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Addendum to: Universal Covering Maps and Radial Variations

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Addendum to: Universal covering maps and Radial variations

by

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In this short note we complement our study of the boundary behaviour of universal covering maps. We give several examples showing that the result in [J-M] is best possible.

It is well known that the boundary behaviour of universal covering maps $P : \mathbb{D} \to \Omega$ can vary considerably, depending on topological and metric properties of the domain Ω and its boundary. To give an example, we let Ω be simply connected. Then the universal covering map is a conformal mapping, and by a result of Beurling, the set of θ where $\lim_{r\to 1} P(re^{i\theta})$ does not exist has vanishing logarithmic capacity. Moreover the set of θ for which $\lim_{r\to 1} P'(re^{i\theta})$ exists and is finite has Hausdorff dimension 1; even the set of θ where $\int_0^1 |P''(re^{i\theta})|dr < \infty$ has Hausdorff dimension equal to one. These are the results of N. Makarov [M] and the present authors [J-M 1] respectively. Similar results hold for the case of universal covering maps onto domains with uniformly perfect boundaries.

A very different picture arises, however, when one considers the twice punctured plane $\Omega = \mathbb{C} \setminus \{-1, 1\}$. In that case the set of angles θ for which the boundary value of the universal covering map, $\lim_{r\to 1} P(re^{i\theta})$, exists is at most countable. This should be compared with the result in [J-M], that for every domain Ω one has

$$\int_0^1 |P''(re^{i\theta})|dr < \infty$$

for at least countably many $\theta \in [0, 2\pi[$. Thus the result of [J-M] is best possible in general – simply because $P : \mathbb{D} \to \mathbb{C} \setminus \{-1, 1\}$ has boundary values for only countably many radii.

We will now consider domains Ω for which the boundary values of the universal covering map P exist almost everywhere. It is natural to conjecture that for this class of domains there exist also non trivial lower bounds on the size of the following sets,

$$\{e^{i\theta} : \lim_{r\to 1} P'(re^{i\theta}) \text{ exists and is finite}\},$$

$$\left\{e^{i\theta} : \int_0^1 |P''(re^{i\theta})|dr < \infty\right\}.$$

Below however we will give an example of a domain Ω_0 with cap $\partial\Omega_0 > 0$ for which these sets are at most countable. Consequently the result of [J-M] is best possible, even when the boundary values of P exist almost everywhere.

Theorem 1 *There exists a domain $\Omega_0 \subseteq \mathbb{C}$, with universal covering map $P : \mathbb{D} \to \Omega_0$ such that*

1. *The boundary of Ω_0 has positive logarithmic capacity (hence the boundary values for the universal covering map exist almost everywhere).*

2. *The set of angles $e^{i\theta}$ for which $\lim_{r\to 1} P'(re^{i\theta})$ exists and is finite, is at most countable.*

We obtain the proof of Theorem 1 by linking lower bounds for $|P'(z)|$ to the geometric estimates for the hyperbolic metric on the domain Ω. By definition the density λ_Ω for the hyperbolic metric is given by

$$\lambda_\Omega(P(z))|P'(z)| = \frac{1}{1-|z|^2},$$

where $P : \mathbb{D} \to \Omega$ is the universal covering map. The result of Beardon and Pommerenke [B-P] provides geometric estimates for λ_Ω as follows:

$$\lambda_\Omega(w) \sim \frac{1}{\operatorname{dist}(w, \partial\Omega)(\beta(w)+1)},$$

where

$$\beta(w) = \inf\left\{\left|\log\left|\frac{w-a}{b-a}\right|\right| : |w-a| = \operatorname{dist}(w,E), a, b \in E\right\}.$$

We will now define the domain Ω_0 for which Theorem 1 holds. We let

$$E_n = \{(1+2^{-2n})e^{2\pi i 8^{-2n} m} : 1 \le m \le 8^{2n}\}.$$

The set E_n consists of equidistributed points on the circle with radius $1+2^{-2n}$. We point out that the (euclidian) distance between neighboring points in E_n is much smaller than the (euclidian) distance between E_n and E_{n+1}. Now we let $E = \bar{\mathbb{D}} \cup \bigcup_{n=1}^{\infty} E_n$ and $\Omega_0 = \mathbb{C} \setminus E$. The boundary of Ω_0 is $\mathbb{T} \cup \bigcup_{n=1}^{\infty} E_n$.

We have defined the domain Ω_0 to be unbounded. This however is not essential for our purposes. We can define a bounded domain for which Theorem 1 holds by applying the inversion $i(z) = 1/z$ to Ω_0. The resulting domain is now containd in the unit disk and the proof given below shows that also $i(\Omega_0)$ satisfies the conclusions of our Theorem 1.

Our proof of Theorem 1 relies on the following Lemma which gives lower bounds for the hyperbolic distance in $\mathbb{C} \setminus E$.

Lemma 1 *Let $\{w_n\}$ be a sequence of points in $\mathbb{C} \setminus E$ such that $|w_n| = 1 + 2^{-2n-1}$. Then the hyperbolic distance in $\mathbb{C} \setminus E$ between w_n and w_{n+1} exeeds $c_1 n$, and more generally*

$$d_{\mathbb{C}\setminus E}(w_{n+k}, w_n) \ge c_1 k n.$$

Comment: The points w_n are chosen such that w_n and w_{n+1} are separated by the "barrier" formed by the points in E_n.

PROOF. We let $B = \{z \in \mathbb{C} : 1 + 2^{-2n-2} \leq |z| \leq 1 + 2^{-2n}\}$. Note that B is the annulus between E_n and E_{n+1}. Next we select disjoint annuli contained in B as follows: We define

$$D_j \subseteq B, \quad j \in \{-4n, \dots, 0, \dots 4n\}$$

such that for $w \in D_j$,

$$2^{-|j|-2n} \leq \text{dist}(w, \partial B) \leq 2^{-|j|+1-2n}.$$

Note that the annuli D_j are chosen in such a way that their thicknes is proportional to their distance to ∂B; the distance of D_j to ∂B is $\geq 8^{-2n}$. By drawing the picture we see the following two estimates for $w \in D_j$:

$$(1 - 8^{-2n})\text{dist}(w, E) \leq \text{dist}(w, \partial B) \leq (1 + 8^{-2n})\text{dist}(w, E),$$

$$\left| \log \left| \frac{w-a}{a-b} \right| \right| \geq 2,$$

whenever $a, b, \in E$ and $|w - a| = \text{dist}(w, E)$.

Combining these observations, with the result of Beardon and Pommerenke we obtain the following lower bound for the density of the hyperbolic metric in $\mathbb{C} \setminus E$: For $w \in D_j$ we have

$$\lambda_{\mathbb{C} \setminus E}(w) \geq c 2^{2n+|j|}.$$

Let now σ be any curve connecting w_{n-1} to w_n. For such a curve the euclidian diameter of $\sigma \cap D_j$ exeeds $2^{-2n-|j|}$. Consequently we have the following minorization for the hyperbolic length of $\sigma \cap D_j$,

$$\int_{\sigma \cap D_j} \lambda_{\mathbb{C} \setminus E}(w)|dw| \geq c,$$

and therefore we can estimate the hyperbolic length of σ in $\mathbb{C} \setminus E$ as follows,

$$\int_\sigma \lambda_{\mathbb{C} \setminus E}(w)|dw| \geq \sum_{j=-4n}^{4n} \int_{\sigma \cap D_j} \lambda_{\mathbb{C} \setminus E}(w)|dw| \geq c(8n-1).$$

■

Proof of Theorem 1 For the domain $\Omega_0 = \mathbb{C} \setminus (\bigcup E_n \cup \bar{\mathbb{D}})$ we have

$$\partial \Omega_0 = \mathbb{T} \cup \bigcup_{n=1}^{\infty} E_n.$$

Clearly $\partial \Omega$ has positive capacity (simply because $\mathbb{T}$ is connected). Hence by the theorem of R. Nevanlinna for the universal covering map $P : \mathbb{D} \to \Omega_0$ the boundary values exist almost everywhere, i.e., the set

$$\{\theta : \lim_{r \to 1} P(re^{i\theta}) \text{ exists}\}$$

has full measure in the interval $[0,2\pi[$. Observe that $\bigcup_{n=1}^{\infty} E_n$ is a countable set of isolated points in $\partial\Omega_0$, hence the set

$$\left\{\theta : \lim_{r\to 1} P(re^{i\theta}) \in \bigcup_{n=1}^{\infty} E_n\right\}$$

is at most countable. To prove the theorem it remains to show now that $\lim_{r\to 1} P(re^{i\theta}) \in \mathbb{T}$ implies that

$$\limsup_{r\to 1} |P'(re^{i\theta})| = \infty.$$

By our assumption that $\lim_{r\to 1} P(re^{i\theta}) \in \mathbb{T}$, we can select a sequence of points $\{z_n\}$ on the ray $[0, e^{i\theta}[$ such that $P(z_n) = w_n$ satisfies

a) $|w_n| = 1 + 2^{-2n-1}$,

b) $1 - |z_n| < 1 - |z_{n-1}|$,

Now we apply Lemma 1 to the sequence of points $\{w_n\}$, and we obtain

$$\begin{aligned} d_{\mathbb{D}}(z_n, z_{n-1}) &\geq d_{\Omega_0}(P(z_n), P(z_{n-1})) \\ &= d_{\Omega_0}(w_n, w_{n-1}) \\ &\geq cn. \end{aligned}$$

Therefore

$$\frac{1-|z_{n-1}|^2}{1-|z_n|^2} \geq c2^{cn},$$

where $c > 0$ is a universal constant and $n \in \mathbb{N}$.

Now we present simple geometric considerations to prepare for the application of the result of Beardon and Pommerenke. First we remark that for points $w_n \in \Omega$, satisfying $|w_n| = 1 + 2^{-2n-1}$ we have

$$\operatorname{dist}(w_{n-1}, E) \leq 5\operatorname{dist}(w_n, E).$$

Next we observe that for any $w \in \Omega_0$ with $|w| = 1 + 2^{-2n-1}$, and any $a \in E$ for which $|w - a| = \operatorname{dist}(w, E)$, there exists $b \in E$ such that

$$\left|\log\left|\frac{w-a}{b-a}\right|\right| \leq 2.$$

It now remains to combine the defining equation

$$\lambda_{\mathbb{C}\setminus E}(P(z))|P'(z)| = \frac{1}{1-|z|^2},$$

with the estimate of Beardon and Pommerenke,

$$\lambda_{\mathbb{C}\setminus E}(w) \sim \frac{1}{\operatorname{dist}(w,E)(\beta(w)+1)}$$

to obtain a lower estimate for $|P'(z)|$. Indeed we have

$$\begin{aligned}\frac{|P'(z_n)|}{|P'(z_{n-1})|} &= \frac{\lambda_{\mathbb{C}\setminus E}(w_{n-1})(1-|z_{n-1}|^2)}{\lambda_{\mathbb{C}\setminus E}(w_n)(1-|z_n|^2)}\\ &\geq c\frac{\operatorname{dist}(w_n,E)(1-|z_{n-1}|^2)}{\operatorname{dist}(w_{n-1},E)(1-|z_n|^2)}\\ &\geq c2^{cn},\end{aligned}$$

or,

$$|P'(z_n)| \geq c2^{cn}|P'(z_{n-1})|.$$

■

We close this note by discussing another domain Ω_1 with properties similar to those of Ω_0. We let $F_n = \{2^{-2n}e^{2\pi i 8^{-2n}m} : 1 \leq m \leq 8^{2n}\}$ be a set consisting of equidistributed points on the circle of radius 2^{-2n}. Then we let $F = \bigcup_{n=1}^{\infty} F_n \cup \{0\}$, and

$$\Omega_1 = \mathbb{C} \setminus F.$$

In the following list of remarks we sketch the boundary behavior of the universal covering map $P : \mathbb{D} \to \Omega_1$.

Remarks:

1. The set $\{\theta : \lim_{r\to 1} P(re^{i\theta}) \text{ exists}\}$ has measure zero in the interval $[0, 2\pi[$. This follows from the fact that the boundary of Ω_1 is countable.

2. The point $\{0\}$ is a cluster point in $\partial\Omega_1$. Hence by the theorem of J. Fernandez and M. Melian [F-M], the set $\{\theta : \lim_{r\to 1} P(re^{i\theta}) = 0\}$ has Hausdorff dimension equal to one.

3. One can now adapt the proof of Theorem 1 to show that

$$\limsup_{r\to 1} |P'(re^{i\theta})| = \infty$$

   whenever $\lim_{r\to 1} P(re^{i\theta}) = 0$.

4. The set $A = \{\theta : \lim_{r\to 1} P(re^{i\theta}) \in \bigcup_{n=1}^{\infty} F_n\}$ is at most countable, and for $\theta \in A$ we have

$$\int_0^1 |P''(re^{i\theta})|dr < \infty.$$

This follows by Proposition 5 in [J-M] from the fact that the points in $\bigcup_{n=1}^{\infty} F_n$ are isolated in $\partial\Omega_1$.

5. Our previous remarks 3) and 4) contain the following dichotomy: The boundary of Ω_1 can be decomposed as $\partial\Omega_1 = X \cup Y$, such that the following holds,

   a) If $\lim_{r\to 1} P(re^{i\theta}) \in X$ then $\limsup_{r\to 1} |P'(re^{i\theta})| = \infty$.

   b) If $\lim_{r\to 1} P(re^{i\theta}) \in Y$ then $\int_0^1 |P''(re^{i\theta})|dr < \infty$.

6. Note that the same dichotomy is also present in the examples Ω_0 and $\mathbb{C} \setminus \{-1, 1\}$.

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