Effective Cartan-Tanaka Connections on $\mathscr{C}^6$-smooth Strongly Pseudoconvex Hypersurfaces M^3 in $\mathbb{C}^2$

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I. Gaussian curvature of surfaces

II. Spherical real analytic hypersurfaces $M^3 \subset \mathbb{C}^2$

III. Cartan connections and curvature functions

IV. Explicit curvatures and coframes

V. Perspectives on explicit Cartan CR connections

"Cartan connections,

Geometry of Homogeneous Spaces, and Dynamics"

Organized by A. Čap, C. Frances and K. Melnick

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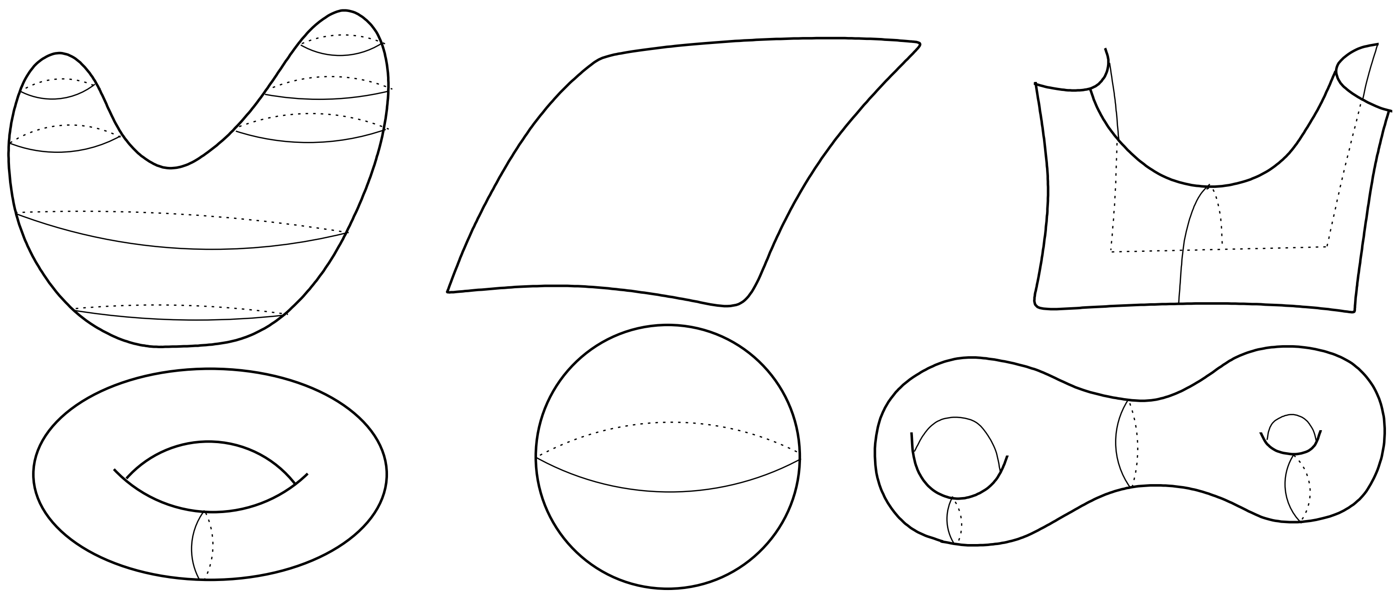
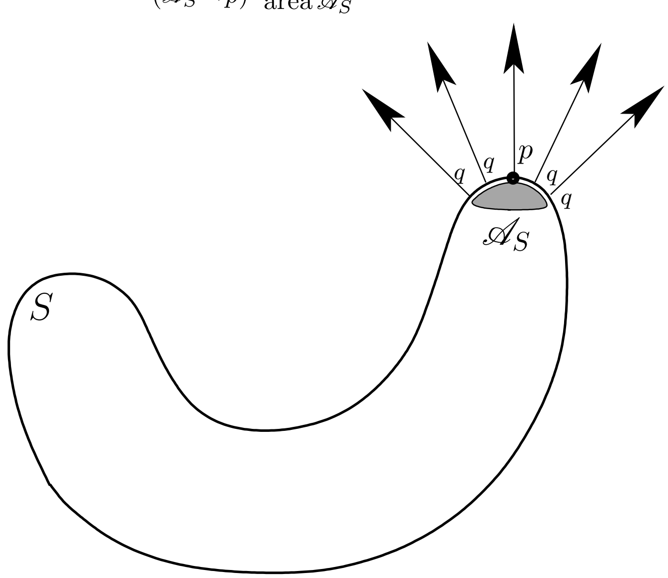
I – Gaussian curvature of surfaces

- **Surfaces S^2 embedded in $\mathbb{R}^3$:**

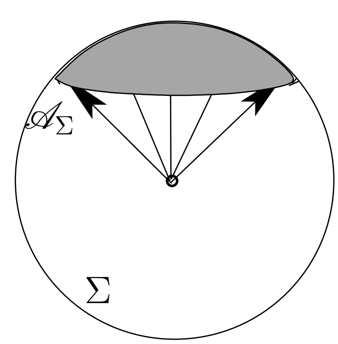


- **Gaussian curvature:** Defined *extrinsically* as the quotient of two infinitesimal areas:

$$\text{Curvature} = \lim_{(\mathscr{A}_S \to p)} \frac{\text{area}\, \mathscr{A}_\Sigma}{\text{area}\, \mathscr{A}_S}$$



Auxiliary unit sphere:



- **Local representation of the surface as a graph:**

$$z = z(x, y)$$

- **Extrinsic formula for the curvature:**

$$\boxed{\text{Gaussian curvature} = \frac{z_{xx}\, z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}}$$

- **Gauss' 1816 Preisschrift:** Using geodesic triangles:

The curvature of a surface remains unchanged when it undergoes any deformation which leaves invariant the length of curves.

[Infinitesimal isometries]

- **Principle of sufficient reason (Leibniz):** Curvature should express in terms of the *intrinsic metric*:

$$ds^2 = E(u,v)\, du^2 + 2F(u,v)\, dudv + G(u,v)\, dv^2.$$

- **Hard calculation performed by Gauss:**
  - ☐ start out from an intrinsic parametrization:

$$(u,v) \longmapsto \big(x(u,v),\ y(u,v),\ z(u,v)\big);$$

  - ☐ express accordingly the metric coefficients:

$$E = x_u^2 + y_u^2 + z_u^2,$$
$$F = x_u x_v + y_u y_v + z_u z_v,$$
$$G = x_v^2 + y_v^2 + z_v^2;$$

  - ☐ **eliminate** $z = z(x,y)$ from **extrinsic** curvature:

$$\frac{z_{xx}\, z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}.$$

• **Theorema Egregium:** The (Gaussian) curvature of a surface is *intrinsic because* it expresses as the following explicit rational differential expression in the second-order jet of the three elements E, F, G:

$$\text{curvature} = \frac{1}{4\,(EG-F^2)^2}\bigg\{ E\left[\frac{\partial E}{\partial v}\cdot\frac{\partial G}{\partial v} - 2\,\frac{\partial F}{\partial u}\cdot\frac{\partial G}{\partial v} + \frac{\partial G}{\partial u}\cdot\frac{\partial G}{\partial u}\right]$$
$$+\,F\left[\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial v} - \frac{\partial E}{\partial v}\cdot\frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial v}\cdot\frac{\partial F}{\partial v} + 4\,\frac{\partial F}{\partial u}\cdot\frac{\partial F}{\partial v} - 2\,\frac{\partial F}{\partial u}\cdot\frac{\partial G}{\partial u}\right]$$
$$+\,G\left[\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial u}\cdot\frac{\partial F}{\partial v} + \frac{\partial E}{\partial v}\cdot\frac{\partial E}{\partial v}\right]$$
$$+\,2\,\big(EG-F^2\big)\left[-\frac{\partial^2 E}{\partial v^2} + 2\,\frac{\partial^2 F}{\partial u\partial v} - \frac{\partial^2 G}{\partial u^2}\right]\bigg\}.$$

• **Cartan's coframe reformulation:**

$$ds^2 = \big(\theta_1\big)^2 + \big(\theta_2\big)^2.$$

thanks to a Gram-Schmidt orthonormalization, with:

$$\theta^1 = A(u,\,v)\,du + B(u,\,v)\,dv,$$
$$\theta^2 = C(u,\,v)\,du + D(u,\,v)\,dv.$$

• **Forget about expliciteness:** $A,\ B,\ C,\ D$ could be computed in terms of $E,\ F,\ G$.

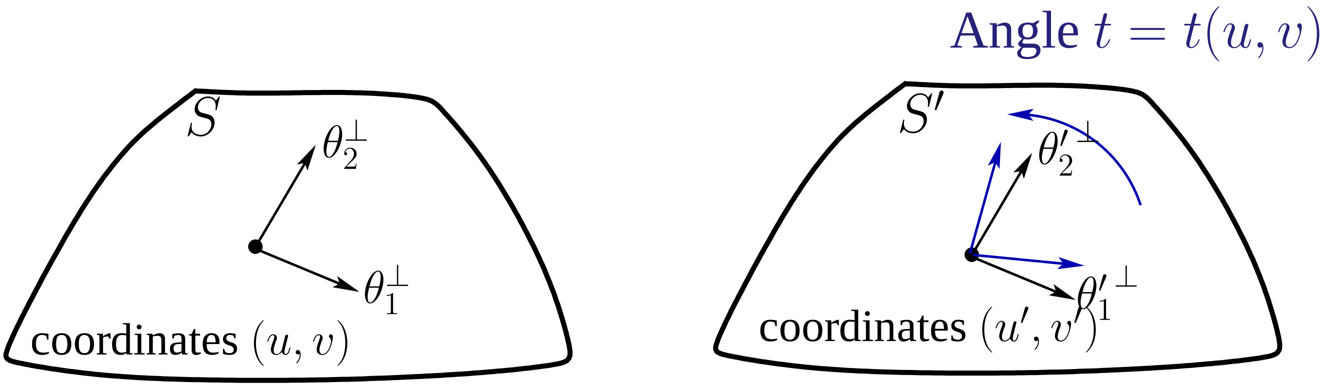
• **Equivalence to another surface metric:**

$$ds'^2 = \big(\theta_1'\big)^2 + \big(\theta_2'\big)^2$$

• **When there is an isometry $S \to S'$ from the surface S to another surface S':**

$$\begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

with $t = t(u, v)$ being a certain (unknown) function.



• **Lifted coframe:** Set t as a new independent variable:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

• **Advantage of Cartan's approach:** Differential invariance is set up at the beginning:

$$\omega, \;\; d\omega \quad \text{similar to} \quad \omega' = \omega, \;\; d\omega' = d\omega.$$

• **Absorption of torsion:**

$$d\omega^1 = -\pi \wedge \omega^2 \quad \text{and} \quad d\omega^2 = \pi \wedge \omega^1.$$

• **Apply differential operator d:**

$$0 = dd\omega^1 = -d\pi \wedge \omega^1 \quad \text{and} \quad 0 = dd\omega^2 = d\pi \wedge \omega^2.$$

- **Deduce from Cartan's lemma:** There exists a certain function κ so that:

$$d\pi = \underbrace{\kappa}_{\text{Gaussian curvature}} \cdot \omega^1 \wedge \omega^2.$$

- **Summary:**

☐ Explicit differential algebra was in Gauss 1827.

☐ Surfaces $S^2 \subset \mathbb{R}^3 =$ easiest case of Cartan's theory of the equivalence problem.

☐ Cartan, Chern, Tanaka: usually, they leave aside **Gaussian-like explicit computations**, which are hard.

☐ Our goal today is to compute explicitly Cartan connections, coframes and curvatures, for known structures.

☐ Our future goal is to construct Cartan-Tanaka connections for several new — yet unstudied — Cauchy-Riemann structures.

- **The plan of the talk is:**

☐ Gauss (done).

☐ Second order differential equations: $y_{xx} = F(x, y, y_x)$.

☐ Hypersurfaces $M^3 \subset \mathbb{C}^2$ equivalent to the sphere.

☐ Cartan-Tanaka connections for such $M^3 \subset \mathbb{C}^2$.

☐ Connections for other Cauchy-Riemann structures.

II – Spherical real analytic hypersurfaces

• **Start out:** A refresher about second order ordinary differential equations.

• **Work with:** Either real or complex numbers:

$$\mathbb{K} := \mathbb{R} \textbf{ or } \mathbb{C}.$$

• **Projective group:** Let $\mathsf{PGL}_2(\mathbb{K})$ be the projective group of (Möbius) transformations of $\mathbb{P}^2(\mathbb{K})$:

$$(x, y) \longmapsto \left(\frac{a_1 + b_1 x + c_1 y}{1 + \lambda\, x + \mu\, y}, \ \frac{a_2 + b_2 x + c_2 y}{1 + \lambda\, x + \mu\, y}\right),$$

and let $\mathfrak{pgl}_2(\mathbb{K})$ be its Lie algebra, of dimension 8.

• **Élie Cartan 1924:** [Bulletin des Sciences Math.]: Construction of a unique $\mathfrak{pgl}_2(\mathbb{K})$-valued (Cartan) connection associated to any second-order differential equation:

$$y_{xx} = F(x, y, y_x),$$

with $x, y \in \mathbb{K}$. [Doubrov-Komrakov].

• **Lie-Tresse two principal differential invariants:**

$$\mathscr{I}^1 := F_{y_x y_x y_x y_x}$$

$$\mathscr{I}^2 := \mathsf{DD}\big(F_{y_x y_x}\big) - F_{y_x}\,\mathsf{D}\big(F_{y_x y_x}\big) - 4\,\mathsf{D}\big(F_{y y_x}\big) + \\ + 6\,F_{yy} - 3\,F_y\,F_{y_x y_x} + 4\,F_{y_x}\,F_{y y_x},$$

where the *total differential operator* is:

$$\mathsf{D} := \partial_x + y_x\,\partial_y + F(x, y, y_x)\,\partial_{y_x}.$$

• **Special case:** When invariants vanish identically:

$$0 \equiv \mathscr{I}^1 \equiv \mathscr{I}^2.$$

• **Equivalently:** The curvature of Cartan's projective connection vanishes identically.

Corollary. [Lie 1883] *Such a second-order differential equation:*

$$y_{xx} = F(x, y, y_x)$$

is equivalent to the Newtonian free particle:

$$Y_{XX} = 0$$

under some point transformation:

$$(x, y) \longmapsto \big(X(x, y), \, Y(x, y)\big)$$

if and only if:

$$0 = \mathscr{I}^1 = \mathscr{I}^2.$$

• **Further explorations/modernizations:**

[Lie; Tresse; Koppisch; Gonzalez-Lopez; Grissom-Thompson-Wilkens; Hsu-Kamran; Romanovsky; Nurowski-Sparling; Crampin-Saunders; Doubrov-Komrakov].

• **Open question in CR geometry:** Characterize local biholomorphic equivalence of a strongly pseudoconvex real hypersurface $M^3 \subset \mathbb{C}^2$ to the standard unit sphere:

$$1 = z\overline{z} + w\overline{w}.$$

explicitly in terms of some defining function for M.

• **Question mentioned/considered by:**

[Vitushkin, Isaev, Ezhov, Schmalz, McLaughlin]

• **Strong mathematical links:**

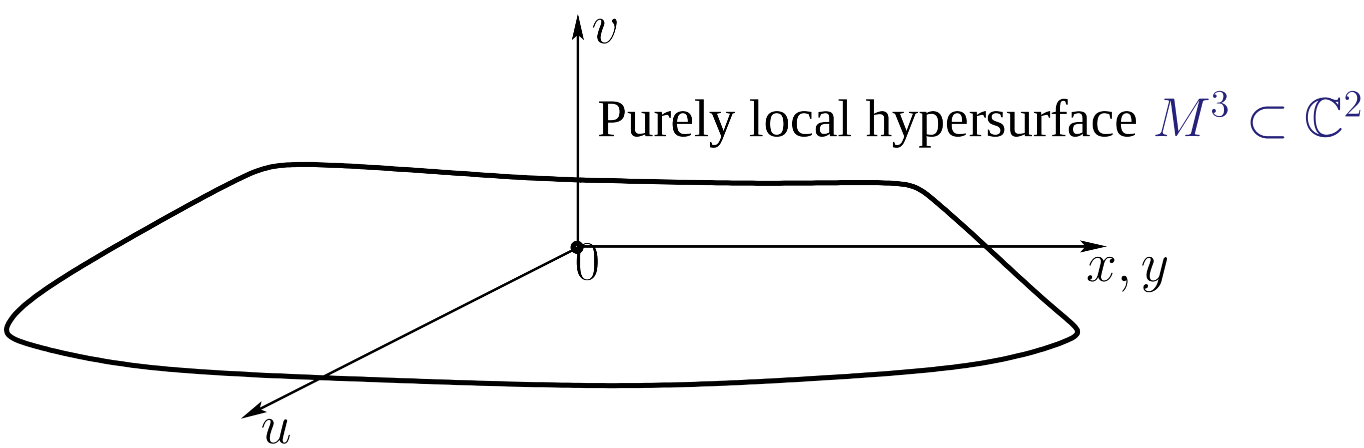
Integrable PDE systems	$\longleftrightarrow$	**Cauchy-Riemann geometry**
Second-order differential equations	$\longleftrightarrow$	**Real hypersurfaces $M^3 \subset \mathbb{C}^2$**

• **Hypersurfaces $M^3 \subset \mathbb{C}^2$ are graphs of the form:**

$$v = \varphi(x, y, u),$$

in some local holomorphic coordinates:

$$(z, w) = (x + iy,\ u + iv).$$



• **Rewrite the equation of M as:**

$$\frac{w-\overline{w}}{2i} = \varphi\Big(x, y, \frac{w+\overline{w}}{2}\Big).$$

- **When the graphing function φ is real analytic:** May solve with respect to w:

$$w = \Theta\big(z, \overline{z}, \overline{w}\big).$$

- **Segre (Beniamino) 1931** [Lie, long before]**:** Consider $w = w(z)$ as a function of z and differentiate:

$$w(z) = \Theta\big(z, \overline{z}, \overline{w}\big)$$
$$w_z(z) = \Theta_z\big(z, \overline{z}, \overline{w}\big).$$

- **Assume M is strongly pseudoconvex at the origin:**

$$w(z) = \overline{w} + i\, z\overline{z} + \mathrm{O}(3),$$
$$w_z(z) = \overline{z} + \mathrm{O}(2).$$

- **Hence may solve using implicit function theorem:**

$$\overline{z} = \zeta\big(z,\, w(z),\, w_z(z)\big),$$
$$\overline{w} = \xi\big(z,\, w(z),\, w_z(z)\big).$$

- **Segre (Beniamino) 1931** [Webster 1977]**:** Associate to any *real analytic* strongly pseudoconvex Cauchy-Riemann hypersurface $M^3 \subset \mathbb{C}^2$ a unique second-order ordinary differential equation by substituting the parameters $\overline{z}$ and $\overline{w}$ in the second derivative:

$$\begin{aligned} w_{zz}(z) &= \Theta_{zz}\big(z,\, \overline{z},\, \overline{w}\big) \\ &= \Theta_{zz}\big(z,\, \zeta\big(z,\, w(z),\, w_z(z)\big),\, \xi\big(z,\, w(z),\, w_z(z)\big)\big) \\ &=: \Phi\big(z,\, w(z),\, w_z(z)\big). \end{aligned}$$

• **Élie Cartan 1932 just after Segre 1931:** Construction of a natural $\mathfrak{pgl}_2(\mathbb{R})$-valued connection associated to any strongly pseudoconvex real hypersurface $M^3 \subset \mathbb{C}^2$.

• **Redone with some variations by:** [Chern-Moser; Jacobowitz; Yamaguchi; Nurowski-Sparling]

• **Fact:** None of these works provide curvatures or coframes explicitly in terms of a graphing function $\varphi(x, y, u)$ for $M^3 \subset \mathbb{C}^2$.

• **Paradox:** The $\mathfrak{pgl}_2(\mathbb{C})$-valued connection, coframe, curvature of the associated second order differential equation are known explicitly in the literature.

• **Reason due to differential algebra swelling:**

□ for a differential equation $w_{zz} = \Phi(z, w, w_z)$, the connection depends upon the 4-th order jet $J^4_{z,w,w_z}\Phi$

□ for a hypersurface $w = \Theta(z, \overline{z}, \overline{w})$, the data depend upon the **sixth-order** jet $J^{\mathbf{6}}_{x,y,u}\Theta$

□ furthermore, computations explode because one has to divide by the Levi-form.

• **Standard unit 3-sphere $S^3 \subset \mathbb{C}^2$:**

$$1 = z\overline{z} + w\overline{w}$$

• **Recall the Cayley transform:**

$$(z, w) \longmapsto \left(\tfrac{i\,z}{1+w},\ \tfrac{i-iw}{1+w}\right) =: (z', w')$$

which has inverse: $\quad (z', w') \longmapsto \left(\tfrac{2z'}{i+w'},\ \tfrac{i-w'}{i+w'}\right)$

• **This transform shows that:** $S^3 \backslash \{\infty\}$ is biholomorphically equivalent to the *Heisenberg sphere*:

$$w' = \overline{w}' + 2iz'\overline{z}'.$$

• **Fact:** This *graphed* model is more convenient to work with.

Proposition. [Easy] *A strongly pseudoconvex local real analytic hypersurface:*

$$w = \Theta\big(z, \overline{z}, \overline{w}\big)$$

is locally biholomorphic to a piece of the Heisenberg sphere:

$$w = \overline{w} + 2i\, z\overline{z}$$

if and only if its associated second-order ordinary complex differential equation:

$$w_{zz}(z) = \Phi\big(z,\ w(z),\ w_z(z)\big)$$

is locally equivalent to the Newtonian free particle:

$$w_{zz}(z) = 0,$$

if and only if:

$$0 \equiv \mathscr{I}^1 \equiv \mathscr{I}^2.$$

Theorem. [M., 2010] *An arbitrary real analytic hypersurface $M \subset \mathbb{C}^2$ which is Levi nondegenerate and has a complex defining equation of the form:*

$$w = \Theta\big(z,\, \overline{z},\, \overline{w}\big)$$

in some system of local holomorphic coordinates $(z, w) \in \mathbb{C}^2$ is equivalent to the Heisenberg sphere if and only if *its graphing complex function Θ satisfies the following explicit sixth-order algebraic partial differential equation:*

$$0 \equiv \left(\frac{-\,\Theta_{\overline{w}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}}\frac{\partial}{\partial \overline{z}} + \frac{\Theta_{\overline{z}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}}\frac{\partial}{\partial \overline{w}}\right)^2 \big[\mathsf{AJ}^4(\Theta)\big]$$

identically in $\mathbb{C}\big\{z, \overline{z}, \overline{w}\big\}$, where:

$$\begin{aligned}
\mathsf{AJ}^4(\Theta) := \frac{1}{[\Theta_{\overline{z}}\Theta_{z\overline{w}} - \Theta_{\overline{w}}\Theta_{z\overline{z}}]^3}\Bigg\{ & \Theta_{zz\overline{z}\overline{z}}\left(\Theta_{\overline{w}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}}\end{vmatrix}\right) - \\
& - 2\Theta_{zz\overline{z}\overline{w}}\left(\Theta_{\overline{z}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}}\end{vmatrix}\right) + \Theta_{zz\overline{w}\overline{w}}\left(\Theta_{\overline{z}}\Theta_{\overline{z}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}}\end{vmatrix}\right) + \\
& + \Theta_{zz\overline{z}}\left(\Theta_{\overline{z}}\Theta_{\overline{z}}\begin{vmatrix}\Theta_{\overline{w}} & \Theta_{\overline{w}\overline{w}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{w}\overline{w}}\end{vmatrix} - 2\Theta_{\overline{z}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{w}} & \Theta_{\overline{z}\overline{w}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\overline{w}}\end{vmatrix} + \Theta_{\overline{w}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{w}} & \Theta_{\overline{z}\overline{z}} \\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\overline{z}}\end{vmatrix}\right) + \\
& + \Theta_{zz\overline{w}}\left(-\,\Theta_{\overline{z}}\Theta_{\overline{z}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}\overline{w}}\end{vmatrix} + 2\Theta_{\overline{z}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{z}\overline{w}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\overline{w}}\end{vmatrix} - \Theta_{\overline{w}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{z}\overline{z}} \\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\overline{z}}\end{vmatrix}\right)\Bigg\}
\end{aligned}$$

- **Proof:** Express the vanishing of the two curvatures:

$$0 \equiv \mathscr{I}^1 \equiv \mathscr{I}^2$$

in terms of $J^6_{z,\overline{z},\overline{w}}\Theta$ thanks to transfer formulas. □

• **Same open problem in higher dimensions:** Characterize when a Levi nondegenerate *real analytic* hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ with $\underline{n \geqslant 2}$:

$$w = \Theta\big(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n, \overline{w}\big)$$

is biholomorphic to the Heisenberg pseudo-sphere:

$$\frac{w-\overline{w}}{2i} = |z_1|^2 + \cdots + |z_q|^2 - |z_{q+1}|^2 - \cdots - |z_n|^2,$$

where $(q, n-q)$ is the signature of the Levi form of M.

• **Expected applications:** Complete classification of tube spherical hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$ whose Levi form has signature $(n, 0)$, $(n-1, 1)$, or $(n-2, 2)$ [Isaev, LNM 2020, Springer, May 2011].

• **Remind Chern-Moser 1974:** Construction of a natural projective $\mathfrak{pgl}_{n+1}(\mathbb{R})$-valued connection associated to such $M^{2n+1} \subset \mathbb{C}^{n+1}$.

• **Differential algebra obstacle:** Basic elements — coframe and curvatures — of this projective connection were never computed explicitly in terms of a defining function for the hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$: **still an open problem!**

• **Hachtroudi 1937** [PhD under Cartan]**:** Construction of a natural $\mathfrak{pgl}_{n+1}(\mathbb{K})$-valued connection associated to any completely integrable second-order PDE system:

$$y_{x_{k_1} x_{k_2}} = F_{k_1 k_2}(x_1, \ldots, x_n, y, y_{x_1}, \ldots, y_{x_n})$$
$$(k_1,\ k_2 = 1 \ldots n),$$

with $n \geqslant 2$.

• **Good news:** Contrary to Chern's, Hachtroudi's results are effective!

Theorem. [Hachtroudi 1937] *The curvature of the projective normal (Cartan) connection associated to the above PDE system vanishes if and only if the right-hand side functions F_{k_1,k_2} satisfy the following explicit differential system, which is* linear *in terms of their second-order derivatives :*

$$0 \equiv \frac{\partial^2 F_{k_1,k_2}}{\partial y_{x^{\ell_1}} y_{x^{\ell_2}}} -$$
$$- \tfrac{1}{n+2} \sum_{\ell'=1}^{n} \bigg(\delta_{k_1,\ell_1} \frac{\partial^2 F_{\ell',k_2}}{\partial y_{x^{\ell'}} \partial y_{x^{\ell_2}}} + \delta_{k_1,\ell_2} \frac{\partial^2 F_{\ell',k_2}}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell'}}} + \delta_{k_2,\ell_1} \frac{\partial^2 F_{k_1,\ell'}}{\partial y_{x^{\ell'}} \partial y_{x^{\ell_2}}} + \delta_{k_2,\ell_2} \frac{\partial^2 F_{k_1,\ell'}}{\partial y_{x^{\ell_1}} \partial y_{x^{\ell'}}} \bigg) +$$
$$+ \tfrac{1}{(n+1)(n+2)} \big[\delta_{k_1,\ell_1} \delta_{k_2,\ell_2} + \delta_{k_2,\ell_1} \delta_{k_1,\ell_2} \big] \sum_{\ell'=1}^{n} \sum_{\ell''=1}^{n} \frac{\partial^2 F_{\ell',\ell''}}{\partial y_{x^{\ell'}} \partial y_{x^{\ell''}}} \qquad \begin{matrix} (1 \leqslant k_1,\ k_2 \leqslant n) \\ (1 \leqslant \ell_1,\ \ell_2 \leqslant n) \end{matrix}.$$

• **Associate a PDE system to** $M^{2n+1} \subset \mathbb{C}^{n+1}$**:**

$$w(z) = \Theta\big(z, \overline{z}, \overline{w}\big),$$
$$w_{z_1}(z) = \frac{\partial \Theta}{\partial z_1}\big(z, \overline{z}, \overline{w}\big),\ \ldots\ldots,\ w_{z_n}(z) = \frac{\partial \Theta}{\partial z_n}\big(z, \overline{z}, \overline{w}\big).$$

- **Use Levi-nondegeneracy of M to solve:**

$$\overline{z}_1 = \zeta_1\big(z,\, w(z),\, w_z(z)\big),\ \dots,\ \overline{z}_n = \zeta_n\big(z,\, w(z),\, w_z(z)\big),$$
$$\overline{w} = \xi\big(z,\, w(z),\, w_z(z)\big).$$

- **Insert in all possible second-order derivatives:**

$$\begin{aligned} w_{z_{k_1}z_{k_2}}(z) &= \frac{\partial^2\Theta}{\partial z_{k_1}\partial z_{k_2}}\big(z,\, \overline{z},\, \overline{w}\big) \\ &= \frac{\partial^2\Theta}{\partial z_{k_1}\partial z_{k_2}}\Big(z,\, \zeta\big(z,\, w(z),\, w_z(z)\big),\ \xi\big(z,\, w(z),\, w_z(z)\big)\Big) \\ &=: \Phi_{k_1,k_2}\big(z,\, w(z),\, w_z(z)\big) \qquad\qquad (k_1, k_2 = 1 \cdots n), \end{aligned}$$

Proposition. [easy] *A Levi nondegenerate local real analytic hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ is locally biholomorphic to a piece of the Heisenberg pseudosphere if and only if its associated second-order PDE system is locally equivalent to the trivial second-order system:*

$$w'_{z'_{k_1}z'_{k_2}}(z') = 0 \qquad\qquad (1 \leqslant k_1,\, k_2 \leqslant n),$$

whose solutions are hyperplanes of $\mathbb{P}^{n+1}(\mathbb{C})$.

- **Summary:**

☐ Nobody yet is able to compute the Cartan-Chern-Moser $\mathfrak{pgl}_{n+1}(\mathbb{R})$-valued connection associated to a Levi nondegenerate real hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ *explicitly* in terms of its defining function.

☐ But for second-order PDE systems, this is done [Lie, Cartan, Hachtroudi] and less difficult.

☐ To know when hypersurfaces are locally equivalent to a piece of the standard unit sphere, it then suffices to express that Hachtroudi's curvature for the associated PDE system vanishes.

☐ When one writes down vanishing of Hachtroudi curvature in terms of Θ, one gets the following.

Theorem. [M., 2010] *An arbitrary local real analytic hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ with $\underline{n \geqslant 2}$ which is Levi nondegenerate is pseudospherical* if and only if *its complex graphing function Θ satisfies the following explicit nonlinear fourth-order system of partial differential equations:*

$$
\begin{aligned}
0 \equiv \sum_{\mu=1}^{n+1}\sum_{\nu=1}^{n+1} \Bigg[\Delta^{\mu}_{[0_{1+\ell_1}]} \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar t_\mu \partial \bar t_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar t^\mu \bar t^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar t^\tau} \bigg\} - \\
- \frac{\delta_{k_1,\ell_1}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell'}]} \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar t_\mu \partial \bar t_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar t^\mu \bar t^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar t^\tau} \bigg\} - \\
- \frac{\delta_{k_1,\ell_2}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell_1}]} \cdot \Delta^{\nu}_{[0_{1+\ell'}]} \bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar t_\mu \partial \bar t_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar t^\mu \bar t^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar t^\tau} \bigg\} - \\
- \frac{\delta_{k_2,\ell_1}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell'}]} \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar t_\mu \partial \bar t_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar t^\mu \bar t^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar t^\tau} \bigg\} - \\
- \frac{\delta_{k_2,\ell_2}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell_1}]} \cdot \Delta^{\nu}_{[0_{1+\ell'}]} \bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar t_\mu \partial \bar t_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar t^\mu \bar t^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{\ell'} \partial \bar t^\tau} \bigg\} + \\
+ \frac{1}{(n+1)(n+2)} \cdot \big[\delta_{k_1,\ell_1}\delta_{k_2,\ell_2} + \delta_{k_2,\ell_1}\delta_{k_1,\ell_2} \big] \cdot \\
\cdot \sum_{\ell'=1}^{n}\sum_{\ell''=1}^{n} \Delta^{\mu}_{[0_{1+\ell'}]} \cdot \Delta^{\nu}_{[0_{1+\ell''}]} \bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{\ell''} \partial \bar t_\mu \partial \bar t_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar t^\mu \bar t^\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{\ell''} \partial \bar t^\tau} \bigg\},
\end{aligned}
$$

for all pairs of indices (k_1, k_2) with $1 \leqslant k_1, k_2 \leqslant n$, and for all pairs of indices (ℓ_1, ℓ_2) with $1 \leqslant \ell_1, \ell_2 \leqslant n$.

III – Cartan connections and curvature functions

• **Summary:**

☐ Using known explicit projective connections on PDE systems, one can characterize local biholomorphic equivalence to the sphere.

☐ Cartan connections in CR geometry are not effective in terms of the graphing function(s). **Open problem!**

• **Ezhov-McLaughlin-Schmalz:**

[Notices of the AMS, **58** (2011), no. 1, 20–27]:

Construction of a normal, regular, Cartan-Tanaka $\mathfrak{pgl}_2(\mathbb{R})$-valued connection associated to any *real analytic* strongly pseudoconvex hypersurface $M^3 \subset \mathbb{C}^2$.

• **Comment 1:** This approach is alternative to Cartan 1932 and to Chern-Moser 1974.

• **Comment 2:** Ezhov-McLaughlin-Schmalz use M is *real analytic*.

• **Today:** **Improve this Notices of the AMS paper:** Joint works with M. Sabzevari (PhD) and M. Aghasi (co-supervisor):

arxiv.org/abs/1104.1509 "[AMS 2011]".

arxiv.org/abs/1104.5300 (joint with B. Alizadeh)

• **Assume only:** M is $\mathscr{C}^6$-smooth, not real analytic.

• **Arbitrary homogeneous space:** Let G be a Lie group with a closed subgroup H, and let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras.

• **Cartan geometry of type** (G, H)**:** A manifold M is a right principal H-bundle:

$$\pi\colon \mathscr{P} \longrightarrow M$$

together with a $\mathfrak{g}$-valued one-form ω on $\mathscr{P}$ satisfying:

(i) $\omega_p\colon T_p\mathscr{P} \longrightarrow \mathfrak{g}$ is an isomorphism for any $p \in \mathscr{P}$;

(ii) if $R_h(p) := ph$ is the right translation on $\mathscr{P}$ by $h \in H$, then for any such h:

$$R_h^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega;$$

(iii) $\omega(H^\dagger) = \mathsf{h}$ for every $\mathsf{h} \in \mathfrak{h}$, where:

$$H^\dagger|_p := \frac{d}{dt}\Big|_0 \big((R_{\exp(t\mathsf{h})}(p)\big)$$

is the left-invariant vector field on $\mathscr{G}$ corresponding to h.

• **Associated curvature** 2**-form:**

$$\Omega(X, Y) := d\omega(X, Y) + \big[\omega(X), \omega(Y)\big]_{\mathfrak{g}},$$

where X, Y are vector (fields) on $\mathscr{P}$.

• $\mathrm{Ad}(h^{-1})$**-equivariancy implies:** $\Omega(X, Y)$ vanishes if either X or Y is vertical.

• **Consequence:** Ω is fully represented by the associated *curvature function*:

$$\kappa \in \mathscr{C}^\infty\big(\mathscr{P},\, \Lambda^2(\mathfrak{g}^*/\mathfrak{h}^*) \otimes \mathfrak{g}\big)$$

which sends a point $p \in \mathscr{P}$ to the map:

$$\kappa(p)\colon\ (\mathfrak{g}/\mathfrak{h}) \wedge (\mathfrak{g}/\mathfrak{h}) \longrightarrow \mathfrak{g}$$

defined by:

$$\begin{aligned}\big(\mathsf{x}' \operatorname{mod} \mathfrak{h}\big) \wedge \big(\mathsf{x}'' \operatorname{mod} \mathfrak{h}\big) \longmapsto -\ \Omega_p\big(\omega_p^{-1}(\mathsf{x}'),\, \omega_p^{-1}(\mathsf{x}'')\big) &= \\ = -\,[\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}} + \omega_p\big(\big[\widehat{X}',\, \widehat{X}''\big]\big)&,\end{aligned}$$

where:

$$\widehat{X} := \omega^{-1}(\mathsf{x})$$

is the *constant field* on $\mathscr{P}$ associated to an $\mathsf{x} \in \mathfrak{g}$.

• **Lie algebra bases:** Denote:

$$r := \dim_{\mathbb{R}} \mathfrak{g}, \qquad n := \dim_{\mathbb{R}} \big(\mathfrak{g}/\mathfrak{h}\big),$$

whence $n - r = \dim_{\mathbb{R}} \mathfrak{h}$.

• **Suppose:** $r \geqslant 2$, $n \geqslant 1$, $n - r \geqslant 1$ so that $\mathfrak{g}$, $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}$ are all nonzero.

• **Pick up an adapted basis:**

$$\begin{aligned}\mathfrak{g} &= \operatorname{Span}_{\mathbb{R}}\big(\mathsf{x}_1, \dots, \mathsf{x}_n, \mathsf{x}_{n+1}, \dots, \mathsf{x}_r\big) \\ \mathfrak{h} &= \operatorname{Span}_{\mathbb{R}}\big(\mathsf{x}_{n+1}, \dots, \mathsf{x}_r\big),\end{aligned}$$

- **Expand accordingly the curvature function:**

$$\boxed{\kappa(p) = \sum_{1\leqslant i_1<i_2\leqslant n} \sum_{k=1}^{r} \kappa^k_{i_1,i_2}(p)\; \mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k.}$$

- **Space of k-cochains:**

$$\mathscr{C}^k := \Lambda^k(\mathfrak{g}^*/\mathfrak{h}^*) \otimes \mathfrak{g}.$$

- **Differential operator:** $\partial^k : \mathscr{C}^k \longrightarrow \mathscr{C}^{k+1}$ defined by:

$$(\partial^k\Phi)\big(\mathsf{z}_0, \mathsf{z}_1, \dots, \mathsf{z}_k\big) := \sum_{i=0}^{k} (-1)^i \big[\mathsf{z}_i,\, \Phi(\mathsf{z}_0, \dots, \widehat{\mathsf{z}}_i, \dots, \mathsf{z}_k)\big]_{\mathfrak{g}} + \\ + \sum_{0\leqslant i<j\leqslant k} (-1)^{i+j}\, \Phi\big([\mathsf{z}_i, \mathsf{z}_j]_{\mathfrak{g}}, \mathsf{z}_0, \dots, \widehat{\mathsf{z}}_i, \dots, \widehat{\mathsf{z}}_j, \dots, \mathsf{z}_k\big).$$

- **Especially for $k = 2$:** Cohomology space

$$\mathscr{H}^2 := \ker(\partial^2)/\mathrm{im}(\partial^1)$$

encode deformations of Lie algebras [Goze] and are central when constructing Cartan connections [Tanaka, Morimoto, Cap-Schichl].

- **Algorithm using Gröbner bases:** Computed these cohomology spaces [Alizadeh-Aghasi-M.-Sabzevari].

Lemma. (Bianchi identity) [Tanaka, Cap-Schichl] *For any three* $\mathsf{x}', \mathsf{x}'', \mathsf{x}''' \in \mathfrak{g}$, *one has at every point* $p \in \mathscr{P}$:

$$0 = (\partial^2 \kappa)(p)(\mathsf{x}', \mathsf{x}'', \mathsf{x}''') + \sum_{\mathrm{cycl}} \kappa(p)\big(\kappa(p)(\mathsf{x}', \mathsf{x}''),\, \mathsf{x}'''\big) + \\ + \sum_{\mathrm{cycl}} \big(\widehat{X}'(\kappa)\big)(p)(\mathsf{x}'', \mathsf{x}''').$$

- **The case of graded Lie algebras:**

$$\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\nu,$$
$$\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\nu,$$

as in Tanaka's theory, with:

$$\big[\mathfrak{g}_{\lambda_1},\, \mathfrak{g}_{\lambda_2}\big]_{\mathfrak{g}} \subset \mathfrak{g}_{\lambda_1 + \lambda_2}$$

- **Second cohomology is graded too:**

$$\mathscr{H}^2 = \oplus_{h \in \mathbb{Z}} \mathscr{H}^2_{[h]},$$

- **Graded Bianchi identities:** [Cap-Schichl]:

$$\partial^2_{[h]}\big(\kappa_{[h]}\big)(\mathsf{x}', \mathsf{x}'', \mathsf{x}''') = -\sum_{\mathrm{cycl}} \sum_{h'=1}^{h-1} \Big(\kappa_{[h-h']}\big(\kappa_{[h']}(\mathsf{x}', \mathsf{x}''),\, \mathsf{x}'''\big)\Big) - \\ - \sum_{\mathrm{cycl}} \big(\widehat{X}'\kappa_{[h+|\mathsf{x}'|]}\big)(\mathsf{x}'', \mathsf{x}''')$$

show that the lowest order nonvanishing curvature must be ∂-closed, and more generally, any homogeneous curvature component is determined by the lower components up to a ∂-closed component.

IV – Explicit curvatures and coframes

• **Three-dimensional Cauchy-Riemann submanifold:** Let now $M^3 \subset \mathbb{C}^2$ be a *local* strongly pseudoconvex $\mathscr{C}^6$-smooth real 3-dimensional hypersurface, represented in coordinates $(z, w) = (x + iy, u + iv)$ as the graph:

$$\begin{aligned} v &= \varphi(x, y, u) \\ &= x^2 + y^2 + \mathrm{O}(3). \end{aligned}$$

• **Such M^3's are** *geometry-preserving* **deformations of the** *Heisenberg sphere* $\mathbb{H}^3$**:**

$$v = x^2 + y^2.$$

• **Study firstly the geometry of this homogeneous model:**

Lemma. [Known] *The Lie algebra:*

$$\mathfrak{hol}(\mathbb{H}^3) := \Big\{ \mathsf{X} = Z(z, w) \frac{\partial}{\partial z} + W(z, w) \frac{\partial}{\partial w} : \\ \mathsf{X} + \overline{\mathsf{X}} \text{ tangent to } \mathbb{H}^3 \Big\}$$

of infinitesimal CR automorphisms of the Heisenberg sphere $\mathbb{H}^3$ in $\mathbb{C}^2$ is **8***-dimensional and generated by:*

$$\begin{aligned} &\mathsf{T} := \partial_w, \quad \mathsf{H}_1 := \partial_z + 2iz\,\partial_w, \quad \mathsf{H}_2 := i\,\partial_z + 2z\,\partial_w, \\ &\mathsf{D} := z\,\partial_z + 2w\,\partial_w, \quad \mathsf{R} := iz\,\partial_z, \\ &\mathsf{I}_1 := (w + 2iz^2)\,\partial_z + 2izw\,\partial_w, \quad \mathsf{I}_2 := (iw + 2z^2)\,\partial_z + 2zw \\ &\mathsf{J} := zw\,\partial_z + w^2\,\partial_w. \end{aligned}$$

- **For general $M^3 \subset \mathbb{C}^2$:** Seek a Cartan-Tanaka connection valued in the 8-dimensional *abstract* real Lie algebra:

$$\mathfrak{g} := \mathbb{R}\,\mathsf{t} \oplus \mathbb{R}\,\mathsf{h}_1 \oplus \mathbb{R}\,\mathsf{h}_2 \oplus \mathbb{R}\,\mathsf{d} \oplus \mathbb{R}\,\mathsf{r} \oplus \mathbb{R}\,\mathsf{i}_1 \oplus \mathbb{R}\,\mathsf{i}_2 \oplus \mathbb{R}\,\mathsf{j}$$
$$\big(\text{with } \mathfrak{h} := \mathbb{R}\,\mathsf{d} \oplus \mathbb{R}\,\mathsf{r} \oplus \mathbb{R}\,\mathsf{i}_1 \oplus \mathbb{R}\,\mathsf{i}_2 \oplus \mathbb{R}\,\mathsf{j}\big)$$

spanned by some eight abstract vectors enjoying the same commutator table as $\mathsf{T}, \dots, \mathsf{J}$:

	t	h_1	h_2	d	r	i_1	i_2	j
t	0	0	0	$2\,\mathsf{t}$	0	h_1	h_2	d
h_1	$*$	0	$4\,\mathsf{t}$	h_1	h_2	$6\,\mathsf{r}$	$2\,\mathsf{d}$	i_1
h_2	$*$	$*$	0	h_2	$-\mathsf{h}_1$	$-2\,\mathsf{d}$	$6\,\mathsf{r}$	i_2
d	$*$	$*$	$*$	0	0	i_1	i_2	$2\,\mathsf{j}$
r	$*$	$*$	$*$	$*$	0	$-\mathsf{i}_2$	i_1	0
i_1	$*$	$*$	$*$	$*$	$*$	0	$4\,\mathsf{j}$	0
i_2	$*$	$*$	$*$	$*$	$*$	$*$	0	0
j	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0;

- **Fact:**

$$\mathfrak{g} \simeq \mathfrak{pgl}_2(\mathbb{R}).$$

- **Natural Tanaka grading [known]:**

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}_{\mathfrak{h}},$$

where:

$$\mathfrak{g}_{-2} = \mathbb{R}\,\mathsf{t}, \qquad \mathfrak{g}_{-1} = \mathbb{R}\,\mathsf{h}_1 \oplus \mathbb{R}\,\mathsf{h}_2,$$
$$\mathfrak{g}_0 = \mathbb{R}\,\mathsf{d} \oplus \mathbb{R}\,\mathsf{r},$$
$$\mathfrak{g}_1 = \mathbb{R}\,\mathsf{i}_1 \oplus \mathbb{R}\,\mathsf{i}_2, \qquad \mathfrak{g}_2 = \mathbb{R}\,\mathsf{j}.$$

Main computational objective: *Provide a Cartan-Tanaka connection all elements of which are completely effective in terms of $\varphi(x,y,u)$ — assuming only $\mathscr{C}^6$-smoothness of M.*

- **Recall the equation of our hypersurface:**

$$v = \varphi(x,y,u).$$

A posteriori fact: *All data of the Cartan-Tanaka connection will depend only upon $\varphi(x,y,u)$.*

- **Complex tangent bundle:**

$$T^c M = TM \cap \sqrt{-1}\,TM$$

generated by the two vector fields:

$$H_1 := \frac{\partial}{\partial x} + \Big(\frac{\varphi_y - \varphi_x\,\varphi_u}{1+\varphi_u^2}\Big)\frac{\partial}{\partial u},$$
$$H_2 := \frac{\partial}{\partial y} + \Big(\frac{-\varphi_x - \varphi_y\,\varphi_u}{1+\varphi_u^2}\Big)\frac{\partial}{\partial u}.$$

- **Levi form-type Lie-bracket:**

$$
\begin{aligned}
T &:= \tfrac{1}{4}\,[H_1, H_2] \\
&= \Big(\tfrac{1}{4}\,\tfrac{1}{(1+\varphi_u^2)^2}\big\{ -\varphi_{xx} - \varphi_{yy} - 2\,\varphi_y\,\varphi_{xu} - \varphi_x^2\,\varphi_{uu} + \\
&\qquad\qquad + 2\,\varphi_x\,\varphi_{yu} - \varphi_y^2\,\varphi_{uu} + 2\,\varphi_y\,\varphi_u\,\varphi_{yu} + \\
&\qquad\qquad + 2\,\varphi_x\,\varphi_u\,\varphi_{xu} - \varphi_u^2\,\varphi_{xx} - \varphi_u^2\,\varphi_{yy}\big\}\Big)\,\tfrac{\partial}{\partial u}.
\end{aligned}
$$

- **Strong pseudoconvexity means:**

$$\big\{H_1, H_2, T\big\} \text{ makes up a frame on } M^3.$$

- **Complicated Levi form factor:** Call Υ the numerator:

$$T = \tfrac{1}{4}\,[H_1, H_2] = \frac{1}{4}\,\frac{\Upsilon}{\Delta^2}\,\frac{\partial}{\partial u}.$$

- **Allow the two notational coincidences:**

$$x_1 \equiv x, \qquad\qquad x_2 \equiv y.$$

- **Introduce the two length-three brackets:**

$$\big[H_i,\, T\big] = \tfrac{1}{4}\,\big[H_i, [H_1, H_2]\big] =: \Phi_i\,T \qquad\qquad (i = 1, 2),$$

- **Fact 1:** These are both multiples of T by means of two functions:

$$\Phi_i := \frac{A_i}{\Delta^2\,\Upsilon} \qquad\qquad (i = 1, 2).$$

- **Fact 2:** Expansions of these numerators A_i are one page long.

- **Fact 3:** Expansions of the numerators A_{i,k_1,k_2,k_3} below are more than one hundred page long.

• **Lastly:** Introduce furthermore the H_k-iterated derivatives of the functions Φ_i up to order 3:

$$H_{k_1}(\Phi_i) = \frac{A_{i,k_1}}{\Delta^4\,\Upsilon^2}$$
$$H_{k_2}(H_{k_1}(\Phi_i)) = \frac{A_{i,k_1,k_2}}{\Delta^6\,\Upsilon^3},$$
$$H_{k_3}(H_{k_2}(H_{k_1}(\Phi_i))) = \frac{A_{i,k_1,k_2,k_3}}{\Delta^8\,\Upsilon^4},$$

where $i, k_1, k_2, k_3 = 1, 2$.

Proposition. [AMS 2011] *All the numerators appearing above are inductively given by:*

$$\begin{aligned}
A_{i,k_1} := {} & \Delta^2\big(\Upsilon\, A_{i,x_{k_1}} - \Upsilon_{x_{k_1}}\, A_i\big) + \Delta\big(-2\,\Delta_{x_{k_1}}\,\Upsilon\, A_i + \Upsilon\,\Lambda_{k_1}\, A_{i,u} - \Upsilon_u\,\Lambda_{k_1}\, A_i\big) - \\
& -2\,\Delta_u\,\Upsilon\,\Lambda_{k_1}\, A_i \qquad (i, k_1 = 1,2), \\
A_{i,k_1,k_2} := {} & \Delta^2\big(\Upsilon\, A_{i,k_1,x_{k_2}} - 2\,\Upsilon_{x_{k_2}}\, A_{i,k_1}\big) + \Delta\big(-3\,\Delta_{x_{k_2}}\,\Upsilon\, A_{i,k_1} + \Upsilon\,\Lambda_{k_2}\, A_{i,k_1,u} - \\
& -2\,\Upsilon_u\,\Lambda_{k_2}\, A_{i,k_1}\big) - 3\,\Delta_u\,\Upsilon\,\Lambda_{k_2}\, A_{i,k_1} \qquad (i, k_1, k_2 = 1,2), \\
A_{i,k_1,k_2,k_3} = {} & \Delta^2\big(\Upsilon\, A_{i,k_1,k_2,x_{k_3}} - \Upsilon_{x_{k_3}}\, A_{i,k_1,k_2}\big) + \Delta\big(-6\,\Delta_{x_{k_3}}\,\Upsilon\, A_{i,k_1,k_2} + \Upsilon\,\Lambda_{k_3}\, A_{i,k_1,k_2,u} - \\
& -3\,\Upsilon_u\,\Lambda_{k_3}\, A_{i,k_1,k_2}\big) - 6\,\Delta_u\,\Upsilon\,\Lambda_{k_3}\, A_{i,k_1,k_2} \qquad (i, k_1, k_2, k_3 = 1,2).
\end{aligned}$$

Furthermore, these iterated derivatives identically satisfy:

$$H_2(\Phi_1) \equiv H_1(\Phi_2)$$

and four third-order relations [new in the subject]*:*

$$\begin{aligned}
0 \equiv {} & -H_1(H_2(H_1(\Phi_2))) + 2\,H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - \\
& -\Phi_2\,H_1(H_2(\Phi_1)) + \Phi_2\,H_2(H_1(\Phi_1)), \\
0 \equiv {} & -H_2(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_2))) - H_1(H_1(H_2(\Phi_2))) - \\
& -\Phi_1\,H_2(H_1(\Phi_2)) + \Phi_1\,H_1(H_2(\Phi_2)), \\
0 \equiv {} & -H_1(H_1(H_1(\Phi_2))) + 2\,H_1(H_2(H_1(\Phi_1))) - H_2(H_1(H_1(\Phi_1))) + \\
& +\Phi_1\,H_1(H_1(\Phi_2)) - \Phi_1\,H_2(H_1(\Phi_1)), \\
0 \equiv {} & H_2(H_2(H_1(\Phi_2))) - 2\,H_2(H_1(H_2(\Phi_2))) + H_1(H_2(H_2(\Phi_2))) - \\
& -\Phi_2\,H_2(H_1(\Phi_2)) + \Phi_2\,H_1(H_2(\Phi_2)).
\end{aligned}$$

Theorem. [AMS 2011] *Associated to such an* $M^3 \subset \mathbb{C}^2$, *there is a unique* $\mathfrak{pgl}_2(\mathbb{R})$*-valued Cartan connection which is normal and regular in the sense of Tanaka. Its curvature function reduces to:*

$$\begin{aligned}\kappa(p) = \underline{\kappa_{i_1}^{h_1 t}(p)}\, \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \underline{\kappa_{i_2}^{h_1 t}(p)}\, \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \\ + \kappa_{i_1}^{h_2 t}(p)\, \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \kappa_{i_2}^{h_2 t}(p)\, \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \\ + \kappa_{j}^{h_1 t}(p)\, \mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{j} + \kappa_{j}^{h_2 t}(p)\, \mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{j},\end{aligned}$$

where the two main curvature coefficients, having homogeneity four, are of the form:

$$\kappa_{i_1}^{h_1 t}(p) = -\,\mathbf{\Delta_1}\, c^4 - 2\,\mathbf{\Delta_4}\, c^3 d - 2\,\mathbf{\Delta_4}\, c d^3 + \mathbf{\Delta_1}\, d^4,$$

$$\kappa_{i_2}^{h_1 t}(p) = -\,\mathbf{\Delta_4}\, c^4 + 2\,\mathbf{\Delta_1}\, c^3 d + 2\,\mathbf{\Delta_1}\, c d^3 + \mathbf{\Delta_4}\, d^4,$$

in which the two functions $\mathbf{\Delta_1}$ *and* $\mathbf{\Delta_4}$ *of only the three variables* (x, y, u) *are* explicitly *given by:*

$$\begin{aligned}\mathbf{\Delta_1} = \tfrac{1}{384}\Big[& H_1(H_1(H_1(\Phi_1))) - H_2(H_2(H_2(\Phi_2))) + 11\, H_1(H_2(H_1(\Phi_2))) - 11\, H_2(H_1(H_2(\Phi_1))) + \\ & + 6\,\Phi_2\, H_2(H_1(\Phi_1)) - 6\,\Phi_1\, H_1(H_2(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_2)) + 3\,\Phi_1\, H_2(H_2(\Phi_1)) - \\ & - 3\,\Phi_1\, H_1(H_1(\Phi_1)) + 3\,\Phi_2\, H_2(H_2(\Phi_2)) - 2\,\Phi_1\, H_1(\Phi_1) + 2\,\Phi_2\, H_2(\Phi_2) - \\ & - 2\,(\Phi_2)^2\, H_1(\Phi_1) + 2\,(\Phi_1)^2\, H_2(\Phi_2) - 2\,(\Phi_2)^2\, H_2(\Phi_2) + 2\,(\Phi_1)^2\, H_1(\Phi_1)\Big],\end{aligned}$$

$$\begin{aligned}\mathbf{\Delta_4} = \tfrac{1}{384}\Big[& -3\, H_2(H_1(H_2(\Phi_2))) - 3\, H_1(H_2(H_1(\Phi_1))) + 5\, H_1(H_2(H_2(\Phi_2))) + 5\, H_2(H_1(H_1(\Phi_1))) + \\ & + 4\,\Phi_1\, H_1(H_1(\Phi_2)) + 4\,\Phi_2\, H_2(H_1(\Phi_2)) - 3\,\Phi_2\, H_1(H_1(\Phi_1)) - 3\,\Phi_1\, H_2(H_2(\Phi_2)) - \\ & - 7\,\Phi_2\, H_1(H_2(\Phi_2)) - 7\,\Phi_1\, H_2(H_1(\Phi_1)) - 2\, H_1(\Phi_1)\, H_1(\Phi_2) - 2\, H_2(\Phi_2)\, H_2(\Phi_1) + \\ & + 4\,\Phi_1\Phi_2\, H_1(\Phi_1) + 4\,\Phi_1\Phi_2\, H_2(\Phi_2)\Big],\end{aligned}$$

and where the remaining secondary curvature coefficients are [use Bianchi identities]*:*

$$\kappa_{i_1}^{h_2 t} = \kappa_{i_2}^{h_1 t}, \qquad \kappa_{i_2}^{h_2 t} = -\,\kappa_{i_1}^{h_1 t},$$

Corollary. [AMS 2011, 113 pages] *A $\mathscr{C}^6$-smooth strongly pseudoconvex local hypersurface $M^3 \subset \mathbb{C}^2$ is biholomorphic to $\mathbb{H}^3$, if and only if:*

$$\mathbf{\Delta_1} \equiv \mathbf{\Delta_4} \equiv 0,$$

identically as functions of (x, y, u).

- **A few formulas from the proofs:**

$$\begin{aligned}
\alpha_{tj} = {} & 3a^4 + 3b^4 - 4e^2 - \Phi_1 a^2 bc + ca\Phi_2 b^2 - \Phi_1 ab^2 d - \Phi_2 a^2 bd - 2\Phi_2 bce - 2\Phi_1 ace - 2\Phi_2 ade + 2\Phi_1 bde - \\
& - \Phi_1 a^3 d + \Phi_2 a^3 c - \Phi_1 b^3 c - \Phi_2 b^3 d + 6a^2 b^2 + \big[\tfrac{3}{16} H_1(\Phi_1) + \tfrac{3}{16} H_2(\Phi_2)\big] b^2 d^2 + \\
& + \big[- \tfrac{11}{1536} H_2(\Phi_2) H_1(\Phi_1) - \tfrac{1}{192} H_1(H_1(\Phi_1))\Phi_1 - \tfrac{11}{3072} H_2({\Phi_2}^2) + \tfrac{1}{384} {\Phi_2}^2 H_2(\Phi_2) - \tfrac{11}{3072} H_1(\Phi_1^2) + \\
& + \tfrac{1}{384} \Phi_1^2 H_1(\Phi_1) + \tfrac{1}{48} H_1(H_2(H_1(\Phi_2))) + \tfrac{1}{384} H_2(H_2(H_2(\Phi_2))) + \tfrac{1}{384} H_1(H_1(H_1(\Phi_1))) + \tfrac{1}{384} {\Phi_2}^2 H_1(\Phi_1) - \\
& - \tfrac{1}{192} H_2(H_2(\Phi_2))\Phi_2 + \tfrac{1}{48} H_2(H_1(H_1(\Phi_2))) + \tfrac{1}{64} H_2(H_1(\Phi_1))\Phi_2 - \tfrac{1}{48} \Phi_1 H_2(H_1(\Phi_2)) + \tfrac{1}{384} {\Phi_1}^2 H_2(\Phi_2) - \\
& - \tfrac{7}{384} H_2(H_2(H_1(\Phi_1))) + \tfrac{1}{64} H_1(H_2(\Phi_2))\Phi_1 - \tfrac{7}{384} H_1(H_1(H_2(\Phi_2))) - \tfrac{1}{48} \Phi_2 H_1(H_1(\Phi_2))\big] d^4 + \\
& + \big[- \tfrac{11}{768} H_2(\Phi_2) H_1(\Phi_1) - \tfrac{7}{192} H_2(H_2(H_1(\Phi_1))) + \tfrac{1}{192} H_2(H_2(H_2(\Phi_2))) + \tfrac{1}{192} H_1(H_1(H_1(\Phi_1))) + \\
& + \tfrac{1}{24} H_1(H_2(H_1(\Phi_2))) - \tfrac{1}{96} H_2(H_2(\Phi_2))\Phi_2 + \tfrac{1}{32} H_1(H_2(\Phi_2))\Phi_1 + \tfrac{1}{192} \Phi_2^2 H_1(\Phi_1) - \tfrac{7}{192} H_1(H_1(H_2(\Phi_2))) + \\
& + \tfrac{1}{192} \Phi_2^2 H_2(\Phi_2) - \tfrac{11}{1536} H_1(\Phi_1^2) - \tfrac{1}{24} \Phi_2 H_1(H_1(\Phi_2)) - \tfrac{11}{1536} H_2(\Phi_2^2) + \tfrac{1}{32} H_2(H_1(\Phi_1))\Phi_2 - \tfrac{1}{96} H_1(H_1(\Phi_1))\Phi_1 + \\
& + \tfrac{1}{192} \Phi_2^2 H_2(\Phi_2) + \tfrac{1}{192} \Phi_1^2 H_1(\Phi_1) - \tfrac{1}{24} \Phi_1 H_2(H_1(\Phi_2)) + \tfrac{1}{24} H_2(H_1(H_1(\Phi_2)))\big] c^2 d^2 + \big[- \tfrac{1}{32} H_1(H_1(\Phi_1)) + \\
& + \tfrac{1}{32} H_2(\Phi_2)\Phi_1 - \tfrac{1}{32} H_1(H_2(\Phi_2)) + \tfrac{1}{32} H_1(\Phi_1)\Phi_1\big] bcd^2 + \big[\tfrac{1}{32} H_2(H_1(\Phi_1)) + \tfrac{1}{32} H_2(H_2(\Phi_2)) - \\
& - \tfrac{1}{32} H_2(\Phi_2)\Phi_2 - \tfrac{1}{32} H_1(\Phi_1)\Phi_2\big] acd^2 + \big[- \tfrac{1}{32} H_1(H_1(\Phi_1)) + \tfrac{1}{32} H_2(\Phi_2)\Phi_1 - \tfrac{1}{32} H_1(H_2(\Phi_2)) + \\
& + \tfrac{1}{32} H_1(\Phi_1)\Phi_1\big] ad^3 + \big[\tfrac{1}{32} H_2(H_1(\Phi_1)) + \tfrac{1}{32} H_2(H_2(\Phi_2)) - \tfrac{1}{32} H_2(\Phi_2)\Phi_2 - \tfrac{1}{32} H_1(\Phi_1)\Phi_2\big] ac^3 + \\
& + \tfrac{3}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big] a^2 d^2 + \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 - H_2(H_1(\Phi_1)) - H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2\big] bd^3 + \\
& + \big[- \tfrac{1}{32} H_1(H_1(\Phi_1)) + \tfrac{1}{32} H_2(\Phi_2)\Phi_1 - \tfrac{1}{32} H_1(H_2(\Phi_2)) + \tfrac{1}{32} H_1(\Phi_1)\Phi_1\big] bc^3 + \\
& + \tfrac{3}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big] a^2 c^2 + \tfrac{3}{16}\big[H_1(\Phi_1) + H_2(\Phi_2)\big] b^2 c^2 + \tfrac{1}{32}\big[H_2(\Phi_2)\Phi_2 - H_2(H_1(\Phi_1)) - \\
& - H_2(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_2\big] dbc^2 + \tfrac{1}{32}\big[- H_1(H_1(\Phi_1)) + H_2(\Phi_2)\Phi_1 - H_1(H_2(\Phi_2)) + H_1(\Phi_1)\Phi_1\big] ac^2 d + \\
& + \big[- \tfrac{11}{1536} H_2(\Phi_2) H_1(\Phi_1) - \tfrac{1}{192} H_1(H_1(\Phi_1))\Phi_1 - \tfrac{11}{3072} H_2(\Phi_2^2) + \tfrac{1}{384} \Phi_2^2 H_2(\Phi_2) - \tfrac{11}{3072} H_1(\Phi_1^2) + \\
& + \tfrac{1}{384} \Phi_1^2 H_1(\Phi_1) + \tfrac{1}{48} H_1(H_2(H_1(\Phi_2))) + \tfrac{1}{384} H_2(H_2(H_2(\Phi_2))) + \tfrac{1}{384} H_1(H_1(H_1(\Phi_1))) + \tfrac{1}{384} \Phi_2^2 H_1(\Phi_1) - \\
& - \tfrac{1}{192} H_2(H_2(\Phi_2))\Phi_2 + \tfrac{1}{48} H_2(H_1(H_1(\Phi_2))) + \tfrac{1}{64} H_2(H_1(\Phi_1))\Phi_2 - \tfrac{1}{48} \Phi_1 H_2(H_1(\Phi_2)) + \tfrac{1}{384} \Phi_1^2 H_2(\Phi_2) - \\
& - \tfrac{7}{384} H_2(H_2(H_1(\Phi_1))) + \tfrac{1}{64} H_1(H_2(\Phi_2))\Phi_1 - \tfrac{7}{384} H_1(H_1(H_2(\Phi_2))) - \tfrac{1}{48} \Phi_2 H_1(H_1(\Phi_2))\big] c^4.
\end{aligned}$$

0	$\kappa_t^{h_1 h_2} = \widehat{T}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	2	$\kappa_{h_2}^{h_1 t} = \widehat{H}_2^*([\widehat{H}_1, \widehat{T}])$	3	$\kappa_r^{h_2 t} = \widehat{R}^*([\widehat{H}_2, \widehat{T}])$
1	$\kappa_{h_1}^{h_1 h_2} = \widehat{H}_1^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	2	$\kappa_{h_1}^{h_2 t} = \widehat{H}_1^*([\widehat{H}_2, \widehat{T}])$	4	$\kappa_j^{h_1 h_2} = \widehat{J}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$
1	$\kappa_{h_2}^{h_1 h_2} = \widehat{H}_2^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	2	$\kappa_{h_2}^{h_2 t} = \widehat{H}_2^*([\widehat{H}_2, \widehat{T}])$	4	$\kappa_{i_1}^{h_1 t} = \widehat{I}_1^*([\widehat{H}_1, \widehat{T}])$
1	$\kappa_t^{h_1 t} = \widehat{T}^*([\widehat{H}_1, \widehat{T}])$	3	$\kappa_{i_1}^{h_1 h_2} = \widehat{I}_1^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	4	$\kappa_{i_2}^{h_1 t} = \widehat{I}_2^*([\widehat{H}_1, \widehat{T}])$
1	$\kappa_t^{h_2 t} = \widehat{T}^*([\widehat{H}_2, \widehat{T}])$	3	$\kappa_{i_2}^{h_1 h_2} = \widehat{I}_2^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	4	$\kappa_{i_1}^{h_2 t} = \widehat{I}_1^*([\widehat{H}_2, \widehat{T}])$
2	$\kappa_d^{h_1 h_2} = \widehat{D}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	3	$\kappa_d^{h_1 t} = \widehat{D}^*([\widehat{H}_1, \widehat{T}])$	4	$\kappa_{i_2}^{h_2 t} = \widehat{I}_2^*([\widehat{H}_2, \widehat{T}])$
2	$\kappa_r^{h_1 h_2} = \widehat{R}^*([\widehat{H}_1, \widehat{H}_2] - 4\widehat{T})$	3	$\kappa_d^{h_2 t} = \widehat{D}^*([\widehat{H}_2, \widehat{T}])$	5	$\kappa_j^{h_1 t} = \widehat{J}^*([\widehat{H}_1, \widehat{T}])$
2	$\kappa_{h_1}^{h_1 t} = \widehat{H}_1^*([\widehat{H}_1, \widehat{T}])$	3	$\kappa_r^{h_1 t} = \widehat{R}^*([\widehat{H}_1, \widehat{T}])$	5	$\kappa_j^{h_2 t} = \widehat{J}^*([\widehat{H}_2, \widehat{T}])$

$$[H_1,[H_1,[H_1,[H_1,[H_1,H_2]]]]] \overset{1}{=} \big(H_1(H_1(H_1(\Phi_1))) + 4\Phi_1 H_1(H_1(\Phi_1)) + 3H_1(\Phi_1)H_1(\Phi_1) + 6(\Phi_1)^2 H_1(\Phi_1) + (\Phi_1)^4\big)[H_1,H_2],$$

$$[H_1,[H_1,[H_1,[H_2,[H_1,H_2]]]]] \overset{2}{=} \big(H_1(H_1(H_1(\Phi_2))) + 3\Phi_1 H_1(H_1(\Phi_2)) + \Phi_2 H_1(H_1(\Phi_1)) + 3H_1(\Phi_1)H_1(\Phi_2) + 3\Phi_1\Phi_2 H_1(\Phi_1) + 3(\Phi_1)^2 H_1(\Phi_2) + (\Phi_1)^3\Phi_2\big)[H_1,H_2],$$

$$[H_1,[H_1,[H_2,[H_2,[H_1,H_2]]]]] \overset{3}{=} \big(H_1(H_1(H_2(\Phi_2))) + 2\Phi_1 H_1(H_2(\Phi_2)) + 2\Phi_2 H_1(H_1(\Phi_2)) + H_1(\Phi_1)H_2(\Phi_2) + 2H_1(\Phi_2)H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2],$$

$$[H_1,[H_2,[H_1,[H_1,[H_1,H_2]]]]] \overset{4}{=} \big(H_1(H_2(H_1(\Phi_1))) + 2\Phi_1 H_1(H_1(\Phi_2)) + \Phi_2 H_1(H_1(\Phi_1)) + \Phi_1 H_2(H_1(\Phi_1)) + 3H_1(\Phi_1)H_2(\Phi_2) + 3\Phi_1\Phi_2 H_1(\Phi_1) + 3(\Phi_1)^2 H_1(\Phi_2) + (\Phi_1)^3\Phi_2\big)[H_1,H_2],$$

$$[H_1,[H_2,[H_1,[H_2,[H_1,H_2]]]]] \overset{5}{=} \big(H_1(H_2(H_1(\Phi_2))) + \Phi_1 H_1(H_2(\Phi_2)) + 2\Phi_2 H_1(H_1(\Phi_2)) + \Phi_1 H_2(H_1(\Phi_2)) + H_1(\Phi_1)H_2(\Phi_2) + 2H_1(\Phi_2)H_1(\Phi_2) + 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2],$$

$$[H_1,[H_2,[H_2,[H_2,[H_1,H_2]]]]] \overset{6}{=} \big(H_1(H_2(H_2(\Phi_2))) + 3\Phi_2 H_1(H_2(\Phi_2)) + \Phi_1 H_2(H_2(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) + 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2],$$

$$[H_2,[H_1,[H_1,[H_1,[H_1,H_2]]]]] \overset{7}{=} \big(H_2(H_1(H_1(\Phi_1))) + 3\Phi_1 H_2(H_1(\Phi_1)) + \Phi_2 H_1(H_1(\Phi_1)) + 3H_1(\Phi_1)H_1(\Phi_2) + 3(\Phi_1)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_1(\Phi_1) + (\Phi_1)^3\Phi_2\big)[H_1,H_2],$$

$$[H_2,[H_1,[H_1,[H_2,[H_1,H_2]]]]] \overset{8}{=} \big(H_2(H_1(H_1(\Phi_2))) + 2\Phi_1 H_2(H_1(\Phi_2)) + \Phi_2 H_2(H_1(\Phi_1)) + \Phi_2 H_1(H_1(\Phi_2)) + 2H_1(\Phi_2)H_1(\Phi_2) + H_2(\Phi_2)H_1(\Phi_1) + 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2],$$

$$[H_2,[H_1,[H_2,[H_2,[H_1,H_2]]]]] \overset{9}{=} \big(H_2(H_1(H_2(\Phi_2))) + 2\Phi_2 H_2(H_1(\Phi_2)) + \Phi_1 H_2(H_2(\Phi_2)) + \Phi_2 H_1(H_2(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) + 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2],$$

$$[H_2,[H_2,[H_1,[H_1,[H_1,H_2]]]]] \overset{10}{=} \big(H_2(H_2(H_1(\Phi_1))) + 2\Phi_1 H_2(H_1(\Phi_2)) + 2\Phi_2 H_2(H_1(\Phi_1)) + 2H_1(\Phi_2)H_1(\Phi_2) + H_1(\Phi_1)H_2(\Phi_2) + 4\Phi_1\Phi_2 H_1(\Phi_2) + (\Phi_1)^2 H_2(\Phi_2) + (\Phi_2)^2 H_1(\Phi_1) + (\Phi_1)^2(\Phi_2)^2\big)[H_1,H_2],$$

$$[H_2,[H_2,[H_1,[H_2,[H_1,H_2]]]]] \overset{11}{=} \big(H_2(H_2(H_1(\Phi_2))) + \Phi_1 H_2(H_2(\Phi_2)) + 3\Phi_2 H_2(H_1(\Phi_2)) + 3H_1(\Phi_2)H_2(\Phi_2) + 3(\Phi_2)^2 H_1(\Phi_2) + 3\Phi_1\Phi_2 H_2(\Phi_2) + \Phi_1(\Phi_2)^3\big)[H_1,H_2],$$

$$[H_2,[H_2,[H_2,[H_2,[H_1,H_2]]]]] \overset{12}{=} \big(H_2(H_2(H_2(\Phi_2))) + 4\Phi_2 H_2(H_2(\Phi_2)) + 3H_2(\Phi_2)H_2(\Phi_2) + 6(\Phi_2)^2 H_2(\Phi_2) + (\Phi_2)^4\big)[H_1,H_2].$$

$$\kappa_{i_2}^{h_2 t} = \widehat{I}_2^*([\widehat{H}_2,\widehat{T}]) = -\widehat{T}(\alpha_{h_2 i_2}) + \alpha_{h_2h_2}H_2(\alpha_{ti_2}) + \alpha_{h_2h_1}H_1(\alpha_{ti_2}) + \beta_{i_2h_1}\big(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_1})}_{\circ} - \alpha_{th_1}\underline{H_1(\alpha_{h_2h_1})}_{\circ} - \alpha_{th_2}\underline{H_2(\alpha_{h_2h_1})}_{\circ} - \alpha_{tt}\underline{T(\alpha_{h_2h_1})}_{\circ} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_1})}_{\circ} - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_1})}_{\circ} - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_1})}_{-\alpha_{h_2h_1}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_1})}_{\alpha_{h_1h_1}} - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_1})}_{\circ} + \alpha_{h_2h_1}\underline{H_2(\alpha_{th_1})}_{\circ}\big) + \beta_{i_2h_2}(\alpha_{h_2h_1}\underline{H_1(\alpha_{th_2})}_{\circ} - \alpha_{th_1}\underline{H_1(\alpha_{h_2h_2})}_{\circ} + \alpha_{h_2h_2}\underline{H_2(\alpha_{th_2})}_{\circ} - \alpha_{th_2}\underline{H_2(\alpha_{h_2h_2})}_{\circ} - \alpha_{tt}\underline{T(\alpha_{h_2h_2})}_{\circ} - \alpha_{td}\underbrace{\widehat{D}(\alpha_{h_2h_2})}_{-\alpha_{h_2h_2}} - \alpha_{tr}\underbrace{\widehat{R}(\alpha_{h_2h_2})}_{\alpha_{h_1h_2}} - \alpha_{ti_1}\underline{\widehat{I}_1(\alpha_{h_2h_2})}_{\circ} - \alpha_{ti_2}\underline{\widehat{I}_2(\alpha_{h_2h_2})}_{\circ} - \alpha_{tj}\underline{\widehat{J}(\alpha_{h_2h_2})}_{\circ}\big) + \beta_{i_2t}\big(4\alpha_{h_2h_1}\alpha_{th_2} + \alpha_{h_2h_1}\alpha_{tt}\Phi_1 - 4\alpha_{h_2h_2}\alpha_{th_1} + \alpha_{h_2h_2}\alpha_{tt}\Phi_2\big),$$

V – Perspectives on explicit Cartan CR connections

• **Today: three (deeper) levels of explicit calculations:**

□ Informative linear algebra: Absorption of torsion; normalization; prolongation of equivalence problems; appearance of curvatures tensors; dimensional counts.

□ Polynomial differential algebra: Expand completely quadratic, cubic, quartic, polynomial remainders.

□ Relations (syzygies): Free and non-free Lie algebra impose nontrivial relations between iterated Lie brackets.

• **Classification problem (still open in dimension 5):** To provide a complete list of all possible (local or global) real analytic submanifolds $M^{2n+1} \subset \mathbb{C}^{n+1}$ up to change of holomorphic coordinates on $\mathbb{C}^{n+1}$.

Joël M., *Sophus Lie, Friedrich Engel et le problème de Riemann-Helmholtz*, Hermann, Paris, 2010, **349** pages.

Joël M., *Sophus Lie and Friedrich Engel's Theory of Transformation Groups (Vol. I, 1888). Modern Presentation and English Translation*, **650** pages, submitted to SV.

• **Cartan connection problem:** To determine classes of homogeneous spaces corresponding to CR submanifolds $M^{2n+1} \subset \mathbb{C}^{n+1}$ of small dimension, and to construct Cartan connections on geometry-preserving deformations of the found homogeneous models.

□ Cartan connections should be widespread in differential geometry, also for *non* semi-simple homogeneous Klein models, especially on CR manifolds.

• **Question still open in CR geometry:** Make Chern-Moser's computations explicit in terms of the defining equation for a Levi nondegenerate $M^{2n+1} \subset \mathbb{C}^{n+1}$ [Isaev, LNM 2020, Springer, May 2011].

• **Beloshapka-Ezhov-Schmalz:** [Russ. J. Math. Phys. 2007] Cartan-Tanaka connection for Engel CR manifolds $M^4 \subset \mathbb{C}^3$ that are geometry-preserving deformations of Beloshapka's cubic:

$$\left[\begin{aligned} v_1 &= z\overline{z} + \mathrm{O}(4) = \varphi_1(x, y, u_1, u_2) \\ v_2 &= 2i\, z\overline{z}(z + \overline{z}) + \mathrm{O}(4) = \varphi_2(x, y, u_1, u_2). \end{aligned}\right.$$

• **M.-Sabzevari:** [in progress; many generalizations]

$$\left[\begin{aligned} v_1 &= 2i\, z\overline{z} + \mathrm{O}(4) = \varphi_1(x, y, u_1, u_2, u_3) \\ v_2 &= 2i\, z\overline{z}(z + \overline{z}) + \mathrm{O}(4) = \varphi_2(x, y, u_1, u_2, u_3) \\ v_3 &= 2z\overline{z}(z - \overline{z}) + \mathrm{O}(4) = \varphi_3(x, y, u_1, u_2, u_3). \end{aligned}\right.$$

$$\begin{aligned} \mathsf{T}_1 &:= \partial_{w_1} \\ \mathsf{T}_2 &:= \partial_{w_2} \\ \mathsf{T}_3 &:= \partial_{w_3} \\ \mathsf{L}_1 &:= \partial_z + (2iz)\,\partial_{w_1} + (2iz^2 + 4w_1)\,\partial_{w_2} + 2z^2\,\partial_{w_3} \\ \mathsf{L}_2 &:= i\,\partial_z + (2z)\,\partial_{w_1} + (2z^2)\,\partial_{w_2} - (2iz^2 - 4w_1)\,\partial_{w_3} \\ \mathsf{D} &:= z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 3w_3\,\partial_{w_3} \\ \mathsf{R} &:= iz\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}. \end{aligned}$$

• **Deformations of the light cone:**

$$w + \overline{w} = \frac{2z_1\overline{z}_1 + z_1^2\overline{z}_2 + \overline{z}_1^2 z_2}{1 - z_2\overline{z}_2}.$$

[Tanaka's prolongation procedure does not apply]

THE END