minority vote, which by the choice of s must happen eventually. It follows that $\mathfrak{b}(s,t,u) = \mathfrak{b}(s',t,u)$.

Now we repeat the argument with $[s',t,u]$. If $t \in S(\nu)$ then let $t' = t$; otherwise $t = \mathfrak{b}(t^1, t^2, t^3)$ for some $t^1, t^2, t^3 \in S(\nu)$.

- As long as $[s',t,u]$ has unanimous vote, we refrain from making a selection among t^1, t^2, t^3 and take $\varphi([t^1, t^2, t^3])$ for the next iterate.
- If t has two-to-one majority vote (*i.e.,* $s'_i \neq t_i = u_i$), disqualify the t^k (if any) with $t^k_i = s'_i$ from being selected. As both t^k and s' will be replaced by ν by the action of φ, t^k and s' will take the same vote ever after, so no other $t^{k'}$ can later share minority vote with s'.
- If t has minority vote eventually, then select $t' = t^{k'}$ for an undisqualified $k' = 1,2,3$ such that t and $t^{k'}$ share vote. We have $\varphi([s',t',u]) = \varphi([s',t,u])$.

We iterate this algorithm until t has reached minority vote, which by the choice of t must happen eventually. It follows that $\mathfrak{b}(s,t,u) = \mathfrak{b}(s',t',u)$.

Finally we turn to u. If $u \in S(\nu)$ then take $u' = u$. Otherwise $u = b(u^1, u^2, u^3)$ for some $u^1, u^2, u^3 \in S(\nu)$. If u is never chopped off, then $[s',t',u]$ is flat with u in the middle, and no new branch point is created. If u is chopped off eventually, then we follow an algorithm as before.

- As long as $[s',t',u]$ has unanimous vote we refrain from making a selection among u^1, u^2, u^3 and take $\varphi([u^1, u^2, u^3])$ for the next iterate.
- If s' has minority vote, disqualify the u^k (if any) voting with s' from being selected. As both u^k and s' will be replaced by ν by the action of φ, u^k and

s' will take the same vote ever after, so no other $u^{k'}$ can later share minority vote with s'.

- If t' has minority vote, disqualify the $u^{k'}$ (if any) voting with t' from being selected. Note that $k \neq k'$ because u^k shares vote with s'. As both $u^{k'}$ and t' will be replaced by ν by the action of φ, $t^{k'}$ and t' will take the same vote ever after.
- If u has minority vote eventually, let u' be the remaining undisqualified $u^{k''}$ which shares vote with u.

In this way we obtain $s', t', u' \in S(\nu)$ such that $\mathfrak{b}(s,t,u) = \mathfrak{b}(s',t',u')$.

Since $[s,t,u]$ behaves as $[s',t',u']$ under φ and whenever s, t or u is chopped off, this sequence is replaced by ν just as s', t' or u' is replaced by ν, it follows from Lemma 20.4 that if all three sequences s, t, u are chopped off once, they will be chopped off infinitely often. If one sequence, say s, is never chopped off, and another sequence, say t is chopped a finite number of times only, some iterate $\varphi^{\circ k}([s,t,u])$ takes the form $[\sigma^{\circ k}, \nu, \tilde{u}]$ form some $\tilde{u} \in V$. In this triod, $\sigma^{\circ k}$ and ν are never chopped off, so they are equal. But this contradicts that $\mathtt{0}\nu, \mathtt{1}\nu \notin V$. This proves that $[s,t,u]$ is not of type (5).

Finally, to prove that $[s,t,u]$ is not of type (4), assume by contradiction that the first sequence is chopped off, and later on reaches the `stop` case with the $\star$ in that sequence. Since $[s',t',u']$ has the same behavior under φ, it would reach `stop` case with the $\star$ in the first sequence as well. This contradicts Lemma 20.3. $\square$

Vertices Along Arcs. For sequences $s, t \in V$, set

$$E(s,t) := \{u \in V : \text{the triod } [s,t,u] \text{ is flat with } u \text{ in the middle}\} \cup \{s,t\} .$$

We call s and t *adjacent* if $E(s,t) = \{s,t\}$. If $u, u' \in E(s,t) \setminus \{s\}$ are different sequences, then at least one of u and u' must be chopped off the triod $[s,u,u']$ under iteration of φ. We write $u \succ u'$ if u' is chopped off, and $u' \succ u$ otherwise. If $[s,u,u']$ reaches the `stop` case with u in the middle, then we write $u \succ u'$ as well. (It follows from Lemma 20.8 item (1) below that the sequence u or u' which is not chopped off, is indeed the middle point of the flat triod $[s,u,u']$.) By convention we take $s \succ u$ for all $u \in E(s,t) \setminus \{s\}$.

20.7. **Lemma ($\succ$ is Transitive and Linear)**
The order $\succ$ is a transitive and linear on $E(s,t)$.

Proof. For unity of exposition, we will say that u is chopped off from $[s,u,u']$ by φ also if $[s,t,u]$ reaches the `stop` case with another sequence than u in the middle. As we will not iterate φ further in this proof once the order of sequences in $[s,t,u]$ has been established, this abuse of terminology is inconsequential.

Suppose that $u, u', u'' \in E(s,t) \setminus \{s\}$ with $u \succ u'$ and $u' \succ u''$, say u'' is chopped off from $[s,u',u'']$ at iterate l and u' is chopped off from $[s,u,u']$ at iterate k. Iterate φ

on the triod $[s, u, u'']$; as long as the second or third sequence is not chopped off from $[s, u, u'']$, the chopping off of the first sequence happens at exactly the same times as for $[s, u, u']$ and $[s, u', u'']$. So let us wait until one of u and u'' is chopped off. Assume by contradiction that u is chopped off first; by the choice of k, this can only happen at an iterate $\geq k$. Since $u_i = u'_i$ for $i < k$, our assumption implies that also $u'_i = u''_i$. By definition of k, the first entries of the first and second sequence of $\varphi^{\circ k}([s, u, u'])$ agree, but the first entry of the third sequence (namely u'_k) is different. Since u' is the middle point of $[s, u', u'']$, $u'_k = u''_k$. But this means that u'' is chopped off from $[s, u, u'']$ at this iterate, and we have a contradiction.

This shows that $u \succ u''$; hence the order $\succ$ is indeed transitive. This shows that the set $E(s, t)$ is linearly ordered by $\succ$, with maximal sequence s and minimal sequence t.

Note that $E(t, s)$ is the same set of vertices, but the ordering goes in the other direction. $\square$

20.8. **Lemma (Properties of $\succ$)**

(1) *If $u, u' \in E(s, t)$ are such that $s \succ u \succ u'$, then $[s, u, u']$ is a flat triod with u as middle point.*
(2) *If s, t, u, v are such that $u \in E(s, v)$ and $t \in E(s, u)$, then $t \in E(s, v)$ and $t \succeq u$ in the $E(s, v)$-order.*
(3) *If s, t, u are different sequences in V and $E(s, u)$ and $E(t, u)$ intersect only in u, then u is the middle point of the flat triod $[s, t, u]$.*
(4) *For all $v \in V$, there exists $s \in S(\nu)$ such that $v \in E(s, \star\nu)$.*

Proof. In this proof, we will iterate triods $[u, u', u'']$ with sequences in $V \setminus S(\nu)$. We have seen in Lemma 20.6 that such a triod is not of type (3)–(5). We always have that $u, u', u'' \in E(s, t)$ for some $s, t \in S(\nu)$, say $u \succ u' \succ u''$ in the $E(s, t)$-order. Therefore iterating φ on $[u, u', u'']$ mimics iterating φ on $[s, u', t]$, and once u and u'' are chopped off, the two triods become identical. Therefore $[u, u', u'']$ is flat.

(1) Let k be the first iterate that u' is chopped off from $[s, u, u']$. At this iterate, $\varphi^{\circ k}([s, u, u']) = \varphi^{\circ k}([s, u, t])$, and because u is the middle point of $[s, u, t]$, u will never be chopped off neither from $[s, u, t]$ nor from $[s, u, u']$.

(2) Let l be the first time that u is chopped off from the triod $[s, t, u]$. The most work goes in proving that $t \in E(s, v)$. Suppose by contradiction that at some iterate k, the second sequence of $[s, t, v]$ is chopped off under iteration of φ. There are three cases:

- If $k > l$. Then $\varphi^{\circ l}([s, t, u]) = \varphi^{\circ l}([s, t, v])$, and since t is the middle point of $[s, t, u]$, it will not be chopped off at all.
- $k = l$. Then the first and second sequence of $\varphi^{\circ k-1}([s, t, u])$ start with the same symbol, the first and second sequence of $\varphi^{\circ k-1}([s, t, v])$ start with different symbols. Yet for both triods, these first symbols are the same, so we have a contradiction.

- $k < l$. Then the first and third sequence of $\varphi^{\circ k-1}([s,t,v])$ start with the same symbol, but the first symbol of the second sequence is different. Also the first symbols of the second and third sequence of $\varphi^{\circ k-1}([s,t,u])$ are the same. This implies that the first symbols of the first and third sequence of $\varphi^{\circ k-1}([s,u,v])$ agree, but disagree from the first symbol of the second sequence. But this implies that u is chopped off from $[s,u,v]$, contradicting our assumption.

This shows that t is never chopped off from $[s,t,v]$, hence $t \in E(s,v)$. Since t is the middle point of $[s,t,u]$, u will be chopped off before t is, so $t \succ u$ in the $E(s,v)$ order.

(3) We iterate the triod $[s,t,u]$. If the triod is branched, then by Lemma 20.5, the branch point $v = \mathfrak{b}(s,t,u)$ belongs to both $E(s,u) \cap E(t,u)$. If $[s,t,u]$ is flat, with s in the middle, then $s \in E(s,u) \cap E(t,u)$. If $[s,t,u]$ is flat, with t in the middle, then $t \in E(s,u) \cap E(t,u)$. So in all three cases, $E(s,u) \cap E(t,u)$ contains more than one point. The remaining possibility is that u is the middle point of $[s,t,u]$.

(4) This is trivial if $v \in S(\nu)$, so assume that $v = \mathfrak{b}(s,t,u) \in V \setminus S(\nu)$ for some $s,t,u \in S(\nu)$. By Lemma 20.5, $v \in E(s,t) \cap E(t,u) \cap E(u,s)$. If $\star\nu$ equals one of s,t,u, any other sequence of s,t,u fulfills the assertion. So we can assume that s,t,u have a common first entry. If $t \in E(s,\star\nu)$, then by item (2) applied to the quadruple $s,v,t,\star\nu$, also $v \in E(s,\star\nu)$ and we are finished. Similarly, if $s \in E(t,\star\nu)$, then $v \in E(t,\star\nu)$ and again we are finished. The remaining case is that $[s,t,\star\nu]$ is a branched triod, say $v' = \mathfrak{b}(s,t,\star\nu)$. If $v = v'$, then we are finished again. If $v \succeq v'$ in the $E(s,t)$-order, then $v \in E(s,\star\nu)$ by item (2) applied to the quadruple $s,v,v',\star\nu$. Similarly, if $v' \succeq v$, then $v \in E(t,\star\nu)$. This proves item (4). $\square$

Edges of the Tree. For any $s \in V$, let $E(\star\nu,s) =: \{s^0, s^1, , \dots, s^{k-1}, s^k\}$ be as above, in decreasing $E(\star\nu,s)$-order, with $s^0 = \star\nu$ and $s^k = s$. Then attach edges $[s^i, s^{i+1}]$ between the vertices s^i and s^{i+1} to the tree, for $i = 0,1,\dots,k-1$. Take the union of such edges for all $s \in V$, omitting repetitions (so that every pair of vertices in V is joined by at most one edge).

20.9. **Proposition (The Union of Edges is a Tree)**
The union T of edges is a tree, and every endpoint of T belongs to $S(\nu)$.

Proof. Since V is finite, T is finite. By construction, each $s \in S(\nu)$ is connected to $\star\nu$, and hence T connects all $s \in S(\nu)$. If $v \in V \setminus S(\nu)$, then by Lemma 20.8 item (4), $v \in E(s,\star\nu)$ for some $s \in S(\nu)$. Therefore T is connected.

Let us prove that T contains no loops. Since we constructed T by attaching strings of edges $E(\star\nu,s)$, T can only have a loop if the following occurs:

> There are $s,s' \in S(\nu)$ such that $t \in E(\star\nu,s) \cap E(\star\nu,s')$, but there is $u \in E(\star\nu,s) \setminus E(\star\nu,s')$ such that $\star\nu \succ u \succ t$ in the $E(\star\nu,s)$-order.

We show that the above cannot happen. Indeed, by Lemma 20.8 item (1), u is the middle point in the flat triod $[\star\nu,u,t]$. Therefore $u \in E(\star\nu,t)$. Next apply Lemma 20.8

item (2) to the quadruple $\star\nu, u, t, s'$ to conclude that $u \in E(\star\nu, s')$, but this contradicts that $u \in E(\star\nu, s) \setminus E(\star\nu, s')$. Hence T contains no loop.

Finally, each $v \in V \setminus S(\nu)$ is obtained as a branch point of a branched triod, and therefore cannot be an endpoint. Hence all the endpoints of T belong to $S(\nu)$. □

20.10. **Lemma (Components of the Tree)**
There are at most two edges in T with $\star\nu$ as endpoint, and $T \setminus \{\star\nu\}$ consist of at most two trees: the sequences starting with `0` *form one tree, and the sequences starting with* `1` *form the other.*

Proof. If $\star\nu$ has the endpoint of at least three edges, then there are $s, t, u \in S(\nu)$ such that $E(\star\nu, s)$, $E(\star\nu, t)$ and $E(\star\nu, u)$ are pairwise disjoint, except for the common endpoint $\star\nu$. At least two, say s and t, share the first symbol. Therefore $\varphi([s, t, u]) = [\sigma(s), \sigma(t), \sigma(u)]$ or $[\sigma(s), \sigma(t), \nu]$. Furthermore, $\sigma(E(\star\nu, s)) = E(\nu, \sigma(s))$ is disjoint from $\sigma(E(\star\nu, t)) = E(\nu, \sigma(t))$, except for the common endpoint ν. Lemma 20.8 item (4) states that ν is the middle point of $[\sigma(s), \sigma(t), \nu]$, but this contradicts Lemma 20.3. Therefore $\star\nu$ has at most two arms, and $T \setminus \{\star\nu\}$ consists of at most two components, each of which is connected and contain no loops.

For the last statement, recall that $E(\star\nu, s) \setminus \{\star\nu, s\}$ contains all sequences in $v \in V$ that are the middle point of a flat triod $[\star\nu, v, s]$. Applying φ to it does not result in the `stop` case, so s and v share the first symbol. Therefore the components of $T \setminus \{\star\nu\}$ are $\cup_{s \in S(\nu), s_1 = \mathtt{0}} E(\star\nu, s) \setminus \{\star\nu\}$ and $\cup_{s \in S(\nu), s_1 = \mathtt{1}} E(\star\nu, s) \setminus \{\star\nu\}$, and all the sequences in one component have the same the first symbol. □

Dynamics of the Tree. In order to define a map $f\colon T \to T$, set $f(s) := \sigma(s)$ for $s \in V$. For any edge $[s, t]$ between vertices $s, t \in V$, define the map $f|_{[s,t]}\colon [s, t] \to [f(s), f(t)] \subset T$ to be an orientation preserving homeomorphism. Since T is a tree, the map $f|_{[s,t]}$ is unique up to homotopy.

20.11. **Lemma (Dynamics Locally Injective)**
For every connected subtree $T' \subset T$ such that $\star\nu$ does not disconnect T', the restriction of f to T' is injective.

Proof. Since T and T' are connected trees, the fact that $f(x) = f(y)$ for $x, y \in T'$ would imply that f was not locally injective on every point in $[x, y] \subset T'$. It therefore suffices to prove that f is locally injective for every $x \in T' \setminus \{\star\nu\}$.

By construction, f is locally injective on every interior point of every edge. It thus suffices to show that f is locally injective at every vertex $s \neq \star\nu$. If this was not the case, then there would be vertices $t, u \in T'$ such that $[s, t]$ and $[s, u]$ were disjoint except for the common endpoint s, while $[f(s), f(t)]$ and $[f(s), f(u)]$ had more points in common than $f(s)$. But Lemma 20.8 item (3) implies that the triod $[s, t, u]$ is flat with s in the middle, and the triod $[f(s), f(t), f(u)] = \varphi([s,t,u])$ is flat with $f(s)$ in the middle, so $f(s) \in [f(t), f(u)]$. This is a contradiction. $\square$

We are now ready to prove Theorem 3.10.

Proof. We start with the existence proof. Set $c_0 := \star\nu$ (the critical point) and $c_k := \sigma^{\circ k}(\star\nu)$ (the critical orbit). Construct a tree T with dynamics as in Proposition 20.9. We check the six properties of a Hubbard tree:

(1) The map f is clearly continuous on T, and it is surjective on $S(\nu) \setminus \{\star\nu\}$. If $\nu = \overline{1^k\star}$ for some k, then f permutes (and hence is surjective on) $S(\nu)$, whereas if $\nu = 1^k 0\ldots$, then $\star\nu$ is obviously the middle point of the triod $(\nu, \star\nu, \sigma^{\circ k}(\nu))$, and hence not an endpoint. By Proposition 20.9, this means that f is surjective on the set of endpoints of T, hence surjective onto all of T.

(2) In $T \setminus \{c_0\}$, any two vertices whose itineraries start with the same entry are in the same connected component by construction of T. Therefore, $T \setminus \{c_0\}$ consists of at most two connected components, and f is injective on each of the by Lemma 20.11. Therefore, each point in T has at most two preimages.

(3) It follows from Lemma 20.11 that f is locally injective at every $s \in T \setminus \{c_0\}$. It is continuous by definition, hence a local homeomorphism everywhere except at the critical point.

(4) It has been shown in Proposition 20.9 that every endpoint of T is in $S(\nu)$, *i.e.*, on the critical orbit.

(5) The critical point is obviously periodic or preperiodic because ν is.

Before proving that the last condition, it is worthwhile to interpret the sequences in V as itineraries. By Lemma 20.10, $T \setminus \{c_0\}$ consists of at most two connected components such that two sequences $s, t \in V \setminus \{\star\nu\}$ are in the same component if and only if their initial entries coincide. Therefore, every vertex $s \in V$ (which is a sequence in $\{0, 1, \star\}$) encodes the itinerary of its own dynamics with respect to the usual partition of T induced by removing $\star\nu$.

(6) Expansivity is now trivial: marked points are in V, and these are distinguished by their itineraries. If the itineraries of s and t first differ in the k-th position, then the arc $[f^{\circ k}(s), f^{\circ k}(t)]$ contains the critical point $\star\nu$.

Uniqueness has already been proved in Proposition 3.15. $\square$

Proof of Corollary 3.11. The uniqueness is proven in Corollary **??**. For the existence, we start by building the Hubbard tree (T, f) for ν, and then add points from $\hat{V}$ and arcs where necessary in an inductive procedure. We describe this procedure in detail, using points on the tree and their symbolic itinerary interchangeably.

For $w \in \hat{V}$, we iterate the triod map φ on $[w, s, t]$ for all marked points $s, t \in V$, to decide how to attach w to the tree. Recall that marked points $s, t \in V$ are adjacent if the arc $E(s,t) = \{s, t\}$. There are three possibilities:

(a) There are adjacent $s, t \in T$ such that $[w, s, t]$ is flat with w in the middle. Put w as a new marked point on the arc $[s, t]$.
(b) We can find $s \in V$ such that for all $t \in V$ adjacent to s, the triod $[w, s, t]$ is flat with s in the middle. In this case, attach an arc $[w, s]$ to s. (The points w and s are adjacent, as long as we don't have to add new marked point on the open arc (w, s) later in the process.)
(c) If neither (a) nor (b) hold, then there is at least one pair of adjacent vertices $s, t \in V$ such that $[w, s, t]$ is a branched triod. Let $b := \mathfrak{b}(w, s, t)$. Note that Lemma 20.3 no longer holds for $V \cup \hat{V}$, and it can indeed happen that $b = b_1 b_2 \dots b_k \mathtt{0}\nu$ or $b_1 b_2 \dots b_k \mathtt{1}\nu$ for some (possibly empty) word $b_1 b_2 \dots b_k \in \{\mathtt{0}, \mathtt{1}\}^k$. In this case, recode b to $b_1 b_2 \dots b_k \star \nu$ and attach an arc $[w, b_1 b_2 \dots b_k \star \nu]$ to $b_1 b_2 \dots b_k \star \nu$. Put b on the arc $[s, t]$ and attach an arc $[w, b]$ to b. (The points w and b are adjacent, as long as we don't have to add new marked point on the open arc (w, b) later in the process.)

Choose the next $w' \in \hat{V}$ and repeat the whole process with the tree with dynamics and marked points created so far. After all of $\hat{V}$ is treated, we check that the resulting graph is indeed a proper extended Hubbard tree.

(i) $(\hat{T}, \hat{f})$ is closed under taking triods, *i.e.,* for any choice of marked points $s, t, u \in \hat{T}$, the branch point $\mathfrak{b}(s, t, u)$ already exists in $\hat{T}$. This is the same proof as in Lemma 20.6, but without the burden to verify that the sequences $\mathtt{0}\nu$ and $\mathtt{1}\nu$ do not appear among the marked points. This is because in the construction of extended Hubbard trees, we replace $\mathtt{0}\nu$ and $\mathtt{1}\nu$ with $\star\nu$ as under case (c) above. Note also that if $b = \mathfrak{b}(s, t, u)$ for some $s, t, u \in V \cup \hat{V}$, then $\sigma(b) = \mathfrak{b}(\varphi[s, t, u])$, and the three sequence of $\varphi[s, t, u]$ belong to $V \cup \hat{V}$ as well. Therefore the set of marked points is σ-invariant.

(ii) $\hat{T}$ has no loops. This is immediate because T has no loops, and only arcs are attached to T in the process of creating $\hat{T}$. As each of these attached arcs has a point in $\hat{V}$ as endpoint, each endpoint of $\hat{T}$ belongs to $V \cup \hat{V}$.

(iii) We define the dynamics of $\hat{f}$ on the (σ-invariant set of) marked points by σ. This shows that $\hat{f}$ is at most two-to-one on the marked points, and (since $\mathtt{0}\nu$, $\mathtt{1}\nu$ and $\star\nu$ are identified in case (c)) also that $\hat{f}$ is expansive. An arc $[s, t]$ between two adjacent marked points will be mapped homeomorphically onto the arc $[\sigma(s), \sigma(t)]$, precise as explained above Lemma 20.11. Lemma 20.11 itself then shows that $\hat{f}$ is locally injective, and in fact that every point in $\hat{T}$ has at most two preimages. This verifies all the conditions of an extended Hubbard tree. $\square$

Remark. Since ν need not be an endpoint in an extended Hubbard tree, it can happen that the critical point $\star\nu$ is a branch point. Figure 20.3 illustrates this for $\nu = \mathtt{10}\overline{\mathtt{1}}$ and the tree is extended to include sequences $w = \overline{\mathtt{1100}}$ and $\beta = \overline{\mathtt{0}}$. In fact, if ν is $\star$-periodic, then c_0 can get any number of arms. For example, if $\nu = \overline{\mathtt{10}\star}$, then

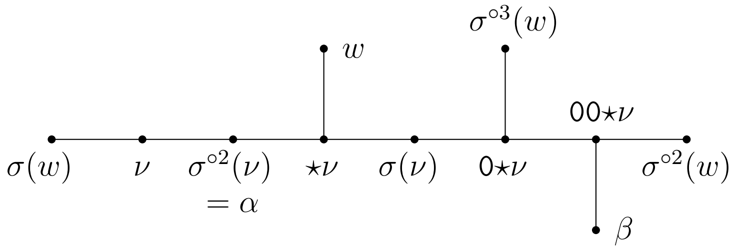


FIGURE 20.1. An example of an extended Hubbard tree where $\star\nu$ is a branch point.

$T = [c_1, c_2]$ is an arc and when $\hat{V} = \{\sigma^{\circ i}(w) : i \geq 0\}$ for

$$w = \underbrace{\mathtt{110110110\ldots110}}_{\mathtt{110}\text{ repeated } n \text{ times}}\mathtt{0}\star\nu,$$

then in the extended tree $(\hat{T}, \hat{f})$, the critical point c_0 has $n+2$ arms and c_1 and c_2 both have $n+1$ arms, see Figure 20.2.

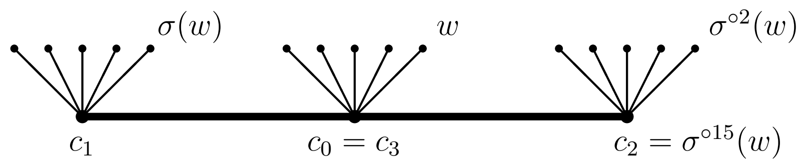


FIGURE 20.2. The extended Hubbard tree for $\nu = \overline{\mathtt{10}\star}$ and $\hat{V} = \{\sigma^{\circ i}(w) : i \geq 0\}$ with $w = (\mathtt{110})^n\mathtt{0}\star\nu$ for $n = 5$. The Hubbard tree is in bold lines.

Sometimes, we need to extend the Hubbard tree by finitely many additional periodic or preperiodic points: for example, we need the existence of a second fixed point with itinerary $\overline{\mathtt{0}}$, in addition to the α fixed point with itinerary $\overline{\mathtt{1}}$. The result is as follows.

20.12. **Theorem (Existence of Extended Hubbard Trees)**
Let ν be a $\star$-periodic or preperiodic kneading sequence, and let V_0 be a finite set of periodic or preperiodic itineraries none of which has a forward shift which equals 0ν or 1ν. Then there is an extended Hubbard tree *T' with a continuous map $f\colon T' \to T'$ with the following properties:*

- *there is a connected f-invariant subtree $T \subset T'$ such that the restriction of f to T is the Hubbard tree for ν;*

- *the map $f\colon T' \to T'$ satisfies all axioms of a Hubbard tree, except that the map need not be surjective onto T' and the endpoints need no longer be on the critical orbit;*
- *all endpoints have finite orbits, and the expansivity condition is extended (beyond branch points and points on the critical orbit) to all endpoints and their forward orbits;*
- *for every $v \in V_0$, there is a point in T' with itinerary v.*

This extended tree is unique (up to equivalence).

Sketch. The proof runs along the same lines as the proof for usual Hubbard trees, except that the initial set $S(\nu)$ is replaced by a set

$$S_0(\nu) := \left\{\star\nu, \nu, \sigma(\nu), \sigma^{\circ 2}(\nu), \dots\right\} \cup \bigcup_{v\in V_0} \bigcup_{k\geq 0} \{\sigma^{\circ k}(v)\} \ .$$

Remark. Since ν need not be an endpoint in an extended Hubbard tree, it can happen that the critical point $\star\nu$ is a branch point. Figure 20.3 illustrates this for $\nu = \mathsf{10\overline{1}}$ and the tree is extended to include sequences $v = \overline{\mathsf{1100}}$ and $\beta = \overline{\mathsf{0}}$.

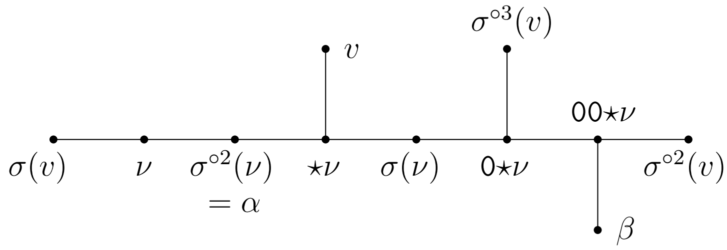


Figure 20.3. An example of an extended Hubbard tree where $\star\nu$ is a branch point.

The current version of this section (Delft, 23 Sept 09) contains both the old version of the existence proof of extended Hubbard trees (with just a sketch of the proof, and with the proof number different from 3.11), and Henk's new version of the proof, but still not quite complete. This needs to be worked out.

There is also an algorithm at the beginning of Henk's version that shows how to compute the external angle from the Hubbard tree. That algorithm should of course appear somewhere in the book, and it should presumably go (or be) in Section 13.

21. Infinite Hubbard Trees and Abstract Julia Sets

We show how the (infinite) Hubbard tree of a non-periodic kneading sequence ν can be obtained as a limit of the Hubbard trees of $\star$-periodic kneading sequences approximating ν. Next we define abstract Julia sets, in analogy to Hubbard trees, and show that they can be obtained as appropriate inverse limit spaces over Hubbard trees.

It is natural to think that if ν is a non-periodic kneading sequence, Algorithm 15.2 can be used as an infinite procedure to find the (infinite) Hubbard tree of ν. In this section, we show that this is indeed true. In Proposition 21.3 we make precise how one can interpret the limit. If ν is strictly preperiodic, even though the internal address is infinite, then a sequence of Hubbard trees with increasing numbers of marked points in the limit yields a finite Hubbard tree, as illustrated in Example 21.4.

21.1. **Definition (Dendrite)**
A topological space X is a dendrite *if it is compact, metrizable, connected and locally connected and if it contains no closed curves.*

By this definition, the topological dimension of X is automatically 1 (unless X is a point or empty).

21.2. **Definition (Infinite Hubbard Tree or Hubbard Dendrite)**
An infinite Hubbard tree *or* Hubbard dendrite *is a tree or dendrite which satisfies all the conditions of Definition 3.2, except for the finiteness of the critical orbit, and for which each branch point has only finitely many arms.*

The last requirement of this definition is not part of the standard definition of dendrite in topology (see *e.g.* Kuratowski [**Ku**]), and we cannot make such a requirement for abstract Julia sets below.

21.3. **Proposition (Hubbard Tree of an Infinite Internal Address)**
For any infinite internal address one can construct the corresponding Hubbard tree (or dendrite) as an inverse limit space.

Proof. Obtaining a dendrite as an inverse limit space over a sequence of monotone sequence of bonding maps is a common procedure in topology [**Na**]. We give the details for our specific case. Assume that we have an infinite internal address $1 \to S_1 \to S_2 \to \dots$ with corresponding kneading sequence ν. Denote by (T_k, f_k) the Hubbard tree for each finite part $1 \to S_1 \to \dots \to S_k$ of the internal address. In Algorithm 15.2 we saw how to construct (T_{k+1}, f_{k+1}) from (T_k, f_k). The tree T_k contains a characteristic periodic point p_1 with itinerary $\overline{\nu_1 \dots \nu_{S_{k-1}}}$. Let

$$U_k = \{x \in T_k \ : \ f_k^{\circ i}(x) \notin (p_1, \dots, p_{S_{k-1}}) \text{ for some } i \geq 0\},$$

where $(p_1, \dots, p_{S_{k-1}})$ denotes the open connected hull of $\mathrm{orb}(p_1)$. Write $x \sim y$ if $x = y$ or x and y belong to the closure of the same connected component of U_k. There is a one-to-one correspondence between $T_k/\sim$ and T_{k-1}. Denote the resulting projection by $\pi_{k-1} : T_k \to T_{k-1}$. Build the inverse limit space

$$T_\infty = (T_k, \pi_k) = \{x = x_1x_2\dots \ : \ \pi_k(x_{k+1}) = x_k \in T_k \text{ for all } k \geq 1\},$$

equipped with product topology. Define $f_\infty : T_\infty \to T_\infty$ by

$$f_\infty(x)_k = \lim_{i\to\infty} \pi_k \circ \dots \circ \pi_i \circ f_{i+1}(x_{i+1}).$$

This is well-defined because the sequence $\{\pi_k\circ\dots\circ\pi_i\circ f_{i+1}(x_{i+1})\}_i$ stabilizes. Note however that $y = f_\infty(x)$ does not imply that $f_i(x_i) = y_i$ for every $i \in \mathbb{N}^*$, as Example 21.4 below shows. For $x \in T_\infty$, define its *kneading symbol* as

$$\begin{cases} \mathtt{0} & \text{if } x_i \text{ is on the } \mathtt{0}\text{-side of } T_i \text{ for } i \text{ sufficiently large,} \\ \star & \text{if } x = (0,0,0,\dots) \text{ is the critical point,} \\ \mathtt{1} & \text{if } x_i \text{ is on the } \mathtt{1}\text{-side of } T_i \text{ for } i \text{ sufficiently large.} \end{cases}$$

Note that π_i preserves the side of the tree, unless $\pi_i(x) = 0$ and $x \neq 0$. Therefore above definition is meaningful. The itinerary and kneading sequence are defined in the usual way. We check that the above construction satisfies the following properties.

(i) T_∞ is a tree or a dendrite.
(ii) $f_\infty : T_\infty \to T_\infty$ is continuous, surjective and at most two-to-one.
(iii) f_∞ has a single critical point $(0,0,0,\dots)$ and is a local homeomorphism onto its image elsewhere.
(iv) The kneading sequence of (T_∞, f_∞) is equal to ν.

(i) We first check the properties of dendrites, including the finiteness of arms at every branch point.
• Compactness follows directly from Tychonov's Theorem, and the metric

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x_k, y_k)$$

for arc-length metric d_k on T_k is compatible with product topology.
• T_∞ is one-dimensional. Let $g_k : T_\infty \to T_k,\ g_k(x_1x_2\dots) = x_k$ be the k-th projection; in the product topology of T_∞, g_k is continuous. Observe that

$$g_k^{-1}(y \in T_k) = \{(\pi_1 \circ \dots \circ \pi_{k-1}(y), \pi_2 \circ \dots \circ \pi_{k-1}(y), \dots, \pi_{k-1}(y), y, y, y, \dots)\} \quad (25)$$

is a singleton whenever y is not precritical in (T_k, f_k). It is easily seen that

$$\mathcal{U} := \{g_N^{-1}(U) \ : \ N \in \mathbb{N}^*,\, U_N \text{ open in } T_N \text{ and } \partial U \cap \cup_i f_N^{-i}(0) = \emptyset\}.$$

is a basis of the product topology. Let $x \in T_\infty$ and $U' = g_N^{-1}(U) \in \mathcal{U}$ be an open neighborhood of x. Without loss of generality, ∂U consists of finitely many non-precritical points in T_N. By (25), $\partial U' = g_N^{-1}(\partial U)$ consists of finitely many points as well. Hence

$\partial U'$ is zero-dimensional. This proves that T_∞ is one-dimensional (in the inductive dimension).

Let

$$T_\infty^* := \{x \in T_\infty \ : \ x_n \text{ is not eventually constant}\}.$$

We claim that T_∞^* contains no arc. Indeed, if $[x, y] \subset T_\infty^*$ for distinct point x and y, find N such that $x_N \neq y_N$. Take $z_N \in [x_N, y_N] \subset T_N$ non-precritical. As $g_N([x, y])$ is connected and compact, it contains the arc $[x_N, y_N]$. But then $g_N^{-1}(z_N) \cap [x, y] \neq \emptyset$, which contradicts (25).

• T_∞ contains no closed curves. Indeed, assume by contradiction that T_∞ contains distinct points x and y and arcs L and L' connecting x and y such that L and L' have disjoint interiors. Find N such that $x_N \neq y_N$ and take $z_N \in (x_N, y_N) \subset T_N$ non-precritical. Then $g_N^{-1}(z_N)$ intersects both L and L', contradicting (25).

• Each branch point in T_∞ has only finitely many arms. There are no branch points in T_∞^*. Indeed, suppose that $[x, y, z]$ forms a non-degenerate triod in T_∞. Find N such that x_N, y_N and z_N are all distinct and form a non-degenerate triod in T_N. Let w_N be its branch point. As $0 \in T_N$ is not a branch point, w_N is not precritical. Therefore $g_N^{-1}(w_N)$ consists of a single point $w \in T_\infty \setminus T_\infty^*$ which is the branch point of $[x, y, z]$. This shows that T_∞ has at most countably many branch points, each of which have only finitely many arms.

• T_∞ is arc-connected and (hence) locally connected. Let $x, y \in T_\infty$ be arbitrary. Construct recursively homeomorphic parametrizations $\gamma_i : [0, 1] \to [x_i, y_i] \subset T_i$ such that $\sup_t |\gamma_i^{-1} \circ \pi_i \circ \gamma_{i+1}(t) - t| \leq 2^{-i}$. Define

$$\gamma : [0, 1] \to T_\infty, \ \gamma(t)_k = \lim_i \pi_k \circ \cdots \circ \pi_{i-1} \circ \gamma_i(t).$$

By construction, $\gamma_k^{-1} \circ \pi_k \circ \cdots \circ \pi_{i-1} \circ \gamma_i$ converges uniformly on $[0, 1]$ for each k. Therefore, the limits

$$\gamma(t)_k = \gamma_k(\lim_i \gamma_k^{-1} \circ \pi_k \circ \cdots \circ \pi_{i-1} \circ \gamma_i(t))$$

exist and depend continuously on $t \in [0, 1]$. Obviously $\gamma(t)_k = \pi_k \circ \gamma(t)_{k+1}$ for all k, and $\gamma(0) = x$ and $\gamma(1) = y$. Hence γ parametrizes an arc connecting x and y. Let $x \in T_\infty$ be arbitrary and $U \in \mathcal{U}$ be a neighborhood of x. The above construction shows that U is arc-connected as well. Thus T_∞ is locally connected.

(ii) The continuity properties of f_∞ are easily decided by means of the basis $\mathcal{U}$. By (25) it follows that for each $U' = g_N^{-1}(U) \in \mathcal{U}$, we have $f_\infty^{-1}(U') = g_N^{-1}(f_N^{-1}(U)) \in \mathcal{U}$. Thus f_∞ is continuous.

(iii) Similarly, if $U' = g_N^{-1}(U) \in \mathcal{U}$ and $0 \notin U$, then $f_\infty(U') = g_N^{-1}(f_N(U)) \in \mathcal{U}$, showing that f_∞ is a local homeomorphism on $T_\infty \setminus \{0\}$. Obviously f_∞ is not one-to-one at any neighborhood of $\{0\}$. If there are distinct point x, y and z such that $f_\infty(x) = f_\infty(y) = f_\infty(z)$, then we can find $N \in \mathbb{N}^*$ and arbitrarily small pairwise disjoint neighborhoods $U(x_N) \ni x_N$, $U(y_N) \ni y_N$, $U(z_N) \ni z_N$ such that $f_N(U(x_N)) \cap$

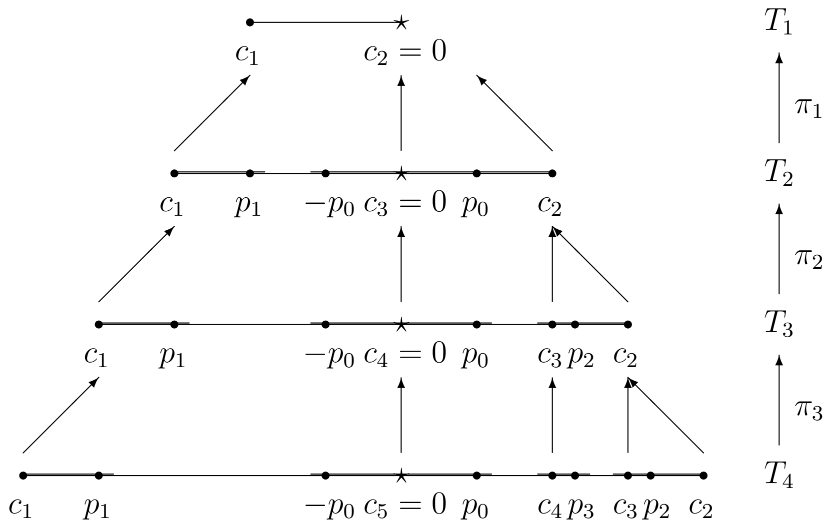


FIGURE 21.1. Construction of the Hubbard Tree with $1 \to 2 \to 3 \to \dots$

$f_N(U(y_N)) \cap f_N(U(z_N)) \neq \emptyset$. This contradicts that f_N is at most two-to-one. Thus f_∞ is at most two-to-one as well.
(iv) Finally, because π_n preserves the symbol of points (except when $\pi_n(x) = 0 \in T_n$), the symbols of $c_n \in T_k$ are the same whenever $S_k > n$. This shows that $\nu(f_\infty)$ coincides with $\nu(f_k)$ up to entry $S_k - 1$. Therefore $\nu(f_\infty) = \lim_{k\to\infty} \nu(f_k)$. $\square$

21.4. **Example (Finite Tree with Infinite Internal Address)** Figure 21.1 shows the construction for the internal address $1 \to 2 \to 3 \to 4 \to \dots$ The thick lines indicate some of the components of U_k which are squeezed to points by π_{k-1} (so c_2 and $c_3 \in T_k$ always belong to the same component of U_k). The map f_∞ on T_∞ is conjugate to $z \mapsto z^2 - 2$ on $[-1, 1]$. Observe that $(c_2, c_2, c_2, \dots)$ is a fixed point of f_∞, while $f_i(c_2) = c_3 \neq c_2$ for each i.

A Hubbard tree in the sense of Douady & Hubbard (Definition 3.4) is a subset of the Julia set J. One can obtain the Julia set from the Hubbard tree by taking all full preimages and then taking the closure. In the remainder of this section, we will use an inverse limit construction to do the same for the Hubbard tree (finite or infinite) constructed in Algorithm 15.2 and Proposition 21.3. Thus we obtain a locally connected model for the Julia set. Note that the construction is purely topological; it does not depend on an embedding of the Hubbard tree into the plane. Another approach to construct Julia sets was given in [**BL**].

21.5. **Definition (Abstract Julia Set)**
An abstract Julia set (J, f) *is a dendrite* J *with a mapping* f *and a distinguished point* 0*, the* critical point *such that*

(1) $f : J \to J$ *is continuous and surjective;*
(2) *every point in* $J \setminus \{f(0)\}$ *has exactly two inverse images under* f*;*
(3) *near every point other than the critical point, the map* f *is a local homeomorphism onto its image;*
(4) *(expansivity) if* z *and* z'*,* $z \neq z'$ *are branch points or points on the critical orbit, then there is an* $n \geq 0$ *such that* $f^{\circ n}([z, z'])$ *contains the critical point.*

Note that we do not require that the critical orbit be finite, nor that branch points have only finitely many arms. As we will in Example 21.8, if 0 is periodic, then every point on $\text{orb}_f(0)$ has infinitely many arms in J. The connected hull T of $\text{orb}_f(0)$ in J is the Hubbard tree (or dendrite) and c_1 is always an endpoint of T. This implies that 0 has only two arms in T, and from this it follows that each branch point has in fact finitely many arms in T. The characterizing difference with the Hubbard tree (or dendrite) is that $f : J \to J$ is two-to-one everywhere, except at the critical value, which has only one preimage. We can equip (J, f) with symbolic dynamics in the usual way: Since f is two-to-one, $J \setminus \{0\}$ has an even number of components (in general more than two, because the critical value is not assumed to be an endpoint of J). The component of $J \setminus \{0\}$ containing the critical value obtains the symbol $\mathtt{1}$. Give the other components symbols in such a way that if $z \neq z'$ and $f(z) = f(z')$, then $\{e_1(z), e_1(z')\} = \{\mathtt{0}, \mathtt{1}\}$.

21.6. **Algorithm (Hubbard Tree to Abstract Julia Set)**
The abstract Julia set can be modeled as inverse limit space over the Hubbard tree. First we introduce some notation: If $0 \neq x \in T$ *then* $-x \neq x$ *is the point such that* $f(-x) = f(x)$*. Obviously* x *and* $-x$ *have different symbols in the symbolic dynamics. For completeness,* $-0 = 0$*. The point* $-x$ *need not exist in* T*; in this case we can extend* T *in the obvious way as to include* $-x$*. (Once the abstract Julia set* (J, f) *is constructed, it will be clear that* $-x$ *exists for every* $x \in J$*.) Let* $(J_1, f) = (T \cup -T, f)$*. Suppose* (J_i, f) *is constructed. Take any point* $x \in J_i$ *such that* $f(x)$ *has more arms than* x *(take also* $x = 0$*, if* c_1 *has more than half the arms at* 0*). Add a decoration at* x *which is homeomorphic to the missing (set of) arm(s) at* $f(x)$ *and extend* f *to this decoration in the obvious way. The symbol of this decoration is the same as the symbol of* x*. If* $x = 0$*, then we add a pair of such decorations, one with symbol* $\mathtt{0}$ *and the other with symbol* $\mathtt{1}$*. (If we consider* J_i *as embedded in the plane, then we can also embed these decorations in such a way that* f *preserves the cyclic order of the (old and new) arms at* x*.) The result of all these additions is called* (J_{i+1}, f)*. Obviously,* $J_i \subset J_{i+1}$ *and there is a natural projection* $\pi_i : J_{i+1} \to J_i$ *defined as* $\pi_i(z) = z$ *if* $z \in J_i$ *and* $\pi_i(z) = x$ *if* z *lies in a decoration attached to* $x \in J_i$*. Finally build the inverse limit*

$$J = (J_i, \pi_i) = \{(z_1, z_2, \dots) \ : \ J_i \ni z_i = \pi_i(z_{i+1}) \text{ for all } i \geq 1\},$$

with the product topology. Define $f : J \to J$ *as* $f((z_1, z_2, \dots)) = (f(z_1), f(z_2), \dots)$. *Note that* $z_i \mapsto (\pi_1 \circ \dots \circ \pi_{i-1}(z_i), \pi_{i-1}(z_i), z_i, z_i, \dots)$ *gives an embedding of* J_i *into* J.

21.7. **Lemma (Properties of Abstract Julia Sets)**
The inverse limit space J *is a dendrite, and if* z *separates* J*, then* $f^{\circ n}(z) \in T$ *for some* $n \geq 0$*. In particular, each branch point in* J *is either preperiodic or precritical.*

PROOF. The proof proceeds analogously to Proposition 21.3. Observe that the decorations of $J_{i+1} \setminus J_i$ are always attached to (preimages of) marked points of J_i, *i.e.,* points on the critical orbit or branch points. It follows by induction that endpoints of J_{i+1} are in the same grand orbit of the critical point. Therefore

$$\mathcal{U} := \{g_N^{-1}(U) \ : \ N \in \mathbb{N}^*, U \text{ open in } J_N \text{ and } \partial U \cap \cup_{i,j \geq 0} f_N^{-i}(f_N^{\circ j}(0)) = \emptyset\}$$

(where $g_N : J \to J_N$ is the N-th projection) is a convenient basis of the product topology of J. The proof of Proposition 21.3 now applies to show that J is a dendrite, and that $J \setminus J^*$ contains no arcs for $J^* := \{z \in J \ : \ z_i \text{ is not eventually constant}\}$. The set $T^* := \{z \in J : z_n \in T \text{ for all } n\}$ is the embedded version of T in J. Since the decorations of $J_{i+1} \setminus J_i$ of J_i do not separate J_i, a point z separates J only if $z \notin J^*$. But then $f^{\circ n}(z) \in T^*$ for some $n \geq 0$. If z is a branch point, then $f^{\circ n'}(z)$ is a branch point in T for some $n' \geq n$. This follows because $f^{\circ n}(z) \in T$ is a branch point in J, and "branch point decorations" of T are attached to 0 or preimages of branch points. $\square$

21.8. **Example (Julia Sets with Branch Points with Infinitely Many Arms)**
Figure 21.2 illustrates the procedure for the case that $\nu = \overline{10\star}$, so 0 has period three. Every third step in the construction gives extra arms to 0, so that in the limit, 0 (as well as all its preimages) has infinitely many arms. It provides a topological model of the Julia set with its period three Siegel disks squeezed to points with infinitely many arms.

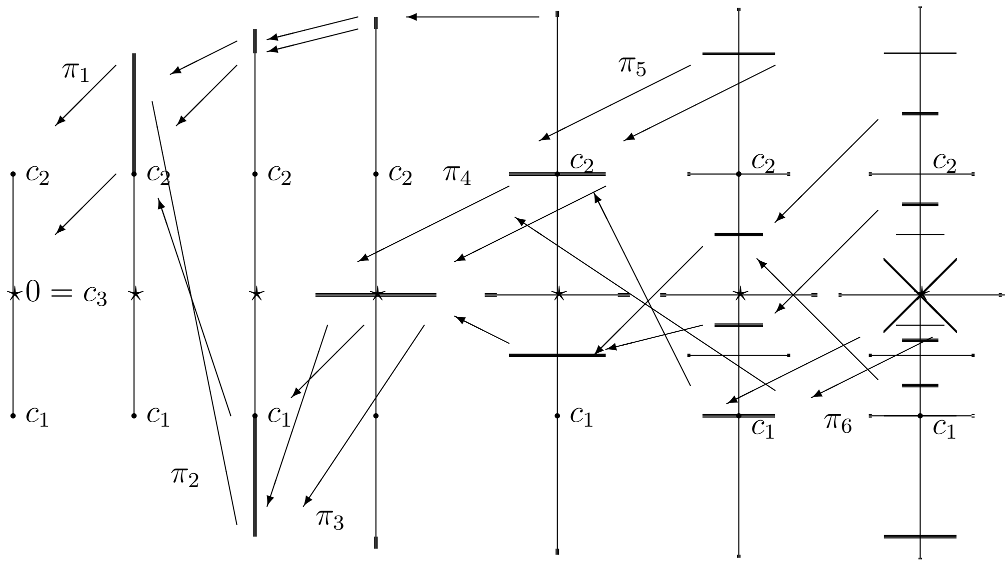


FIGURE 21.2. Inverse limit construction of the abstract Julia set with kneading sequence $\nu = \overline{10\star}$. Arrows indicate the action of π_k; not all are drawn as not to clutter the figure. Thick lines indicate the parts of $J_j \setminus J_{j-1}$.

22. Symbolic Dynamics of Unicritical Polynomials

In this section we extend the discussions of Section 3–8 to the unicritical setting. In fact, the statements and arguments of these sections carry over to the unicritical situation with only little modification. We give a summary of the most important results for unicritical Hubbard trees, kneading sequences and the parameter tree of degree $d > 1$ and outline their proofs; for more details we refer to [**Kaf1**]. We stick to the reasoning of the quadratic case as closely as possible, while in [**Kaf1**] sometimes slightly different approaches were taken.

This section is structured the following way: we first introduce unicritical Hubbard trees and kneading sequences of degree $d \geq 2$ and show that these concepts are equivalent. Then we investigate the dynamical properties of Hubbard trees more closely. As in the quadratic case (Sections 6 and 7), the obtained results give structure to the parameter tree of degree d, i.e. to the space of kneading sequences of degree d.

22.1. Hubbard Trees and Kneading Sequences of Degree d. Hubbard trees and kneading sequences of degree $d > 2$ are defined analogously to the quadratic case. Note however that unlike quadratic Hubbard trees, Hubbard trees of degree $d > 2$ do not come with a partition that is unique in a natural way.

22.1. **Definition (Hubbard Tree of Degree d)**
A Hubbard tree of degree d *is a triple* $\mathcal{T} := (T, f, \mathcal{P})_d$ *consisting of a tree* T*, a map* $f\colon T \to T$ *with a distinguished point* c_0*, the* critical point *and a partition* $\mathcal{P}$ *of* T *that satisfy the following conditions:*

1. $f : T \to T$ *is continuous and surjective;*
2. *every point in* T *has at most d inverse images under* f*;*
3. *at every point other than* c_0*, the map* f *is a local homeomorphism onto its image;*
4. *all endpoints of* T *are on the critical orbit;*
5. *the critical point* c_0 *is periodic or preperiodic;*
6. *(Expansivity) if* $x \neq y$ *are branch points or points on the critical orbit, then there is an* $n \geq 0$ *such that* $c_0 \in f^{\circ n}([x, y])$*;*
7. $\mathcal{P} = \{\{c_0\}, T_0, \ldots, T_{d-1}\}$*, where each* T_i *is either a connected component of* $T \setminus \{c_0\}$ *or empty, and the indices are such that* $f(c_0) \notin T_0$*.*

Note that according to this definition the triple $(\{c_0\}, \mathrm{id}, \{\{c_0\}\})_d$ is also a Hubbard tree. As in the quadratic case, we exclude this trivial Hubbard tree in some statements if it provides a silly counterexample.

We call a point $p \in T$ a *marked point* if it is either a branch point or a point on the critical orbit. The marked points are exactly the *vertices* of the Hubbard tree.

Remark. A Hubbard tree $\mathcal{T}$ as defined above should actually be called a *unicritical Hubbard tree* because it contains only one critical point. In general, a Hubbard tree in the sense of [**DH1**] which is generated by a degree d polynomial can have up to $d - 1$

pairwise distinct critical points. For convenience, we drop the term "unicritical" in this section because we only consider Hubbard trees with exactly one critical point.

22.2. **Definition (Kneading Sequence of Degree d)**
Let us define

$$\begin{array}{rcl} \Sigma_d & := & \{0,\dots,d-1\}^{\mathbb{N}^*} \\ \Sigma_d^+ & := & \{\nu \in \Sigma_d \colon \textit{the first entry in } \nu \textit{ is unequal } 0\}, \\ \Sigma_d^\star & := & \Sigma_d^+ \cup \{\textit{all } \star\textit{-periodic sequences}\}, \\ \Sigma_d^{\star\star} & := & \{\nu \in \Sigma_d^+ \colon \nu \textit{ is non-periodic}\} \cup \{\textit{all } \star\textit{-periodic sequences}\}. \end{array}$$

A kneading sequence of degree d *is an element of the set* $\Sigma_d^\star$*.*

We define the ρ-function just as in the quadratic setting.

22.3. **Definition (ρ-Function (Unicritical Case))**
For a sequence $\nu \in \Sigma_d^\star$*, define*

$$\rho_\nu : \mathbb{N}^* \to \mathbb{N}^* \cup \{\infty\}, \quad \rho_\nu(n) = \inf\{k > n : \nu_k \neq \nu_{k-n}\}.$$

We usually write ρ *instead of* ρ_ν*. For* $k \geq 1$*, we call*

$$\mathrm{orb}_\rho(k) := k \to \rho(k) \to \rho^{\circ 2}(k) \to \rho^{\circ 3}(k) \to \dots$$

the ρ*-orbit of* k*. We call* $\mathrm{orb}_\rho(1)$ *the* ρ-orbit of the kneading sequence ν*. If there is a* k *such that* $\rho^{\circ k+1}(1) = \infty$*, then we say that the* ρ*-orbit of* ν *is* finite *and write* $1 \to \rho(1) \to \dots \to \rho^{\circ k}(1)$*. As a result, the orbit* $\mathrm{orb}_\rho(1)$ *is a finite or infinite sequence that never contains* ∞*.*

The *kneading sequences of a Hubbard tree* is the itinerary of its critical value $c_1 := f(c_0)$. (The itinerary of a point $p \in T$ is defined analogously to Definition 3.5, and hence an element of $\{0,\dots,d-1,\star\}^{\mathbb{N}^*}$.) Thus every Hubbard tree generates a $\star$-periodic or preperiodic kneading sequence in a natural way. Note that the kneading sequence $\overline{\star}$ belongs to the trivial Hubbard tree $(\{c_0\}, \mathrm{id}, \{\{c_0\}\})_d$. In the remainder of this section, we will give a short outline of the proof that every $\star$- and preperiodic kneading sequences in $\Sigma_d^{\star\star}$ is realized by a Hubbard tree of degree d. Indeed, there is a bijection between the set of $\star$- or preperiodic kneading sequences and the set of Hubbard trees of degree d up to the following equivalence relation.

Let $(T, f, \mathcal{P})_d$ and $(T', f', \mathcal{P}')_d$ be two Hubbard trees of degree d and let V, V' be the set of their marked points. $(T, f, \mathcal{P})_d$ and $(T', f', \mathcal{P}')_d$ are *equivalent* if there is a bijection between V and V' which is respected by the dynamics such that for all $v \in V$, $v \in T_i \subset T$ if and only if $\varphi(v) \in T'_i \subset T'$, and such that adjacent edges stay adjacent under φ.

22.4. **Theorem (Existence and Uniqueness of Hubbard Trees)**
Every $\star$*-periodic or preperiodic kneading sequence is realized by a Hubbard tree; this tree is unique (up to equivalence).*

The remainder of the subsection is devoted to prove this theorem. We proceed just as in the quadratic case and consider the set $S(\nu) = \{\star\nu, \nu, \sigma(\nu), \sigma^{\circ 2}(\nu), \dots\}$ for any $\star$-or preperiodic kneading sequence ν . We extend this set to $V(\nu) = \bigcup_{[s,t,u]} \mathfrak{b}(s,t,u) \cup S(\nu)$, which gives rise to the set of vertices of the yet to be constructed Hubbard tree. Here the union runs over all branched (formal) triods $[s,t,u]$ generated by $S(\nu)$, and $\mathfrak{b}(s,t,u)$ denotes the branch sequence of the branched triod $[s,t,u]$ (for a definition see page **??**). Using the triod map, we determine the relative location of any three elements $s,t,u \in V(\nu)$, that is, whether they form a flat or a branched (formal) triod. We join the points in $V(\nu)$ accordingly by straight lines. This yields a topological tree and by the method we constructed this tree, it follows that it is indeed a Hubbard tree. Let us discuss the construction in more detail.

22.5. **Definition (Formal Triod)**
Any triple of pairwise different sequences $s,t,u \in S(\nu) \cup \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}^*}$ is called a formal triod $[s,t,u]$.

22.6. **Definition (The Formal Triod Map)**
If s,t,u are pairwise different such that $i\nu \notin \{s,t,u\}$ for all $i = \mathtt{0},\dots,d-1$, then we define the formal triod map as follows:

$$\varphi([s,t,u]) := \begin{cases} [\sigma(s),\sigma(t),\sigma(u)] & \text{if } s_1 = t_1 = u_1; \\ \mathtt{stop} & \text{if } s_1, t_1, u_1 \text{ are pairwise distinct}; \\ [\sigma(s),\sigma(t),\nu] & \text{if } s_1 = t_1 \neq u_1 \ ; \\ [\sigma(s),\nu,\sigma(u)] & \text{if } s_1 = u_1 \neq t_1 \ ; \\ [\nu,\sigma(t),\sigma(u)] & \text{if } t_1 = u_1 \neq s_1 \ . \end{cases} \tag{26}$$

A triod $[s,t,u]$ can be iterated until time k if for all $0 \le i < k$, $\varphi^{\circ i}([s,t,u]) \neq \mathtt{stop}$ and the three obtained sequences are pairwise distinct. The triod $[s,t,u]$ can be iterated *infinitely often* if it can be iterated until time k for all $k \in \mathbb{N}$.

We say that $[s,t,u]$ is *flat* if exactly one of the following holds:

- $[s,t,u]$ can be iterated under φ infinitely often such that exactly one sequence is never chopped off;
- there is an iterate k such that $\varphi^{\circ k}([s,t,u]) = \mathtt{stop}$ and one of the sequences starts with the symbol $\star$. We call this event a `stop` *with* $\star$.

The triod $[s,t,u]$ is *branched* if exactly one of the following holds:

- $[s,t,u]$ can be iterated under φ infinitely often such that each sequence is chopped off once;
- there is an iterate k such that $\varphi^{\circ k}([s,t,u]) = \mathtt{stop}$ and none of the sequences starts with the symbol $\star$. We call this event a `stop` *without* $\star$.

Unicritical versions of the combinatorial Lemmas 20.1 and 20.3 imply that each $[s,t,u] \in V(\nu)$ is either a flat or branched triod. The proofs of these lemmas carry over to the unicritical case. [2]

The major difference to the quadratic setting is that now also branched triods may reach $\texttt{stop}$. Therefore, Lemma 20.4 reads as follows. The reasoning of the quadratic case carries over, yet one has to keep in mind that now a $\texttt{stop}$ without $\star$ might occur.

22.7. **Lemma (Branched Triods, Unicritical Case)**
Let $\nu \in \Sigma_d^\star$ be $\star$-periodic or preperiodic. If $[\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \sigma^{\circ m}(\nu)]$ is such that each of the three sequences is chopped off at least once, then either each sequence is chopped off infinitely often or eventually, a $\texttt{stop}$ without $\star$ occurs. □

22.8. **Lemma (Branch Points of Branched Triods)**
Fix a $\star$-periodic or preperiodic ν. Suppose that $s,t,u \in S(\nu) \cup \{\texttt{0},\texttt{1}\}^{\mathbb{N}^*}$ are such that $[s,t,u]$ is a branched triod; set $v := \mathfrak{b}(s,t,u)$. If $\sigma^{\circ k}(w) \notin \{\texttt{0}\nu, \texttt{1}\nu\}$ for all $w \in \{s,t,u,v\}$ and all $k \geq 0$, then $[s,t,v]$, $[s,u,v]$ and $[t,u,v]$ are flat with v in the middle.

Proof. By assumption, the triod map can either be iterated forever for $[s,t,u]$ or a $\texttt{stop}$ without $\star$ occurs, say at time n. In the second case, the triod $[s,t,v]$ is mapped to $\texttt{stop}$ after n or less iterations. If it is mapped to $\texttt{stop}$ after exactly n steps, then the image of v equals $\star\nu$ and the images of s,t start with different symbols in $\{0,\ldots,d-1\}$ because $\nu = \mathfrak{b}(s,t,u)$. It follows that $[s,t,v]$ is a flat triod. If the iteration of $[s,t,v]$ stops before a $\texttt{stop}$ is reached in $[s,t,u]$, say at time k, then either the iteration reaches one of the sequences $\texttt{0}\nu$ and $\texttt{1}\nu$ (which is excluded by hypothesis), or $[s,t,v]$ is mapped to $\texttt{stop}$; but since the first k entries $v \in \{\texttt{0},\texttt{1}\}^{\mathbb{N}^*}$ are constructed by majority vote among the sequences s,t,u, this is not possible. The majority vote also assures that v never can be chopped off before time k and for higher iteration steps this is guaranteed by (the unicritical version of) Lemma 20.3, so that $[s,t,v]$ is flat with v in the middle. The reasoning for the other two sequences $[s,u,v]$ and $[t,u,v]$ is the same. □

The remaining statements and proofs of Section 20 carry over literally. Note however that the role of $\texttt{0}\nu, \texttt{1}\nu$ is now played by the d sequences $i\nu$, where $i = \texttt{0},\ldots,d-1$:

In Lemma 20.6, it is essential to determine for any given time which sequence forms a minority vote. While in the quadratic case there was just one possible symbol for this vote, in the unicritical case of degree d we have $d-1$ options that all yield a minority

[2]In the proof of Lemma 20.1, the behavior of ν under the ρ-map is compared with the modified sequence $\tilde{\nu}$. For any quadratic kneading sequence ν, $\tilde{\nu}$ is defined in a unique way by specifying a position m and taking the opposite symbol there. In the unicritical case, we have $d-2$ choices for this symbol at position m. So we have to pay more attention when defining the sequences $\tilde{\nu}$ in the different parts of the proof. In particular, we have to choose $\nu'_m = \nu_{\rho(k)}$, and furthermore in **Case 1**, $\nu'_{m_0} = \nu_{\rho(k)}$ if $m_0 = m$ and $\nu'_{m_0} = \nu_{\rho(m_1)-m_1}$ if $m_0 > m$, where $m_1 \in \operatorname{orb}_\rho(\rho(k)-k)$ such that $\rho(m_1) = m_0$. In **Case 2**, we choose $\nu'_{n_0} = \nu_{\rho(n_1)-n_1}$, where $n_1 \in \operatorname{orb}_\rho(1)$ such that $\rho(n_1) = n_0$.

vote. However, for the argument it is only important to know whether a minority vote occurs or not, the specific symbol does not matter.

Lemmas 20.7–20.11 deal with flat triods. We only have a slightly differing statement for Lemma 20.10:

22.9. **Lemma (Components of the Tree of Degree d)**
There are at most d edges in T with $\star\nu$ as endpoint, and $T \setminus \{\star\nu\}$ consist of at most d connected components: for each $i \in \{0, \dots, d-1\}$, the sequences starting with i form a subtree of T.

Proof. To prove the corresponding Lemma in the quadratic case, we showed that there are no two edges $E(\star\nu, s), E(\star\nu, t)$ which intersect only in $\star\nu$ such that s and t start with the same symbol. Since now d symbols are at our disposal, the arguments of the quadratic setting prove the lemma. $\square$

To finish the proof of Theorem 22.4, we have to show that two Hubbard trees which generate the same kneading sequence are equivalent. For this it is enough to determine the relative location of any three points on the critical orbit. As in the quadratic case, the type of a triod $[c_j, c_k, c_l]$ (flat or branched) can be determined by iterating it under the triod map. The behavior under this iteration is completely given by the itineraries of the points c_j, c_k, c_l. So the relative location is completely encoded in the itineraries of the points on the critical orbit and thus in the kneading sequence of the Hubbard tree.

22.2. Dynamical Properties of Hubbard Trees. In this section we summarize the most important properties of Hubbard trees of degree d focusing on their periodic orbits. It turns out that the behavior is widely the same as in the quadratic case. This is due to the following equivalence which is the basis for all proofs in Sections 3 and 4. Let x, y be two distinct points in a Hubbard tree $\mathcal{T}$ and τ_x, τ_y their itineraries. Then

$$\tau_1(x) = \tau_1(y) \text{ or } c_0 \in \{x, y\} \iff c_0 \notin (x, y) \iff f|_{[x,y]} \text{ is injective.}$$

If x, y are contained on the same orbit, then the first equivalence says that the ρ-map can be used to find precritical points in $[x, y]$, see e.g. the proof of Lemma 4.8.

From the definition of Hubbard trees of degree d, it follows immediately that the critical value c_1 is an endpoint, that $f(c_1) \neq c_1$, and that each branch point is periodic or preperiodic. Unlike in the quadratic case, now a branch point might very well be precritical. If $c_1 \in T_i$, then the subtree T_i contains a unique fixed point, which is usually called the *α-fixed point*. It lies in (c_0, c_1).

By the above equivalence and the definition of the subtrees T_i, we have that for any itinerary τ, the set T_τ consisting of all points with itinerary τ is connected. If a periodic itinerary τ is realized by some point $p \in T$, then T_τ contains also a periodic point p'

such that the exact period of τ and p coincide. This is Lemma 3.9; its statement is very useful to find periodic points in a Hubbard tree, e.g. in the proof of Proposition 22.14.

The next important step to determine the behavior of subsets of T under f is the existence of *characteristic* points. For a definition, we refer to Definition 4.2. Just as in the quadratic case, any periodic orbit of a Hubbard tree of degree d has a characteristic point with one exception: if c_0 is preperiodic and p has the same itinerary as a endpoint of T, then $\mathrm{orb}(p)$ need not have a characteristic point.

As a consequence we derive the main result of our investigation of the dynamical plane, which provides a classification of the behavior of global and local arms at periodic points.

22.10. **Proposition (Behavior of Global/Local Arms)**
Let z be the characteristic point of periodic orbit of exact period n which is disjoint from $\mathrm{orb}(c_0)$*. If G is a global arm of z and L is its associated local arm, then either $c_0 \notin f^{\circ i}(G)$ for all $i = 0, \ldots, n$ or $f^{\circ n}$ maps L to the local arm of z pointing to c_0 or c_1.*

The map $f^{\circ n}$ permutes all local arms at z transitively or it fixes the local arm pointing to c_0 and permutes the remaining local arms transitively. □

22.3. Admissibility. Let us turn to the question which (equivalence classes of) Hubbard trees and, equivalently, which kneading sequences are realized by unicritical polynomials.

By Proposition 22.10, there are exactly two types of periodic branch points which are disjoint from the critical cycle, namely branch points with none of their local arms fixed under the first return map and those which have exactly one local arm that is fixed under the first return map. We call a periodic branch point $b \notin \mathrm{orb}(c_0)$ with one fixed local arm *evil.*

Topological admissibility. If a Hubbard tree $\mathcal{T}$ of degree d contains no evil branch point, then we can turn it into a Hubbard tree in the sense of [**Poi**] 23.1 by defining an *angle function* for each marked point of $\mathcal{T}$. Therefore, each equivalence class of Hubbard trees without evil branch points contains a representative which is generated by some unicritical polynomials of degree d.

Combinatorial admissibility. Just as in the quadratic case, one can read off from the itinerary of a periodic point disjoint from $\mathrm{orb}(c_0)$ whether one of its local arms is fixed under the first return dynamics or not. Indeed, if z is a characteristic point of exact period n and itinerary τ, then n is contained in the ρ-orbit of τ if and only if none of its local arms are fixed. This is Lemma 5.17 for quadratic sequences. The proof carries over to the unicritical case because it only uses that $\tau_1(x) \neq \tau_1(y)$ if and only if $c_0 \notin [x, y]$, and thus the fact that the ρ-map can be used to locate precritical points. This is the basic properties for all proofs in Section 5. Thus the statements of this section extend to the unicritical setting, where the term “internal address” has to be replaced by "ρ-orbit of ν". For the estimates on the number of arms of periodic points, we again need the technical Lemma 20.2, whose statement and proof are literally the same in the quadratic setting. As main result in this subsection, we have the following statement.

22.11. **Theorem (Thm:AdmissibilityAlex)**
A Hubbard tree of degree d contains an evil branch point of exact period m if and only if its associated kneading sequence fails the Admissibility Condition 5.1 for the integer m. □

Remark. If ν is a $\star$-periodic kneading sequence of exact period n, then replacing each $\star$ consistently by some symbol $i \in \{0, \ldots, d-1\}$ yields d distinct periodic kneading sequence, exactly one of which does not contain n in its ρ-orbit. This sequence is called *lower kneading sequence* of ν and denoted by $\overline{\mathcal{A}}(\nu)$. The remaining $d-1$ sequences are all *upper kneading sequences* of ν and denoted by $\mathcal{A}_i(\nu)$, where i it the symbol replacing $\star$. Whenever we talk about some $\mathcal{A}_i(\nu)$, we assume that the i is chosen such that $\mathcal{A}_i(\nu)$ exists.

The statements of Section 5 involving lower and upper kneading sequence also hold in the unicritical setting.

22.4. The Parameter Planes. In the remainder of this section we investigate the structure of the sets $\Sigma_d^{\star\star}$ and $\Sigma_d^{\star}$. All statement of Section 6 and 7 together with their proofs carry over to the unicritical setting if one adapts the statements accordingly: instead of the internal address one has to consider the ρ-orbit of ν, and the role of $\mathcal{A}(\nu)$ is played by the $d-1$ sequences $\mathcal{A}_i(\nu)$.

As in the quadratic case, we define a relation $<$ on $\Sigma_d^{\star\star}$ and $\Sigma_d^{\star}$ and show that it is a partial order. The set $\Sigma_d^{\star\star}$ together with the partial order $<$ is called the *parameter tree of degree d*. The following observation guarantees that $<$ is not reflexive; it is also needed to transfer some of the reasoning of Section 6 and 7 to the unicritical setting.

22.12. **Lemma (Unique Characteristic Point for ν)** *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and $\nu \in \Sigma_d^{\star}$ be $\star$-periodic. Suppose that T has a characteristic point y with $\tau(y) = \mathcal{A}_i(\nu)$. Then T does not contain a characteristic points z such that $\tau(z) \in \{\mathcal{A}_j(\nu), \nu\}$ for any $j \neq i$.*

Proof. We assume by way of contradiction that there is a characteristic point z with $\tau(z) \in \{\mathcal{A}_j(\nu), \nu\}$. Without loss of generality we can assume that $z \in]y, c_1]$ and that the exact period n of z coincides with the one of its itinerary, and thus with the one of ν. The precritical point $\xi \in]y, z]$ with smallest step has $\mathrm{STEP}(\xi) = n$ (if $z = c_1$, then $\xi = c_1$). Hence, $f^{\circ n}(L_y(c_1)) = L_y(c_1)$. This implies that the local arm at z pointing towards c_0 must also be fixed under $f^{\circ n}$. Therefore $\tau(y) = \overline{\mathcal{A}}(\nu)$ because the exact periods of z and $\tau(z)$ coincide (this is Lemma 5.17 which also holds for the unicritical case). But this contradicts our hypothesis. $\square$

22.13. **Definition (Order for Unicritical Kneading Sequences)**
Let ν and ν' be two $\star$-periodic kneading sequences. We say that $\nu > \nu'$ if the Hubbard tree for ν contains a characteristic periodic point with itinerary $\mathcal{A}_i(\nu')$ for some $i \in \{0, \dots, d-1\}$.

For all non-periodic sequences $\nu \in \Sigma_d^\star$, we set $\nu > \nu^k$ for all k, where the truncated sequences *ν^k are defined the same way as in the beginning of Section 7 for quadratic sequences. If a $\star$-periodic kneading sequence ν' has $\nu' > \nu^k$ for all k, then we say that $\nu' > \nu$. We extend this relation to the set $\Sigma_d^{\star\star}$ by taking the transitive hull.*

One can further extend $<$ to a relation on all of $\Sigma_d^\star$: if $\nu = \mathcal{A}_i(\nu_\star)$ for a $\star$-periodic sequence ν_*, then we define $\nu > \nu_\star$, and ν compares to all sequences $\nu' \neq \nu_\star$ just as $\nu_\star$ does. If $\nu = \overline{\mathcal{A}}(\nu_\star)$, then $\nu < \nu_\star$, and $\nu < \nu'$ if the kneading sequence ν' is non-admissible and the associated Hubbard tree has an evil branch point with itinerary $\overline{\mathcal{A}}(\nu_\star)$; for all other kneading sequences ν', ν compares to ν' just as $\nu_\star$ does.

We are going to show that the relation $<$ is a partial order on the set $\Sigma_d^\star$. Let us first focus on $\star$-periodic kneading sequences:

Lemma 22.12 implies that $\nu \not> \nu$. Transitivity follows by item (a) of the orbit forcing Lemma for Hubbard trees of degree d below. For all other sequences in $\Sigma_d^\star$, transitivity is built into the definition. Observe also that the extensions of $<$ are compatible with the partial order defined in each previous step; in particular, they do not add any new relations to the set on which $<$ has already been defined.

22.14. **Proposition (Orbit forcing)**
Let $(T, f, \mathcal{P})_d$ and $(\widetilde{T}, \tilde{f}, \widetilde{\mathcal{P}})_d$ be two Hubbard trees with $\star$-periodic kneading sequences ν and $\tilde{\nu}$.

(a) *Let $p \in T$, $\tilde{p} \in \widetilde{T}$ be two periodic characteristic points of exact period n such that $\tau_i(p) = \tau_i(\tilde{p})$ for all $i < n$ ($p = c_1$ or $\tilde{p} = \tilde{c}_1$ is allowed). If $z \in (c_0, p) \subset T$ is a characteristic point, then there is a characteristic point $\tilde{z} \in (\tilde{c}_0, \tilde{p}) \subset \widetilde{T}$ such that z and $\tilde{z}$ have the same itinerary, the same type and the same number of arms.*

(b) *Suppose that $p \in T$ is a periodic non-precritical point of exact period n and itinerary $\mathcal{A}_i(\tilde{\nu})$. Let $b \in T$ be the point where the arc containing p branches off from $[0, c_1]$, i.e., $[0, b] = [0, c_1] \cap [0, p]$ ($b = p$ is allowed). Denote by $\widetilde{T}$ the Hubbard tree associated to $\tilde{\nu}$. If $z \in]0, b[$ is a characteristic point, then there is a characteristic point $\tilde{z}$ in $\widetilde{T}$ such that z and $\tilde{z}$ have the same itinerary and are of the same type. Moreover, if z is chosen so that (z, b) contains a further characteristic point, then z and $\tilde{z}$ also have the same number of arms.*

$\square$

The proof of the above proposition works the same way as in the quadratic case. Again the crucial ingredient is that for two points x, y disjoint from the critical orbit, we have that $\tau_1(x) = \tau_1(y)$ if and only if $c_0 \notin [x, y]$ if and only if $f|_{[x,y]}$ is injective. The same holds true for the following lemma, which is needed in the proof of the Branch Theorem 22.16 to show that for any two not comparable kneading sequences ν, $\tilde{\nu}$, there is indeed a *maximal* sequence which forms their branch point.

22.15. **Lemma (Not Forced Characteristic Points)**
Suppose that in the situation of Proposition 22.14, item (b), the characteristic point z is contained in $(b, c_1]$ and that either there is a characteristic point in $[b, z)$ or b is the limit of characteristic periodic points. Then z is not forced in the Hubbard tree $\widetilde{T}$, i.e., $\widetilde{T}$ contains no characteristic point $\tilde{z}$ such that the itinerary $\tau(\tilde{z})$ of $\tilde{z}$ equals $\mathcal{A}_j(\nu)$ for some j, where ν is $\star$-periodic such that $\tau(z) = \mathcal{A}_i(\nu)$ for some i. □

Let us summarize the results on the structure of $\Sigma_d^\star$:

- $\nu > \overline{\star}$ for any kneading sequence $\nu \in \Sigma_d^\star$.
- If ν is $\star$-periodic of exact period n then
  (1) $\mathcal{A}_i(\nu) \not> \mathcal{A}_j(\nu)$ for all $i, j \in \{0, \dots, d-1\}$
  (2) $\mathcal{A}_i(\nu) > \overline{\mathcal{A}}(\nu)$ for every i
  (3) There is no element of $\Sigma_d^\star$ between $\overline{\mathcal{A}}(\nu)$ and ν, between ν and $\mathcal{A}_j(\nu)$, and between $\mathcal{A}_j(\nu)$ and $\mathcal{A}_{qn}^{-1}(\mathcal{A}_j(\nu)) = \overline{(\nu_1 \dots \nu_{n-1} j)^{q-1} \nu_1 \dots \nu_{n-1} \star}$.

If we define a kneading sequence $\nu \in \Sigma_d^{\star\star}$ to be *admissible* if ν does not violate the Admissibility Condition 5.1 for any $m \in \mathbb{N}$, then we get the following two statement for the space $(\Sigma_d^{\star\star}, <)$:

- If ν is admissible then any $\nu' < \nu$ is also admissible.
- If ν is not admissible then there is a unique admissible $\star$-periodic kneading sequence μ such that $\mu \not< \nu$ but for any $\mu' < \mu$, we have that $\mu' < \nu$.

Our main result on the parameter level shows that the sets $\Sigma_d^{\star\star}$ and $\Sigma_d^\star$ have the structure of a tree.

22.16. **Theorem (Branch Theorem for $\Sigma_d^{\star\star}$)**
Let $\nu, \tilde{\nu} \in \Sigma_d^{\star\star}$. Then there is a unique kneading sequence μ such that exactly one of the following cases holds:

(1) $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = [\overline{\star}, \mu]$, *where μ is either $\star$-periodic or preperiodic.*
(2) $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = [\overline{\star}, \mu] \setminus \{\mu\}$, *where μ is a $\star$-periodic such that the exact period of μ and $\overline{\mathcal{A}}(\mu)$ coincide.* □

Missing picture here!

FIGURE 22.1. An illustration of the local structure of $\Sigma_d^\star$ as given by Theorem 22.17. Pictured is a neighborhood of a $\star$-periodic sequence ν. At $\mathcal{A}_i(\nu)$ the sequences $\mathcal{A}_{qn}^{-1}(\mathcal{A}_i(\nu))$ branch off and at $\overline{\mathcal{A}}(\nu)$, the sequences $\mathcal{A}_{qn}^{-1}(\overline{\mathcal{A}}(\nu))$. In the picture, the first ones are denoted by B_i^q, the latter ones by $\overline{B}^q$.

22.17. **Theorem (Branch Theorem for $\Sigma_d^\star$)**
Let $\nu, \tilde{\nu} \in \Sigma_d^\star$. Then $[\overline{\star}, \nu] \cap [\overline{\star}, \tilde{\nu}] = [\overline{\star}, \mu]$, where $\mu \in \{\nu, \tilde{\nu}\}$ is possible. Moreover, μ is either preperiodic or $\mu \in \{\mu^\star \mathcal{A}_i(\mu^\star), \overline{\mathcal{A}}(\mu^\star)\}$[3] for some $\star$-periodic sequence $\mu^\star$. □

[3] This is different to the quadratic case where the branch point cannot be $\star$-periodic. This reflects that in the Multibrot sets, branching might occur within a hyperbolic component W, namely if the sequences $\nu, \tilde{\nu}$ lie in the wakes of different sectors of W.

B. Further Work and Other Languages

There are various descriptions of quadratic Julia sets, studied by many authors. In this appedix, we summarize work of several authors that initiated these various descriptions, and draw parallels with the main text and sometimes add results ourselves. Sometimes we alter the notation from the original papers in order to be consistent with the main text.

23. Hubbard Trees and the Orsay Notes

23.1. The Orsay Notes. Douady, Hubbard, Posdronasvili

23.2. Poirier's Approach to Hubbard Trees. The work of Douady and Hubbard [**DH1**] shows that the filled-in Julia set of any postcritically finite polynomial contains a special topological tree, its *Hubbard tree*. This Hubbard tree is the minimal graph in the filled-in Julia set which connects all points on the critical orbits, where edges within any Fatou component U correspond to *inner rays* of U. Note that this graph sometimes is referred to as the *regulated convex hull* of all points on the critical orbits. Thus, the Hubbard trees in [**DH1**] are defined in a less abstract way than in Definition 3. If all critical points are strictly preperiodic then Douady and Hubbard show that the Hubbard tree determines the polynomial completely. AK: The first part of this paragraph is about the Orsay notes. This part could either be extended or deleted depending on where and how the section about the Orsay notes is going to appear.

Motivated by this result, Poirier introduces *abstract Hubbard trees* as topological trees with dynamics meeting certain properties which mimic the dynamic behavior of Hubbard trees of polynomials. The fundamental idea of his approach is similar to our definition of Hubbard trees in Section 3. However, Hubbard trees in the sense of Poirier and Definition 3.1 are not equivalent.

Starting out with a topological tree T, Poirier adds step by step further properties to get a more and more special tree, yielding in the end a tree with dynamics that is realizable as Hubbard tree of some postcritically finite polynomial.

23.1. **Definition (Poirier's Abstract Hubbard Trees and Hubbard Forests)** *A tree T with vertex set V is a Hubbard tree in the sense of Poirier if the following properties are satisfied.*

- *There is a map $\tau : V \to V$ on the set of vertices V of T which can be extended to all of T such that τ restricted to any edge is a homeomorphism.*
- *There is a local degree function $\delta : V \to \mathbb{Z}$ such that $d = 1 + \sum_{v\in V}(\delta(v) - 1)$ and $d > 1$.*
- *There is an angle function $\angle(e, e')$ which assigns a rational angle to any pair of edges incident at a common vertex v. The function $\angle$ is skew-symmetric (i.e., $\angle(e, e') = -\angle(e', e)$) and it assigns the angle 0 if and only if $e = e'$; furthermore, $\angle(e, e'') = \angle(e, e') + \angle(e', e'')$ for three edges incident at a vertex v and $\angle(\tau(e), \tau(e')) = \delta(v) \cdot \angle(e, e')$. Also $\angle$ induces a cyclic order on the edges incident at common vertex.*
- *(Expansiveness) For any adjacent* Julia vertices *v, v' there is an m such that the arc from $\tau^{\circ m}(v)$ to $\tau^{\circ m}(v')$ contains a further vertex.*
- *If v is a Julia vertex where m edges meet at then the angle function at v has values in $\{i/m : i \in \mathbb{N}\}$.*

HB: What is a Julia vertex. It is explained below when you already have a Julia set, and that seems a bit circular for the present exposition. I think the last bullet point needs rewriting.

The main result of [**Poi2**] states that, indeed, any abstract Hubbard tree can be realized by some polynomial. Furthermore, any postcritically finite polynomial is uniquely determined by its Hubbard tree.

23.2. **Theorem (Realizability of Abstract Hubbard Trees)**
Any abstract Hubbard tree *is realized by a postcritically finite polynomial which is unique up to affine conjugation.*

The proof of Theorem 23.2 is based on critical portraits studied in [**Poi1**]. In fact, each abstract Hubbard tree gives rise to a unique critical portrait, which is admissible and thus is realized by a unique postcritically finite polynomial (up to affine conjugation).

Poirier also observes that for any Hubbard tree T of a polynomial, the number of rays landing at a periodic Julia vertex v, *i.e.,* a vertex of T that lies in the Julia set of the corresponding polynomial, equals the number of edges of T incident at v, a result which is essentially due to Douady and Hubbard.

In [**Poi3**] Poirier considers maps $f : \mathbb{C}^{\cup} \to \mathbb{C}^{\cup}$ from a finite disjoint union $\mathbb{C}^{\cup}$ of copies of $\mathbb{C}$ to itself such that the restriction $f : \mathbb{C}_u \to \mathbb{C}_{u'}$ to any of these copies is a complex postcritically finite polynomial of degree at least two. The *filled-in Julia set* K_f is by definition the set of all points in $\mathbb{C}^{\cup}$ whose orbits remain bounded, the *Julia set* J_f is the boundary of K_f.

For each copy $\mathbb{C}_u$, Poirier defines a *regulated tree* T_u by taking the regulated convex hull of the postcritical set contained in $\mathbb{C}_u$. This construction is analogous to the definition of Hubbard trees for postcritically finite polynomials [**DH1**].

23.3. **Definition (Hubbard Forests)**
An abstract Hubbard forest *is an (in general not connected) graph H together with a function φ from the set of vertices V of H to itself, a degree function $d : V \to \mathbb{N}^*$ and an angle function $\angle_v(e, e')$ which is defined for edges incident at a common vertex v and which takes values in $\mathbb{R}/\mathbb{Z}$. The graph and the specified functions must meet analogous requirements as specified for abstract Hubbard trees as in Definition 23.1.*

Two Hubbard forests are *isomorphic* if they are isomorphic as graphs such that the additional properties which turn them into Hubbard forests are preserved.

23.4. **Theorem (Realizability of Hubbard Forests)**
If $f : \mathbb{C}^{\cup} \to \mathbb{C}^{\cup}$ is a postcritically finite proper holomorphic map and M is a finite forward invariant set containing all critical points of f, then the regulated hull of M can be given the structure of a Hubbard forest in a natural way.

Conversely, every abstract Hubbard forest is isomorphic to one constructed in this way. It is unique up to component-wise affine conjugation.

24. Laminations

Thurston [**Th**] developed the theory of (quadratic) laminations of which the study of quadratic Julia sets is just one, but for us most important, application. Every quadratic dendrite Julia set is the quotient space of some quadratic lamination, and Thurston's Nonwandering Triangle Theorem implies that the only branchpoint in the Julia set are pre(periodic) or (pre)critical. The question whether the same result is true in higher order lamination was answered in the negative by Blokh and Oversteegen, [**BO**], after partial results by Blokh and Levin [**BL**] and Kiwi [**Ki1**]. The contributions of Kiwi also touches on the theory of portraits and puzzles, see further sections. In this section, we also discuss Keller's axiomatic approach to laminations (Julia and Mandelbrot equivalences) from [**Ke**].

24.1. Thurston Laminations.

Recall that $\mathbb{D}$ denotes the complex unit disk.

24.1. **Definition (Quadratic Lamination)**
A quadratic lamination *is a decomposition of* $\mathbb{D}$ *into leaves and gaps as follows:*

- *A* leaf *spanned by two angles* $\vartheta, \vartheta' \in \mathbb{S}^1$ *is a geodesic in* $\mathbb{D}$ *connecting the boundary points* ϑ *and* ϑ'*;*
- *a* gap *is a component of the complement of all leaves in* $\mathbb{D}$*;*

subject to the following conditions:

gap boundaries: *for every gap, the set of angles in* $\mathbb{S}^1$ *on its boundary is totally disconnected;*
lamination unlinked: *no two leaves intersect within* $\mathbb{D}$*;*
forward invariance: *for every leaf* (ϑ, ϑ')*, there is a leaf* $(2\vartheta, 2\vartheta')$ *(the* image leaf*) unless* $2\vartheta' = 2\vartheta$*;*
backwards invariance: *for every leaf* (ϑ, ϑ')*, there are two leaves* $(\vartheta_1, \vartheta_1')$ *and* $(\vartheta_2, \vartheta_2')$ *(the* preimage leaves*) so that* $\vartheta = 2\vartheta_1 = 2\vartheta_2$ *and* $\vartheta' = 2\vartheta_1' = 2\vartheta_2'$*;*
lamination closed: *the set of leaves is closed: if* $(\vartheta_n, \vartheta_n')$ *is a sequence of leaves so that* $\vartheta_n \to \vartheta$ *and* $\vartheta_n' \to \vartheta'$*, then* (ϑ, ϑ') *is also a leaf.*

The *diameter* or *length* of a leaf or gap is the length of the shortest interval on $\mathbb{S}^1$ containing the boundary points in $\mathbb{S}^1$ of the leaf or gap (normalized so that the diameter of $\mathbb{S}^1$ is 1). The set of boundary points in $\mathbb{S}^1$ of any gap may be finite (at least three points), countable or uncountable. The metric on $\mathbb{D}$ defining the geodesic leaves is usually taken to be the hyperbolic metric in the Poincaré model, but the Euclidean metric works just as well (it yields the same leaves as the hyperbolic metric in the Klein model). A *critical leaf* is a leaf of diameter $1/2$; by the unlinking property, a critical leaf is necessarily unique (it need not exist). If one exists, then all other leaves have diameter strictly less than $1/2$, and every gap (if any) has diameter at most $1/2$. A *precritical leaf of* k *steps* is a leaf (α, β) so that $(2^{k-1}\alpha, 2^{k-1}\beta)$ is a critical leaf.

24.2. **Lemma (Lamination From Non-Periodic Angle)**
For every non-periodic angle ϑ, there is a unique minimal quadratic lamination which contains the leaf $(\vartheta/2, (\vartheta+1)/2)$.

Proof. Consider a leaf (α, β). Then exactly one of $(\alpha/2, \beta/2)$ and $(\alpha/2, (\beta+1)/2)$ intersects the critical leaf; the other one must be a leaf by backwards invariance, and there must be a second preimage leaf containing the angle $(\alpha+1)/2$. The same applies to these two preimage leaves, and so on. The choice is always unique unless $\alpha = 2^k\vartheta$ or $\beta = 2^{k'}\vartheta$ for some $k, k' \geq 0$.

The critical leaf $(\vartheta/2, (\vartheta+1)/2)$ has exactly 2^k precritical leaves of $k+1$ steps. Their construction is uniquely possible because ϑ is non-periodic. This cannot destroy the unlink property: the construction assures that no preimage leaf is linked with the critical leaf, and if two non-critical leaves are linked, they are both contained in the same complementary component of the critical leaf, and then their image leaves are also linked.

Countable repetition of taking preimages yields a collection of leaves which is unlinked and forward and backward invariant. It also satisfies the condition on gap boundaries because for every single leaf, the backwards orbit has endpoints that are dense in $\mathbb{S}^1$.

Finally, we take the closure of the set of all leaves. This yields a quadratic lamination as specified, and every quadratic lamination containing the leaf $(\vartheta/2, (\vartheta+1)/2)$ must contain all leaves of this lamination. $\square$

HB: I suspect that this part, including the lemma below and its proof, is not meant to go in here.

We construct a quadratic lamination with a periodic gap bounded by infinitely many leaves (this gap models a Siegel disk). For every k, we define gaps V_k as follows: let $I_{k,j}$ be the collection of complementary intervals of S_k, and let their endpoints be $\vartheta_{k,j}$ and $\vartheta'_{k,j}$. Introduce leaves $(\vartheta_{k,j}, \vartheta'_{k,j})$ for all j. These leaves bound a unique gap V_k, and $\partial V_k \cap \mathbb{S}^1 = S_k$. The critical leaf $(\vartheta/2, (\vartheta+1)/2)$ is disjoint from all leaves V_k: by construction, it does not disconnect any set S_k. Similarly, no precritical leaf can disconnect any S_k (the precritical leaves up steps up to m separate the regions of points in $\mathbb{S}^1$ with identical first m entries in the itineraries). For the same reason, the precritical leaves of steps up to n separate the sets S_k from each other, and hence they separate the gaps V_k. As a result, the gaps V_k are disjoint. By Lemma 14.4, one gap has a critical leaf on its boundary, and this critical leaf cannot intersect $((\vartheta/2, (\vartheta+1)/2)$, so this gap must have the leaf $(\vartheta/2, (\vartheta+1)/2)$ on its boundary. The image gap then has ϑ on its boundary, so ϑ is a limit point on the orbit of φ and, as above, the itinerary of ϑ is a periodic shift of τ. □

24.3. **Lemma (Only Upper Periodic Sequences are Generated)**
If a non-periodic angle ϑ has a periodic itinerary (with respect to the partition generated by any angle), then the itinerary is an upper sequence (its internal address is finite and ends with the exact period).

The only claim that remains to show is that τ is an upper sequence.

We continue the construction of preimage leaves from Lemma 24.2. Suppose ϑ is not periodic. Then for every $m \geq 0$, the 2^m precritical leaves of $m+1$ steps disconnect $\mathbb{D}$ into $2^m + 1$ domains with the following property: external angles on the boundary of the same domain have the same initial m entries in their itineraries, and conversely: any two angles on the boundary of the same domain have itineraries that coincide for m entries (excluding angles on the backwards orbit of ϑ). Since the itineraries on the boundary of different gaps V_k and V'_k differ within the initial n entries, they are separated by a precritical leaf of at most n steps. This proves that leaves on the boundary of gaps do not intersect with each other or with precritical leaves.

As in Lemma 24.2, we continue to add preimages of all leaves countably often. This satisfies backwards invariance, it preserves the unlink property, and it satisfies the gap condition. Forward invariance is clear by construction, using that $S_{k+1} = 2S_k$: angle doubling sends every boundary leaf of V_k to a boundary leaf of V_{k+1}. Finally, we take the closure of all leaves.

Every V_k is still a gap in this lamination: no precritical leaf can disconnect a V_k (all points in V_k have identical itineraries), and no preimage leaf of some $V_{k'}$ can disconnect V_k (the V_k are unlinked, and hence their preimages are unlinked too).

The sequence of gaps V_k is periodic with period n. By Lemma 14.4, we can renumber the V_k cyclically so that every V_k (with $k \in \{1, 2, \dots, n\}$) has a boundary leaf which is precritical of $n + 1 - k$ steps; in particular, a boundary leaf of V_n is the critical leaf, and $\vartheta \in \partial V_1$. We claim that for V_1, the longest leaf is precritical with n steps.

For every k, let $\ell_k \geq \ell'_k$ be the diameters of the two longest complementary intervals: clearly $\ell_k \leq 1/2$, and $\ell'_k < \ell_k$ because V_k contains infinitely many boundary points.

If $\ell_k \leq 1/4$, then $\ell_{k+1} = 2\ell_k$ and $\ell'_{k+1} = 2\ell'_k$. If $\ell_k > 1/4$, then also $\ell'_k \geq 1/4$ (otherwise, the gap V_{k+1} would contain the critical leaf); in this case, $\ell_k \in (1/4, 1/2]$ and all other boundary leaves of V_k combined have length $\ell_k - \ell'_k \leq 1/4$, so each individual boundary leaf of V_k (other than the two longest ones) has diameter less than $1/4$.

The two longest leaves in $V_0 = V_n$ have lengths $1/4 \leq \ell'_0 < \ell_0 = 1/2$, and their images are one leaf with length $1 - 2\ell'_0$ and a single point. Let b_1 be the longest leaf of V_1; it has diameter $\ell_1 = 1 - 2\ell'_0$: the boundary of V_0 is contained in two short intervals, and their images under doubling meet at ϑ with combined length $1 - 2\ell'_0$. Therefore, b_1 is the image of the second longest leaf of V_0.

After $n - 1$ iterations, b_1 maps to a boundary leaf of $V_n = V_0$. If, during this iteration, an intermediate leaf ever had length less than ℓ_1, then in the step before the first time, the length had to be greater than ℓ'_0, and the only candidate is the critical leaf. Since this does not happen, the leaf $2^{n-1}b_1$ is a boundary leaf of V_n of length at least ℓ_1. We claim that this is the critical leaf.

All boundary leaves of V_0, except the longest two, have combined lengths $\ell_1/2$, so they cannot be $2^{n-1}b_1$. Therefore, if $2^{n-1}b_1$ is not the critical leaf, then it is the second longest leaf, which is thus periodic with period n. Exclude this case!

Therefore, the longest leaf of V_1 is precritical with n steps as claimed. We define the internal address of ϑ in this context as the sequence $S_1, S_2, \dots, S_k$ of strictly increasing integers so that there is a sequence of precritical leaves a_i of S_i steps separating V_1 from a_{i-1} (where a_1 is the critical leaf of $S_1 = 1$ step), so that there is no precritical leaf of less than S_i steps between a_i and V_1. This sequence clearly terminates with the entry n for the leaf b_1. It remains to show that the internal address of ϑ in this context equals the internal address of the kneading sequence $\nu(\vartheta)$ associated to ϑ. This resembles the proof from Proposition 6.8 that the internal address of ν equals the internal address by closest precritical points. One possibility is to slightly perturb ϑ so as to become preperiodic.

What if ϑ is periodic?

HB: Keller's text does cover periodic ϑ. How should we use that?

24.2. No Wandering Gaps for Quadratic Laminations. HB: any precursor to Penrose's proof of no wandering gaps in unicritical case?

24.3. Wandering gaps. Thurston introduced laminations of degree $d > 1$ as a tool to study polynomial dynamics [**Th**]. Recall from Section 24.1 that a lamination is a special equivalence relation on the circle $\mathbb{S}^1$ which is σ_d-invariant, where $\sigma_d : \mathbb{S}^1 \to \mathbb{S}^1$ is the multiplication of angles by d. For $n > 2$, an *n-gon* (or n-gap) is an equivalence class containing n points. It is called *wandering* if all its σ_d-images are n-gons and their convex hulls (taken within the unit disk) are pairwise disjoint. One central observation of Thurston for quadratic laminations, see Section 24.1, is that there are no wandering n-gons. For a long time, one of the major questions concerning lamination had been whether this result also holds true for higher degree laminations. After a series of papers on upper bounds for the number of such wandering n-gons, Blokh and Oversteegen showed in 2004 that for any degree $d \geq 3$, there exist uncountably many laminations with *wandering triangles*, that is, wandering 3-gons [**BO**]. Let us first point out some of the result obtained on upper bounds.

In his thesis [**Ki1**], Kiwi showed that for any polynomial p with connected Julia set and any $z \in \mathbb{C}$ with infinite orbit, there are at most $\delta + 1$ rays landing at z, where δ is the number of distinct critical values of p. Moreover, if z is any point with infinite forward orbit, then the number of rays landing at z is at most 2^d.

Since external rays of a polynomial with connected Julia set give rise to a Thurston lamination, Kiwi's results are directly related to the study of wandering n-gons. Following Goldberg and Milnor [**GM**], his proofs are based on orbit portraits and the study of *angular sizes* of the sectors spanned by external rays defining the orbit portrait under consideration.

A similar result is the following inequality, due to Bloh and Levin [**BL**], relating the number of points in wandering gaps to the degree of the lamination. We give some more details on this paper in Section 24.4 below. Note that an equivalence class g of the lamination is called critical if $\sigma_d|_g$ is not injective.

24.4. **Theorem (Upper Bound on Wandering n-gons)**
If $\Gamma \neq \emptyset$ is a collection of equivalence classes g with $\#(g) > 2$ which are neither preperiodic nor precritical and belong to pairwise disjoint orbits, then

$$\sum_{g \in \Gamma} (\#(g) - 2) \leq k_\sim - 1 \leq d - 2.$$

Here d is the degree of the lamination and $k_\sim$ is the maximal number of critical equivalence classes h belonging to pairwise disjoint orbits such that $\sigma_d(h)$ is a single point with infinite σ-orbit.

Furthermore, $\#(g') \leq 2^d$ for any non(pre)periodic equivalence class g'.

After these results on an upper bound for the number of wandering triangles, Blokh and Oversteegen [**BO**] prove the existence of laminations of degree d with wandering triangles by explicitly constructing such a lamination for degree 3. They pick two critical leaves together with their preimages and one triangle T and then add iteratively all forward images of T. This defines a map on a subset of the circle which can be extended to a σ_3-invariant lamination. Their construction leaves enough choices to yield uncountably many laminations with wandering triangles. Further results on wandering gaps were obtained by Childers, [**Ch**].

24.4. Blokh and Levin's Growing Trees. Blokh and Levin provide a new approach to the problem of wandering gaps by their study of *growing trees.*

24.5. **Definition (Growing Trees)**
Let T be a tree contained in a topological space X, let $f : X \to X$ be a continuous map and set $T_\infty := \bigcup_{i=0}^{\infty} f^{\circ i}(T)$. Assume that

(i) *$f(T) \cap T \neq \emptyset$;*
(ii) *$T_n := \bigcup_{i=0}^{n} f^{\circ i}(T)$ is a tree for all n;*
(iii) *the set C_f of all critical points in T is finite and $f|_{T_\infty \setminus C_f}$ is locally injective.*

Then the sequence $T_0 \subset T_1 \subset \cdots$, as well as the set T_∞, is called a growing tree.

Following Douady [**D3**], they consider the quotient space $J = \mathbb{S}^1/\sim$ under the equivalence relation $\sim$ given by a lamination, and define within this space a growing tree. Indeed, if β and $-\beta = \gamma_1$ are the fixed point (with external angle 0) and its preimage on J, or more generally for a degree d lamination, $\sigma_d^{-1}(\beta) \setminus \{\beta\} = \{\gamma_1, \ldots, \gamma_{d-1}\}$, they take the initial tree $T = T_0 := \cup_i [\gamma_i, \beta] \subset J$. For quadratic laminations, this gives $T_1 = [-\beta, \beta] \cup [\beta, c_1]$, $T_2 = [-\beta, \beta] \cup [\beta, c_1] \cup [\beta, c_2]$, etc. For finite critical orbits, $T_\infty = [-\beta, \beta] \cup$ Hubbard tree, but the theory treats more general cases as well, including what we called Hubbard dendrite in Section 21.

24.6. **Example (The Growing Tree for the Golden Ratio Siegel Disk)**
Page 86 of [**BL**] gives the description of the growing tree associated to the polynomial $f(z) = z^2 + e^{2\pi i \gamma} z$ for the golden ratio $\gamma = \frac{1}{2}(\sqrt{5} - 1)$. This polynomial has a Siegel disk with rotation number γ. The growing tree T_∞ becomes a star centered at the α-fixed point with countably many arms that are rotated by angle γ under the dynamics, see Figure 24.1.

A growing tree is called *strongly recurrent* if for every $m \in \mathbb{N}$ and $x \in T_m$ which is not an endpoint of T_m, there is $k \geq 0$ such that $f^{\circ k}(x) \in T_0$. For strongly recurrent growing trees, ever local arm at every point $x \in T_\infty$ will intersect T_0 infinitely often under iteration of f. Note that the tree in Example 24.6 is not strongly recurrent. We quote part of Proposition 2.7 from [**BL**] which is instrumental in the proof of the non-existence of wandering continua:

24.7. **Proposition (Continua Map into T_0 Infinitely Often)**
If $M \subset J$ is a non-singleton continuum or $M = \{x\}$ has at least two external angles, then there is $i \geq 0$ such that $f^{\circ i}(M) \cap T_0 \neq \emptyset$. More specifically:

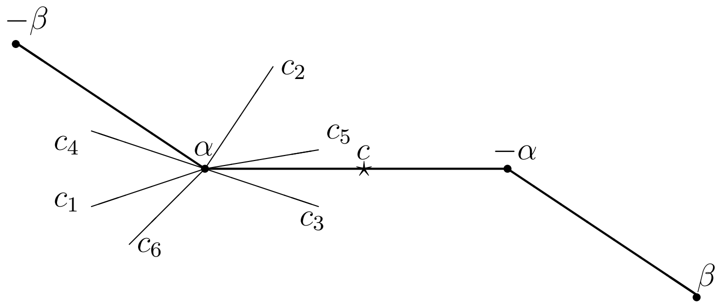


FIGURE 24.1. Towards the growing tree for a Julia set with a golden ratio Siegel disk. Depicted is T_6 and $T_0 = [-\beta, \beta]$.

(a) *if M is a non-singleton continuum, then $f^{\circ i}(M) \cap T_0 \neq \emptyset$ infinitely often, and $\cup_{i\geq 0} f^{-i}(T_0)$ is dense in M;*
(b) *if $M = \{x\}$ is not a preimage of a critical point or of β, then $f^{\circ i}(x) \in T_0 \setminus \{\beta\}$ infinitely often if and only if x has at least two external angles.*

Given two trees $T \subset T'$, any endpoint of a component of $T' \setminus T$ that is not contained in T' is called *outer endpoint*. Blokh and Levin give estimates on the number $\mathrm{oen}(T_{n+1}, T_n)$ of newly generated outer endpoints at any step n and show that $\lim_{n\to\infty} \mathrm{oen}(T_{n+1}, T_n) =: \mathrm{oen}(T_\infty)$ exists. This number is then related to the number of arms at certain branch points in T_m and T_∞. Moreover, they give an upper bound for $\mathrm{oen}(T_\infty)$, from which an upper bound for the number of vertices with infinite orbit in T_∞ is derived. This ultimately leads to the proof of Theorem 24.4.

24.5. Rational and $\mathbb{R}$eal Laminations. HB: This subsection should maybe come later, in the chapter on Portraits

Closely related to the work of Thurston on quadratic lamination is the sequence of papers by Kiwi on rational and real laminations [**Ki2, Ki3, Ki4**]. These combinatorial objects comprise the landing pattern of external rays of complex polynomials.

In an expository account [**Ki2**], Kiwi characterizes the equivalence relation on $\mathbb{Q}/\mathbb{Z}$ that are generated by the landing pattern of external rays of polynomials with connected Julia set.

24.8. **Definition (Rational Lamination)**
Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial with connected Julia set $J(f)$. Then the rational lamination of f, *$\lambda_{\mathbb{Q}}(f)$, is the equivalence relation on $\mathbb{Q}/\mathbb{Z}$ given by $s \sim t$ if and only if the external rays at angle s and t land at the same point $z \in J(f)$.*

A formal rational lamination *is an equivalence relation $\lambda_{\mathbb{Q}}$ on $\mathbb{Q}/\mathbb{Z}$ that meets the following five requirements:*

(1) *$\lambda_{\mathbb{Q}}$ is closed in $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$;*

(2) *each equivalence class is finite;*
(3) *if A is an equivalence class so is $d \cdot A$;*
(4) *for any equivalence class A, the map $A \mapsto d \cdot A$ is consecutive preserving;*
(5) *equivalence classes are pairwise unlinked.*

24.9. **Theorem (Existence of Rational Lamination)**
The rational lamination $\lambda_{\mathbb{Q}}(f)$ of any polynomial f with connected Julia set has the five properties of a formal rational lamination given in Definition 24.8.

Conversely, for any formal rational lamination $\lambda_{\mathbb{Q}}$ there is a complex polynomial with connected Julia set such that $\lambda_{\mathbb{Q}} = \lambda_{\mathbb{Q}}(f)$.

To prove that every formal rational lamination is realized by some polynomial, Kiwi extends $\lambda_{\mathbb{Q}}$ to an equivalence relation $\lambda_{\mathbb{R}}$ on the circle $\mathbb{R}/\mathbb{Z}$ and, later on, $\lambda_{\mathbb{R}}$ to an equivalence relation $\lambda_{\mathbb{C}}$ on the complex plane. More precisely, $\lambda_{\mathbb{R}}$ is the smallest equivalence relation on the circle that contains $\overline{\lambda_{\mathbb{Q}}}$, and as such, $\lambda_{\mathbb{R}}$ inherits the properties of $\lambda_{\mathbb{Q}}$ stated in Definition 24.8. Any equivalence class on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ meeting these five properties is a $\mathbb{R}$*eal lamination* of degree d. Note that there are $\mathbb{R}$eal laminations that are not the extension of any formal rational lamination.

A thorough investigation of λ-unlinked classes and gaps is the basis for constructing a branched covering $P_\lambda : \mathbb{C} \longrightarrow \mathbb{C}$, called *topological realization of* λ, that realizes the lamination λ. For any (rational or $\mathbb{R}$eal) lamination λ, two points s, t are in the same λ-unlinked class if for all λ-classes A, $\{s,t\}$ and A are disjoint and $\{s,t\}$ is contained in the same connected component of $(\mathbb{R}/\mathbb{Z}) \setminus A$ (*i.e.,* they are unlinked). For any λ-class A, let Convex(A) denote the Euclidean convex hull of A in the closed unit disk $\overline{\mathbb{D}}$. Then any component of $\overline{\mathbb{D}} \setminus$Convex$(A)$ is called a *gap.* (Note that this definition differs from [**Th**].) HB: And from Definition 24.1?

To finish the proof of Theorem 24.9, Kiwi relies on the characterization of critical portraits in the postcritically finite setting [**BFH, Poi**]. He extends the notion of critical portraits to branched covers and shows that there is a sequence of admissible critical portraits Θ_n (admissible in the sense of Poirier, see Definition 23.1) that converge in some sense to the critical marking of P_λ. By work of Poirier, there are postcritically finite polynomials f_n with critical portrait Θ_n. It remains to show that the f_n converge to f and f, indeed, generates the lamination λ.

In [**Ki3**], Kiwi focuses on polynomials with connected Julia set whose periodic points are all repelling and describes how they are arranged in the shift locus according to their real lamination. He shows that each critical portrait Θ as introduced in [**BFH**], see Section 25.1, generates an equivalence relation on the circle which is the real lamination of a polynomial with all periodic points repelling and connected Julia set under the assumption that Θ has *aperiodic kneading*.

A critical portrait Θ has aperiodic kneading if none of the angles forming Θ has periodic itinerary with respect to the partition induced by Θ on the circle.

The $\mathbb{R}$*eal lamination* $\lambda(f)$ *of a polynomial* f with connected Julia set and without irrationally neutral cycles is an equivalence class on the circle such that $s \sim t$ if the *prime end impression* of s and t are not disjoint. Observe that $\lambda(f)$ is a closed equivalence relation, each class is finite, distinct classes are pairwise unlinked, and if d is the degree of f, then multiplication by d preserves classes and is consecutive preserving restricted to any class. A *degree* d $\mathbb{R}$*eal lamination* is an equivalence relation on the circle meeting these five properties.

The *visible shift locus* S_d^{vis} is the set of polynomials f of degree d in the shift locus such that for each critical point c of f, either exactly k external radii land at c, where k is the local degree of c, or $f(c)$ lies on an external radius. Observe that the notion of *external radii* generalizes the idea of external rays to disconnected Julia sets. If the Julia set J_f is connected, then the Böttcher coordinate at ∞ extends to a conformal isomorphism φ_f of the whole basin of infinity Ω_f. However, if J_f is disconnected, then the Böttcher coordinate extends only to a subset Ω_f^*. The image $U_f := \varphi_f(\Omega_f^*)$ is a starlike domain around infinity. The external radius at angle t is the φ_f-preimage of the largest part of $\{re^{2\pi it} \ : \ r \in (1,\infty)\}$ that is contained in U_f. Thus, an external radius either terminates at a point z (at which the gradient flow of the Green function equals 0) or is in fact an external ray.

Kiwi extends the notion of impression to parameter space: he defines the *impression* of a critical portrait Θ to be the set of polynomials f such that there is a sequence of polynomials $f_n \in S_d^{vis}$ and $\Theta(f_n)$ converges to Θ. Let $\lambda(\Theta)$ denote the equivalence relation on the circle induces by the critical portrait Θ. With these definitions the main result of [**Ki3**] reads as follows.

24.10. **Theorem (Critical Portraits, $\mathbb{R}$eal Lamination)**
Let f be a polynomial of degree d contained in the impression of the critical portrait Θ. If Θ has aperiodic kneading then all cycles of f are repelling and $\lambda(f) = \lambda(\Theta)$. If Θ has periodic kneading then at least one cycle of f is non-repelling.

In [**Ki4**], Kiwi investigates fibers and laminations associated to polynomials without any irrationally neutral cycle which have connected Julia sets.

For such polynomials, he defines *fibers* as follows (which generalizes the definition of [**Sch3**] to disconnected Julia sets): let $J_{fin}(f)$ be the set of all periodic and preperiodic points in $J(f)$ which are not in the grand orbit of a Cremer point. The *fiber* of a point $\zeta \in J(f)$ is the set of points ξ that lie in the same connected components of $J(f) \setminus Z$ for all finite subsets $Z \subset J_{fin}(f) \setminus \{\zeta, \xi\}$.

Let d be the degree of f and Σ_f be backward orbit under $\sigma_d : t \mapsto d \cdot t$ of the set of argument of external radii that terminate at critical points of f. Then S_f denotes the topological space obtained from the circle $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ by replacing each point $t \in \mathbb{S}^1 \cap \Sigma_f$ by two points $t^- < t^+$ together with the local order topology. Given a polynomial f without neutral cycles and connected Julia set, the *impression* of $t \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the set of points $z \in J_f$ such that there is a sequence of z_n converging to z such that $\varphi_f(z_n) \to e^{2\pi it}$, where φ_f is the (extended) Böttcher coordinate of the basin of ∞.

The lamination of a polynomial without neutral cycles is given by an equivalence relation $\sim$ on the circle $\mathbb{S}^1$, where $s \sim t$ if there are finitely many elements $s_1 = s, \dots, s_n = t$ of the circle such that the impression of s_i and s_{i+1} intersect for all $i = 1, \dots, n-1$.

The main results in [**Ki4**] read as follows.

HB: In the below theorems I changed δJ into ∂J. Same for ∂U, which I assume is the Siegel disk of just a bounded Fatou component. Fiber$(\zeta) = \{\zeta\}$ was changed into "the fiber of ζ, as it was not defined anyway.

24.11. **Theorem (Triviality of Fibers)**
Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial with no irrational neutral cycle. Then, if $\zeta \in \partial J$ is preperiodic or periodic then its fiber is the singleton $\{\zeta\}$.

Given a lamination λ, a *rotation curve* is a simple closed curve in $\mathbb{S}^1/\lambda$ that is periodic under the action on $\mathbb{S}^1/\lambda$ induced by multiplication by d.

24.12. **Theorem (Realizability of Real Laminations)**
An equivalence relation λ on the circle $\mathbb{S}^1$ is the lamination of a polynomial f without irrationally neutral cycles and with connected Julia set $J(f)$ if and only if λ is a real lamination such that if $\gamma \in \mathbb{S}^1/\lambda$ is a rotation curve then the return map is not a homeomorphism.

HB: this part needs better explanation. A picture would also be very helpful.

To prove Theorem 24.11, Kiwi follows theory of puzzles introduced Branner-Hubbard [**BH**] and Yoccoz [**HY**], see Section 27. He shows the theorem for polynomials whose periodic Fatou components are all fixed by defining puzzle pieces of depth n as connected components of the complement of a graph Γ_n. At each level n, the graph Γ_n is obtained by putting together graphs for each Fatou component U of f.

The graph for an attracting fixed point z consists of (i) a Jordan curve β winding around z within the basin of attraction, (ii) closed arcs connecting the preimages in ∂U of some boundary fixed point of U to β, (iii) the parts of the external rays landing at these preimages points up some equipotential line and (iv) this equipotential line. The definition for parabolic fixed points is similar. The corresponding graphs of level n are obtained by pull backs of the original graphs.

For the basin of ∞, the graph at level n consists of the set $Z_n = \{z \in J_f : f^{\circ n}(z)$is periodic of period at most n and the landing point of at least 2 rays$\}$ together with the external rays landing at points in Z_n and the equipotential line at height d^{-n}.

To prove the "if"-direction of Theorem 24.12, Kiwi shows that the laminations generated by f and a branched covering obtained via collapsing fibers of f are identical and that the latter one meets all requirements of a $\mathbb{R}$eal lamination.

For the "only if"-direction, Kiwi follows the proof of Theorem 24.9. He first shows that there is a unique (up to topological conjugacy) branched covering P_λ, the topological realization of λ, which generates λ and is obtained by extending the equivalence relation λ to the complex plane. Again there is a sequence of admissible critical portraits Θ_n, realized by the postcritically finite polynomials f_n, that converge to the critical marking of P_λ (we refer to [**Ki4**, Def. 6.23] for the definition of this convergence), and the f_n converge to f which, indeed, generates the lamination λ.

24.6. Keller's Approach: Julia Equivalences. Keller [**Ke**] takes an axiomatic approach to laminations, identifying them as equivalence relations of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ satisfying certain axioms.

24.13. **Definition (Julia Equivalence)**
A Julia equivalence is an equivalence relation $\approx$ such that

(a) $\approx$ *is completely invariant;*
(b) *if* $\alpha_1 \approx \alpha_2$ *then* $\alpha_1 + \frac{1}{2} \approx \alpha_2 + \frac{1}{2}$*;*
(c) $\approx$ *is unlinked (also called* planar*) in the sense of* HB: Still needs reference *;*
(d) $\approx$ *is* non-degenerate *(or non-periodic) in the sense that there is no* $\beta \in \mathbb{S}^1$ *and* $n \in \mathbb{N}^*$ *such that* $2^n\beta \approx \beta \approx \beta + \frac{1}{2}$.

The *diameter* of the equivalence relation is $\mathrm{diam}((\approx) = \sup\{d(\beta_1, \beta_2) : \beta_1 \approx \beta_2\}$, where $d(x, y)$ is the length of the shortest arc in $\mathbb{S}^1$ connecting x and y. We call $\alpha \in \mathbb{S}^1$ a *main point* of $\approx$ if there are $\beta_1 \approx \beta_2$ such that $\mathrm{diam}(\approx) = d(\beta_1, \beta_2)$ and $\alpha = 2\beta_1$. The equivalence relation with main point α is denoted by $\approx^\alpha$.

We have seen from Lemma 24.2 that if α is not periodic under the angle doubling map, then there is a Thurston lamination with diagonal leaf $(\frac{\alpha}{2}, \frac{\alpha+1}{2})$, and this lamination has main point α. If α is periodic, then the construction of Lemma 24.2 cannot be carried out, and one of the aims of [**Ke**] is to obtain a sensible equivalence relation also in this case.

Degeneracy (*i.e.,* the failure of condition (d) of Definition 24.13) also occurs if the external angle α corresponds to a Siegel disk. In this case, α is a non-periodic main point, but its equivalence class is periodic. Chapters 2.3.1 and 3.2.4 of [**Ke**] explains the connection of symbolic dynamics of this case with Sturmian sequences, see also Section 26.1.

With these special cases in mind, we state the following theorem characterizing Julia equivalences. (This is Theorem 1.8 in [**Ke**], and Chapter 2 culminates in its proof.)

24.14. **Theorem (Characterization of Julia Equivalences)**
Each $\alpha \in \mathbb{S}^1$ *is the main point of a unique Julia equivalence. (However, a Julia equivalence can have more than one main point.)*

Conversely, if $\approx$ *is an equivalence relation of on* $\mathbb{S}^1$ *satisfying conditions (a)-(c) of Definition 24.13, then the following are equivalent:*

- *Each $\approx$-equivalence class is finite;*
- *There is $\alpha \in \mathbb{S}^1$ such that $\approx = \approx^\alpha$;*
- *The equivalence $\approx$ is a Julia equivalence (i.e., it also satisfies condition (d) of Definition 24.13).*

In particular, $\approx = \approx^\alpha$ for each main point of $\approx$.

As an analogue of Julia equivalences $\approx^\alpha$ (which lead to abstract Julia sets as quotient spaces $\mathbb{S}^1/\approx^\alpha$), a single "Mandelbrot" equivalence $\sim$ is introduced to provide an abstract Mandelbrot set $\mathbb{S}^1/\sim$. This quotient space is in the spirit of the pinched disk model of the Mandelbrot set.

24.15. **Definition (Equivalence Relation for the Abstract Mandelbrot Set)**
In the "Mandelbrot" equivalence, $\alpha \sim \beta$ if and only if $\approx^\alpha = \approx^\beta$.

In terms of parameter rays, but modulo local connectivity of $\mathcal{M}$, $\alpha \sim \beta$ if and only if the external parameter rays R_α and R_β land at the same point. Let $e^\alpha(\beta)$ denote the itinerary of β with respect to the partition given by $\{\frac{\alpha}{2}, \frac{\alpha+1}{2}\}$. Then the following properties are derived in [**Ke**, Theorem 3.1].

24.16. **Theorem (Properties of the "Mandelbrot" Equivalence)**

a) The equivalence classes of $\sim$ restricted to the periodic angles under the angle doubling map consists of exactly two points, except for $\alpha = 0$ which is only equivalent to itself [1].

b) The following are equivalent:

(i) $\alpha \sim \gamma$;
(ii) $\alpha \approx^\alpha \gamma$ *and* $\gamma \approx^\gamma \alpha$;
(iii) $e^\alpha(\alpha) \simeq e^\alpha(\gamma)$ *and* $e^\alpha(\gamma) \simeq e^\gamma(\gamma)$, *where we write $x \simeq y$ for two itineraries x and y if $x_i = y_i$ or $x_i = \star$ or $y_i = \star$ for all $i \in \mathbb{N}^*$;*
(iv) $e^\beta(\alpha) = e^\beta(\gamma)$ *for all β between α and γ;*
(v) $\alpha \approx^\beta \gamma$ *for all β between α and γ.*

Conditions (iv) and (v) refer to the limbs in the Mandelbrot set emerging from the equivalence class of α, and Keller continues to describe these limbs and its hyperbolic components in symbolic terms. Also symbolic characterizations of *visible* hyperbolic components are given, in the sense of Definition **??**, HB: need reference to Def. of Visible *i.e.,* hyperbolic component W' is visible from W if the path connecting W and W' contains no hyperbolic component of lower period. Tuning and renormalization are discussed at length, in terms both of symbolic dynamics and of equivalences, leading to a characterization of the kneading sequence present in little copies of the Mandelbrot set in itself.

A further topic is the Translation Principle How is the Translation Principle dealt with in Lau and Schleicher? see also part on Kauko which expresses the similarities between

[1]This corresponds to external angles of the roots of hyperbolic components in $\mathcal{M}$.

different limbs in the abstract Mandelbrot set. Given a hyperbolic component W of period m, the *visible tree* $\mathcal{V}is_{\frac{p}{q}}(W)$ at angle $\frac{p}{q}$ is a set of visible hyperbolic components within the $\frac{p}{q}$-wake of W. Two visible trees $\mathcal{V}is_{\frac{p_1}{q_1}}(W)$ and $\mathcal{V}is_{\frac{p_2}{q_2}}(W)$ are *translation equivalent* if there is a bijection between them that extends homeomorphically to the plane and that increases the periods of the hyperbolic components of $\mathcal{V}is_{\frac{p_1}{q_1}}(W)$ by $(q_2-q_1)m$. The Translation Principle does hot hold in general, see the counter-example below Example 5.3, but the following partial version is valid.

24.17. **Theorem (Partial Translation Principle)**
For each hyperbolic component W, each visible tree $\mathcal{V}is_{\frac{p}{q}}(W)$, except possibly the one with angle $\frac{1}{2}$, is translation equivalent to either $\mathcal{V}is_{\frac{1}{3}}(W)$ or $\mathcal{V}is_{\frac{2}{3}}(W)$.

In term of internal addresses, the following conclusion is drawn:

24.18. **Corollary (Partial Translation Principle for Internal Addresses)**
Given an admissible internal address $1 \to S_1 \to \ldots \to S_k$ and $0 \leq r < S_k$, if $q \geq 2$ is such that $1 \to S_1 \to \ldots \to S_k \to qS_k + r$ is admissible, then $1 \to S_1 \to \ldots \to S_k \to q'S_k + r$ is admissible for all $q' \geq 2$.

25. Portraits

Portraits refers to sets of external angles whose dynamical rays land a special points in the Julia set; thus we have critical portraits and fixed point portraits. This section discusses various approaches taken in the literature. In addition, to determining the properties of the Julia set from their portraits, the question whether every abstract portrait can be realized by a complex polynomial is central in this area.

25.1. Bielefeld, Fisher and Hubbard's Approach to Critical Portraits. Building upon the thesis of Fisher [**Fi**], Bielefeld, Fisher and Hubbard show in [**BFH**] that any polynomial p with all critical points preperiodic is characterized by its *critical marking*. Bielefeld, Fisher and Hubbard define *marked polynomials* $\{p, R_{\vartheta_1}, \dots, R_{\vartheta_n}\}$ as follows: given a polynomial p, one adds for each critical point c_i the angle $\vartheta_i \in \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ of some external ray R_{ϑ_i} that lands at the critical value $p(c_i)$. Thus in general, a polynomial p generates several marked polynomials. The set $\mathcal{P}_d$ is the set of all marked polynomials of degree d with all critical points strictly preperiodic. The critical marking (sometimes also called *critical portrait*, compare Section 25.2 and [**Poi**]) for a marked polynomial $\{p, R_{\vartheta_1}, \dots, R_{\vartheta_n}\}$ is its image under the map PA from $\mathcal{P}_d$ into a finite collection of sets of angles Θ_i:

$$PA(\{p, R_{\vartheta_1}, \dots, R_{\vartheta_n}\}) = (\Theta_1, \dots, \Theta_n),$$

where Θ_i consists of the angles of the preimage rays of R_{ϑ_i} landing at c_i. Here PA stands for *preangle*. If $\vartheta \in \mathbb{S}^1$ and $\sigma_d : \mathbb{S}^1 \to \mathbb{S}^1, \vartheta \mapsto d\vartheta$, then any subset of $m_d^{-1}(\vartheta)$ that contains at least two elements is a *d-preangle* of ϑ. Thus, PA maps a critically marked polynomial to a finite collection of preangles.

A *PPDFA* (that is a *preperiodic polynomial determining family of angles*) of degree d is a finite collection $\mathbf{\Theta}$ of d-preangles Θ_i of rational angles which meet the following conditions:

- for all i, all angles in Θ_i are rational;
- $\sum(\#(\Theta_i) - 1) = d - 1$;
- for any $i \neq j$, Θ_i, Θ_j are unlinked;
- for all i, there is no $\vartheta \in \Theta_i$ which is periodic;
- for any $i \neq j$, $\Theta_i \not\sim_{\mathbf{\Theta}} \Theta_j$[2].

Indeed, a PPDFA is a formal critical portrait which is the candidate critical portrait for some polynomial p with all critical points preperiodic. The set of all PPDFAs of degree d is denoted by $\mathcal{A}_d$.

Theorem II and Theorem III of [**BFH**] prove that there is a bijection between the sets $\mathcal{P}_d$ and $\mathcal{A}_d$ and that there is a necessary and sufficient condition for two PPDFAs determining the same polynomial.

[2]We refer to [**BFH**, Def. 2.7] for a definition of this equivalence relation.

25.1. **Theorem (PA is Bijective)**
The map $PA : \mathcal{P}_d \to \mathcal{A}_d$ is a bijection.

$\{\Theta_1, \ldots, \Theta_n\}$ *and* $\{\Theta'_1, \ldots, \Theta'_n\}$ *determine the same polynomial if and only if the* Θ'_n *can be renumbered such that* $\Theta_1 \sim_{\boldsymbol{\Theta}} \Theta'_1, \ldots, \Theta_n \sim_{\boldsymbol{\Theta}} \Theta'_n$.

25.2. Poirier's Approach to Critical Portraits. In [**Poi1**] Poirier extends [**BFH**] to the set of all postcritically finite polynomials. Thus, critical points are now allowed to be periodic or map onto other periodic critical points. A more careful definition of marked polynomials allows to extend this concept to postcritically finite polynomials with one or more periodic critical points.

Let U be a Fatou component of the polynomial p and $z \in \partial U$. Let us denote the rays within the unbounded Fatou component that land at z by $R_{\vartheta_1}, \ldots R_{\vartheta_k}$. These rays partition $\mathbb{C}$ into k regions. If the rays are labeled counterclockwise such that U is contained in the region bounded by R_{ϑ_1} and R_{ϑ_2}, then ϑ_1 is called the *(left) supporting argument* and ϑ_2 the *(right) supporting argument* of U. A *critically marked polynomial* is the triple $(p, \mathcal{F}, \mathcal{J})$, where p is postcritically finite and $\mathcal{F} = \{F_k\}_{k=1}^m, \mathcal{J} = \{J_k\}_{k=1}^n$ are collections of subsets of $\mathbb{Q}/\mathbb{Z}$ each of which is associated to one critical point of p as follows.

Periodic critical points: Let z_1^F be an n-periodic critical point contained in a Fatou component U of p and $z \in \partial U$ be a fixed point of $f^{\circ n}$. If ϑ is the left supporting argument at z then F_1 consists of the arguments of the preimage rays of $p(R_\vartheta)$ that land at z. The remaining sets F_k for periodic critical points z_k^F are constructed in an analogous way, where restrictions apply if the critical cycle of z_k^F contains a critical point z_i^F for which the set F_i has already been constructed.

Preperiodic critical points: The set F_k for a preperiodic critical point z_k^F that maps onto a periodic critical point is obtained the following way: if $z_k^F \in U$ and $f^{\circ n}(z_k^F) \in U_n$ is the earliest iterate of z_k^F that is periodic, then let R_ϑ be the left supporting ray of U_n. The set F_k consists of the arguments of the preimage rays in $p^{-n}(R_\vartheta)$ that land at U.

For critical points z_k^J in the Julia set, the J_k are constructed similarly by taking any ray landing at z_k^J.

Note that by construction, the number of elements contained in some F_k or J_k corresponds to the local degree of p at the associated critical point.

25.2. **Theorem (Critical Marking Determines Polynomial)**
The tuple $(\mathcal{F}, \mathcal{J})$ determines the postcritically finite polynomial, that is, if p, q are two postcritically finite polynomial with critical marking $(p, \mathcal{F}, \mathcal{J})$ and $(q, \mathcal{F}, \mathcal{J})$, then $p \equiv q$.

If $\mathcal{F} = \{F_k\}, \mathcal{J} = \{J_k\}$ are finite families of sets of rational d-preangles (as defined in the previous Subsection 25.1), then the tuple $(\mathcal{F}, \mathcal{J})$ is called an *admissible critical portrait of degree d* if it meets certain requirements among them that

$d-1 = \sum_k(\#(F_k)-1) + \sum_k(\#(J_k)-1)$ and that $\mathcal{F}, \mathcal{J}$ are unlinked and hierarchical families [**Poi1**, Chapter 3].

25.3. **Theorem (Admissible Critical Portraits Realized)**
If $(\mathcal{F}, \mathcal{J})$ is an admissible critical portrait of degree d then there is a unique postcritically finite polynomial p of degree d such that $(p, \mathcal{F}, \mathcal{J})$ is its critical marking.

Roughly, the proof goes as follows. One constructs a topological graph, a so-called *web*, from the given collection of angles. Then one embeds the web into the plane. (There are several possibilities which all yield Thurston equivalent maps.) Now one defines a dynamics f on the web which extends to $\hat{\mathbb{C}}$ as a branched covering. The final step is to show that this map is actually a polynomial. Here Thurston's characterization for rational functions comes into play. By the work of Bielefeld, Fisher and Hubbard [**BFH**], it is enough to show that there are no Levy cycles in order to finish the proof. HB: A reference, within the book preferably, to Levy cycles is necessary

25.3. Fixed Point Portraits. A concept similar to critical portraits are *fixed point portraits* studied by Goldberg and Milnor in [**GM**]. A fixed point portrait of a polynomial p with connected Julia set is the collection of non-empty sets Θ_i consisting of the angles of all external rays that land at the fixed point z_i of p. Note that two non-conjugate polynomials may share the same fixed point portrait. The sets Θ_i are *rational rotation sets* of the circle, which are investigated in [**Go**] in detail. In the following we give a short summary of the results in [**GM**].

If p is a polynomial of degree d then there are $d-1$ fixed rays, which partition the complex plane in, say, m regions U_i. (Note that n equals $d - \#\{z : p(z) = z\}|$.) Each U_i contains exactly one fixed point or *virtual fixed points* and at least one critical point. The *critical weight* of U_i is the number of critical points in U_i, counting multiplicity, and equals the number of fixed points in ∂U_i. Similarly, let S_i be one of the sectors in $\mathbb{C}$ generated by the rays defining Θ_i. The critical weight of S_i equals the number of fixed rays in S_i and also the number of fixed points or virtual fixed points in S_i. These results form the basis of Goldberg's and Milnor's characterization of fixed point portraits of polynomials.

More precisely, they show that given a collection of non-empty sets of rational angles $\{\Theta_1, \dots, \Theta_d\}$ which satisfy four properties is realized as a fixed point portrait of some polynomial p of degree d (with connected Julia set). The polynomial p necessarily has d pairwise distinct fixed points z_i such that at each of them at least one external ray is landing. In [**?**], HB: should this be **[Poi4]**? Poirier extends this result to any collection of rational angles which satisfy the four properties specified in [**GM**].

To prove their realization theorem, Goldberg and Milnor link fixed point portraits and critical portraits developed by Fisher [**Fi**]. They first show that the fixed point portrait is uniquely determined by the critical portrait and then that the candidate fixed point portrait $\{\Theta_1, \ldots, \Theta_d\}$ generates a critical portrait which is realized by some polynomial of degree d by [**BFH**].

26. Circle Maps and Siegel Disks

The key observation in the analysis of the dynamics of Siegel disks U is that points on ∂U are landing points of both internal and external rays. As a consequence, the corresponding set of angles is both invariant under a rotation and the angle doubling map. Bullett and Sentenac [**BuS**] obtain many results on the set of external angles associated to Siegel disks, include very detailed number-theoretic properties.

26.1. Bullett and Sentenac. Suppose the Julia set of a quadratic polynomial $f : z \mapsto z^2 + c$ is locally connected and surrounds a Siegel disk U, say $f(U) = U$ for simplicity. The dynamics on a Siegel disk are conjugate to a rotation, say over an angle $2\pi\alpha$ with $\alpha \in [0,1] \setminus \mathbb{Q}$, on the open unit disk $\mathbb{D}$. Let $\varphi : \mathbb{D} \to U$ denote the conjugacy, so $f \circ \varphi(z) = \varphi(e^{2\pi i\alpha})$ and $\varphi(\{re^{2\pi i\eta} : r \in (0,1)\})$ is called the *internal ray* of *internal angle* η. The *landing point* of the internal ray $\lim_{r\to 1} \varphi(re^{2\pi i\eta})$ exists and belongs to ∂U for every η. Conversely, every point on ∂U is the landing point of an internal rays and also of at least one external ray (at least when the corresponding Julia set is locally connected).

The set of external angles with rays landing at ∂U is invariant under the angle doubling map g and the set of internal angles with internal rays landing on ∂U is invariant under the rotation $\eta \mapsto \eta + \alpha$ (in fact, this is the whole circle $\mathbb{S}^1$). Since the f-images of landing points are the landing points of image rays, this raises the question which sets of external angles are both invariant under the angle doubling map and a version of rotation. This was one of the motivations that led Bullett and Sentenac to write [**BuS**], drawing from ideas of Douady, *e.g.* [**D4**]. A key observation is that $g|_C$ must preserve the *circular order*: if three distinct angles $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{S}^1$ are ordered in counter-clockwise direction, then the image angles $g(\varphi_1), g(\varphi_2), g(\varphi_3)$ are distinct and ordered in counter-clockwise direction as well.

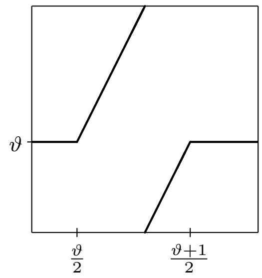
26.1. **Lemma (Orientation of Angle Doubling)**
The angle doubling maps preserves the circular order of $\varphi_1, \varphi_2, \varphi_3$ if and only if $\varphi_1, \varphi_2, \varphi_3$ belong to a half-open semi-circle.

Proof. We follow Bullett and Sentenac [**BuS**, Lemma 1]. The "if"-direction is trivial. For the "only if"-direction, first observed that if two angles, say φ_1 and φ_2 are diametrically opposite (*i.e.,* $\varphi_3 = \varphi_1 + \frac{1}{2}$), then $g(\varphi_1) = g(\varphi_3)$. Suppose that no two angles are diametrically opposite and that $\varphi_1, \varphi_2, \varphi_3$ do not belong to any semi-circle. Then for one of the angles, say φ_2, $\varphi_2 + \frac{1}{2}$ belongs to the shorter arc between φ_1 and φ_3. Then $\varphi_1, \varphi_2 + \frac{1}{2}, \varphi_3$ belong to a semi-circle, and have opposite order to $\varphi_1, \varphi_2, \varphi_3$. It follows that $g(\varphi_1), g(\varphi_2), g(\varphi_3) = g(\varphi_1), g(\varphi_2 + \frac{1}{2}), g(\varphi_3)$ has the same order as $\varphi_1, \varphi_2 + \frac{1}{2}, \varphi_3$, which is opposite to $\varphi_1, \varphi_2, \varphi_3$. $\square$

We can crop the angle doubling map g to an order preserving circle map

$$g_\vartheta(x) = \begin{cases} 2x & \text{if } x \in [\frac{\vartheta}{2}, \frac{\vartheta+1}{2}]; \\ \vartheta & \text{otherwise.} \end{cases}$$

see Figure 26.1). The set of C_ϑ of angles whose forward orbits avoid $(\frac{\vartheta+1}{2}, \frac{\vartheta}{2})$ has the



$$g_\vartheta(x) = \begin{cases} 2x & \text{if } x \in [\frac{\vartheta}{2}, \frac{\vartheta+1}{2}]; \\ \vartheta & \text{otherwise.} \end{cases}$$

FIGURE 26.1. The *cropped* angle doubling map g_ϑ.

same behavior under g as under g_ϑ. To study the behavior of C_ϑ the rotation number is of interest. For the order preserving map g_ϑ, the rotation number

$$r_\vartheta = \lim_{n\to\infty} \frac{G_\vartheta^n(x) - x}{n}$$

is well-defined and independent of x. Here $G_\vartheta : \mathbb{R} \to \mathbb{R}$ is the *lift* of g_ϑ satisfying $G_\vartheta(x) = g_\vartheta$ for $x \in [0,1)$ and $G_\vartheta(x+1) = G_\vartheta(x) + 1$ for all $x \in \mathbb{R}$. The rotation number depends continuously and monotonically (because $\vartheta \mapsto G_\vartheta$ is monotone) on ϑ, see [**ALM1**, Chapter 3]. The first two main results from [**BuS**] are as follows:

26.2. **Theorem (Structure of C_ϑ)**
Let r_ϑ be the rotation number of g_ϑ.

(a) *If $r_\vartheta \in [0,1] \setminus \mathbb{Q}$, then the set C_ϑ is a Cantor set of upper box dimension* 0*, consisting of the closure of any of its points.*
(b) *If $r_\vartheta \in [0,1] \cap \mathbb{Q}$, say $r_\vartheta = \frac{p}{q}$ in reduced form, then C_ϑ is a finite orbit of period q.*

The map $r : \vartheta \mapsto r_\vartheta$ is a Devil's staircase*. More precisely, r is continuous and non-decreasing from $[0,1]$ onto itself such that r is locally constant whenever $r_\vartheta \in \mathbb{Q}$. The set $C = \{\vartheta \in [0,1] : r_\vartheta \notin \mathbb{Q}\}$ has Hausdorff dimension* 0*.*

PROOF OF THE FIRST PART. Let $I_n = g^{-n}([\frac{\vartheta}{2}, \frac{\vartheta+1}{2}])$ for $n \geq 0$. Since $r_\vartheta \notin \mathbb{Q}$, the intervals I_n are pairwise disjoint and have length $|I_n| = 2^{-(n+1)}$. This already shows that $\cup_{n\geq 0} I_n$ has Lebesgue measure $\sum_{n\geq 0} 2^{-(n+1)} = 1$, so $\mathrm{Leb}(C_\vartheta) = 0$. Let G_n be any component of $\mathbb{S}^1 \setminus \cup_{k<n} I_k$. If $|G_n| > 2^{-n}$, then $\mathrm{Leb}(G_n \setminus \cup_k I_k) > 2^{-n} - \sum_{k\geq n} 2^{-k+1} = 0$, contradicting that $\mathrm{Leb}(C_\vartheta) = 0$. Therefore C_ϑ is covered by n intervals of length at most 2^{-n}. Since n is arbitrary, it follows that C_ϑ is totally disconnected, and has upper box dimension $\lim_{n\to\infty} -\frac{\log n}{\log 2^{-n}} = 0$.

The dynamics of $g_\vartheta : C_\vartheta \to C_\vartheta$ is semiconjugate to the circle rotation over angle r_ϑ. The semiconjugacy collapses the endpoints of each interval I_n, but is bijective elesewhere. Since none of the I_ns are adjacent (as $r_\vartheta \notin \mathbb{Q}$), C_ϑ has no isolated points and thus is a Cantor set, and equal to the orbit closure of each of its points. □

We can associate symbolic dynamics to g_ϑ using the partition $[\vartheta/2, \vartheta]$ (with symbol `0`) and $[\vartheta, (1+\vartheta)/2]$ (with symbol `1`). The resulting subshift is well-studied. Morse and Hedlund [**MH**] were the first to study such sequences, calling them *Sturmian*. They are *balanced* in the sense that the number of `0`s in a subword of length n differs from the number of `0`s in another subword of length n by at most 1. Later studies tied Sturmian sequences firmly to symbolic dynamics of rotations, but there is a wide range of contexts in which Sturmian sequences emerge naturally. Since g_ϑ is semi-conjugate to the rotation over r_ϑ and Lebesgue measure is the only invariant probability measure of rotations, it follows immediately that

$$r_\vartheta = \lim_{n\to\infty} \frac{1}{n}\{0 < i \le n : g_\vartheta^{\circ i}(x) \in [\vartheta/2, \vartheta]\}$$

for each $x \in C_\vartheta$. If ϑ is such that C_ϑ is a cantor set, then this implies that $r_\vartheta = \lim_{n\to\infty} \frac{1}{n}\{0 < i \le n : b_i = 1\}$ where $0.b_1b_2b_3b_4\dots$ is the binary expansion of ϑ. This is in fact the inverse of the staircase algorithm below, but there are more sophisticated methods of finding (the continued fraction of) r_ϑ from g_ϑ-orbits.

Since the graph of the function $\vartheta \mapsto r_\vartheta$ is a Devil's staircase, it does not have a proper inverse. However, regardless of whether C_ϑ is finite or a Cantor set, its convex hull $\mathrm{conv}\,(C_\vartheta)$ can be determined algorithmically. We first give the main result:

26.3. **Theorem (Further Structure of C_ϑ)**
Suppose that $r \in (0,1)$ is the rotation number of g_ϑ.

(a) *If $r \in \mathbb{Q}$, say $r = \frac{p}{q}$ in reduced form, then there are two unique adjacent points $0 < \vartheta_r^- < \vartheta_r^+ < 1$ in C_ϑ such that $conv(C_\vartheta) = [\vartheta_r^+/2, (\vartheta_r^- + 1)/2]$. The length of this interval is $(2^{q-1}-1)/(2^q-1)$.*
(b) *If $r \in [0,1] \setminus \mathbb{Q}$, then $\vartheta = \vartheta_r$ is exactly the point such that $conv(C_\vartheta) = [\vartheta/2, (\vartheta+1)/2]$, and $\mathrm{orb}_g(\vartheta)$ is a dense subset of C_ϑ.*

For rotation numbers $r < \sigma < r'$ with $\sigma \in \mathbb{Q}$ and $r, r' \in [0,1] \setminus \mathbb{Q}$, we have $\vartheta_r < \vartheta_\sigma^- < \vartheta_\sigma^+ < \vartheta_{r'}$.

26.4. **Algorithm (Staircase Algorithm)**
One algorithm to compute $\vartheta_r^{(\pm)}$ from r is as follows. Draw the line ℓ from the origin in the plane $\mathbb{R}^2$ with slope r. Starting from the point $(1,0)$, draw a staircase taking unit length steps to the right or vertically upwards, staying as close under ℓ as possible (and

possibly touching ℓ), see Figure 26.2. Encode the staircase by

$$\begin{cases} 0 \; : \; \textit{a single step to the right;} \\ 1 \; : \; \textit{a single step to the right followed by a single step upwards.} \end{cases}$$

If $r \notin \mathbb{Q}$, then the resulting 0-1-*sequence is the binary expansion of ϑ_r. If $r \in \mathbb{Q}$, then the staircase will periodically touch ℓ; the binary expansion of its code is ϑ_r^+. The staircase just below it (i.e., which avoids touching ℓ) yields the binary expansion of ϑ_r^-.*

HB: Relate this to the algorithms in section 13!?

26.5. **Example (Two possibilities for C_ϑ to have period 4)**

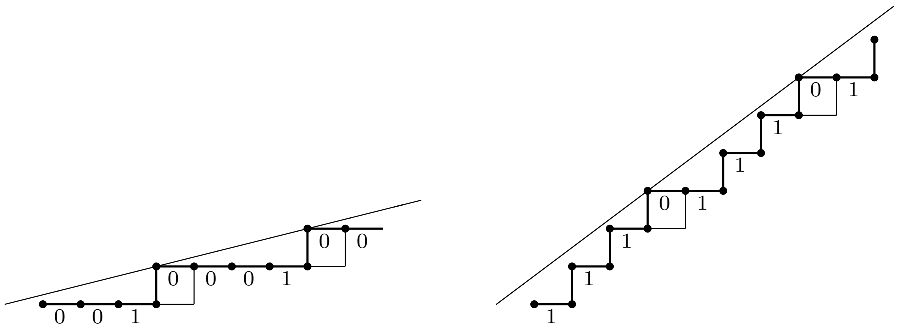


FIGURE 26.2. Staircases for $r = \frac{1}{4}$ (left) and $r = \frac{3}{4}$ (right). The symbols indicate the code for the upper staircase (thick lines) to find ϑ_r^+. The thin lines indicate the lower staircase to find ϑ_r^-; its code differs from the code of the upper staircase by exactly two digits in every period.

If C_ϑ is a periodic orbit of period 4, then r can take values $\frac{1}{4}$ and $\frac{3}{4}$. For $r = \frac{1}{4}$, we find the staircases in the left panel of Figure 26.2:

$$\vartheta_{\frac{1}{4}}^- = 0.\overline{0001} \text{ (binary) } = \frac{1}{15} \quad \text{ and } \quad \vartheta_{\frac{1}{4}}^+ = 0.\overline{0010} \text{ (binary) } = \frac{2}{15},$$

and $C_\vartheta = \{\frac{1}{15} = \vartheta_{\frac{1}{4}}^+/2, \ \frac{2}{15}, \ \frac{4}{15}, \ \frac{8}{15} = (\vartheta_{\frac{1}{4}}^- + 1)/2\}$.

For $r = \frac{3}{4}$, we find the staircases in the right panel of Figure 26.2:

$$\vartheta_{\frac{3}{4}}^- = 0.\overline{1101} \text{ (binary) } = \frac{13}{15} \quad \text{ and } \quad \vartheta_{\frac{3}{4}}^+ = 0.\overline{1110} \text{ (binary) } = \frac{14}{15},$$

and $C_\vartheta = \{\frac{7}{15} = \vartheta_{\frac{3}{4}}^+/2, \ \frac{11}{15}, \ \frac{13}{15}, \ \frac{14}{15} = (\vartheta_{\frac{3}{4}}^- + 1)/2\}$.

The third g-orbit of period 4 is $\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$; it is not contained in a semi-circle and hence g does not preserve its circular order.

If $r \notin \mathbb{Q}$, then one can establish number theoretic properties of ϑ_r. Recall that an irrational x is of *bounded type* if the entries a_i in its continued fraction expansion

$$x = [a_0; a_1, a_2, a_3, \dots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}, \quad a_0 \in \mathbb{Z}, a_i \in \mathbb{N}^* \text{ for } i \geq 1$$

form a bounded sequence. It is of *constant type* if $\{a_i\}_i$ is eventually constant, and x is *noble* if a_i is eventually constant 1. Clearly, the golden ratio $\gamma = \frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, 1, \dots]$ is noble.

An irrational x is *Diophantine* of order $\beta \geq 2$ if there is $c > 0$ such that

$$|x - \frac{p}{q}| \geq \frac{c}{q^\beta} \quad \text{for all} \;\; p, q \in \mathbb{Z} \setminus \{0\},$$

and *Liouville* if it is not Diophantine of any order. Whereas numbers of constant type are known to be Diophantine of order 2, Roth's Theorem [**Ro**] states that every algebraic irrational is Diophantine of order β for every $\beta > 2$.

26.6. **Theorem (Transcendence of ϑ_r)**
Assume that the rotation number $r = r_\vartheta \notin \mathbb{Q}$.

(a) *If r is not of constant type, then ϑ_r is Liouville.*
(b) *If r is of constant type, but not noble, then ϑ_r not Diophantine of order* 3.
(c) *If r is noble, then ϑ_r not Diophantine of order $2 + \gamma - \varepsilon$ for the golden ratio $\gamma = \frac{\sqrt{5}-1}{2}$ and every $\varepsilon > 0$.*

It follows from Roth's Theorem that ϑ_r is transcendental for every irrational r.

This theorem shows that the function $\vartheta \mapsto r_\vartheta$ is locally constant at every algebraic number ϑ.

The set C_ϑ can also be obtained by iterating the square root map $\mathrm{Sqrt}_\vartheta : \mathbb{C} \to \mathbb{C}$, $z \mapsto \sqrt{z}$, taking the branch of $\sqrt{z}$ with argument $\vartheta/2 \leq \sqrt{z} \leq (1+\vartheta)/2$. Bullett and Sentenac [**BuS**, Theorem 5] show that for each $\vartheta \in [0,1]$, there is a unique attractor A_ϑ in the sense that A_ϑ is a minimal closed set having a neighborhood U_ϑ such that $\mathrm{Sqrt}_\vartheta(U_\vartheta) \subset U_\vartheta$. This attractor belongs to $\{z \in \mathbb{C} : |z| = 1\}$ and Sqrt_ϑ preserves the circular order on A_ϑ. In fact, A_ϑ is the set of unit vectors with arguments in C_ϑ.

The itinerary described above should not be confused with the binary expansion of ϑ. Binary expansions play a role in *Douady tuning*. Douady tuning [**D4**] expresses external angles whose ray lands on the boundary of a smaller copy of the Mandelbrot set ∂W in $\mathcal{M}$ by taking the corresponding angle on $\mathcal{M}$ and replacing the 0s and 1s in the binary expansion by by periodic blocks d^- and d^+ of the pair of external angles whose rays land at the root of W. First observe that if $X \subset \mathbb{S}^1$ is g^n-invariant and contained in an arc of length 2^{-n} then g^n preserves the circular order on A (cf. Lemma 26.1) but the converse is not true. If A is a finite g-orbit, then it may occur that A does not have a single rotation number, but rather a sequence of rotation numbers as a result of tuning.

26.7. **Definition (Rotation Sequence)**
A closed subset $A \subset \mathbb{S}^1$ has rotation sequence $(\nu_1, \nu_2, \dots, \nu_n)$*, where $\nu_i \in \mathbb{Q} \cap [0,1]$ for $i < n$ and $\nu_n \in [0,1]$ if*

- *$n = 1$ and A has rotation number ν_1 for g, or*
- *$n > 1$, $\nu_1 = p_1/q_1$ and $A \subset I_0^\circ \cup I_1^\circ \cup \dots \cup I_{q_1-1}^\circ$ (with intervals $I_k = g(I_{k-1})$ for $1 \le k \le q_1 - 1$), g maps $A \cap I_k$ bijectively to $A \cap I_{k+1 \bmod q_1}$ and $g^{\circ q_1}$ acts on $A \cap I_0$ with rotation sequence $(\nu_2, \dots \nu_n)$.*

The main theorem (see [**BuS**, Theorem 7]) on this topic is:

26.8. **Theorem (Existence of Sets with Rotation Sequences)**
For each rotation sequence $(\nu_1, \dots \nu_n)$ there exists a unique minimal closed g-invariant set $A \in \mathbb{S}^1$ with rotation sequence $(\nu_1, \dots \nu_n)$.

A partial proof is contained in Corollary 14.8. The rotation sequences of length two are associated to hyperbolic components attached to (*i.e.,* bifurcating from) the main cardioid of the Mandelbrot set, in the sense that if c is chosen in the boundary of such a hyperbolic component, and $z \mapsto z^2 + c$ has a Siegel disk, then the dynamics on this disk is described by a rotation sequence of length 2. Similarly, the rotation sequences of length 3 are associated to hyperbolic components attached to hyperbolic components of the previous kind. For example, if the rotation sequence is $(\frac{1}{2}, \frac{1}{3}, \gamma)$, where $\gamma = \frac{\sqrt{5}-1}{2}$ is the golden ratio, then the binary sequence of ϑ is obtained as follows. Start at the end of the rotation sequence and write down the binary expansion of the number with rotation number γ:

$$\vartheta_\gamma = 0.7098034 \dots = 0.1011010110 \dots \text{(binary)}.$$

The angle $\vartheta = \frac{1}{7} = 0.\overline{001}$ has rotation number $\frac{1}{3}$. Replace in the above sequence (after the decimal point) every 0 by 001 and every 1 by 010 (*i.e.,* 001 with the last two symbols swapped) to obtain

$$\vartheta_{(\frac{1}{3}, \gamma)} = 0.010001010010001010001010010001 \dots$$

The angle $\vartheta_{\frac{1}{2}} = \frac{1}{3} = 0.\overline{01}$. Replace in the above sequence (after the decimal point) every 0 by 01 and every 1 by 10:

$$\vartheta_{(\frac{1}{2}, \frac{1}{3}, \gamma)} = 0.011001010110011001011001010110011001010110011001011001010110 \dots$$

This is the decimal expansion of the angle whose external ray lands at $c \in \partial\mathcal{M}$ in a secondary hyperbolic component, in the $\frac{1}{3}$-wake of the period 2 hyperbolic component in the $\frac{1}{2}$-wake of the main cardioid. The polynomial $z \mapsto z^2 + c$ has a Siegel disk of rotation number γ and period $6 = 3 \cdot 2$.

Rotation sequences can be infinite, when expressing infinitely renormalizable maps. For example $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$ is related to the Morse sequence

$$01101001100101101001011001101001\dots,$$

and the angle ϑ with this binary expansion has a ray landing at the Feigenbaum parameter, *i.e.,* the limit parameter of the period doubling cascade along the real axis.

27. Yoccoz Puzzles

Puzzles were developed by Branner and Hubbard's and by Yoccoz to prove local connectivity of Julia sets, and has since been taken up by many authors to address a much wider range of questions. In this section we present puzzles and the corresponding critical tableaux. This gives a symbolic description of the dynamics of Julia sets, alternative to the kneading theory that is predominant in this book.

27.1. Puzzle pieces and (critical) tableaux. *Puzzles* were designed by Yoccoz as a tool to prove local connectivity of, originally, quadratic Julia sets, but a paper very influential on the development of puzzles was written by Branner and Hubbard [**BH**] on cubic polynomials. By now, puzzles (and parapuzzles in parameters spaces) are commonly used for various maps and purposes. In this section we focus on the combinatorial description of the dynamics emerging from puzzles, phrased in the language of *tableaux*. The main early references are [**BH, HY, Mi0**].

27.1. **Definition (Puzzles and Puzzles Pieces)**
Take $c \in \mathcal{M}$ and $G_c : \mathbb{C} \setminus J_c \to (0, \infty)$ be the corresponding Green function. Fix $R > 0$ and let

$$U_0 := \{z \in \mathbb{C} : z \in J_c \text{ or } G_c(z) < R\}.$$

The fixed point α has $q \geq 2$ external rays, say $\gamma_0, \dots, \gamma_{q-1}$, landing on it. Let

$$\Gamma_0 = (\{\alpha\} \cup \gamma_0 \cup \dots \cup \gamma_{q-1}) \cap U_0.$$

The closures of the components of $U_0 \setminus \Gamma_0$ are the puzzle pieces *of* depth 0*.*
Continue inductively:

$$U_n = p_c^{-1}(U_{n-1}) \quad \text{and} \quad \Gamma_n = p_c^{-1}(\Gamma_{n-1}),$$

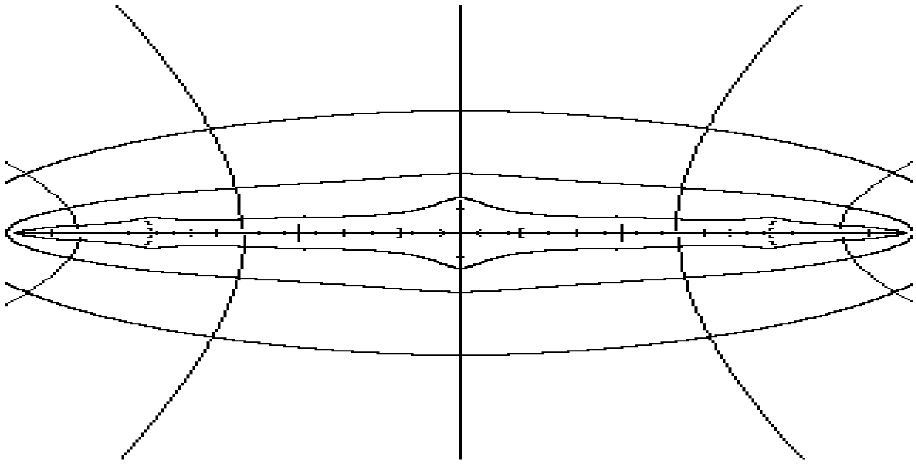
and the closures of the components of $U_n \setminus \Gamma_n$ are the puzzle pieces *of* depth n*.*

Obviously $U_0 \supset U_1 \supset U_2 \supset \dots$ and since $p_c(\Gamma_0) \supset \Gamma_0$, it follows that the number of puzzle pieces of depth n increases with n. Also $\Gamma_0 \cap J_c = \{\alpha\}$, so $\Gamma_n \cap J_c$ consists of all preimages of α of order less than n. Whenever x is not eventually mapped on α, there is a nest of puzzle pieces $X_0(x) \supset X_1(x) \supset \dots \supset X_k(x) \supset \dots$, where $X_k(x)$ is the unique puzzle piece of depth k containing x. The *impression* of x is

$$\mathcal{I}(x) = \cap_{k \geq 0} X_k(x).$$

For non-renormalizable polynomials, this definition of impression $\mathcal{I}(x)$ has the same outcome as the usage in Section 9. If p_c is renormalizable, say of period N, then for instance the critical impression $\mathcal{I}(0)$ contains the little Julia set $\tilde{J}$ of $p_c^{\circ N}$. In this case, one could start the puzzle construction for $\tilde{J}$. Observe that

$$p_c(X_k(x)) = X_{k-1}(p_c(x)) \text{ for all } x \in J_c \setminus \cup_{n \geq 0} p_c^{-n}(\alpha).$$



I hope to steal a better picture somewhere sometime

FIGURE 27.1. The Yoccoz puzzle for the quadratic Fibonacci map with internal address $1 \to 2 \to 3 \to 5 \to 8 \to 13 \to 21 \to \dots$

If $\mathcal{I}(x) = \{x\}$, this means effectively that arbitrary small neighborhoods of x can be mapped onto J_c by a branched covering map $p_c^{\circ n} : X_n(x) \to$ component of $U_0 \setminus \Gamma_0$, whose covering degree can be determined from the number of puzzle pieces $X_{n-k}(p_c^{\circ k}(x))$ that contain the critical point. This is expressed by the following theorem, cf. Propositions 5.6 and 5.7 in [**HY**]:

27.2. **Theorem (Impressions and Local Connectivity)**
Take $c \in \mathcal{M}$.

(i) *If p_c is non-renormalizable, then $\mathcal{I}(x) = \{x\}$ for all $x \in J_c \setminus \cup_{k \geq 0} p_c^{-k}(\alpha)$. In this case J_c is locally connected.*

(ii) *If $p = p_c$ is renormalizable, say its renormalization is hybrid equivalent to $\tilde{p}$, then the critical impression $\mathcal{I}(0)$ is homeomorphic to the Julia set of $\tilde{p}$, and the same is true for the impressions of all $x \in J_c$ that eventually map into $\mathcal{I}(0)$. For all other points $x \in J_c \setminus \cup_{k \geq 0} p_c^{-k}(\alpha)$, the impression $\mathcal{I}(x) = \{x\}$.*

Instrumental in the use of this theorem is the sequence of (possibly degenerate) annuli $A_n(x) = X_n(x) \setminus X_{n+1}(x)$.

27.3. **Lemma (Divergence of Moduli Sums)**
If the series of moduli $\sum_{n=0}^{\infty} \mathrm{mod}\, A_n(x)$ diverges, (where the modulus of a degenerate annulus is 0), then $\operatorname{diam} X_n(x) \to 0$ as $n \to \infty$ and $\mathcal{I}(x) = \{x\}$.

We will not prove this lemma, nor discuss its role in Theorem 27.2, but instead present the notion of tableaux and critical tableaux. These were primarily designed to determine for which annuli, $p_c : A_n(x) \to A_{n-1}(p_c(x))$ is a univalent maps, a branched covering or proper covering map (the *off-critical*, *semi-critical* and *strictly critical* case). They also give a way to describe the combinatorial properties of Julia sets as a whole.

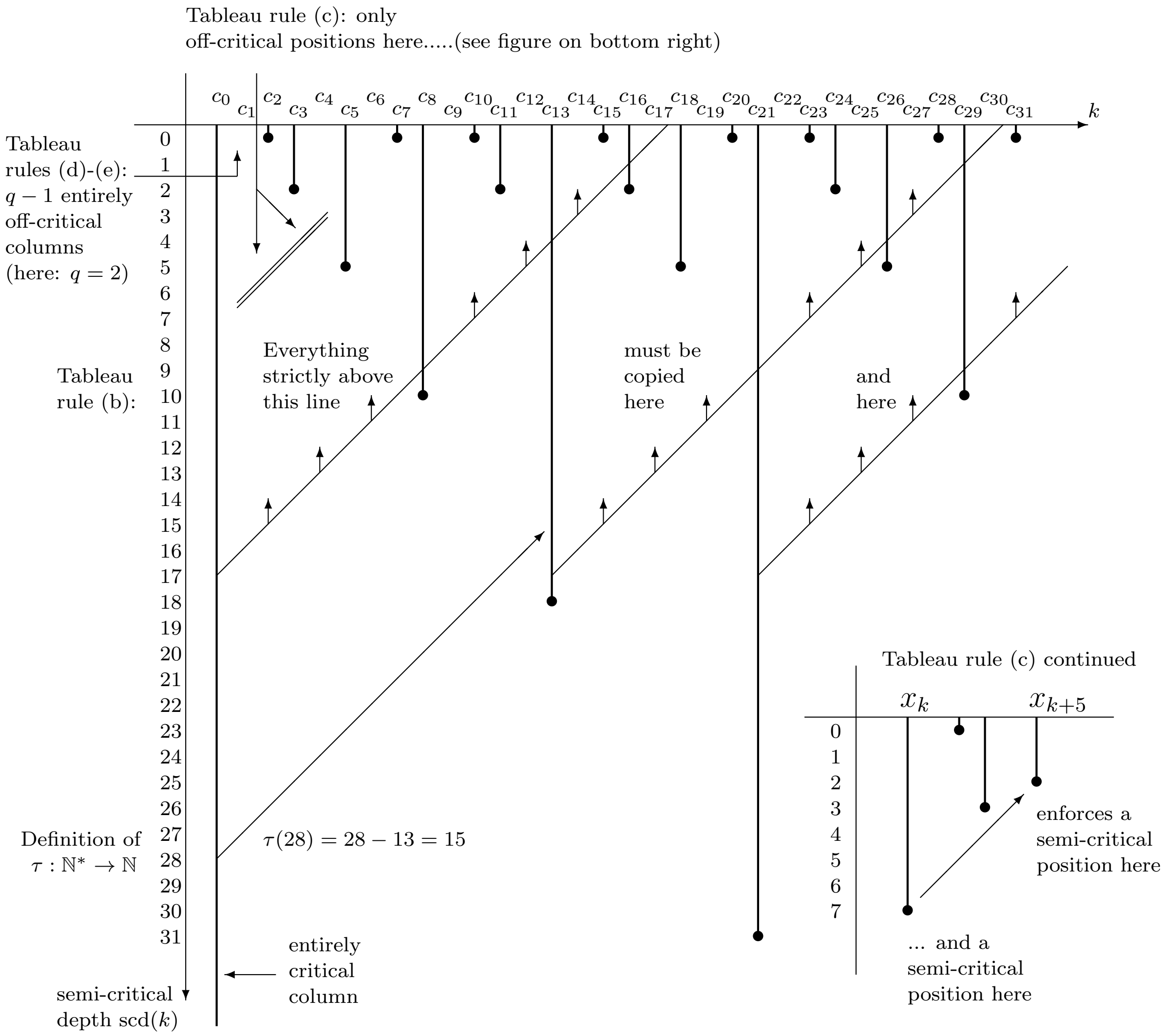


FIGURE 27.2. An example of a marked grid (namely of the Fibonacci map with internal address $1 \to 2 \to 3 \to 5 \to 8 \to 13 \to 21 \to \dots$), with $|$ for strictly critical and $\bullet$ for semi-critical positions. The texts illustrate tableau rules (b)-(e), and how to compute the Yoccoz' τ-function.

27.4. **Definition ((Critical) Tableaux and Marked Grids)**
Fix $c \in \mathcal{M}$. *The* tableau $T(x)$ *is the two-dimensional array*

$$T(x) := (T_{n,k}(x))_{n,k\geq 0} = \left(X_n(p_c^{\circ k}x)\right)_{n,k\geq 0}.$$

If $x = 0$, then we call $T(0)$ the critical tableau. *In the tableau $T(x)$, position*

$$(n,k) \text{ is called } \begin{cases} \text{off-critical} & \text{if } T_{n,k}(x) \not\ni 0; \\ \text{semi-critical} & \text{if } T_{n,k}(x) \setminus T_{n+1,k}(x) \ni 0; \\ \text{strictly critical} & \text{if } T_{n+1,k}(x) \ni 0. \end{cases}$$

The position (n,k) is critical *if it is semi-critical or strictly critical.*

A marked grid *is the two-dimensional array $\mathbb{N}^2 = (n,k)_{n,k \geq 0}$ where is each (n,k) is marked according to whether it is off-critical, semi-critical or strictly critical, see Figure 27.2. The marked grid of the critical point is the* critical marked grid.

Tableau rules. The marked grid of every quadratic polynomial satisfies the following rules, see Figure 27.2.

(a) Each column is either entirely critical, entirely off-critical, or has a unique semi-critical position, above which all positions are strictly critical. Thus the *semi-critical depth* $\mathrm{scd}(k) \in \{-1, 0, 1, 2, 3, \dots, \infty\}$ is uniquely determined as the position $(\mathrm{scd}(k), k)$ at which the k-th column is semi-critical, with entirely off-critical resp. entirely critical columns denoted as $\mathrm{scd}(k) = -1$ resp. ∞.

(b) If $(n,k) \in T(x)$ is critical, then $(i, k+j) \in T(x)$ has the same nature as $(i,j) \in T(0)$ whenever $i + j < n$.

(c) If $(n,k) \in T(0)$ is strictly critical and $(n+i, i) \in T(0)$ is off-critical for $0 < i < k$, and if $(n+k, l) \in T(x)$ is semi-critical, then $(n, l+k) \in T(x)$ is semi-critical as well.

(d) There are at most $q - 1$ consecutive columns in $T(x)$ that are entirely off-critical.

(e) The columns $1, \dots, q-1$ in the critical tableau $T(0)$ are entirely off-critical. (Rules (d) and (e) refer to the $q-1$ components of $U_0 \setminus \Gamma_0$ that do not contain the critical point, see Lemma 3.7.)

Whether p_c is renormalizable or not can be read off from the tableau, using tableau rule (b).

27.5. **Lemma (Periodic Tableaux)**
Suppose that $c \in \mathcal{M}$ is such that p_c has only repelling periodic points. If in the critical tableau $T(0)$, column $k > 0$ is entirely off-critical, then $T(0)$ is periodic. In fact, if $k_0 > 0$ is the smallest such column number, then p_c is renormalizable of period k_0.

Since the 'movement' in the tableau is 'north-east' we can draw lines from position (n,k) in this direction and count how many critical positions it intersects. If this number is s, then the covering degree of $p_c^{\circ n} : X_n(x) \to X_0(p_c^{\circ n}(x))$ is 2^s, and the number of branching points is determined by the number of semi-critical intersections.

This motivates the *Yoccoz' τ-function* $\tau : \mathbb{N}^* \to \mathbb{N}$ for the critical tableau $T(0)$:

$$\tau(n) = \begin{cases} i & \text{if } i < n \text{ is maximal such that } (i, n-i) \text{ is critical;} \\ 0 & \text{if no such } i < n \text{ exists.} \end{cases}$$

Further investigations on the function τ lead to weights functions $(w_k)_{k\in\mathbb{N}}$ which are used to determine the degree and number of branching points of $p_c^{\circ n}(X_n(0))$ exactly, see [**HY**]. This topic is beyond the scope of this introduction.

Yoccoz' τ-function together with tableau rules (a) and (b) completely determine the critical tableau. However, they do not completely determine p_c, or even its internal address, and, contrary to internal addresses, the lack of uniqueness of polynomials with the same tableau is not reflecting certain symmetries in the Mandelbrot set. The problem is that the semi-critical depth does not determine on which side of 0 forward images c_k land, as the following example shows.

27.6. **Example (Multiple polynomials with the same critical tableau)**
The kneading sequences $\nu = \mathtt{100}\overline{\mathtt{10}}$ and $\tilde{\nu} = \mathtt{101}\overline{\mathtt{10}}$ have the same critical tableau with semi-critical depth

$$\mathrm{scd}(k) = \begin{cases} \infty & \text{if } k = 0; \\ -1 & \text{if } k = 1; \\ 0 & \text{if } k = 2; \\ 2 & \text{if } k \in 2\mathbb{N}^* + 3; \\ -1 & \text{if } k \in 2\mathbb{N}^* + 4. \end{cases}$$

An algorithm how to derive the critical tableau from the internal address (or kneading invariant) effectively has to our knowledge not been devised.

HB: What did Faught do in his thesis? Relevant for this section?

We conclude this section with the definition of persistent recurrence. It may be clear that if $c \in \mathcal{M}$ is such that p_c is non-renormalizable, has only repelling periodic points and 0 is non-recurrent, then the critical tableau $T(0)$ has bounded semi-critical depth $\mathrm{scd}(k)$ for $k \geq 0$, or equivalently, $\tau(n)$ is bounded.

27.7. **Definition (Persistent Recurrence)**
If a non-renormalizable quadratic polynomial p_c has a recurrent critical point 0, then it is called persistently recurrent *if $\tau(n) \to \infty$.*

Equivalently, for every neighborhood $U \ni 0$ there are only finitely many n such that if $V \ni c_1$ is the pull-back of U along the orbit $c_1, c_2, \dots, c_{n-1}, c_n$ (which naturally only applies if $c_n \in U$), then $p_c^{\circ n-1} : V \to U$ is a univalent. This formulations extends to the (infinitely) renormalizable case as well: infinitely renormalizable maps have persistently recurrent critical points.

If 0 is persistently recurrent, then the critical omega-limit set $\omega(0)$ is a minimal Cantor set. The property has been investigated for real parameters c in [**Bru2**, Section 3], where it was shown that the condition $S_k - S_{k-1} \to \infty$ implies (but is not equivalent to) persistent recurrence of the critical point. For complex parameters, not even this is true, as the next example demonstrates.

27.8. **Example ($S_k - S_{k-1} \to \infty$ does not imply minimality of $\omega(0)$)**
The internal address

$$1 \to 2 \to 4 \to 10 \to 22 \to \ldots \to S_k \to S_{k+1} \to \ldots$$

with $S_{k+1} = 2S_k + 2$ for $k \geq 2$ does not yield a persistently recurrent critical point. In fact, $\omega(0) \ni \alpha$ (and therefore $\omega(0)$ is not even minimal), as can be seen from the fact that the kneading sequence

$$\nu = \mathtt{101110}\underbrace{\mathtt{11111}}_{\text{five 1s}}\mathtt{01110}\underbrace{\mathtt{1111111}}_{\text{seven 1s}}\mathtt{0111011111101110}\underbrace{\mathtt{111111111}}_{\text{nine 1s}}\mathtt{0}\ldots$$

contains arbitrarily long blocks of 1s.

27.2. Maybe here work by Petersen and Roesch?

28. Iterated Monodromy Groups and Kneading Automata

HB: I didn't understand too much of this section. It seems a good idea to illustrate it with a good example, followed through the entire section

This is a powerful and rather new approach to study dynamical system uses methods from group theory. Should we mention work by Grigorchuk, Pilgrim, ...HB: I suppose so, at least the references In this section we give a short overview following [**IMG**], on iterated monodromy groups and kneading automata, over their interdependence and how they relate to kneading sequences of Definition 2.1 (compare also Proposition 11.6). We will define the iterated monodromy group for coverings between spaces although it can be defined more generally for orbispaces, too.

28.1. **Definition (Iterated Monodromy Group)**
Let $\mathcal{M}$ be an arcwise and locally arcwise connected space and $\mathcal{M}_1$ an arcwise connected subspace of $\mathcal{M}$. If $f : \mathcal{M}_1 \to \mathcal{M}$ is a covering, then the iterated monodromy group IMG(f) *associated to f us given by*

$$\mathrm{IMG}(f) := \pi_1(\mathcal{M})/\bigcap_{n>1} K_n,$$

where K_n is the kernel of the action $\pi_1(\mathcal{M})$ by the natural monodromy action on the nth iterate $f^{\circ n} : \mathcal{M}_n \to \mathcal{M}$ of f.

HB: Using the same symbol and font $\mathcal{M}$ as the Mandelbrot set seems confusing. The action $\pi_1(\mathcal{M})$, should that be the fundamental group? Maybe rewrite that sentence. $\mathcal{M}_n$ is not defined. The intersection $\bigcap_{n>1} K_n$ should be over $n \geq 1$ maybe?

IMG(f) acts naturally on a rooted tree T, which is defined the following way: choose any base point $t \in \mathcal{M}$. Then the nth level of the tree is the set $f^{-n}(t)$ and a vertex $z \in f^{-n}(t)$ is connected by an edge with $f(z) \in f^{-n+1}(t)$. The *monodromy action* of a loop $\gamma \in \pi_1(\mathcal{M}, t)$ starting and ending in t on the level f^{-n} is the permutation on f^{-n} induced by the map

$$z \mapsto \gamma(z),$$

where $\gamma(z)$ is the end of the unique $f^{\circ n}$-preimage of γ starting at z. This way, the fundamental group $\pi_1(\mathcal{M}, t)$ acts on the whole tree T. The action is called *iterated monodromy action of* $\pi_1(\mathcal{M})$ and is independent from the choice of the base point t.

HB: I don't understand this last sentence. Is t still the base point? I don't see the connection between the loop γ and the points in $f^{-n}(t)$.

Now let us turn to automata: we start out with some basic definitions. Let X be a finite set, which will be called the *alphabet* in the following. Then

$$X^\star = \{x_1 \cdots x_n : x_i \in X,\, n \in \mathbb{N}\}$$

HB: You say $\mathbb{N}$, so the empty word is included? Just to make sure.

represents the set of all finite words over the alphabet X. The set $X^\star$ is the vertex set of a *rooted tree* in a natural way, *i.e.,* two vertices are joined by an edge if and only if they are of the form v and vx with $v \in X^\star$ and $x \in X$. A map $f : X^\star \to X^\star$ is an endomorphism of this rooted tree $X^\star$ if f preserves the root and adjacency of vertices.

28.2. **Definition (Automaton)**
An automaton (A, X) *over the alphabet* X *is given by*

(1) *its set of states* A *and*
(2) *a map* $\tau : A \times X \to X \times A$.

If $\tau(q, x) = (y, p)$, *then the projection to the two coordinates* $y(x, q)$ *and* $p(x, q)$ *are called the* output *and* transition function.

Usually the following two notations are used for the action of τ *on* $(A \times X)$*:*

$$q \cdot x = y \cdot p \qquad \text{or equivalently} \qquad y = q(x) \quad \text{and} \quad p = q|_x.$$

The Moor diagram *of* (A, X) *is the labeled directed graph so that its vertices are identified with the states of the automaton and for any* $\tau(x, q) = (y, p)$ *there is an directed edge (sometimes also called arrow) from* q *to* p *that is labeled by* (y, x).

The dual Moor diagram *of* (A, X) *is the labeled directed graph with the set of vertices identified with the elements of the alphabet* X. *Its arrows are elements of* $A \times X$, *where an arrow* (g, x) *starts in* x *and ends in* $g(x)$ *and is labeled by* $(g, g|_x) \in A \times A$.

HB: What exactly are states?

Remark. Observe that an automaton can also process any word $v \in X^n$ by processing step by step the letters of v. That is, (A, X) induces an automaton (A, X^n) in a natural way. The states of (A, X) act on $X^\star$ in a natural way, or in other words, they induce an endomorphism of the rooted tree.

Two automata (A, X), (B, X) can be composed so that the set of states of the resulting automaton $(A \cdot B, X)$ is the direct product of the states of the sets A, B.

The *dual automaton* (X', A') of an automaton (A, X) is the automaton over the alphabet A' with states X' such that X, X' and A, A' are bijective and such that for (X', A'), $x' \cdot q' = y' \cdot p'$ if and only if for (A, X), $q \cdot x = p \cdot y$.

An automaton is *invertible* if each of its states defines an invertible transformation of X^*. If (A^{-1}, X) is the inverse automaton of (A, X) then the set of states A^{-1} and A are bijective and we have that $g^{-1} \cdot x = y \cdot h^{-1}$ if and only if $g \cdot y = x \cdot h$.

Every automaton generates a group in the following way:

28.3. **Definition (Group Generated by an Automaton)**
Let (A, X) *be an invertible automaton. Then the group generated by* (A, X) *is the group* $\langle A \rangle$ *of transformation generated by all states in* A.

The iterated monodromy groups $\mathrm{IMG}(f)$ of postcritically finite polynomials can be obtained as groups generated by special automata, so called *kneading automata*. Before we give a definition of these special automata, we introduce a method to picture a set of permutations of a finite set X.

A multi-set $T = \{\pi_i\}_{i \in I}$ is a map from a finite set of indices I into the set of permutations of X. The *cycle-diagram* of T is an oriented two-dimensional CW-complex whose 0-cells coincide with X and whose 2-cells are defined as follows: for any permutation $\pi_i \in T$ and any cycle $(x_1, \dots, x_m)$ in π_i there is a polygon with vertices $x_1, \dots, x_m$ so that the order of the x_i in the cycle is equal to the order of the vertices of the oriented 2-cell. Moreover, different cells are only allowed to intersect at 0-cells.

28.4. **Definition (Kneading Automaton)**
A finite permutational automaton is called a kneading automaton *if the following three conditions hold:*

(1) *Every state $g \in A$ has a unique incoming arrow, i.e., there is a unique $(h, x) \in A \times X$ such that $g = h|_x$.*
(2) *For every cycle $(x_1, \dots, x_m)$ of the action of any state g on X, there is at most on letter x_i such that the state $g|_{x_i}$ is non-trivial.*
(3) *The multi-set of permutations defined by A on X is tree-like, that is it cycle-diagram is connected and contractible.*

Note that if (A, X) is a kneading automaton, so is (A, X^n) for any n. Could not find definition of permutational automaton

Nekrashevych HB: needs a reference defines the kneading automaton for a postcritically finite polynomial f via its spider. HB: clearly we need a section on and reference to spiders. Recall that the spider S of a topological polynomial p is a collection of disjoint curves that connect each point in the postcritical set of f to infinity, see Section 13. The *critical portrait* $\mathcal{C}$ associated to the spider S is the set $\{X_z : z \text{ is a critical point}\}$, where X_z is the subset of elements in $\{f^{-1}(\gamma_{f(z)}) : \gamma_{f(z)} \in S\}$ that land at z. The critical portrait partitions the complex plane into d sectors, where d equals the degree of f. Let us label the sectors S_x by some index set X. HB: Is X the alphabet as before?

Then the *kneading automaton $K_{\mathcal{C},f}$ of a topological polynomial* f is the automaton with states $\{g_z : z \text{ is a postcritical point }\} \cup \{1\}$ such that output and transition function have the following properties. Let z be any postcritical point.

(1) Suppose first that $y \in f^{-1}(z)$ is not critical then $g_z \cdot x = x \cdot g_y$, where $g_y = 1$ if y is not a postcritical point.
(2) If $y \in f^{-1}(z)$ is critical point let d' be its degree. In this case, y is contained in the boundary of d' sectors $S_{x_1}, \dots, S_{x_{d'}}$.

    If y is not a postcritical point then let α be the preimage loop around y of a small loop around z and suppose that the d' sectors S_{x_i} are labeled according to the cyclic order in which α intersects them. Set $g_z \cdot x_i = x_{i+1} \cdot 1$ for all $i = 1, \dots, d'$.

If y is a postcritical point then there is a path γ_y contained in the spider S and we label the d' sectors such that γ_y is adjacent to the sectors $S_{x_{d'}}$ and S_{x_1}. Set $g_z \cdot x_i = x_{i+1} \cdot 1$ for $i = 1, \ldots, d'-1$ and $g_z \cdot x_{d'} = x_1 \cdot g_y$.

$K_{\mathcal{C},f}$ meets the three requirements of Definition 28.4, and hence, is a kneading automaton.

28.5. **Definition (Planar Kneading Automaton)**
A kneading automaton (A, X) with $d = \#(X)$ is planar *if there exists a cyclic order $g_1, \ldots, g_n$ of its set of non-trivial states such that for every letter $x \in X$, $(g_1 \cdots g_n)^d|_x$ is a cyclic shift of the word $g_1 \cdots g_n$.*

28.6. **Theorem (Realization of Planar Kneading Automaton I)**
If $\mathcal{C}$ is the critical portrait of a topological polynomial f, then the associated kneading automaton $K_{\mathcal{C},f}$ is planar.

Conversely, every planar kneading automaton (A, X) is realized by a topological polynomial, i.e., there is a branched cover $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that (A, X) is isomorphic to $K_{\mathcal{C},f}$.

The obstruction for a topological polynomial to be Thurston equivalent to a postcritically finite polynomial is the existence of a Levy cycle. The equivalent of a Levy cycle in the language of kneading automaton (A, X) is that the set A of state of (A, X) has *bad isotropy groups.* For an exact definition, we refer to [**IMG**, Section 6.9].

Thus, we get the following characterization of kneading automata:

28.7. **Theorem (Realization of Planar Kneading Automaton II)**
Let (A, X) be a kneading automaton. Then the following are equivalent:

(1) *There exists a postcritically finite complex polynomial f with invariant spider $\mathcal{S}$ and associated critical portrait $\mathcal{C}$ such that (A, X) is isomorphic to $K_{\mathcal{C},f}$.*
(2) (A, X) *is planar and A does not have bad isotropy groups.*

28.8. **Corollary (Characterization of Iterated Monodromy Groups)**
A group is isomorphic to the iterated monodromy group of a postcritically finite polynomial if and only if it is isomorphic to a group generated by a planar kneading automaton without bad isotropy groups.

For more information about the groups generated by the kneading automaton $K_{\mathcal{C},f}$ of a postcritically finite quadratic polynomial, we refer to [**IMG**, Section 6.10].

Note that for a postcritically finite polynomial, the limit space $\mathcal{J}_{\mathrm{IMG}(f)}$ of the iterated monodromy group is homeomorphic to the Julia set $J(f)$ of f [**IMG**, Section 5.4]. On the other hand, the limit space of the group generated by (A, X) is approximated by the dual Moor diagram of the automaton (A, X^n), see [**IMG**, Subsection 3.4.3]. Thus, the dual Moore diagrams approximate the Julia set of the given polynomial.

Finally, we link the concept of kneading sequences to kneading automata. Let (A, X) be a kneading automaton over the binary alphabet $\{0, 1\}$. Then A contains only one active state b_1 (*i.e.,* only one state b_1 such that b_1 as a function on X is non-trivial). By the definition of kneading automata, every non-trivial state has only one incoming arrow and since b_1 is the only active state, b_1 can be reached from every non-trivial state by running along the arrows. Thus, the states of (A, X) can be ordered in a (pre-)periodic sequence $b_1, \ldots, b_n$ such that for every $i > 1$, either $b_i|_0$ equals b_{i-1} and $b_i|_1$ is trivial or vice versa.

28.9. **Definition (Kneading Sequence of** (A, X)**)**
The kneading sequence of a kneading automaton (A, X) *with* $X = \{0, 1\}$ *is the sequence of the labels of the arrows in the associated Moor diagram along the path* $(b_i)_{i=1}^{\infty}$.

Observe that by definition, if $K_{\mathcal{C},f}$ is the kneading automaton of a postcritically finite quadratic polynomial f, then the kneading sequence of $K_{\mathcal{C},f}$ coincides with the kneading sequence of Definition 2.1.

29. Maps of the Real Line

The symbolic description of real quadratic maps (*i.e.,* on the interval) goes back further than the complex counterpart. The term *kneading theory* was coined in this context. The admissibility question has long been solved for such maps, and the literature on the (combinatorial) properties of interval maps is very extensive. Apart from real admissibility, we describe in this section a few topics relating to topological entropy, patterns of periodic orbits, orbit forcing and also Sharkovskiy's Theorem.

29.1. Kneading Theory. If $c_1 \in \mathbb{R}$, $f_{c_1} : z \mapsto z^2 + c_1$ preserves the real line. The real quadratic (or logistic) map f_{c_1} is the prototype of a *unimodal map*, *i.e.,* a real map f with a single critical point, such that f is decreasing resp. increasing on the two complementary domains of this critical point. As analog of the partition $\{T_{\mathtt{1}}, \{0\}, T_{\mathtt{0}}\}$ of the Hubbard tree, let us use the partition $\{(-\infty, 0), \{0\}, (0, \infty)\}$, with symbols $\mathtt{1}$, $\star$ and $\mathtt{0}$ respectively, to define symbolic dynamics. The *itinerary* $e(x)$ of $x \in \mathbb{R}$ is defined by

$$e_i(x) = \begin{cases} \mathtt{0} & \text{if } f^{\circ(i-1)}(x) > 0, \\ \star & \text{if } f^{\circ(i-1)}(x) = 0, \\ \mathtt{1} & \text{if } f^{\circ(i-1)}(x) < 0. \end{cases}$$

The itinerary of the critical value c_1 is the *kneading sequence* of fc_1.

29.1. **Definition (Real Admissibility)**
A sequence ν is real admissible *if there is $c_1 \in \mathbb{R}$ such that ν is the kneading sequence of f_{c_1}.*

For instance, the sequences $\overline{\mathtt{0}}$, $\overline{\star}$, $\overline{\mathtt{1}}$ and $\overline{\mathtt{1}\star}$ are all admissible, as they are realized by $c_1 = 0.5$, $c_1 = 0$, $c_1 = -0.5$ and $c_1 = -1$ respectively. All other admissible sequences start with $\mathtt{10}$ and are assumed by some $c_1 \in [-2, -1)$. For these values of c_1, the interval $[c_1, c_2]$ (where we write $c_k = f^{\circ k}(0)$) is forward invariant and contains 0; it is called the *core* of the unimodal map. Note that if $c_1 \in \mathbb{R}$ belongs to the interior of the Mandelbrot set, then the above partition gives rise to itineraries and kneading sequences that may be different from those that arise from the partition by external rays landing at the root of the hyperbolic component containing c_1, which were introduced in Section 1 and were used throughout this text. The standard example, achieved for $c_1 = -1.75$, is the kneading sequence $\overline{\mathtt{101}}$, which is real admissible, but does not appear as complex kneading sequence as defined in Definition 3.5.

Real admissibility conditions go back to [**MSS, DGP, MiT**], and are based on the parity-lexicographic order.

29.2. **Definition (Parity-Lexicographical Order)**
Given $e, \tilde{e} \in \Sigma$ distinct, and $n = \min\{i \geq 1 : e_i \neq \tilde{e}_i\}$, the parity-lexicographical order *between a and b is given by*

$$e <_p \tilde{e} \textit{ if } \begin{cases} e_n < \tilde{e}_n \textit{ and } \#\{i < n : e_i = \mathtt{1}\} \textit{ is even,} \\ e_n > \tilde{e}_n \textit{ and } \#\{i < n : e_i = \mathtt{1}\} \textit{ is odd.} \end{cases}$$

Here $\mathtt{0} < \mathtt{1}$.

Remark. By setting $\mathtt{0} < \star < \mathtt{1}$ we can extend $<_p$ to sequences in $\{\mathtt{0}, \star, \mathtt{1}\}^{\mathbb{N}^*}$ with the property that if $e_m = \star$, then $\sigma^{\circ m}(e) = \nu$ is a fixed kneading sequence $\nu \in \Sigma^\star$. We call this space of itineraries Σ^r.

Remark. Milnor and Thurston [**MiT**] use formal power series rather than symbolic dynamics to phrase their kneading theory. This is a more involved, but for many purposes very powerful method. Let $T_\mathtt{0}$ and $T_\mathtt{1}$ be formal unit vectors associated to our symbols $\mathtt{1}$ and $\mathtt{0}$. Let for $j \geq 0$

$$\varepsilon_j(x) = \begin{cases} +1 & \text{if } f^{\circ j}(x) \in T_\mathtt{0}, \\ -1 & \text{if } f^{\circ j}(x) \in T_\mathtt{1}, \\ 0 & \text{if } f^{\circ j}(x) = 0, \end{cases}$$

expressing the fact that $f|_{T_\mathtt{0}}$ preserves and $f|_{T_\mathtt{1}}$ reverses orientation. Then the product

$$\vartheta_n(x) = \varepsilon_0(x) \cdots \varepsilon_{n-1}(x) \cdot \begin{cases} T_k & \text{if } f^{\circ n}(x) \in T_k \text{ for } k = \mathtt{0}, \mathtt{1}, \\ \frac{1}{2}(T_\mathtt{0} + T_\mathtt{1}) & \text{if } f^{\circ n}(x) = 0 \end{cases}$$

is a formal vector expressing both where $f^{\circ n}(x)$ is situated and whether $f^{\circ n}$ is locally increasing, decreasing or has an extremum point at x. The vector-valued formal power series

$$\vartheta(x, t) = \sum_{n \geq 0} \vartheta_n(x) t^n$$

is called the *invariant coordinate* of x.

29.3. **Lemma (Coefficients of Kneading Coordinate Sum to One)**
The sum of the coefficients of T_j, $j = 1, 2$, satisfy

$$\sum_{j=1}^{2} (1 - \varepsilon(T_j)t) \cdot \delta_j(\vartheta(x, t)) = 1 \tag{27}$$

for every point x. Here the Dirac delta ($\delta_i(T_j) = 1$ if $i = j$ and $\delta_i(T_j) = 0$ otherwise) is extended by linearity to vectors with $\mathbb{Q}[t]$-valued coefficients.

29.4. **Example (Kneading Coordinates of Fixed Points)**
Before proving this lemma, let us see how this works out for the fixed points of f. The orientation reversing fixed point $\alpha \in T_\mathtt{1}$ has $\vartheta(\alpha) = (1 - t + t^2 - t^3 + \dots)T_\mathtt{1}$, so formally

$(1-\varepsilon(T_1)t)\delta_1(\vartheta(x)) = (1+t)(1-t+t^2-t_3\dots) = 1$, whereas $\delta_0(\vartheta(x)) = 0$, because T_0 doesn't appear in $\vartheta(x)$.

For the orientation preserving fixed point $\beta \in T_0$ it works similarly with indices and signs reversed: $\vartheta(\beta) = (1+t+t^2+t^3+\dots)T_0$, so $(1-\varepsilon(T_0)t)\delta_0(\vartheta(x)) = (1-t)(1+t+t^2+t_3\dots) = 1$.

Proof. We write $\sum_{j=0}^{1}(1-\varepsilon(T_j)t)\delta_j(\vartheta(x,t))$ as a double sum, and assume for simplicity that $\mathrm{orb}_f(x) \not\ni 0$:

$$
\begin{aligned}
\sum_{j=0}^{1}\sum_{n\geq 0}&(1-\varepsilon(T_j)t)\delta_j(\vartheta_n(x))t^n \\
&= \sum_{j=0}^{1}\sum_{\substack{n\geq 0\\ f^{\circ n}(x)\in T_j}}(1+\varepsilon(T_j)t)\left(\prod_{k=0}^{n-1}\varepsilon_k(x)\right)t^n \\
&= \sum_{\substack{n\geq 0\\ f^{\circ n}(x)\in T_0}}\left(\prod_{k=0}^{n-1}\varepsilon_k(x)\right)t^n - \sum_{\substack{n\geq 0\\ f^{\circ n}(x)\in T_0}}\left(\prod_{k=0}^{n}\varepsilon_k(x)\right)t^{n+1} \\
&\quad + \sum_{\substack{n\geq 0\\ f^{\circ n}(x)\in T_1}}\left(\prod_{k=0}^{n-1}\varepsilon_k(x)\right)t^n - \sum_{\substack{n\geq 0\\ f^{\circ n}(x)\in T_1}}\left(\prod_{k=0}^{n}\varepsilon_k(x)\right)t^{n+1}.
\end{aligned}
$$

Formally, all positive powers t^n cancel, leaving only $\sum_{x\in T_0} t^0 + \sum_{x\in T_1} t^0 = 1$. If $f^{\circ n}(x) = 0$ for some n, then the mixed definition of $\vartheta_n(x)$ allows a similar proof. □

The qualitative behavior of the entire interval map is given by the invariant coordinate of the critical value. In this terminology, the *kneading increment*

$$\nu(t) = \lim_{x\nearrow 0}\vartheta(x,t) - \lim_{x\searrow 0}\vartheta(x,t)$$

is the object closest to our kneading sequence. [3] This formula obviously expresses the change of kneading coordinate $\vartheta(x)$ as the point x moves across 0, but it can also be used to express change of kneading coordinate $\vartheta(x)$ as the point x moves across z for any precritical point z, say $f^{\circ n}(z) = 0$ for some minimal $n \geq 0$:

$$\lim_{x\nearrow z}\vartheta(x,t) - \lim_{x\searrow z}\vartheta(x,t) = t^n\nu(t).$$

---

[3] We changed the sign from the definition on page 483 of [**MiT**] because in our setup f_c has a minimum rather than a maximum at the critical point. The same construction, with d formal unit vectors T_k, $k = 0, 1 \dots, d\}$, can be carried out for a $d-1$-modal interval map (*i.e.,* with d critical points, and also (although not covered in [**MiT**]) for piecewise continuous maps.

Milnor and Thurston continue to define kneading matrices and *kneading determinants* $D(t)$ which in the unimodal case is equal to

$$D(t) = \delta_1(\nu(t)) - \delta_0(\nu(t)) = 1 + \varepsilon_1(c_1)t + \varepsilon_1(c_1)\varepsilon_2(c_1)t^2 + \dots \tag{28}$$

29.5. **Corollary (Rational Kneading Determinant)**
If the critical point 0 *is periodic of period* p*, then the kneading determinant* $D(t)$ *is a polynomial of degree* $p-1$*. If* 0 *attracted to a periodic orbit* $\mathrm{orb}(x)$*, then the kneading determinant is rational:*

$$D(t) = \frac{P(t)}{1 \pm t^p},$$

where p *is the period of* x*,* P *is a polynomial of degree* $< p$ *and the sign* $\pm$ *is according to whether* $f^{\circ p}$ *reverses resp. preserves orientation at* x*.*

Proof. If $f^{\circ p}(0) = p$, then $\varepsilon_p = 0$, so $D(t)$ is truncated at the $p-1$st term. If 0 is attracted to a periodic attractor, rather than periodic itself, then 0is in the immediate basin of some periodic point x. Writing $\varepsilon_0 = 1$, we obtain $\varepsilon_0 \cdot \varepsilon_1 \cdots \varepsilon_{p-1} = \varepsilon_{kp} \cdot \varepsilon_{kp+1} \cdot \dots \cdot \varepsilon_{kp+p-1} = \pm 1$ for all k, where the sign only depends on whether $f^{\circ p}$ preserves or reverses orientation. Hence $D(t) = P(t)\sum_{k \geq 0} (\pm t)^k = \frac{P(t)}{1 \mp t^p}$ for the polynomial $P(t) = 1 + \varepsilon_1(c_1)t + \varepsilon_1(c_1)\varepsilon_2(c_1)t^2 + \dots + \varepsilon_1(c_1)\varepsilon_2(c_1)\cdots\varepsilon_{p-1}(c_1)t^{p-1}$. □

For an open interval J and $n \geq 0$, let $\gamma_n(J)$ be the number of precritical points z such that n is indeed the minimal integer such that $f^{\circ n}(z) = 0$, forming the formal power series $\gamma(J) = \sum_{n \geq 0} \gamma_n(J)t^n$.

29.6. **Lemma (Difference in Kneading Coordinates in Terms of γ)**
For the interval $J = (a, b)$*, the difference*

$$\lim_{x \nearrow b} \vartheta(x,t) - \lim_{x \searrow a} \vartheta(x,t) = \gamma(J)\nu(t).$$

If $J = (-\beta, \beta)$*, where* β *is the orientation preserving fixed point, is the entire interval, then* $\gamma(J) = (1-t)^{-1}D(t)^{-1}$*.*

Proof. The difference $\lim_{x \nearrow b} \vartheta(x,t) - \lim_{x \nearrow b} \vartheta(x,t)$ is comprised of the increments of all precritical points z. Each precritical point of order n gives a contribution of $t^n\nu(t)$, and $\gamma(J)$ counts how many order n precritical points there are, giving them weight t^n. So the first formula follows.

Since $\vartheta(\beta) = T_0(1 + t + t^2 + \dots) = \frac{1}{1-t}T_0$ and $\vartheta(-\beta) = T_1 - T_0(t + t^2 + t^3 + \dots) = T_1 - \frac{1}{1-t}T_0$, we can use Dirac delta δ_1 and formula (28) to simplify for $J = (-\beta, \beta)$:

$$-1 = \delta_1(\lim_{x \nearrow \beta} \vartheta(x,t) - \lim_{x \searrow -\beta} \vartheta(x,t)) = \delta_1(\gamma(J)\nu(t)) = \gamma(J)(-1-t)D(t).$$

This gives $\gamma(J) = (1-t)^{-1}D(t)^{-1}$. □

A major result is that the *reduced dynamical ζ-function*[4]

$$\zeta(t) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k}\#\{p \ : \ p \text{ is } k\text{-periodic}\}\, t^k\right)$$

of $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\zeta(t)^{-1} = \begin{cases} (1-t)D(t) & \text{if } c \text{ is non-periodic;} \\ (1-t)(1-t^p)D(t) & \text{if } c \text{ is periodic of period } p. \end{cases}$$

One of the consequences is that the first zero of $D(t)$ on $[0, 1)$ is $t_0 = \exp(-h_{top}(f_c))$ provided f_c has positive topological entropy $h_{top}(f_c)$. This amounts to a result from [**DGP**] which states that the slope λ of the tent-map $T_\lambda(x) = \min\{\lambda x, \lambda(1-x)\}$ satisfies $\lambda = \sum_{j=0}^{\infty} \varepsilon_j(c_1)\lambda^{-j}$.

29.2. Lap-numbers and Topological Entropy. Topological entropy was introduced by Adler, Konheim, and McAndrew [**AKM**] in 1965, and made more accessible by Bowen [**Bow**]. For real one-dimensional maps, the seminal paper is by Misiurewicz and Szlenk [**MSz**] (see also [**ALM1**, Chapter 4] and [**MiT**, Section 6]). If $f : X \to X$ is a piecewise monotone map on a real one-dimensional space (*i.e.,* X is an interval, circle, tree, etc.), then we can define the *lap-number* $l(f^{\circ n})$ of $f^{\circ n}$ as the cardinality of the coarsest partition of X into connected subset X_i such that $f^{\circ n}|_{X_i}$ is a homeomorphism for each i. Note that f is called piecewise monotone if $l(f) < \infty$.

29.7. **Theorem (Topological Entropy of One-Dimensional Maps)**
The topological entropy of a piecewise monotone one-dimensional map is given by the exponential growth rates of lap-numbers as well as periodic points:

$$\begin{aligned} h_{top}(f) &= \max\left\{0, \lim_{n\to\infty} \frac{1}{n}\log l(f^{\circ n})\right\} \\ &= \max\left\{0, \limsup_{n\to\infty} \frac{1}{n}\log \#\{p \ : \ p \textit{ is } n\textit{-periodic }\}\right\}. \end{aligned}$$

Here n-periodic can be interpreted as exact period n or not; it makes no difference for the exponential growth rate.

Given an interval map f, it is possible to find a conjugate map (cf. the minimal Hubbard trees of Definition 8.5) which is piecewise affine with constant slope. This result was first proven by Parry [**Pa**] for maps semiconjugate to β-transformations[5]

---

[4]where we count at most one k-periodic point in each lap of $f^{\circ k}$; if there two such orbits with the period of one twice the period of the other (as is the case shortly after a period doubling bifurcation), then only the orbit of the smaller period is counted. Note, however, that the period need not be minimal: a k-periodic orbit also counts for $2k, 3k, \dots$

[5]That is: $x \mapsto \beta x \mod 1$ for $\beta > 1$.

and for general piecewise monotone interval maps by Milnor and Thurston in [**MiT**, Section 7].

29.8. **Theorem (Semiconjugacy to Map with Constant Slope)**
If f is a continuous piecewise monotone interval map with $h_{top}(f) > 0$, then there is an interval map g with constant slope $\pm s$ for $s = e^{h_{top}(f)}$ which is semiconjugate to f, i.e., $g \circ h = h \circ f$.

In fact, the result holds for more complicated graphs X as well, [**BdC**], but one must be aware that the constant slope map g may act on a simpler graph than X. In Theorem 30.6 Penrose gives a version for abstract Julia sets.

A precise formula for the lap-numbers was given in [**MiT**] in terms of the kneading determinant. Write $L(f|_J)$ for the power series $\sum_{n\geq 1} l(f^{\circ n}|_J)t^{n-1}$.

29.9. **Proposition (Formula for Lap-numbers)**
Using the $\gamma(J)$ of Lemma 29.6, we have $L(f|_J) = \frac{1+\gamma(J)}{1-t}$ for every open interval J. For the interval $J = (-\beta, \beta)$, where β is the orientation preserving fixed point, we have $L(f|_J) = \frac{1}{1-t} + \frac{1}{(1-t)^2 D(t)}$.

Proof. We rearrange the sum

$$\begin{aligned}(1-t)L(f|_J) &= \sum_{n\geq 1} l(f^{\circ n}|_J)t^{n-1} - \sum_{n\geq 1} l(f^{\circ n}|_J)t^n \\ &= l(f|_J)t^0 + \sum_{n\geq 1}\left(l(f^{\circ n+1}|_J) - l(f^{\circ n}|_J)\right)t^n.\end{aligned}$$

The factor $l(f^{\circ n+1}|_J) - l(f^{\circ n}|_J) = \gamma_n(J)$ because it is exactly the precritical points of order n that produce the laps of $f^{\circ n+1}$ that were not laps of $f^{\circ n}$ yet. Since also $l(f|_J) = 1 + \gamma_0(J)$, we obtain $L(f|_J) = \frac{1+\gamma(J)}{1-t}$. Applied to $J = (-\beta, \beta)$ and invoking Lemma 29.6, this gives $L(f|_J) = \frac{1}{1-t} + \frac{1}{(1-t)^2 D(t)}$. □

29.3. Real Admissibility.

29.10. **Lemma (Itinerary Map Reverses Order)**
The map $e : \mathbb{R} \to \Sigma^r$ is order reversing: $x \leq y$ implies $e(y) \leq_p e(x)$.

The proof is straightforward, see *e.g.* [**CoE**] or Milnor and Thruston, [**MiT**, Lemma 3.1], in whose formal power series approach this statement becomes: $x \mapsto \vartheta(x,t)$ is decreasing in x for all $t > 0$ sufficiently close to 0. The first characterization of which sequences are real admissible appears in [**DGP, MiT**]:

29.11. **Corollary (First Admissibility Condition)**
If $c_1 \in [-2, -1]$ and ν is the kneading sequence of f_{c_1}, then

$$\sigma(\nu) \leq_p \sigma^{\circ n}\nu \leq_p \nu \text{ for all } n \in \mathbb{N}. \tag{29}$$

Proof. For $c_1 \in [-2,-1]$, the interval $[c_1, c_2]$ is invariant. As c_1 is the infimum and c_2 the supremum of this interval, the corollary follows directly from Lemma 29.10. $\square$

Cutting times in the real setting were introduced in the late 1970s by Hofbauer [**Ho**]. They are given recursively by

$$S_0 = 1, \quad S_{k+1} = \rho(S_k),$$

so the cutting times are exactly the entries of the internal address for a real map. Hofbauer showed that the difference between subsequent cutting times (when finite) is always a cutting time. (This follows easily from the fact the Hubbard tree T is an arc $[c_1, c_2]$: since for each $z \in T$ there is a unique k such that $f(z) \in [\zeta_k, \zeta_{k+1})$, the itinerary $e(z)$ must coincide with ν up to a cutting time.) Hence we can define the *kneading map* $Q : \mathbb{N}^* \to \mathbb{N} \cup \{\infty\}$ by

$$S_k = S_{k-1} + S_{Q(k)}. \tag{30}$$

If $\rho(S_k) = \infty$, then Q is only defined on $\{1, \ldots, k\}$. Based on the ρ-function and cutting times several further admissibility conditions were formulated [**Bru1, Ho, Pen1, Tb**] of which we mention two in the next theorem.

29.12. **Theorem (Real Admissibility Theorem)**
A sequence $\nu = \mathtt{10}\ldots$ is real admissible if and only if any (and therefore all) of the following conditions is satisfied.

(a) $\sigma(\nu) \leq_p \sigma^{\circ n}\nu \leq_p \nu$ *for all* $n \in \mathbb{N}$.
(b) *The kneading map is well-defined by* (30) *above, and*

$$\{Q(k+j)\}_{j\geq 1} \geq_{lex} \{Q(Q^{\circ 2}(k)+j)\}_{j\geq 1} \tag{31}$$

for all $k \geq 1$, where $\geq_{lex}$ stands for the lexicographical order on sequences.
(c) *If $\rho(m) < \infty$, then $\rho(m) - m$ is a cutting time.*

Before proving this theorem, we prove Lemma 29.13 on properties of the parity of symbols $\mathtt{1}$ of strings in real admissible kneading sequences, and Proposition 29.14 which relates real admissibility and admissibility as in Definition 5.1.

29.13. **Lemma (Parity of $\mathtt{1}$s and Real Admissibility)**
Let $\nu \in \Sigma^\star$ be such that $\rho(m) - m$ is a cutting time whenever $\rho(m) < \infty$. Write

$$\vartheta(m) = \#\{0 < j \leq m : \nu_j = \mathtt{1}\}.$$

Then the following properties hold.

(1) *$\vartheta(S_k)$ is odd for every finite cutting time S_k.*
(2) *If $\rho(m) < \infty$, then $\#\{m < j \leq \rho(m) : \nu_j = \mathtt{1}\}$ is even.*
(3) *If $S_k \leq m < S_{k+1}$ and $\vartheta(m)$ is odd, then $S_{k+1} \in \mathrm{orb}_\rho(m)$.*
(4) *Assume $\kappa := \min\{i > 1 : \nu_i = \mathtt{1}\}$ exists. If $\rho^{\circ k}(\kappa) \leq m < \rho^{\circ k+1}(\kappa)$ and $\vartheta(m)$ is even, then $\rho^{\circ k+1}(\kappa) \in \mathrm{orb}_\rho(m)$.*

Proof. (1) We prove the statement by induction on k. Since $S_0 = 1$ and $\nu = \mathtt{1}\dots$, the statement holds for $k = 0$. Suppose it holds for all $j \le k$, then

$$\begin{aligned}\vartheta(S_{k+1}) &= \vartheta(S_k) + \#\{S_k < j \le S_{k+1} : \nu_j = \mathtt{1}\} \\ &= \vartheta(S_k) + \#\{0 < j \le S_{Q(k+1)} : \nu_j = \mathtt{1}\} \pm 1 \\ &= \vartheta(S_k) + \vartheta(S_{Q(k+1)}) \pm 1 \quad \text{is even.}\end{aligned}$$

This completes the induction.

(2) Since $\rho(m) - m$ is a cutting time, say S_k, $\#\{m < j \le \rho(m) : \nu_j = \mathtt{1}\} = \vartheta(S_k) \pm 1$ is even.

(3) We proceed by induction on k again. Since $\nu_1 \dots \nu_{S_1} = \mathtt{10}\dots\mathtt{0}$, it is straightforward to check that the statement holds for $k = 0, 1$. Suppose that it holds for all $j \le k$ and assume that $S_k \le m < S_{k+1}$ and $\vartheta(m)$ is odd. Therefore $\vartheta(m - S_k) = \vartheta(m) - \vartheta(S_k)$ is even, and by part (2), and the fact that $m - S_k < S_{Q(k+1)}$, we have that $S_{Q(k+1)} \notin \operatorname{orb}_\rho(m - S_k)$. But $\nu_{m+1} \dots \nu_{S_{k+1}-1}\nu_{S_{k+1}} = \nu_{m-S_k+1} \dots \nu_{S_{Q(k+1)}-1}\nu'_{S_{Q(k+1)}}$ (where ν'_i denotes the opposite symbol of ν_i), so $S_{k+1} \in \operatorname{orb}_\rho(m)$.

(4) The proof of this statement is analogous to the previous part. □

29.14. **Proposition (Hubbard Trees for Real Quadratic Maps)**
Let ν be preperiodic or $\star$-periodic. Then the Hubbard tree of ν is an arc (and hence without branch points) if and only if ν is real admissible (and in particular, the kneading map Q is well-defined and (31) *holds).*

Remark. For $\nu \in \Sigma^1$ with infinite internal address $S_0 \to S_1 \to S_2 \to \dots$, we can approximate the infinite Hubbard tree by the finite Hubbard trees T^k associated to the truncated internal addresses $S_0 \to S_1 \to \dots \to S_k$. The above proposition extends to: ν is real admissible if and only if each of the T^k is an arc.

Proof. The "if"-direction is obvious, because if f is real admissible, the Hubbard tree is the arc $[c_1, c_2]$. To prove the "only if"-direction, assume that ν fails the real admissibility condition, and that S_{k+1} is the first cutting time where this becomes apparent. Let T be the Hubbard tree associated to ν; we need to show that T contains a periodic branch point. We divide the argument into three cases:

- If $S_{k+1} > 2S_k$ or $\operatorname{orb}_\rho(S_{k+1} - S_k) \not\ni S_k$ then Proposition 5.19 supplies the branch point.
- If $S_l < S_{k+1} - S_k < S_{l+1}$ for some $l < k$, and $S_k \in \operatorname{orb}_\rho(S_{k+1} - S_k)$, then ν fails the admissibility condition of Definition 5.1 for period $m := S_k + S_l$. Indeed, by part (2) of Lemma 29.13, $\vartheta(S_{k+1} - S_k)$ is odd, and

  $$\#\{S_k < j \le m : \nu_j = \mathtt{1}\} = \vartheta(S_{k+1} - S_k) - \vartheta(S_l) \text{ is even.}$$

  Let $r := (S_{k+1} - S_k) - S_l$, so $\rho(m) - m = r$ and $\vartheta(r)$ is even too. As $k + 1$ is minimal such that $\nu_1 \dots \nu_{S_{k+1}}$ fails Definition 5.1, $\operatorname{orb}_\rho(1)$ and $\operatorname{orb}_\rho(r)$ do not intersect before or at entry S_k. Let $\tilde\nu = \nu_1 \dots \nu_{m-1}\nu'_m \dots$ with corresponding ρ-function $\tilde\rho$. Then $\tilde\rho(S_k) = m$, so $m \in \mathrm{orb}_{\tilde\rho}(1)$ and since $\vartheta_{\tilde\nu}(r) = \vartheta(r)$ is even, part (2) of Lemma 29.13 implies that $m \notin \mathrm{orb}_{\tilde\rho}(r)$. But this means that $m \in \mathrm{orb}_\rho(r)$, and m indeed fails Definition 5.1. Hence T must contain an evil m-periodic branchpoint.

- Condition (31) fails, say $Q(k+j) < Q(Q^{\circ 2}(k)+j)$ for some minimal $k, j \geq 1$. Let $m = S_k - S_{Q^{\circ 2}(k)}$ $(= S_{k-1} + S_{Q(k)-1})$. Then

$$\rho(m) = m + S_{Q^{\circ 2}(k)} + S_{Q^{\circ 2}(k)+1} + \dots + S_{Q^{\circ 2}(k)+j} = S_{k+j}.$$

Because $\vartheta(m) = \vartheta(S_k) - \vartheta(S_{Q^{\circ 2}(k)})$ is even, we have in this case that $\#\{m < j \leq \rho(m) : \nu_j = \mathtt{1}\}$ is odd (which would incidentally violate part (2) of Lemma 29.13). Take $r = \rho(m) - m$, then $\vartheta(r)$ is even. If $r = m$, then m fails Definition 5.1, so assume that $r < m$. As in the previous part, take $\tilde\nu = \nu_1 \dots \nu_{m-1}\nu'_m \dots$ with corresponding ρ-function $\tilde\rho$ and kneading map $\tilde Q$. Then $\tilde\rho(S_{k-1}) = S_{k-1} + S_{Q(k)-1} = S_{k-1} + S_{\tilde Q(k)}$. Therefore $\#\{0 < j \leq m : \tilde\nu_j = \mathtt{1}\}$ is odd and $m \notin \mathrm{orb}_{\tilde\rho}(r)$. This implies that $m \in \mathrm{orb}_\rho(r)$, so again m indeed fails Definition 5.1, and T must have an evil m-periodic branchpoint.

This completes the proof. □

Remark. If ν is real admissible and different from $\overline{\star}$ and $\overline{\mathtt{1}\star}$, then its Hubbard tree can be extended to the arc $[-\beta, \beta] \supset [c_1, c_2]$, where β is the β-fixed point with itinerary $e(\beta) = \overline{\mathtt{0}}$. If also $\nu \neq \mathtt{1}\overline{\mathtt{0}}$, then $-\beta < c_1$ and we can define $\kappa = \min\{i > 1 : \nu_i = 1\}$ and there is a closest precritical point $\zeta_\kappa \in (-\beta, c_1)$. By Lemma 5.7, there are sequences

$$\zeta_\kappa < \zeta_{\rho(\kappa)} < \zeta_{\rho^{\circ 2}(\kappa)} < \dots < c_1 < \dots < \zeta_{\rho^{\circ 2}(1)} < \zeta_{\rho(1)} < \zeta_1$$

and hence $\mathrm{orb}_\rho(1) \cap \mathrm{orb}_\rho(\kappa) = \emptyset$. Thunberg [**Tb**, Paper IV, Theorem 1] notes that this property is equivalent to real admissibility. Thus $\nu = \mathtt{10}\cdots \in \Sigma^r$ is real admissible if and only if $\nu = \mathtt{1}\overline{\mathtt{0}}$ or $\mathrm{orb}_\rho(1) \cap \mathrm{orb}_\rho(\kappa) = \emptyset$ for $\kappa = \min\{i > 1 : \nu_i = 1\}$. One can compare this to Penrose's real admissibility results from [**Pen1**, Theorem 2.6.1], see Section 30.1.

Let us now prove Theorem 29.12

Proof. We divide the proof into three steps. First we show that real admissibility implies conditions (a)-(c). Then we use Proposition 29.14 to show that (b) implies real admissibility. In the final step we prove the remaining implications.

Step 1: (a) The necessity of condition (a) is shown in Corollary 29.11.

(b) Formula (31) can be interpreted geometrically as $c_{1+S_k} \in [c_1, c_{1+S_{Q^{\circ 2}(k)}}]$, see Figure 29.1. To see this, notice that $f^{\circ S_{k-1}}$ maps $[\zeta_{S_{k-1}}, c_1]$ homeomorphically onto $[c_1, c_{1+S_{k-1}}]$, and since $S_k = S_{k-1} + S_{Q(k)}$, $c_{1+S_{k-1}} \in U_{Q(k)} := [\zeta_{S_{Q(k)}}, \zeta_{S_{Q(k)-1}})$. Take the $f^{\circ S_{Q(k)}}$-image of this interval to find $c_{1+S_k} \in [c_1, c_{1+S_{Q^{\circ 2}(k)}}]$. Now $c_{1+S_k} \in U_{Q(k+1)}$ and

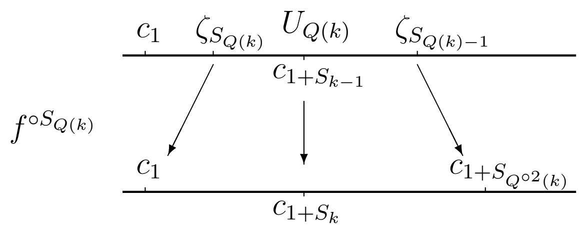


FIGURE 29.1. The position of $c_{1+S_{k-1}}$ in $U_{Q(k)} := [\zeta_{S_{Q(k)}}, \zeta_{S_{Q(k)-1}})$ and the image under $f^{S_{Q(k)}}$.

$c_{1+S_{Q^{\circ 2}(k)}} \in U_{Q(Q^{\circ 2}(k)+1)}$, so $Q(k+1) \geq Q(Q^{\circ 2}(k)+1)$. If the inequality is strict, then (31) holds. Otherwise, *i.e.*, if $Q(k+1) = Q(Q^{\circ 2}(k)+1)$, then both c_{1+S_k} and $c_{1+S_{Q^{\circ 2}(k)}} \in U_{Q(k+1)}$ and we apply $f^{\circ S_{Q(k+1)}}$, which sends $U_{Q(k+1)}$ in an orientation preserving way to $[c_1, c_{1+S_{Q^{\circ 2}(k+1)}}]$. Therefore $f^{\circ S_{Q(k+1)}}(c_{1+S_k}) = (c_{1+S_{k+1}}) \leq f^{\circ S_{Q(k+1)}}(c_{1+S_{Q^{\circ 2}(k)}}) = c_{1+S_{Q^{\circ 2}(k)+1}}$. This shows that $Q(k+2) \geq Q(Q^{\circ 2}(k)+2)$. If the inequality is strict, then again (31) holds; otherwise both $c_{1+S_{k+1}}$ and $c_{1+S_{Q^{\circ 2}(k)+1}} \in U_{Q(k+2)}$ and we can apply $f^{\circ S_{Q(k+2)}}$. Repeating the argument shows that (31) holds in any case.

(c) Condition (c) follows easily from the fact the Hubbard tree T is an arc $[c_1, c_2]$: since for each $z \in T$ there is a unique k such that $f(z) \in [\zeta_k, \zeta_{k+1})$, the itinerary $e(z)$ must coincide with ν up to a cutting time.

Step 2: Every (pre)periodic sequence $\nu \in \Sigma^\star$ comes with a Hubbard tree. If ν satisfies (b), then according to Proposition 29.14, this tree is an arc. Hence ν is real admissible.

Step 3: Finally we prove that (a) and (c) imply (b).

(a) $\Rightarrow$ (b): First observe for $\nu \in \Sigma^1$ that if $\rho(m) < \infty$, then

$$\#\{m < j \leq \rho(m) : \nu_j = 1\} \text{ is odd if and only if } \sigma^{\circ m}(\nu) >_p \nu. \tag{32}$$

From the special case

$$\#\{S_k < j \leq S_{k+1} : \nu_j = 1\} \text{ is odd if and only if } \sigma^{\circ m}(\nu) >_p \nu,$$

it is easy to derive that the map

$$\{\text{kneading maps }, \leq_{lex}\} \mapsto \{\Sigma^1, \leq_p\}$$

is order reversing. Suppose by contradiction that (31) fails at entry k. Let

$$\tilde{Q}(l) = \begin{cases} Q(l) & \text{if } l \leq Q^{\circ 2}(k); \\ Q(k + (l - Q^{\circ 2}(k))) & \text{if } l > Q^{\circ 2}(k). \end{cases}$$

Then $\tilde{Q}$ is the kneading map of $\sigma^{S_k - S_{Q^{\circ 2}(k)}}(\nu)$ and $\tilde{Q} <_{lex} Q$. Therefore $\sigma^{S_k - S_{Q^{\circ 2}(k)}}(\nu) >_p \nu$, contradicting (a).

(c) $\Rightarrow$ (a): Since $\rho(m) - m$ is a cutting time, $\#\{m < j \le \rho(m) : \nu_j = \mathtt{1}\}$ is even by part (2) of Lemma 29.13. Hence $\sigma^{\circ m}(\nu) \le_p \nu$ by (32). Since $\nu_1 = \mathtt{1}$, the parity-lexicographical order implies that $\sigma^{\circ m+1}(\nu) \ge_p \sigma(\nu)$ for all m. $\square$

29.4. $*$-Products. Renormalization for unimodal maps has been described symbolically by means of so-called $*$-products, see [**DGP**] and [**CoE**, page 72]. Suppose that f is a renormalizable unimodal interval map, say it has periodic interval $J \ni 0$ of period l and the itinerary of orbits in $f(J)$ start with the block $\beta = [\beta_1 \dots \beta_{l-1} \cdot]$,[6] and that the renormalization $f^{\circ l}|_J$ is conjugate to a unimodal map $\tilde{f}$ with kneading sequence $\tilde{\nu}$. Then the kneading sequence ν of f has the form $\nu = \beta *_p \tilde{\nu}$, where the *parity $*$-product* $*_p$ is defined as

$$(\beta *_p \tilde{\nu})_j = \begin{cases} \beta_{j \bmod l} & \text{if } l \text{ does not divide } j; \\ \tilde{\nu}_{j/l} & \text{if } l \text{ divides } j \text{ and } \#\{k < l : \beta_k = \mathtt{1}\} \text{ is even;} \\ \tilde{\nu}_{j/l} & \text{if } l \text{ divides } j \text{ and } \#\{k < l : \beta_k = \mathtt{1}\} \text{ is odd.} \end{cases}$$

The parity of $\mathtt{1}$s in the block β determines whether the orientation of the map $f^{\circ l}|_J$ is reversed w.r.t. the orientation of original map f or not. For complex maps, $*$-products are equally useful, but the orientation of the renormalized map makes no sense. Therefore we use $*$-product without parity in the definition.

$$\beta * \tilde{\nu} = \beta_1\beta_2 \dots \beta_{l-1}\nu_1\beta_1\beta_2 \dots \beta_{l-1}\nu_2\beta_1\beta_2 \dots \beta_{l-1}\nu_3 \dots \tag{33}$$

If ν can be written non-trivially in the form of a $*$-product, then we call ν *composite* or *factorizable*. Clearly ν is a multiple $*$-product if f is multiply renormalizable. The following terminology comes from [**Pen1**]: $\beta = [\beta_1 \dots \beta_{l-1} \cdot]$ is called

$$\begin{cases} \textit{atomic} & \text{if } \beta_1 = \beta_2 = \dots = \beta_{l-1}; \\ \text{a } \textit{continent block} & \text{if it is the } *\text{-product of finitely many atomic blocks;} \\ \text{an } \textit{island block} & \text{if } \beta = \alpha * \gamma \text{ where } \alpha \neq [\ \cdot\ ] \text{ is not atomic.} \end{cases}$$

Remark. The kneading sequence of the Feigenbaum map is an infinite continent block, and the only such infinitely block in the quadratic case. For unicritical maps of degree $d \ge 3$, there are uncountably many different infinite continent blocks.

The next proposition is proved in [**Pen1**].

29.15. **Proposition (Unique Decomposition of $*$-Products)**
Every block β has a unique decomposition of the form $\beta = \alpha * \gamma$ where α is an island block and γ a continent block.

[6]We don't specify the final symbol, since it is not the same for every point in $f(J)$.

29.5. Orbit Forcing and Entropy Forcing. *Orbit forcing* refers to the property that any continuous map having a periodic orbit of some type must also have a periodic orbit of another *forced* type. Theorem 6.2 is an example of this phenomenon, but whereas in that theorem, maps come with their own Hubbard tree to act upon, orbit forcing commonly applies to a fixed space X (an interval, circle, tree etc.) on which the class of all continuous functions $f : X \to X$ is considered.

The starting point of orbit forcing is Sharkovskiy's theorem [**Sha1**] (translated in [**Sha2**] and alternative proofs given in *e.g.* [**Št, ALM1, BGMY**]):

29.16. **Theorem (Sharkovskiy's Theorem)**
Consider the following Sharkovskiy ordering $\triangleright$ *on* $\mathbb{N}^*$*:*

$$\begin{array}{l} 3 \,\triangleright\, 5 \,\triangleright\, 7 \,\triangleright\, 9 \,\triangleright\, \dots \\ \quad \dots \,\triangleright\, 2\cdot 3 \,\triangleright\, 2\cdot 5 \,\triangleright\, 2\cdot 7 \,\triangleright\, \dots \\ \quad \dots \,\triangleright\, 4\cdot 3 \,\triangleright\, 4\cdot 5 \,\triangleright\, 4\cdot 7 \,\triangleright\, \dots \\ \qquad\qquad \vdots \qquad\quad \vdots \\ \quad \dots \,\triangleright\, 2^n\cdot 3 \,\triangleright\, 2^n\cdot 5 \,\triangleright\, 2^n\cdot 7 \,\triangleright\, \dots \\ \qquad\qquad \vdots \qquad\qquad \vdots \\ \quad \dots \,\triangleright\, 64 \,\triangleright\, 32 \,\triangleright\, 16 \,\triangleright\, 8 \,\triangleright\, 4 \,\triangleright\, 2 \,\triangleright\, 1. \end{array}$$

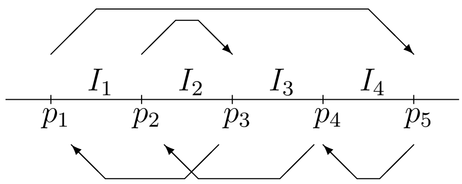
If f is a continuous map on $[0,1]$ or $\mathbb{R}$, and f has a periodic point of period p, then f has a periodic point of period q for every q with $p \triangleright q$.

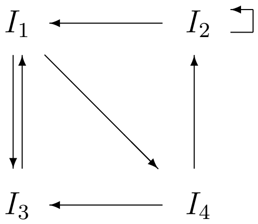
Sharkovskiy's Theorem achieved its reputation in the West only after the publication of Li and Yorke's paper [**LY**], which showed that if a continuous map $f : \mathbb{R} \to \mathbb{R}$ has an orbit of period 3, then periodic orbits of every orbit must exist. This is a special case of Theorem 29.16, but [**LY**] also introduced the notion what is now known as *Li-Yorke chaos*.[7]

In addition to Theorem 29.16, Sharkovskiy's paper includes further information on the *orbit pattern*, *i.e.,* the spacial order in which the periodic orbit is arranged in the interval. On the interval this order can be expressed by a permutation. For example, for a period 5 orbit which is spatially ordered, *i.e.,* $p_1 < p_2 < p_3 < p_4 < p_5$, the permutation (15432) indicates that $f(p_1) = p_5$, $f(p_5) = p_4$, $f(p_4) = p_3$, $f(p_3) = p_2$ and $f(p_2) = p_1$. There are two more period 5 patterns: (15423) and (15324). Whereas period 3 forces period 5, the (only existing) period 3 pattern (132) forces (15324), but (15432) and (15423) force (132). This shows that pattern forcing is a more subtle concept than period forcing. The monographs [**MN**] and [**ALM1**, Chapter 2] give much more information about orbit and pattern forcing.

[7] This is the existence of an uncountable set of points such that for any two of them, say x and y, $\liminf_n |f^{\circ n}(x) - f^{\circ n}(y)| = 0$ and $\limsup_n |f^{\circ n}(x) - f^{\circ n}(y)| > 0$. A pair of points (x, y) with this property is called a *Li-Yorke pair* ; a set in which every pair is Li-Yorke is called *scrambled*

A basic idea in orbit and entropy forcing are *transition graphs* and *matrices*. Let us use the pattern (15423) to explain the idea. The vertices of the transition graph are the interval $I_i = (p_i, p_{i+1})$ for $i \in \{1,2,3,4\}$. We write $I_i \to I_j$ whenever $f(I_i) \supset I_j$, which can be derived from the orbit of p_1, see Figure 29.2. The associated transition matrix $A = (a_{i,j})_{i,j=1}^4$ is defined by $a_{i,j} = 1$ if $I_i \to I_j$ and $a_{i,j} = 0$ otherwise. Each closed





$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

FIGURE 29.2. The transition graph and matrix of the pattern (15423).

path of length n in the graph represents a periodic orbit of period n. For example $I_1 \to I_4 \to I_2 \to I_1$ represents a period 3 orbit, and as we can insert as many I_2s as we like, we can get all periods $n \geq 3$. Periods 1 and 2 are represented by $I_2 \to I_2$ and $I_1 \to I_3 \to I_1$ respectively.

29.17. **Definition (Entropy of a Pattern)**
The entropy $h_{top}(P)$ *of a pattern* P *is* $\max\{0, \log \sigma(A)\}$, *where* $\sigma(A)$ *is the spectral radius of the transition matrix associated to* P.

Entropy forcing is the property that any continuous map having a certain orbit pattern P must have entropy $h_{top}(f) \geq h_{top}(P)$. In fact, for interval maps f, $h_{top}(f) = \sup\{h_{top}(P) : f \text{ has pattern } P\}$, see Theorem 4.4.10 of [**ALM1**]. An important result in this direction, proved in full in [**MSz**], is the following:

29.18. **Theorem (Boundary of Entropy** 0**)**
A continuous interval map f *has* $h_{top}(f) > 0$ *if and only if* f *has a periodic point of period* $n \neq 2^k$.

The boundary of entropy 0 is therefore given by the *type* 2^∞ *maps*, that is the maps with periodic points of period 2^k for each $k \geq 0$, but no other periods. In the quadratic family, the Feigenbaum map is the representative of this. One result from [**MiT**] is the following:

29.19. **Proposition (Polynomial Growth of Lap-numbers)**
The lap-number $l(f^{\circ n})$ grows polynomially in n if and only if f has finitely many periodic points.

For the Feigenbaum map, $l(f^{\circ n})$ grows super-polynomially, but subexponentially, see [**BB**, page 134] for the exact growth and [**MiT**, Table 1.5] for the kneading determinant of this map. More results on the entropy forced by periods can be found in [**ALM1**, Section 4.4].

The theory of period forcing, pattern forcing and entropy forcing has been extended to maps on trees. For example, generalizations of Sharkovskiy's Theorem to the n-od and more complicated trees were obtained in [**ALM2, Bal1**]. It is a natural step to focus this theory on Hubbard trees, and this is the content of [**AF**]. With respect to topological entropy, it is shown that a degree d Hubbard tree T has entropy at most $\log d$, with equality if and only if T equals the (abstract) Julia set. A lower bound for quadratic non-renormalizable Hubbard trees is given by $h_{top}(f|_T) \geq (\log 2)/\#\{\text{endpoints of } T\}$. In fact, if the α-fixed point has q arms, then $h_{top}(f|_T) \geq (\log 2)/(q-1)$, so that there are non-renormalizable Hubbard trees with entropy arbitrarily close to 0.

A further topic in [**AF**] is the effect of renormalization on the dynamics on the Hubbard tree. To this end, recall from Definition 10.3 that a quadratic map is called *renormalizable of disjoint type* if it is renormalizable, say of period n, and the images of the little filled-in Julia sets $K, f(K), \ldots, f^{\circ(n-1)}(K)$ are disjoint. It is shown that a quadratic Misiurewicz polynomial is not renormalizable of disjoint type if and only if $f|_T$ is topologically transitive, *i.e.,* there is a dense orbit in T. Furthermore, f is not renormalizable at all if and only if $f|_T$ is topologically mixing[8], *i.e.,* every iterate $f^{\circ n}$, $n \geq 1$, has a dense orbit in T.

[8]This property is also called *totally transitive*.

30. Maps on Dendrites

The dynamics on (quadratic) Julia sets has motivated various symbolic approaches other than those discussed in this book. In this section, we discuss work from Chris Penrose' thesis [**Pen1**] on quotient spaces of the full shift, and Stewart Baldwin's itinerary topology, which is another way to convert the shift space into a continuum. We also discuss work by various authors, including Bandt, Bousch and Solomyak, on a linear two-to-one iterated function system with a fractal geometry quite distinct from quadratic Julia sets (and the Mandelbrot set), but that does allow a comparable combinatorial approach.

30.1. Penrose's Glueing Spaces. In his Ph.D. thesis [**Pen1**], Chris Penrose takes a symbolic approach to Julia sets of unicritical polynomials $f = f_{c_1} : z \mapsto z^d + c_1$ of degree d, describing them as quotient spaces of $\Sigma_d = \{\mathtt{0}, \dots, d-1\}^{\mathbb{N}^*}$ (where Σ_d is equipped with product topology). Let R be the image of a embedding $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(0) = c_1$ and $\gamma(t) \to \infty$ as $t \to \infty$. Its d preimages meet at 0 and partition $\mathbb{C}$ into d sectors $\mathbb{C}_i$, $i \in \{\mathtt{0}, \dots, d-1\}$. (This is true also if c_1 does not belong to the Multibrot set $\mathcal{M}_d$, and since R is not necessarily an external ray, this procedure can also be carried if c_1 belong to the interior of the Multibrot set.) Assuming that the forward images of R are pairwise disjoint, Penrose defines itineraries $e(z) \in \Sigma_d$ for $z \in K(f)$, the filled-in Julia set, in a way that the wild card symbol $\star$ associated to the critical point is not needed. To this end, a relation $\mathcal{S}_N$ between $z \in \mathbb{C}$ and $e \in \Sigma_d$ is defined by

$$(z, e) \in \mathcal{S}_N \text{ if } f^{\circ n-1}(z) \in \mathbb{C}_{e_n} \text{ for all } 1 \leq n \leq N,$$

and $(z, e) \in \mathcal{S}$ if $(z, e) \in \mathcal{S}_N$ for all $N \in \mathbb{N}^*$. Since this excludes points z that map onto R for some iterate, the relation is extended to $\mathcal{S}'$ defined by

$$(z, e) \in \mathcal{S}' \text{ if } f^{\circ n-1}(z) \in \overline{\mathbb{C}_{e_n}} \text{ for all } n \in \mathbb{N}^*,$$

i.e., $(z, e) \in \overline{\mathcal{S}_N}$ for all $N \in \mathbb{N}^*$. Among the properties shown for the relation $\mathcal{S}'$ are the following, cf. Theorem 1.3.4. in [**Pen1**].

30.1. **Theorem (Coding by the Relation $\mathcal{S}'$)**
Assume that $f^{\circ n}(R) \cap R = \emptyset$ for all $n \geq 1$. Then

- *for every $z \in K(f)$ there is $e \in \Sigma_d$ such that $(z, e) \in \mathcal{S}'$;*
- *if $(z, e) \in \mathcal{S}$ and $(z, \tilde{e}) \in \mathcal{S}'$, then $e = \tilde{e}$;*
- *for all $e \in \Sigma_d$, there is $z \in K(f)$ such that $(z, e) \in \mathcal{S}'$.*

The equivalence relation on Σ_d given by $e \sim \tilde{e}$ if there is $z \in K(f)$ such that $(z, e), (z, \tilde{e}) \in \mathcal{S}$ is a closed equivalence relation, and the coding map

$$\begin{aligned} e \ : \ & K(f) \to \Sigma_d / \sim \\ & z \mapsto [e(z)] \quad \textit{such that } (z, \tilde{e}) \in \mathcal{S}' \textit{ for each } \tilde{e} \in [e(z)], \end{aligned}$$

is continuous when $\Sigma_d / \sim$ is equipped with the quotient topology.

The *kneading sequence of f with respect to R* is the itinerary of c_1, and if $f^{\circ n}(R) \cap R = \emptyset$, then the equivalence class of the kneading sequence consists of a single sequence ν. In this case the relation $\mathcal{S}'$ is easy to describe. If $e, \tilde{e} \in \Sigma_d$, then $e \sim \tilde{e}$ if $e = \tilde{e}$ or there is $n \in \mathbb{N}^*$ such that

$$\begin{cases} e_j = \tilde{e}_j & \text{if } j < n; \\ e_j = \tilde{e}_j = \nu_{j-n} & \text{if } j > n. \end{cases} \tag{34}$$

If $c_1 \in K(f)$ and ν is nonrecurrent under the leftshift σ, then $e : K(f) \to \Sigma_d$ is a homeomorphism, cf. [**Pen1**, Theorem 1.5.3.].

The main part of [**Pen1**] is concerned with the opposite direction: starting with an equivalence relation $\sim$ on Σ_d, describe the topology and further properties of the quotient space $\Sigma_d / \sim$. The type of equivalence relation is more general than the one from (34), and the resulting quotient space is called *glueing space*. It is explained in Chapter 2 of [**Pen1**].

Let Cyl_n be the collection of *n-cylinder sets* in Σ_d, *i.e.,* $C_{e_1 \dots e_n} \in \mathrm{Cyl}_n$ denotes the set $\{\tilde{e} \in \Sigma_d : \tilde{e}_j = e_j \text{ for } 1 \le j \le n\}$.

30.2. **Definition (Glueing Point Map and Glueing Space)**
A glueing point map *is a map*

$$\Gamma : \cup_{n \ge 1} Cyl_n \to \Sigma_d \text{ such that } \Gamma(C) \in C \text{ for all } C \in Cyl_n, n \ge 1.$$

Let $\sim_\Gamma$ be the minimal closed equivalence relation on Σ_d that for each $C_{e_1 \dots e_{n-1}} \in Cyl_{n-1}$, $n \ge 1$, identifies all points $\Gamma(C_{e_1 \dots e_{n-1} j})$, $j \in \{0, \dots, d-1\}$. The quotient space $\Sigma_d / \sim_\Gamma$ is called the glueing space *of Γ.*

30.3. **Example (Arcs as Glueing Spaces)**
For $d = 2$ and

$$\begin{cases} \Gamma(C_{e_1 \dots e_{n-1} \mathtt{0}}) = e_1 \dots e_{n-1} \mathtt{0}\overline{\mathtt{1}}, \\ \Gamma(C_{e_1 \dots e_{n-1} \mathtt{1}}) = e_1 \dots e_{n-1} \mathtt{1}\overline{\mathtt{0}}, \end{cases}$$

for all $n \ge 1$ and $e_1, \dots, e_{n-1} \in \{\mathtt{0}, \dots, d-1\}$, the glueing procedure it the standard way of factorizing the Cantor set Σ_2 onto the unit interval via the map $e \mapsto \sum_{n \ge 1} e_n 2^{-n}$. Thus the glueing space $\Sigma_d / \sim_\Gamma = [0, 1]$ in this case.

An interval as glueing space is also obtain by the glueing point map

$$\begin{cases} \Gamma'(C_{e_1 \dots e_{n-1} \mathtt{0}}) = e_1 \dots e_{n-1} \mathtt{01}\overline{\mathtt{0}}, \\ \Gamma'(C_{e_1 \dots e_{n-1} \mathtt{1}}) = e_1 \dots e_{n-1} \mathtt{11}\overline{\mathtt{0}}. \end{cases}$$

This corresponds to the equivalence relation in (34) when $\nu = \mathtt{1}\overline{\mathtt{0}}$ for the Chebyshev polynomial $f : z \mapsto z^2 - 2$.

Thus the glueing point map Γ' above is a special case of the map

$$\Gamma_\nu(C_{e_1 \dots e_{n-1} j}) = e_1 \dots e_{n-1} j \nu \quad \text{for } j \in \{\mathtt{0}, \dots, d-1\}$$

based on an arbitrary (kneading) sequence ν. Most of the glueing space results in [**Pen1**] concern this map Γ_ν, but there are other possibilities. For instance, the glueing point map (see Example 3 in [**Pen1**])

$$\Gamma_{\Sigma_d}(C_{e_1\dots e_n}) = \overline{e_1\dots e_n}$$

for all $n \geq 1$ and $C_{e_1\dots e_n} \in \mathrm{Cyl}_n$ is associated to renormalizable Julia sets.

The main topological result of Chapter 2 (see Corollary 2.1.1, Proposition 2.3.7 and Corollary 2.3.7 in [**Pen1**]) amounts to the existence of abstract Julia sets as dendrites.

30.4. **Theorem (Glueing Spaces are Uniquely Arcwise Connected)** *A glueing space $\Sigma_\Gamma := \Sigma_d/\sim_\Gamma$ is uniquely arcwise connected; for every $x, y \in \Sigma_\Gamma$, there is a unique arc Γ-arc(x, y) connecting x and y.*

If $\Gamma = \Gamma_\nu$ for some $\nu \in \Sigma_d$, then we abbreviate $\Sigma_\nu := \Sigma_d/\sim_{\Gamma_\nu}$.

We will not discuss the details of the proof here, but an important tool is the *branch* $B[C_{e_1\dots e_k}]$ of a k-cylinder which roughly describes the collection of r-cylinders for $r \leq k$ that are accessible from $C_{e_1\dots e_k}$ via the glueing relation, but not via other k-cylinders. It reflects the component of $\Sigma_\Gamma \setminus C_{e_1\dots e_{k-1}}$ "on the side" of $C_{e_1\dots e_k}$. Also, for each $e \in \Sigma_d$, a successor function on $\mathbb{N}^*$ is introduced that in the special case that $\Gamma = \Gamma_\nu$ coincides with our $\rho_{\nu,x}$-function (see formula (9))

$$\rho_{\nu,e}(n) = \min\{j > n : e_j \neq \nu_{j-n}\}.$$

The *valency* of e is the number of disjoint orbits under this successor map, and it is shown to be equal to the number of components of $\Sigma_\Gamma \setminus [e]$, see Lemma 16.2. In [**Pen2**, Theorem 4.2], it is shown that if e has valency at least 3, then e is (pre)periodic, so this amounts to the Nonwandering Triangle Theorem for unicritical polynomials of degree d. See Section 24.3 for results on the failure of the Nonwandering Triangle Theorem failing for general laminations of degree d (*i.e.,* with more than one critical gap).

Branches are also used to define the arc connecting x and $z \in \Sigma_\Gamma$

$$\Gamma\text{-arc}(x, z) := \{y \in \Sigma_\Gamma : B[C_{y_1\dots y_k}] \cap \{x, z\} \neq \emptyset \text{ for all } n \in \mathbb{N}^*\}.$$

The internal address features as the orbit of 1 under the successor function of ν. It is not interpreted geometrically as in *e.g.* Proposition 11.6, but it is used to prove the results (in [**Pen1**, Theorem 2.5.3 and Corollary 2.5.3.1]) that appear as Lemmas 20.1 and its conclusion that ν is an endpoint of the Hubbard tree or dendrite, cf. Lemma 20.3. Hence all points $\sigma^{\circ n}(\nu)$, $n \geq 0$ lie in the same component of $\Sigma_\Gamma \setminus \nu$ as the "critical point" $\sigma^{-1}(\nu)$, and the connected hull of $\{\sigma^{-1}(\nu), \nu, \sigma(\nu)\sigma^{\circ 2}(\nu), \dots\}$ is a Hubbard tree or dendrite. (Expansivity of σ is shown later on in Theorem 30.6 for "non-renormalizable" sequence ν.)

It is shown that not every $\nu \in \Sigma_d$ is complex admissible; Penrose presents the same simplest example $\nu = \mathtt{101100}\dots$ as Example 5.3. Also *real admissibility* is defined as $\sigma^{\circ 2}(\nu)$ belonging to Γ_ν-arc$(\nu, \sigma(\nu))$., and in [**Pen1**, Theorem 2.6.1.], this is shown to be equivalent to the existence of $a \neq b \in \{0, \dots, d-1\}$ such that $\nu \in \Gamma_\nu\text{-arc}(\overline{a}, a\overline{b})$. In the case $d = 2$ this reduces to: ν belongs to the arc connecting the β-fixed point and its preimage.

Recall the terminology of $*$-products, atomic blocks and island blocks from Section 29.4 If $\nu = \beta * \tilde{\nu}$, where $\beta = [\beta_1 \dots \beta_{l-1} \cdot]$ is a block of length l, then one can define a map

$$H_m : \Sigma_d \to \Sigma_d, \quad e \mapsto \sigma^{\circ m}(\beta * e) \tag{35}$$

for $0 \leq m < l$.

30.5. **Theorem (Small Julia Sets in Σ_ν)**
If $\nu = \beta * \nu$ for a nontrivial block β, then Σ_ν contains sets $H_m(\Sigma_{\tilde{\nu}})$, $m = 0, \dots, l-1$ which are copies of $\Sigma_{\tilde{\nu}}$ embedded in Σ_ν and cyclically permuted by σ.

The sets $H_m(\Sigma_{\tilde{\nu}})$ are pairwise disjoint if β is an island block. If on the other hand β is atomic, then all $H_m(\Sigma_{\tilde{\nu}})$ have a single common (fixed) point of intersection.

Chapter 5 of [**Pen1**] is devoted to a result that is a complex version of a result by Milnor and Thurston [**MiT**] (after an earlier result by Parry [**Pa**]), namely that interval maps $f : I \to I$ are semiconjugate to a piecewise linear map with slope $\pm s$ where $s = \log h_{top}(f)$, see Theorem 29.8. In this text, it is phrased somewhat differently in terms of the existence of a metric which expands distances by a constant factor s.

30.6. **Theorem (Affine metric on Glueing Space Σ_ν)**
If $\nu \in \Sigma_d$ is not the $*$-product of infinitely many atomic blocks, then there is a pseudo-metric $\mathfrak{d} \not\equiv 0$ on Σ_ν and $s > 1$ such that

- $\mathfrak{d}(\sigma(x), \sigma(y)) = s\mathfrak{d}(x, y)$ *whenever* $x_1 = y_1$*;*
- $\mathfrak{d}(x, z) = \mathfrak{d}(x, y) + \mathfrak{d}(y, z)$ *whenever z belongs to the arc connecting x and y.*

The constant s is uniquely determined, and $\mathfrak{d}$ is unique up to a multiplicative constant. If ν is not of $*$-product form $\beta * \tilde{\nu}$ where β is an island block of length $l > 1$, then $\mathfrak{d}$ is a proper metric (i.e., $\mathfrak{d}(x, y) > 0$ whenever $x \neq y$), and the topology inherited from $\mathfrak{d}$ is the same as the quotient topology of Σ_ν. Otherwise, $\mathfrak{d}(x, y) = 0$ implies that there are $0 \leq m < l$ and block $e_1 \dots e_n \in \{0, \dots, d-1\}^n$ and $\tilde{x}, \tilde{y} \in \Sigma_d$ such that

$$x = e_1 \dots e_n H_m(\tilde{x}) \text{ and } y = e_1 \dots e_n H_m(\tilde{y}),$$

where H_m is defined in (35)*.*

An important interpretation of this theorem (not mentioned in [**Pen1**]) is that $s = h_{top}(\sigma|_T)$, the topological entropy of the map restricted to the Hubbard tree T. This follows because with respect to the metric $\mathfrak{d}$, σ is affine on each component of $T \setminus \sigma^{-1}(\nu)$ with constant expansion factor s, and self-maps $f : T \to T$ on trees with constant slopes $\pm s$ have entropy $h_{top}(f|_T) = \max\{\log s, 0\}$, see Theorem 29.8. The same result holds by continuous approximation if ν has a Hubbard dendrite rather than tree. Note however, that $s \neq h_{top}(\sigma|_{\Sigma_\nu}) = \log d$ (which is the topological entropy of the shift on the abstract Julia set Σ_ν) whenever $\nu \neq 1\overline{0}$. Theorem 29.8 fails in general for piecewise affine maps on dendrites, and in this case, the entropy is "carried" by the endpoints of Σ_ν (provided $\nu \neq 1\overline{0}$), whereas all non-endpoints are eventually mapped into the Hubbard dendrite T. It seems that the system $(\Sigma_\nu, \sigma, \mathfrak{d})$ can be represented by a piecewise affine map $(A_\lambda, q_\lambda, d)$ as in (36) with $\lambda = 1/s$, see Section 30.3.

In the light of s being the topological entropy of the Hubbard dendrite, the following results can be interpreted as continuity and monotonicity of entropy as function of the kneading sequence ν.

30.7. **Theorem (Entropy Depends Continuously and Monotonically on ν)** *The map $\nu \mapsto s(\nu)$, where s is as in Theorem 30.6, is continuous. Furthermore, if $\nu, \tilde\nu \in \Sigma_d$ are such that $\tilde\nu \in \Gamma_\nu$-arc$(\nu, \overline{\nu_1})$,*[9] *then $s(\nu) \geq s(\tilde\nu)$.*

The proof of Theorem 30.6 is based on a linear operator φ_ν acting on the Banach space of pseudo-metrics on Σ_ν with supremum norm, as

$$(\varphi_\nu h)(x,y) = \begin{cases} h(\sigma(x), \sigma(y)) & \text{if } x_1 = y_1, \\ h(\sigma(x), \nu) + h(\nu, \sigma(y)) & \text{if } x_1 \neq y_1. \end{cases}$$

It is shown that $\mathfrak{d}$ is the eigenvector of φ_ν corresponding to the leading eigenvalue s. It is observed that this only depends on $x, y \in \{\sigma^{-1}(\nu), \nu, \sigma(\nu), \sigma^{\circ 2}(\nu), \dots\}$, and replacing such x by the integer m such that $x = \sigma^{\circ m} \circ \sigma^{-1}(\nu)$, one can rewrite φ_ν to an operator Φ_ν acting on ℓ^∞-functions on $\mathbb{N} \times \mathbb{N}$:

$$(\Phi_\nu h)(m-1, n-1) = \begin{cases} h(m,n) & \text{if } \nu_m = \nu_n, \\ h(m,0) + h(0,n) & \text{if } \nu_m \neq \nu_n, \end{cases}$$

and $m, n \geq 1$. Using infinite transition graphs and theory of infinite matrices (due to Vere-Jones [**V-J1, V-J2**]), it is shown that Φ_ν is quasi-compact, *i.e.,* it has finitely many eigenvalues of modulus s, each of finite multiplicity, and the rest of the spectrum of Φ_ν is contained in a disk of radius $< s$. A $*$-product structure of ν (*i.e.,* renormalizability of (Σ_ν, σ)) is reflected by the existence of multiple eigenvalues on the circle $\{|z| = s\}$. For real admissible ν, these ideas are closely related to kneading matrices from [**MiT**].

30.2. Baldwin's Continuous Itinerary Functions. Let $f : D \to D$ be continuous map on a dendrite D (see Definition 21.1), having a single critical point 0 where f is not locally homeomorphic onto its image. We coded this point by $\star$ and assume that $f(0) \neq 0$. The components of $D \setminus \{0\}$ are labeled by symbols in $\{1, 2, \dots, d\}$ according to the order by which the critical orbit visits them. So denoting the component with label n by L_n we have $f(0) \in L_1$ and for $n > 2$, $f^{\circ k}(0) \in L_n$ only if there is $l < k$ such that $f^{\circ l}(0) \in L_{n-1}$. Let us assume that f has the *unique itinerary property*, *i.e.,* the itinerary map $i : D \to \Sigma_B := \{\star, 1, \dots, d\}^{\mathbb{N}^*}$ is injective.

[9] that is: $\tilde\nu$ belongs to the arc connecting critical value ν with the α-fixed point

30.8. **Example (A non-acceptable kneading sequence)**
Not every Hubbard tree satisfies the unique itinerary property. For example, if $\nu = \overline{\mathtt{10010}\star}$ (or $\tau = i(0) = \overline{\star\mathtt{12212}}$ in the coding convention of this section), then all points in the open arc (c_1, c_4) have the same itinerary, see Figure 30.1. In the language of Definition 30.9 below, τ is not acceptable, because $\sigma^{\circ 3}(\tau)_j = \tau_j$ for every j, except when j is a multiple of 3. But then either $\sigma^{\circ 3}(\tau)_j = \star$ or $\tau_j = \star$.

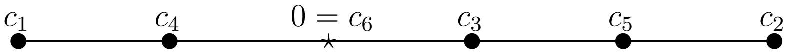


FIGURE 30.1. The Hubbard tree of a non-acceptable kneading sequence $\nu = \overline{\mathtt{10010}\star}$

If the symbol space $\{\star, \mathtt{1}, \mathtt{2}, \dots, d\}$ has the discrete topology, then the itinerary map is discontinuous. Baldwin's idea is to make the itinerary map continuous by giving the symbol space a new topology; namely the one generated by open sets $\{\mathtt{1}\}, \{\mathtt{2}\}, \dots, \{d\}$, but not $\{\star\}$. Thus the only open neighborhood of $\star$ is the whole symbol space $\{\star, \mathtt{1}, \dots, d\}$. This topology is not Hausdorff, and neither is the product topology on Σ_B it generates. This topology is called the *itinerary topology*. For example, Σ_B is the only open neighborhood of $\overline{\star}$.

But the itineraries obtained from points in the connected hull of $\mathrm{orb}_f(0)$ (that is the Hubbard tree or dendrite), or more generally, the itineraries of points in the abstract Julia set, can generate a subset of Σ_B that *is* Hausdorff. The combinatorial description of such subsets are in terms of acceptibility conditions

30.9. **Definition (Acceptable and τ-admissible sequences)**
A sequence $\tau = \star\nu \in \Sigma_B$ is called acceptable *if*

- *If $\tau_n = \star$, then $\tau_{n+j} = \tau_j$ for all $j \in \mathbb{N}^*$;*
- *If $\sigma^{\circ k}(\tau) \neq \tau$ for some k then there is j such that $\star \neq (\sigma^{\circ k}(\tau))_j \neq \tau_j \neq \star$.*

Denote the set of acceptable sequences in Σ_B by Acc_d. Given an acceptable sequence τ, a sequence $x = x_0 x_1 \cdots \in \Sigma_B$ is called τ-admissible *if*

- *If $x_n = \star$, then $\sigma^{\circ n}(x) = \tau$;*
- *If $\sigma^{\circ k}(\tau) \neq \tau$ for some k then there is j such that $\star \neq (\sigma^{\circ k}(x))_j \neq \tau_j \neq \star$.*

The set of all τ-admissible sequences is denoted as D_τ.

The main result can now be summarized as follows:

30.10. **Theorem (D_τ is a Dendrite)**
If $\tau \neq \overline{\star}$ is acceptable, then D_τ as subset of Σ_B with inherited itinerary topology, is a dendrite. In particular, it is Hausdorff, compact separable, metrizable, uniquely arcwise connected, and locally connected. Furthermore, the shift $\sigma : D_\tau \to D_\tau$ is

continuous and self-similar, *i.e., for each symbol $k \neq \star$, the set $\{x \in D_\tau : x_0 = k\}$ is mapped homeomorphically onto D_τ by σ. Also, if $X \subset D_\tau$ is connected, then $\sigma^{-1}(X)$ is connected if and only if it contains τ.*

Consequently, if the itinerary $\tau = i(0)$ of the unique critical point of a dendrite map $f : D \to D$ is acceptable, and all components L_k of $D \setminus \{0\}$ map onto $D \setminus \{f(0)\}$, then D_τ is a homeomorphic copy of D. Furthermore the itinerary map i conjugates (D, f) and (D_τ, σ): $\sigma \circ i = i \circ f$. This leads to abstract Julia sets, provided $\star\nu$ is acceptable, and the Hubbard trees (or Hubbard dendrites for non(pre)periodic kneading sequences ν) is obtained by taking the connected hull of the critical orbit.

The sequence $\tau = \overline{\star 12212}$ in Example 30.8 above corresponds to a renormalizable Hubbard tree. Baldwin approaches renormalization (see Definition 10.3) in a combinatorial way, namely via $*$-products. Given an n-periodic sequence $x \in \Sigma_B$ and another sequence $y \in \Sigma_B$, the $*$-product $x * y$ is a new sequence defined by (see (33))

$$(x * y)_j = \begin{cases} x_{j \bmod n} & \text{if } n \text{ does not divide } j; \\ y_{j/n} & \text{if } n \text{ divides } j. \end{cases}$$

Any sequence z that can be written non-trivially as a $*$-product is called *composite*. Sequences can be multiply composite and even infinitely composite. For example, the kneading sequence of the Feigenbaum map is the infinite $*$-product of $\mathtt{12}$ with itself. Any sequence $z = x * y$, where $y = \star\overline{k}$ or $y = \overline{\star kkk \dots k}$ for some $k \neq \star$ is called a *tupling*. Thus, the kneading sequence of the Feigenbaum map is an ∞-tupling. The sequence in Example 30.8 is a tupling too, because $\tau = \overline{\star 12212} = (\overline{\star 12}) * (\overline{\star 2})$. Tuplings decide on acceptability, see Theorems 3.8 and 3.21 in [**Bal2**].

30.11. **Theorem (Tuplings, Acceptability and Renormalization)**
Take $\tau = \star\nu \in \Sigma_B \setminus \{\overline{\star}\}$. Then

(1) *τ is acceptable if and only if it is not a tupling;*
(2) *if τ is acceptable, then (D_τ, σ) is renormalizable if and only if τ is composite.*

Further topics in [**Bal2**] include arc-transitivity of D_τ, the structure of the inverse limit space over D_τ with the shift σ as bonding map, and the topological structure of the parameter space Acc_d. It would go too far to mention any of the details, but an appropriate 'Hausdorffication' of Acc_d leads to another abstract approach to the Multibrot set $\mathcal{M}_d$.

Another extension is the introduction of an infinite symbol space to express itineraries in, see also [**Bal3**]. This allows the phase space to be a *dendroid* which may fail to be locally connected, and allows branchpoints to have infinite order.

30.3. Pairs of Linear Maps. The nature of the set

$$A_\lambda := \left\{ \sum_{n \geq 0} a_n \lambda^n : a_n \in \{0, 1\} \right\}$$

has been studied at least since Erdös [**Er**]. For $\lambda \in \mathbb{D}$, the set A_λ plays the role of quadratic Julia set, and is also self-similar in the sense that it is the limit set of the *iterated function systems* (IFS)

$$\begin{cases} f_0(z) = \lambda z, \\ f_1(z) = \lambda z + 1, \end{cases}$$

that is, it is the unique non-empty compact set A such that $A = f_0(A) \cup f_1(A)$, see [**Hut, BaHa**]. Note also that

$$A_0 := f_0(A_\lambda) = \left\{ \sum_{n \geq 0} a_n \lambda^n : a_n \in \{0,1\}, a_0 = 0 \right\}$$

and

$$A_1 := f_1(A_\lambda) = \left\{ \sum_{n \geq 0} a_n \lambda^n : a_n \in \{0,1\}, a_0 = 1 \right\}.$$

Considering the λ-plane $\mathbb{C}$ as parameter space, there is the same dichotomy as for quadratic maps $z \mapsto z^2 + c$: either A_λ is a Cantor set or A_λ is connected. Define the analogue of the Mandelbrot set as

$$\mathcal{M}_\times := \{\lambda \in \mathbb{C} : A_\lambda \text{ is connected}\},$$

see Figure 30.2. One can check that $\{|\lambda| \geq \frac{1}{\sqrt{2}}\} \subset \mathcal{M}_\times \subset \{|\lambda| \geq \frac{1}{2}\}$, so the interesting part of the parameter space is the annulus $\{\frac{1}{2} \leq |\lambda| < 1\}$. So from now on, assume that λ belongs to this annulus and also $\lambda \notin \mathbb{R}$. The involutions $\lambda \mapsto -\lambda$ and $\lambda \mapsto \overline{\lambda}$ preserve $\mathcal{M}_\times$, and the two *antennae* $[\frac{1}{2}, 1]$ and $[-1, -\frac{1}{2}]$ belong to $\mathcal{M}_\times$ as well. Hence it suffices to consider the positive quadrant of the annulus, see Figure 30.2.

Bousch [**Bou2**] proved that $\mathcal{M}_\times$ is connected and locally connected, but $\mathcal{M}_\times$, is not self-similar. For example, $\partial\mathcal{M}_\times$ becomes smoother as $\arg\lambda$ increases to $\pi/2$. Furthermore, Bandt [**Ban**] proved that $\mathcal{M}_\times$ contains holes, see the zoom-in of Figure 30.2.

It is easy to see that $\lambda \in \mathcal{M}_\times$ if and only if

$$I_\lambda := A_0 \cap A_1 \neq \emptyset.$$

Figure 30.3 depicts the set of interest to us:

$$\mathcal{M}_1 := \{\lambda \in \mathbb{C} : I_\lambda =: \{c_\lambda\} \text{ is a singleton}\},$$

or equivalently $\mathcal{M}_1 = \{c \in \mathbb{C} : A_\lambda \text{ is a dendrite}\}$, as was proven in [**BK**]. If $A_0 \cap A_1$ consists of exactly one point, then there is a sequence $\{a_n\}_{n \geq 0}$, with $a_0 = 1$ say, such that $c_\lambda = \sum_n a_n \lambda^n = \sum_n (1 - a_n)\lambda^n$. Subtracting these sequences we get that

$$\mathcal{F}_\lambda := \left\{ g(z) = \sum_{n \geq 0} b_n z^n : b_n \in \{-1, 0, 1\}, b_0 \geq 0, g(\lambda) = 0 \right\}$$

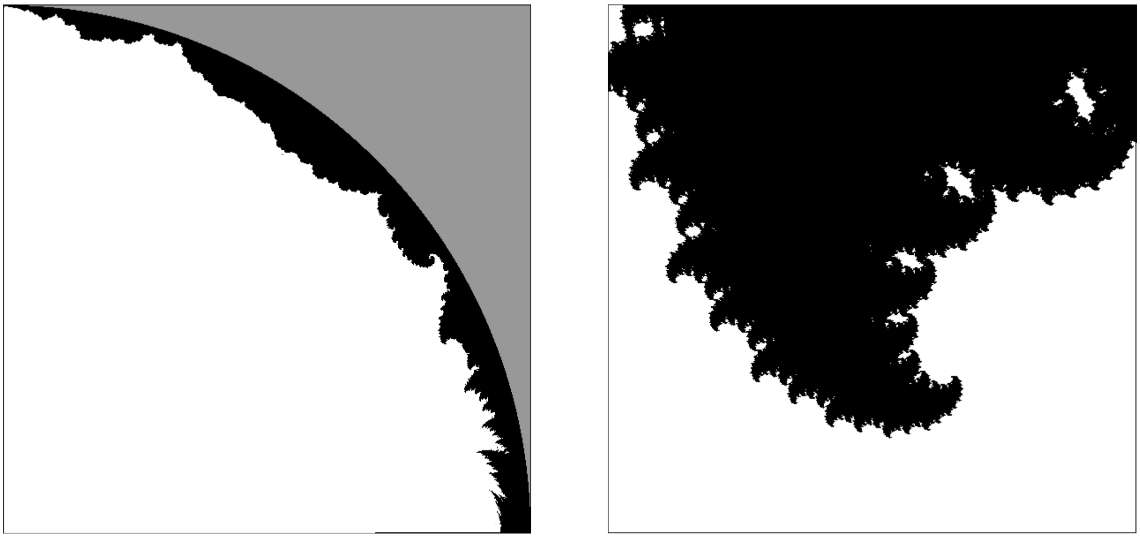


FIGURE 30.2. Left: The top-right quarter of the set $\mathcal{M}_\times$, with $\lambda \geq \frac{1}{\sqrt{2}}$ in grey. Right: A zoom-in near $\lambda = 0.59 + 0.25i$ shows that $\mathcal{M}_\times$ contains holes. For this and the other figures in this section, we are indebted to Christoph Bandt.

consists of exactly one function, say $\mathcal{F}_\lambda = \{g_\lambda\}$, and all the coefficients of g_λ are ± 1.

REMARK. Incidentally, the set

$$\mathcal{M}_2 = \{\lambda \in \mathbb{C} : \# I_\lambda = 2\}$$

is of interest too. If $\lambda \in \mathcal{M}_2$, then the set A_λ is still connected and one-dimensional, but it contains infinitely many topological circles (comparable to the Sierpinski triangle), see the right panel of Figure 30.4. In this case $\mathcal{F}_\lambda$ is a singleton too, but it has exactly one zero coefficient.

If $\sum_n a_n \lambda^n \in A_\lambda$, also $\sum_n (1 - a_n)\lambda^n \in \mathcal{A}_\lambda$, so the involution

$$i : z \mapsto \sum_n (1 - a_n)\lambda^n = \frac{1}{1-\lambda} - z$$

preserves A_λ. This allows us to define dynamics on A_λ, which to some extent is the inverse of the IFS $\{f_0, f_1\}$:

$$q_\lambda(z) := \begin{cases} f_0^{-1}(z) = \frac{z}{\lambda} & \text{if } z \in A_0, \\ i \circ f_1^{-1}(z) = \frac{1-z}{\lambda} + \frac{1}{1-\lambda} & \text{if } z \in A_1. \end{cases} \tag{36}$$

This map is 2-to-1, except at the single critical point $c_\lambda = \frac{1}{2(1-\lambda)}$; at all other points in A_λ, q_λ is a local homeomorphism. It follows that (A_λ, q_λ) satisfies all the axioms of

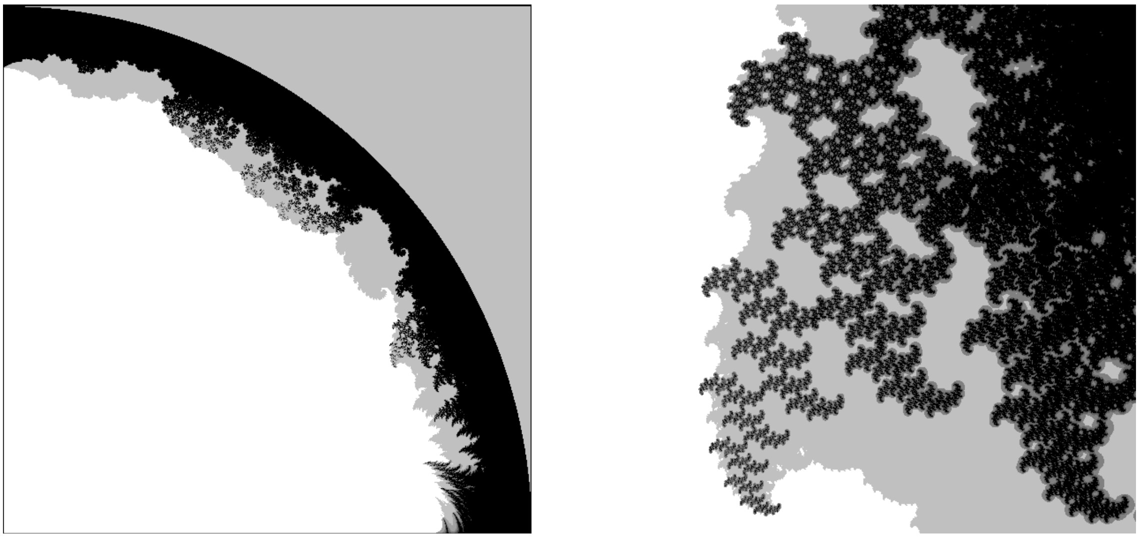


FIGURE 30.3. Left: Quarters of the sets $\mathcal{M}_\times$, in grey, and the set of zeroes of power series with coefficients in $\{-1, 0, 1\}$ superimposed in black. This latter set contains $\mathcal{M}_1$, but as yet it is not known whether they coincide. Right: Zoom-in for the region $0.57 \le \operatorname{Re} \lambda \le 0.65$ and $0.25 \le \operatorname{Im} \lambda \le 0.33$.

an abstract Julia set, and the connected hull of $\operatorname{orb}(c_\lambda)$ in A_λ is the Hubbard tree or Hubbard dendrite according to whether $\operatorname{orb}(c_\lambda)$ is finite or infinite (and its connected hull is indeed a dendrite and not a tree). The fixed points are $\alpha = \frac{1}{1-\lambda^2}$ and $\beta = 0$ (which is always an endpoint of the dendrite A_λ).

Bandt [**Ban**] showed that $\mathcal{M}_1 \neq \emptyset$; he gave algebraic parameters $\lambda \in \mathcal{M}_1$ for which c_λ is strictly preperiodic. In fact, it was proved in [**BR**] that c_λ is nonrecurrent for every $\lambda \in \mathcal{M}_1$. Solomyak [**Sol**] showed that $\mathcal{M}_1$ is uncountable, and hence contains transcendental parameters λ for which $\operatorname{orb}(c_\lambda)$ is infinite. He also showed that for strictly preperiodic parameters, A_λ at $q_\lambda(c_\lambda)$ and $\mathcal{M}_1$ at λ are locally self-similar in the sense of Tan Lei, [**Ta1**]. Together with Ilgar Eroglu and Steffen Rohde [**ERS**], he showed that (A_λ, q_λ) is quasi-conformally conjugate with (J_c, f_c) where c is chosen that q_λ and f_c have the same kneading sequence. However, since not every $c \in \partial\mathcal{M}$ corresponds to some $\lambda \in \mathcal{M}_1$, the question remains.

- For which $c \in \mathcal{M}$ is there a $\lambda \in \mathcal{M}_1$ such that (A_λ, q_λ) and (J_c, f_c) are conjugate systems?

Further questions in this direction are

- What is the topology of $\mathcal{M}_1$? There are reasons to believe that $\mathcal{M}_1$ is (contained in) a Cantor set.
- Do 'most' points $\lambda \in \partial\mathcal{M}_\times$ have a unique power series in $\mathcal{F}_\lambda$, see [**Sol**]?

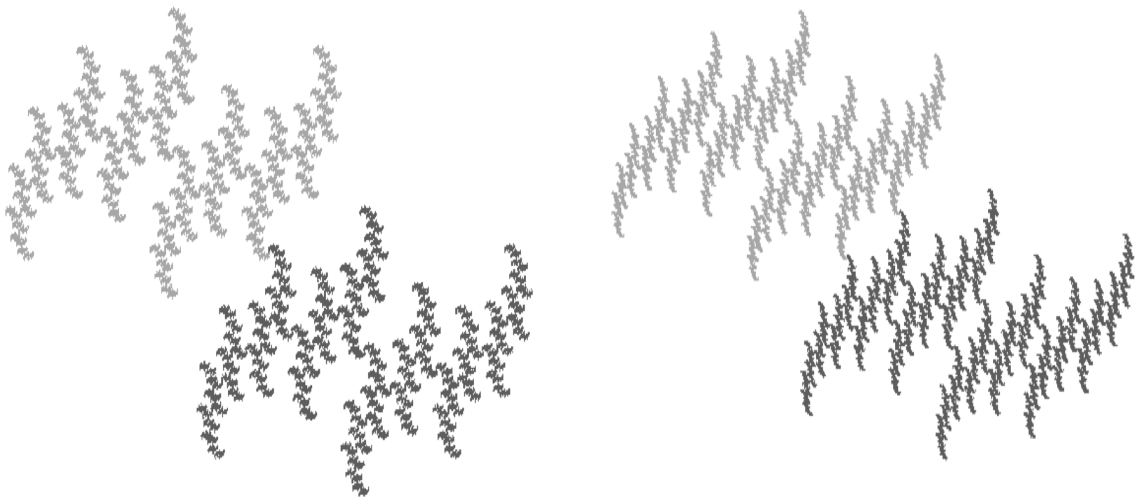


FIGURE 30.4. Examples of sets A_λ when $A_0 \cap \mathcal{A}_1$ is a singleton (left: $\lambda \approx 0.59 + 0.25i$ is the solution of $2\lambda^3 - 2\lambda + 1 = 0$), or consists of two points (right: $\lambda \approx 0.62 + 0.18i$ is a solution of $\lambda^4 + \lambda^3 - 2\lambda + 1 = 0$). Details on these examples can be found in [**Ban**].

In the rest of this section, we discuss the symbolic dynamics and combinatorics of (A_λ, q_λ) in more detail. If we assign the symbols $\mathtt{0}$, $\mathtt{1}$ and $\star$ to $A_0 \setminus \{c_\lambda\}$, $A_1 \setminus \{c_\lambda\}$ and $\{c_\lambda\}$ respectively, we can define the itineraries and kneading invariant ν, in the same way as for f_c on the ordinary Julia set. Note also that q_λ preserve cyclic orders of branches at branchpoints, so A_λ cannot contain any evil branchpoints. It follows that ν always satisfies the admissibility condition of Definition 5.1.

For $z = \sum a_n \lambda^n$, let us denote how q acts on the sequence $\{a_n\}_{n\ge0}$ by Q:

$$Q(\{a_n\}_{n\ge0}) = \begin{cases} \{a_{n+1}\}_{n\ge0} & \text{(left shift)} \qquad \text{if } a_0 = 0, \\ \{1 - a_{n+1}\}_{n\ge0} & \text{(left shift with `flip')} \quad \text{if } a_0 = 1. \end{cases} \tag{37}$$

It follows that the itinerary $e(z) = e_0e_1e_2 \in \Sigma$ of $z = \sum_{n\ge0} a_n\lambda^n$ satisfies

$$e_0 = a_0 \quad \text{and for } n \ge 1: \quad e_n = \begin{cases} \mathtt{1} & \text{if } a_n \ne a_{n-1}, \\ \mathtt{0} & \text{if } a_n = a_{n-1}. \end{cases}$$

Conversely, $a_n = \mathtt{0}$ or $\mathtt{1}$ according to whether $e_0 \dots e_n$ contains an odd or even number of $\mathtt{1}$s[10]. Thus we obtain the following commutative diagram: If $\{a_n\}_{n\ge0}$ corresponds to

[10]This relates to the two standard ways of expressing itineraries real unimodal maps: by recording on which side of c the n-th image of z lies (as we do) or according to where $f^{\circ n}$ is locally increasing or decreasing, as is done in [**MiT**], see also [**CoE**]. The same difference we encounter for the itinerary

$$\begin{array}{ccc}
z \in A_\lambda & \xrightarrow{q_\lambda} & A_\lambda \\
e \big\downarrow & & \big\downarrow \; e \text{ is itinerary map} \\
e(z) \in \Sigma & \xrightarrow{\sigma} & \Sigma \\
\varphi \big\downarrow & & \big\downarrow \; \varphi(e)_n = \begin{cases} 1 & \text{if } \#\{i \le n : e_i = 1\} \text{ is odd}\} \\ 0 & \text{if } \#\{i \le n : e_i = 1\} \text{ is even}\} \end{cases} \\
\{a_n\}_{n\ge 0} \in \Sigma & \xrightarrow{Q} & \Sigma
\end{array}$$

the critical point, *i.e.*, $c_\lambda = \sum_{n\ge 0} a_n\lambda^n = \sum_{n\ge 0}(1-a_n)\lambda^n$, we have the itinerary

$$\star\nu_1\nu_2\cdots = e_0(c_\lambda)e_1(c_\lambda)e_2(c_\lambda)\dots, \text{where } \nu_n = \begin{cases} 1 & \text{if } a_n \ne a_{n-1}, \\ 0 & \text{if } a_n = a_{n-1}. \end{cases}$$

Let us relate this to the single function $g_\lambda(z) = \sum_n b_n z^n = \sum_n a_n z^n - \sum_n (1-a_n)z^n$ of $\mathcal{F}_\lambda$. As $b_n = 2a_n - 1 \in \{-1,1\}$, we conclude that

$$b_0 = 1 \quad \text{ and } \quad b_n = (-1)^{\#\{1\le i\le n \,:\, \nu_i = 1\}}. \tag{38}$$

In other words, if $\{b_n\}_{n\ge 0} \in \{-1,1\}^{\mathbb{N}^*}$ with $b_0 = 1$ is such that $g_\lambda(\lambda) = \sum_{n\ge 0} b_n\lambda^n = 0$, and $\sum_{n\ge 0}\tilde{b}_n\lambda^n \ne 0$ for all other sequence $\{\tilde{b}_n\}_{n\ge 0} \in \{-1,1\}^{\mathbb{N}^*}$ with $\tilde{b}_0 = 1$, then A_λ is a dendrite, and its kneading sequence is related to $\{b_n\}_{n\ge 0}$ by (38).

Solomyak [**Sol**] proved the following theorem:

30.12. **Theorem ($\mathcal{M}_1$ is uncountable)**
For every sequence $\{b_n\}_{n\ge 0}$ of the form

$$1,-1,1,1,1,-1,-1,-1,1,1,1,\underbrace{-1,\dots,-1}_{r_1},\underbrace{1,\dots,1}_{r_2},\underbrace{-1,\dots,-1}_{r_3},\underbrace{1,\dots,1}_{r_4},\dots$$

where each $r_i \in \{3,4\}$, the function $g_b(z) = \sum_{n\ge 0} b_n z^n$ has a zero λ with belongs to $\mathcal{M}_1$, and every different sequence $\{b_n\}_{n\ge 0}$ has a different $\lambda \in \mathcal{M}_1$.

Computing the kneading sequences of the maps q_λ for λ as in this proposition, we find that

$$\nu_\lambda = 1100100100100\dots$$

$e(x)$ for a point $x \in [0,1]$ with respect to the tent map $T(x) = \min\{2x, 2(1-x)\}$ and the dyadic expansion of x. See [**BB**, Sections 12.2-12.5] for some further interesting applications.

satisfies $\sigma^{\circ k}\nu_\lambda <_{lex} \nu_\lambda$ for all $k \geq 1$ and lexicographical order $<_{lex}$. Combining Lemma 5.20[11], and especially formula (3), with Proposition 5.19 and Lemma **??** we can derive that

- the α-fixed point in A_λ has three arms;
- there are no other periodic branch points;
- the critical value $q_\lambda(c_\lambda)$ is an endpoint of A_λ.

This suggests following questions for arbitrary $\lambda \in \mathcal{M}_1$:

- Is $q_\lambda(c_\lambda)$ always an endpoint of A_λ?
- Is there always exactly one cycle of periodic branchpoints in A_λ (on which all other branchpoints are eventually mapped)?

[11]Eroglu, Rohde and Solomyak showed that the property $\sigma^{\circ k}\nu_\lambda <_{lex} \nu_\lambda$ for all $k \geq 1$ implies that ν is admissible, giving another proof of Lemma 5.20. Namely, if $\vartheta = \sum_{n\geq 1} 2^{-n}(1-\nu_n)$, then $g^{\circ k}(\vartheta) \in (\vartheta, \frac{1}{2}) \cup (\frac{1}{2}+\vartheta, 1) \subset \mathbb{S}^1$ for all $k \geq 1$ for the angle doubling map g. In particular, ν is equal to the itinerary of ϑ with respect to the partition $\{(\frac{\vartheta}{2}, \frac{\vartheta+1}{2}), (\frac{\vartheta+1}{2}, \frac{\vartheta}{2})\}$ of $\mathbb{S}^1$. In other words, ν is the itinerary of an external parameter angle ϑ.

31. Trees and the Mandelbrot Set

As we have seen in the main text, not every abstract Hubbard tree is realized by a complex polynomial. This gives rise to virtual parts of the Mandelbrot trees, for which a description by internal address etc. is still possible. Work by Kauko specifies where these virtual limbs branch off from the Mandelbrot (and Multibrot) set, and describes the similarities of virtual and true limbs of the Mandelbrot set.

31.1. Kauko. HB: Definition Multibrot set, picture of visible trees? I'm also not sure yet how to either include or refer to the appropriate parts of the Lau & Schleicher paper.

In [**Kau1, Kau2**], Kauko is investigating the Multibrot sets by looking at their tree structure from a combinatorial point of view. She continues the study of *visible trees* of [**LS**], which is based on a partial order on hyperbolic components. A hyperbolic component W_1 is smaller than a hyperbolic component W_2, written as $W_1 \prec W_2$, if W_2 is contained in the wake of W_1 (see Subsection 9.2 for the definition of wakes). The *combinatorial arc* $[W_1, W_2]$ between W_1 and W_2 is the set of all hyperbolic components W with $W_1 \prec W \prec W_2$ including W_1 and W_2. The wake W_2 is called *visible* from W_1 if none of the hyperbolic components in $[W_1, W_2]$ has period less than W_2. The set of hyperbolic components visible from a base component W form a *visible tree*. The tree starting at internal angle p/q of W is denoted by $\mathcal{T}_{p/q}$. Two visible trees $\mathcal{T}_{p/q}$, $\mathcal{T}_{p'/q'}$ are *combinatorially equivalent* if there is a homeomorphism between them such that each element of $\mathcal{T}_{p/q}$ with period n is mapped to an element of $\mathcal{T}_{p'/q'}$ with period $(q'-q)k+n$ for some $k \in \mathbb{N}$. Note that the definitions as stated above refer to the Mandelbrot set, for Multibrot sets of arbitrary degree $d > 2$, one has to replace the term "component" by "sector".

The main result in [**Kau1**] is that in general two visible trees of a given base sector are not combinatorially equivalent, answering a long standing conjecture in the negative. Kauko also gives a new proof of the *weak translation principle* in the quadratic case, which says that if W is a hyperbolic component of period k and m is the smallest period in its wake then the minimal period of components in any visible tree $\mathcal{T}_{p/q}$ is $(q-2)k+m$. Her proofs are based on calculating the *widths* of wakes of hyperbolic components (see Subsection 9.2 and in particular Proposition 9.24 for a definition).

The paper [**Kau2**] is closely related to the investigation of kneading sequences of Sections 5 to 8. Indeed, Kauko is studying formal kneading sequences and formal addresses, where a *formal kneading sequence* is any infinite sequence $(a_i)_{i\geq 1}$ with $a_i \in \{0, \dots, d-1\}$ and $a_1 \neq 0$, and a *formal address* is a (finite or infinite) sequence $n_1(s_1) \mapsto n_1(s_1) \mapsto \dots$ with $n_1 = 1$, $n_i < n_{i+1}$ and $s_i \in \{1, \dots, d-1\}$ for all i. Kauko determines a class of kneading sequences that are not generated by any unicritical polynomial. She calls these kneading sequences *shadow components* because they are bifurcation sequences similar to the kneading sequences generated by satellite components in the

Multibrot sets but not contained in there. In Section 22, we show that in fact all non-realizable sequences are of this type.

Kauko defines a partial order on the set of all formal addresses which allows her to extend the concept of visible trees to this abstract setting. Using the algorithm given in Corollary 11.7 to determine the hyperbolic components of lowest period between $W_1 \prec W_2$, Kauko shows that in this abstract setting the translation principle holds true, *i.e.*, for any address any two visible trees $\mathcal{T}_{\cdot/q}$ and $\mathcal{T}_{\cdot/q'}$ [12] are combinatorially equivalent.

The second part of the paper is devoted to estimate the width of wakes of hyperbolic sectors. From these results she derives the following statements about the realization of kneading sequences and formal addresses:

- If $\overline{c_1 \dots c_k}$ is the kneading sequence of an (existing) narrow hyperbolic component W, then any sequence $\overline{(c_1 \dots c_k)^{q-1} b_1 \dots b_n}$ for $n \leq k$ is realized above a q-satellite of W.
- Every formal address with at most four entries is realized.
- Under the assumption that all non-existing kneading sequences are shadow components, it holds true that for any strictly increasing sequence of integers $1 \mapsto n_2 \mapsto \dots$, there are integers $s_i \in \{1, 2\}$ such that $1(s_1) \mapsto n_2(s_2) \mapsto \dots$ is realized as internal address in the Multibrot set $\mathcal{M}_3$ of degree 3.

[12] In the abstract setting, it does not make sense to consider different numerators to a given denominator because they cannot be distinguished in terms of kneading sequences.

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[Mix] John Milnor, *Local connectivity of Julia sets: expository lectures.* In: Tan Lei (ed.), *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, Cambridge (2000), 67–116.

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[IMG] This is used by Alex' "Other Further" section; give precise reference!

[ERS] Ilgar Eroglu, Steffen Rohde, Boris Solomyak, In preparation.

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[S4] Dierk Schleicher, *Internal addresses in the Mandelbrot set and Galois groups of polynomials*, manuscript (2007). ArXiv arXiv:math/9411238.

[JR] Leo Jonker, David Rand, *The periodic orbits and entropy of certain maps of the unit interval,* J. London Math. Soc. **22** (1980), 175–181. HB: Used for $*$-products, but I doubt if this is the best reference

Update references.

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C. Old Versions

32. Other Languages and Further Results

32.1. Still to do. Growing trees: Blokh/Levin: AK -¿ HB
Douady: algorithms: DS
Thurston's theorem: DS
Atela: DS (also appendix of Goldberg/Milnor and appendix of [**Mi2**])
Maps on trees; Misiurewicz, Llibre, Alseda: HB (Paper von Nuria und Alseda?)
Im Anhang: jeweils Namen der Pioniere in den Titeln erwhnen oder nicht?

HB: New are the sections on Keller, Penrose, pairs of linear maps, Yoccoz puzzles and $*$-products (this last one is really very tentative). The bit on real admissibility is rewritten. For some reason I couldn't gt some of the pictures on the right spot. I have the following list of queries, wishes and things to do still:

- The single text needs to be split up into separate chapters.
- To be added: text on Thurston's proof of No Nonwandering Triangles. Apparently, it also holds for **uni**critical polynomials. Was that proved elsewhere, other than/before Penrose in an unpublished 1994 preprint [**Pen2**]?
- To be added: Thurston's approach to rational maps, Thurston obstructions, and what Levy cycles are.
- To be added: section on Orsay notes, Posdronasvili.
- Where to put Kiwi's work: in the section on Laminations or the one on Portraits?
- The little section on $*$-products needs to be expanded, I suppose. Are $*$-products to used in Section 10 on renormalization? At any rate, I better wait for Section 10 to be finished before deciding what to do with the bit on $*$-products
- I included some more Milnor & Thurston type of kneading theory ($i.e.,$ the formal power series approach), as it is, on the face of it, much such a powerful method. Not sure if this is really restricted to interval maps, or if many of their results carry over to Hubbard trees. So I put some remarks in, but it could always be ore, or less...
- I find that the parts on the Partial Translation Principle (Kauko, Keller) should have proper references to the Lau & Schleicher paper, or however Lau & Schleicher is going to be covered in our book. Same for visibility. I'd be happy to hear your opinion.
- Still need the picture for Yoccoz puzzles: I'd like the one of the puzzle for the Fibonacci unimodal map, in [**HY**]. Can any of you make it?
- I've never seen Faught's work on Yoccoz Tableaux, but maybe it needs to be included in the current Section 27.1 on Tableaux.
- Should we be using Milnors notation for $f^{\circ n}$ for negative n, i.e., $f^{\circ -n}$ or so? Currently we don't.
- Was Penrose thesis 1990 or 1994? Our list of ref. says 1994 (probably correct), but my copy says 1990, and so does the Math Genealogy Project
- Always in demand: more pictures. Especially the more technical bits from the Portraits Section need some
- Who was the first to come with the non-admissible kneading sequence $101100\ldots$? It's in Penrose, Keller, Kauko, etc. Also in the Orsay notes???
- In Section 16 (on biaccessibility) I want a lemma added along the lines: If $x = e(p)$ is the itinerary of p, and the ρ_e-function admits n disjouint grant orbits in $\mathbb{N}^*$, then $T \setminus \{p\}$ consists of exactly n components. This n is actually what Penrose calls the *valency*.

- I wish, now that thre are so many subsections in this text, that subsection had a different, somewhat more conspicuous format. Maybe Section should get a bigger font, so that subsctions ca get the size of current section

Symbol $\boxed{\mathbb{N}}$ used: 1 times.