

PS Topologie und Funktionalanalysis

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Note: References refer to the lecture notes.

1. Suppose $\sum_{n=1}^{\infty} |c_n| < \infty$. Show that

$$u(t, x) := \sum_{n=1}^{\infty} c_n e^{-(\pi n)^2 t} \sin(n\pi x),$$

is continuous for $(t, x) \in [0, \infty) \times [0, 1]$ and solves the heat equation for $(t, x) \in (0, \infty) \times [0, 1]$. (Hint: Weierstrass M-test. When can you interchange the order of summation and differentiation?)

2. Show that for $n, m \in \mathbb{N}$ we have

$$2 \int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Conclude that the Fourier sine coefficients are given by

$$\hat{u}_{0,n} = 2 \int_0^1 \sin(n\pi x) u_0(x) dx$$

provided the sum in (*) converges uniformly. Conclude that in this case the solution can be expressed as

$$u(t, x) = \int_0^1 K(t, x, y) u_0(y) dy, \quad t > 0,$$

where

$$\begin{aligned} K(t, x, y) &:= 2 \sum_{n=1}^{\infty} e^{-(\pi n)^2 t} \sin(n\pi x) \sin(n\pi y) \\ &= \frac{1}{2} \left(\vartheta\left(\frac{x-y}{2}, i\pi t\right) - \vartheta\left(\frac{x+y}{2}, i\pi t\right) \right). \end{aligned}$$

Here

$$\vartheta(z, \tau) := \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau + 2\pi i n z} = 1 + 2 \sum_{n \in \mathbb{N}} e^{i\pi n^2 \tau} \cos(2\pi n z), \quad \text{Im}(\tau) > 0,$$

is the Jacobi theta function.

3. Show that $|||f| - |g||| \leq \|f - g\|$.
4. Let X be a Banach space. Show that the norm, vector addition, and multiplication by scalars are continuous. That is, if $f_n \rightarrow f$, $g_n \rightarrow g$, and $\alpha_n \rightarrow \alpha$, then $\|f_n\| \rightarrow \|f\|$, $f_n + g_n \rightarrow f + g$, and $\alpha_n g_n \rightarrow \alpha g$.

5. While $\ell^1(\mathbb{N})$ is separable, it still has room for an uncountable set of linearly independent vectors. Show this by considering vectors of the form

$$a^\alpha = (1, \alpha, \alpha^2, \dots), \quad \alpha \in (0, 1).$$

(Hint: Recall the Vandermonde determinant.)

6. Prove Young's inequality

$$\alpha^{1/p} \beta^{1/q} \leq \frac{1}{p} \alpha + \frac{1}{q} \beta, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha, \beta \geq 0.$$

Show that equality occurs precisely if $\alpha = \beta$. (Hint: Take logarithms on both sides.)

7. Show that $\ell^\infty(\mathbb{N})$ is a Banach space.
8. Is $\ell^1(\mathbb{N})$ a closed subspace of $\ell^\infty(\mathbb{N})$ (with respect to the $\|\cdot\|_\infty$ norm)? If not, what is its closure?
9. Consider $\ell^1(\mathbb{N})$. Show that $\|a\| := \sup_{k \in \mathbb{N}} |\sum_{j=1}^k a_j|$ is a norm. Is $\ell^1(\mathbb{N})$ complete with this norm?
10. Show that $\ell^\infty(\mathbb{N})$ is not separable. (Hint: Consider sequences which take only the value one and zero. How many are there? What is the distance between two such sequences?)
11. Let X be a normed space. Show that the following conditions are equivalent.
- (i) If $\|x + y\| = \|x\| + \|y\|$ then $y = \alpha x$ for some $\alpha \geq 0$ or $x = 0$.
 - (ii) If $\|x\| = \|y\| = 1$ and $x \neq y$ then $\|\lambda x + (1-\lambda)y\| < 1$ for all $0 < \lambda < 1$.
 - (iii) If $\|x\| = \|y\| = 1$ and $x \neq y$ then $\frac{1}{2}\|x + y\| < 1$.
 - (iv) The function $x \mapsto \|x\|^2$ is strictly convex.

A norm satisfying one of them is called strictly convex.

12. Consider $X = C([-1, 1])$. Which of the following subsets are subspaces of X ? Which of them are closed?
- (i) monotone functions
 - (ii) even functions
 - (iii) polynomials
 - (iv) polynomials of degree at most k for some fixed $k \in \mathbb{N}_0$
 - (v) continuous piecewise linear functions
 - (vi) $C^1([-1, 1])$
 - (vii) $\{f \in C([-1, 1]) \mid f(c) = f_0\}$ for some fixed $c \in [-1, 1]$ and $f_0 \in \mathbb{R}$
13. Let I be a compact interval. Show that the set $Y := \{f \in C(I, \mathbb{R}) \mid f(x) > 0\}$ is open in $X := C(I, \mathbb{R})$. Compute its closure.
14. Compute the closure of the following subsets of $\ell^2(\mathbb{N})$: (i) $B_1 := \{a \in \ell^2(\mathbb{N}) \mid \sum_{j \in \mathbb{N}} |a_j| \leq 1\}$. (ii) $B_\infty := \{a \in \ell^2(\mathbb{N}) \mid \sum_{j \in \mathbb{N}} |a_j| < \infty\}$.

15. Show that, in a Hilbert space,

$$\sum_{1 \leq j < k \leq n} \|f_j - f_k\|^2 + \left\| \sum_{1 \leq j \leq n} f_j \right\|^2 = n \sum_{1 \leq j \leq n} \|f_j\|^2$$

for every $n \in \mathbb{N}$. The case $n = 2$ is the parallelogram law.

16. Show that the maximum norm on $C[0, 1]$ does not satisfy the parallelogram law.
17. Let X and Y be real normed spaces. Show that an additive (i.e. $T(x+y) = T(x) + T(y)$) continuous map $T : X \rightarrow Y$ is linear. What about complex spaces?
18. Suppose $\mathfrak{Q}$ is a complex vector space. Let $s(f, g)$ be a sesquilinear form on $\mathfrak{Q}$ and $q(f) := s(f, f)$ the associated quadratic form. Prove the **parallelogram law**

$$q(f+g) + q(f-g) = 2q(f) + 2q(g)$$

and the **polarization identity**

$$s(f, g) = \frac{1}{4} (q(f+g) - q(f-g) + i q(f-ig) - i q(f+ig)).$$

Show that $s(f, g)$ is symmetric if and only if $q(f)$ is real-valued.

Note, that if $\mathfrak{Q}$ is a real vector space, then the parallelogram law is unchanged but the polarization identity in the form $s(f, g) = \frac{1}{4}(q(f+g) - q(f-g))$ will only hold if $s(f, g)$ is symmetric.

19. Show that the integral defined in Example 1.15 satisfies

$$\int_c^e f(x) dx = \int_c^d f(x) dx + \int_d^e f(x) dx, \quad \left| \int_c^d f(x) dx \right| \leq \int_c^d |f(x)| dx.$$

How should $|f|$ be defined here?

20. Is it possible to define $\sin(f)$ for $f \in L^1(I)$? If yes, how should this be done? What about $\exp(f)$?
21. Show that in a metric space X a totally bounded set U is bounded.
22. Find a compact subset of $\ell^\infty(\mathbb{N})$ which does not satisfy (ii) from Theorem 1.12.
23. Which of the following families are relatively compact in $C[0, 1]$?
- (i) $F := \{f \in C^1[0, 1] \mid \|f\|_\infty \leq 1\}$
 - (ii) $F := \{f \in C^1[0, 1] \mid \|f'\|_\infty \leq 1\}$
 - (iii) $F := \{f \in C^1[0, 1] \mid \|f\|_\infty \leq 1, \|f'\|_2 \leq 1\}$

24. Consider $X = \mathbb{C}^n$ and let $A \in \mathfrak{L}(X)$ be a matrix. Equip X with the norm (show that this is a norm)

$$\|x\|_\infty := \max_{1 \leq j \leq n} |x_j|$$

and compute the operator norm $\|A\|$ with respect to this norm in terms of the matrix entries. Do the same with respect to the norm

$$\|x\|_1 := \sum_{1 \leq j \leq n} |x_j|.$$

25. Let $X := C[0, 1]$. Investigate the operator $A : X \rightarrow X$, $f(x) \mapsto x f(x)$. Show that this is a bounded linear operator and compute its norm. What is the closure of $\text{Ran}(A)$?
26. Show that the **Fredholm integral operator**

$$(Kf)(x) := \int_0^1 K(x, y)f(y)dy,$$

where $K(x, y) \in C([0, 1] \times [0, 1])$, defined on $\mathfrak{D}(K) := C[0, 1]$, is a bounded operator both in $X := C[0, 1]$ (max norm) and $X := L^2(0, 1)$. Show that the norm in the $X = C[0, 1]$ case is given by

$$\|K\| = \max_{x \in [0, 1]} \int_0^1 |K(x, y)|dy.$$

27. Let $A \in \mathfrak{L}(X)$ be a bijection. Show

$$\|A^{-1}\|^{-1} = \inf_{x \in X, \|x\|=1} \|Af\|.$$

28. Prove Lemma 1.19. (Hint: Hölder's inequality in $\mathbb{C}^n$ and note that equality is attained.)
29. Let X, Y be normed spaces and suppose $A \in \mathfrak{L}(X, Y)$. Show that $\text{Ker}(A)$ is closed. Suppose $M \subseteq \text{Ker}(A)$ is a closed subspace. Show that the induced map $\tilde{A} : X/M \rightarrow Y$, $[x] \mapsto Ax$ is a well-defined operator satisfying $\|\tilde{A}\| = \|A\|$ and $\text{Ker}(\tilde{A}) = \text{Ker}(A)/M$, $\text{Ran}(\tilde{A}) = \text{Ran}(A)$. In particular, $\tilde{A}$ is injective for $M = \text{Ker}(A)$.
30. Given some vectors $f_1, \dots, f_n$ we define their **Gram determinant** as

$$\Gamma(f_1, \dots, f_n) := \det(\langle f_j, f_k \rangle)_{1 \leq j, k \leq n}.$$

Show that the Gram determinant is nonzero if and only if the vectors are linearly independent. Moreover, show that in this case

$$\text{dist}(g, \text{span}\{f_1, \dots, f_n\})^2 = \frac{\Gamma(f_1, \dots, f_n, g)}{\Gamma(f_1, \dots, f_n)}$$

and

$$\Gamma(f_1, \dots, f_n) \leq \prod_{j=1}^n \|f_j\|^2.$$

with equality if the vectors are orthogonal. (Hint: First establish $\Gamma(f_1, \dots, f_j + \alpha f_k, \dots, f_n) = \Gamma(f_1, \dots, f_n)$ for $j \neq k$ and use it to investigate how Γ changes when you apply the Gram-Schmidt procedure?)

31. Let $\{u_j\}$ be some orthonormal basis. Show that a bounded linear operator A is uniquely determined by its matrix elements $A_{jk} := \langle u_j, Au_k \rangle$ with respect to this basis.
32. Let $I = (a, b)$ be some interval and consider the scalar product

$$\langle f, g \rangle := \int_a^b f(x)^* g(x) w(x) dx$$

associated with some positive weight function $w(x)$. Let $P_j(x) = x^j + \beta_j x^{j-1} + \gamma_j x^{j-2} + \dots$ be the corresponding monic orthogonal polynomials obtained by applying the Gram-Schmidt procedure (without normalization) to the monomials:

$$\int_a^b P_i(x) P_j(x) w(x) dx = \begin{cases} \alpha_j^2, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{P}_j(x) := \alpha_j^{-1} P_j(x)$ be the corresponding orthonormal polynomials and show that they satisfy the three term recurrence relation

$$a_j \bar{P}_{j+1}(x) + b_j \bar{P}_j(x) + a_{j-1} \bar{P}_{j-1}(x) = x \bar{P}_j(x),$$

or equivalently

$$P_{j+1}(x) = (x - b_j) P_j(x) - a_{j-1}^2 P_{j-1}(x),$$

where

$$a_j := \int_a^b x \bar{P}_{j+1}(x) \bar{P}_j(x) w(x) dx, \quad b_j := \int_a^b x \bar{P}_j(x)^2 w(x) dx.$$

Here we set $P_{-1}(x) = \bar{P}_{-1}(x) \equiv 0$ for notational convenience (in particular we also have $\beta_0 = \gamma_0 = \gamma_1 = 0$).