G_2-STRUCTURES AND TWISTOR THEORY

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University of Cambridge

- Joint with Tod, Godliński, Sokolov, Doubrov.
- Bulids on Calyey, Sylvester, Penrose, Hitchin, Bryant, Bailey&Eastwood, Doubrov, Godliński&Nurowski, Kryński.

Geometry of plane conics

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    - $SL(2)\subset GL(5)$. Invariant $I=12a_4a_0-3a_3a_1+(a_2)^2$.

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$$x = \frac{1-\lambda^2}{1+\lambda^2}, \quad y = \frac{2\lambda}{1+\lambda^2}.$$

  - Conformal structure on M: $V \in \Gamma(TM)$ is null iff $I(V) = 0$.

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  - Mixture of *old* and *new*: Classical invariant theory (Young, Sylvester), algebraic geometry, twistor theory (Penrose, Hitchin).

G_2 structures and Fernandez–Gray types

- $G_2 \subset SO(7), \quad g = (e^1)^2 + \cdots + (e^7)^2,$

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

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- Transvectants (Grace, Young 1903), or two component spinors (Penrose).

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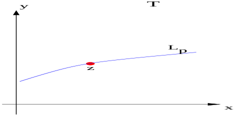
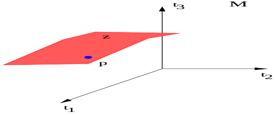
$GL(2,\mathbb{R})$ structures from ODEs.

- Assume that the space of solutions M to the 7th order ODE

$$y^{(7)} = F(x, y, y', \dots, y^{(6)})$$

has a $GL(2,\mathbb{R})$ structure such that normals to surfaces $y = y(x;t)$ in M have root with multiplicity 6. Then F satisfies five contact–invariant conditions $W_1[F] = \cdots = W_5[F] = 0$.





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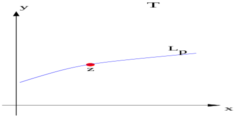

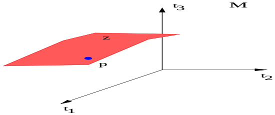

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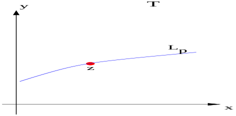

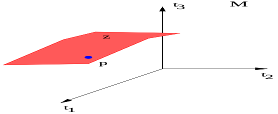

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TWISTOR THEORY

- Family of rational curves L_t parametrised by $t \in M$. $x \rightarrow (x, y(x;t))$ with self–intersection number six in a complex surface Z. Normal vector

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- $H^1(L, N(L)) = 0$. Kodaira Theory: $T_t M \cong H^0(L_t, N(L_t))$.

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- In practice: $f(x, y, t_\alpha) = 0$ with rational parametrisation $x = p(\lambda, t_\alpha), y = q(\lambda, t_\alpha)$. Polynomial in λ giving rise to a null vector is given by

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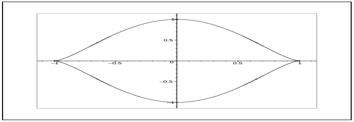

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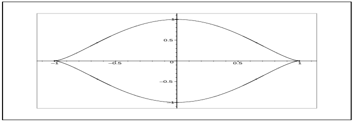

- Two double points and one irregular quadruple point at ∞. $g = 0$.

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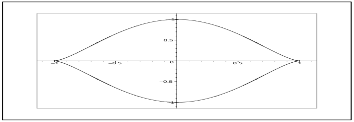

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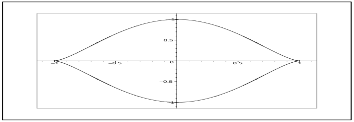

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- Closed Riemannian G_2 structure - explicit but messy.

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- Contact geometry: $(x, y) \in Z$, $(x, y, z) \in P(TZ)$, contact form $\omega = dy - zdx$. Generators of contact transformations

$$X_H = -(\partial_z H)\partial_x + (H - z\partial_z H)\partial_y + (\partial_x H + z\partial_y H)\partial_z,$$

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- 7th order ODE with 10D contact symmetries (submaximal ODE)

$$10(y^{(3)})^3 y^{(7)} - 70(y^{(3)})^2 y^{(4)} y^{(6)} - 49(y^{(3)})^2 (y^{(5)})^2$$
$$+280(y^{(3)})(y^{(4)})^2 y^{(5)} - 175(y^{(4)})^4 = 0, \quad \text{(Noth 1904).}$$

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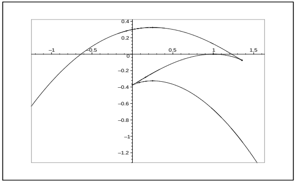
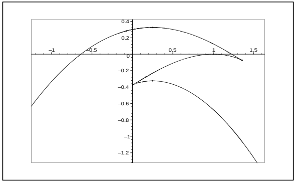
- How about G_2 structure? Two real forms of $Sp(4)/SL(2)$, one of which is a Riemannian homogeneous space $SO(5)/SO(3)$ (Bryant 1987).

$$\left(c_4y + c_1 + c_2x + c_3x^2\right)^3 + 3\,\left(c_4y + c_1 + c_2x + c_3x^2\right)$$
$$\left(3\,(c_5x + c_6)^4 - 6\,(c_5x + c_6)^2 (1 - c_7x)^2 - (1 - c_7x)^4\right)$$
$$+12\,(c_5x + c_6)\left(3\,(c_5x + c_6)^4 (1 - c_7x) + (1 - c_7x)^5\right) = 0.$$



Example 3: Weak G_2 from submaximal ODE

$$\left(c_4y+c_1+c_2x+c_3x^2\right)^3+3\left(c_4y+c_1+c_2x+c_3x^2\right)$$
$$\left(3\left(c_5x+c_6\right)^4-6\left(c_5x+c_6\right)^2\left(1-c_7x\right)^2-\left(1-c_7x\right)^4\right)$$
$$+12\left(c_5x+c_6\right)\left(3\left(c_5x+c_6\right)^4\left(1-c_7x\right)+\left(1-c_7x\right)^5\right)=0.$$



Discriminant of this cubic (in y) is a 3rd power of a quartic with equianharmonic cross–ratio.

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  - Sextic (relevant in this talk) -??