ENUMERATION OF ad-NILPOTENT $\mathfrak{b}$-IDEALS FOR SIMPLE LIE ALGEBRAS

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Dedicated to the memory of Rodica Simion

Abstract. We provide explicit formulas for the number of ad-nilpotent ideals of a Borel subalgebra of a complex simple Lie algebra having fixed class of nilpotence.

§1 Introduction

In this paper we provide formulas for the number of ad-nilpotent ideals of a Borel subalgebra of a complex simple Lie algebra $\mathfrak{g}$ having fixed class of nilpotence. Our results continue and complete the analysis of [OP] and [AKOP], where Lie algebras of type A have been treated.

The basic framework is the following. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n. Let $\mathfrak{h} \subset \mathfrak{g}$ be a fixed Cartan subalgebra, Δ the corresponding root system of $\mathfrak{g}$. Fix a positive system Δ^+ in Δ. For each $\alpha \in \Delta^+$ let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ relative to α, $\mathfrak{n} = \bigoplus\limits_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, and $\mathfrak{b}$ be the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$.

Let $\mathcal{I}^n$ denote the set of ad-nilpotent ideals (i.e., consisting of ad-nilpotent elements) of $\mathfrak{b}$. Then $\mathfrak{i} \in \mathcal{I}^n$ if and only if $\mathfrak{i}$ decomposes as $\mathfrak{i} = \bigoplus\limits_{\alpha \in \Phi} \mathfrak{g}_\alpha$, with $\Phi \subseteq \Delta^+$ being a dual order ideal in the poset $(\Delta^+, \leq)$ of positive roots (i.e., a subset of the positive roots with the property that whenever $\alpha \geq \beta$ with $\beta \in \Delta^+$, then also $\alpha \in \Delta^+$; the partial order $\leq$ being the restriction to Δ^+ of the usual partial order on the root lattice). Clearly, an ideal $\mathfrak{i} \in \mathcal{I}^n$ is nilpotent. We denote its class of nilpotence by $n(\mathfrak{i})$. (By definition, the class of nilpotence of an ideal $\mathfrak{i}$ is the smallest number m such that m-fold bracketing of $\mathfrak{i}$ with itself gives the zero ideal. Thus, Abelian ideals are exactly those with class of nilpotence at most 1.)

ad-Nilpotent ideals were studied by Kostant in [K1], [K2]. He introduced this notion in [K1], where he used it to obtain a useful equivalence criterion for certain decomposably-generated simple G-submodules of $\Lambda(\mathfrak{g})$ in terms of $\mathcal{I}^n$ (here, G is a compact semi-simple Lie group, and $\mathfrak{g}$ is the complexification of its Lie algebra). Moreover, he gave an explicit description of the eigenspace relative to the maximal

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX eigenvalue of the Laplace-Beltrami operator of G by means of the set of Abelian ideals. This subclass has been studied much more recently in [K2] in connection with discrete series representations. The latter paper was partly motivated by a striking enumerative result due to D. Peterson: the Abelian ideals are 2^n in number, independently of the type of $\mathfrak{g}$. (In contrast, as we shall see below, the cardinality of $\mathcal{I}^n$ depends on the type). Peterson's result has a twofold interpretation: on the one hand, looking at its proof one gets an encoding of the Abelian ideals by means of elements of the affine Weyl group of $\mathfrak{g}$; on the other hand it is in its own right an elegant enumerative result for ideals whose class of nilpotence is at most 1.

These results lead to ask similar questions for ad-nilpotent ideals in general: find an appropriate encoding and enumerate $\mathcal{I}^n$ by class of nilpotence. The first problem was settled in [CP], where an encoding in the spirit of Kostant–Peterson was found. The second question, for $\mathfrak{g}$ of type A, was addressed and given a solution in [OP] and [AKOP]. Interestingly, it turns out that in type A ideals in $\mathcal{I}^n$ with class of nilpotence K are equinumerous with Dyck paths of length $2n+2$ and of height $K+1$ (see [AKOP, Theorem 4.4]). In fact, in [AKOP, Sec. 5] an explicit bijection between the former set of ideals and the latter set of Dyck paths is constructed. Several explicit formulas for the number of these ideals can be given (see [OP] and [AKOP, Theorems 4.2, 4.5, 4.6]).

In the present paper we resolve the enumerative question in all other cases. In types B, C, and D we give explicit formulas, expressed in terms of Chebyshev polynomials of the second kind, for the generating function for ideals in $\mathcal{I}^n$ with class of nilpotence K (see the theorems labelled B, C, and D in Section 2). In types E, F, and G we have computed the number of ideals in $\mathcal{I}^n$ with a given class of nilpotence in a rather straightforward way, using a computer (see Section 8).

In type C we are again able to establish a relationship between the ideals in $\mathcal{I}^n$ and certain paths: in Theorem C.3 we prove that in type C ideals in $\mathcal{I}^n$ with class of nilpotence K are equinumerous with paths with step vectors $(1,1)$ and $(1,-1)$ of length $2n$ and height $K+1$, which start at the origin and never pass below the x-axis. However, in contrast to type A, we are not able to provide an explicit bijection between this set of ideals and this set of paths. In types B and D we do not even know any relationship between the ideals in $\mathcal{I}^n$ with class of nilpotence K and any set of paths. It seems indeed that paths are not the right objects to consider in the case of type B and type D. We believe that a fruitful direction for further investigation is to try to relate ideals in $\mathcal{I}^n$ in any of the types A, B, C, or D to non-crossing partitions of the corresponding types. Indeed, in any of these types, the sequence of numbers of ideals in $\mathcal{I}^n$ is identical with the sequence of numbers of non-crossing partitions (cf. [Si] and [R, Cor. 10]). We speculate that there is a naturally defined statistics on non-crossing partitions, such that, in any of the types A, B, C, or D, the ideals in $\mathcal{I}^n$ with class of nilpotence K are equinumerous with the non-crossing partitions (in the set corresponding to $\mathcal{I}^n$) on which this statistics has the value K. (In fact, a rougher version of that problem was already stated in Remark 2 of [R]. There, the problem is posed to relate non-crossing partitions to *non-nesting* partitions. However, non-nesting partitions are by definition antichains of roots, which in turn are in bijection with our ad-nilpotent ideals. See also the final paragraph of the introduction.)

Our paper is organized as follows. In the next section we first summarize the enumerative results from [OP] and [AKOP] in type A, in order to make a comparison with our results in types B, C, and D easy. We state the results in types B, C

and D subsequently, also in Section 2. In Section 3 we review the encoding of ad-nilpotent $\mathfrak{b}$-ideals in type A in terms of subdiagrams of the staircase shape, and the geometric algorithm to determine class of nilpotence in type A given in [AKOP]. This algorithm is of particular importance for our analysis in types B, C, and D. For in Section 4 we show that ad-nilpotent $\mathfrak{b}$-ideals in types B, C, and D can be encoded as certain subdiagrams of a *shifted* staircase (as done originally by Shi [Sh]), and, for the determination of class of nilpotence, we may in fact apply the type A algorithm to diagrams that are obtained by conjugation operations from the subdiagrams of the shifted staircase. We use these observations to prove the enumerative results of Section 2 in the subsequent sections. Section 5 is devoted to the proofs in type C, Section 6 to the proofs in type B, and Section 7 to the proofs in type D. While the results follow quite easily in type C, types B and D need a more refined (though still elementary) analysis. Finally, our enumerative results in the exceptional types are reported in Section 8.

It is interesting to note that ad-nilpotent ideals in a Borel subalgebra have appeared in various (equivalent) disguises in the literature. In [Sh] Shi studied *$\oplus$-sign types*, a relevant notion in the theory of *cells* for Weyl groups. As he shows these are in bijection with dual order ideals in the set of positive roots, which in turn are in bijection with our ad-nilpotent ideals, as we have argued right at the beginning. In another direction, Postnikov introduced, for any of the classical and exceptional root systems, *non-nesting partitions* (see [R, Remark 2]). As we already mentioned, these are by definition antichains of positive roots (a definition that translates to a nice combinatorial interpretation in the classical types), which in turn are in bijection with dual order ideals in the set of positive roots, and, hence, again with our ad-nilpotent ideals. Postnikov also showed that non-nesting partitions are in bijection with the regions of the *Catalan arrangement* contained in the fundamental chamber. These hyperplane arrangements have for instance been investigated by Athanasiadis [A1, A2]. Interestingly, Reiner [R, Remark 2] states a uniform formula for the total number of non-nesting partitions, and, thus, for the *total number* of ad-nilpotent $\mathfrak{b}$-ideals in a Borel subalgebra for $\mathfrak{g}$ of any type, namely

$$\prod_{i=1}^{n} \frac{h+e_i+1}{e_i+1}, \tag{1.1}$$

where h is the Coxeter number and $e_1, e_2, \dots, e_n$ are the exponents of Δ (cf. [Hu; Sec. 3.19]). This formula can easily be checked case by case, by comparing it with the known results on these numbers. For the classical types one may consult [CP, Theorem 3.1] or [Sh, Sec. 3], and for the exceptional types [Sh, Sec. 3], except that the number for E_6 in [Sh] has to be corrected; see the last paragraph and Table 1 in Section 8. Alternatively, for the classical types, one may consult [A1, Theorem 5.5] (cf. also [A2, Theorem 3.1]). However, it would be desirable to find a uniform proof for this formula.

§2 The enumeration of ad-nilpotent $\mathfrak{b}$-ideals in the classical types

We begin by recalling the enumerative results from [OP] and [AKOP] in type A.

Theorem A.1. [OP], [AKOP, Theorem 4.2] *Let $\mathfrak{g}$ be of type A_n. Let $\alpha_n(K)$ denote* *the number of ideals in $\mathcal{I}^n$ with class of nilpotence K. Then*

$$\alpha_n(K) = \sum_{0=i_0<i_1<\cdots<i_K<i_{K+1}=n+1} \prod_{j=0}^{K-1} \binom{i_{j+2}-i_j-1}{i_{j+1}-i_j}. \tag{A.1}$$

The combinatorial meaning of the previous formula has been analysed in detail in [AKOP]. In particular, it connects to the enumeration of Dyck paths. Recall that a Dyck path is a lattice path with diagonal step vectors $(1,1)$ and $(1,-1)$, which starts at the origin and returns to the x-axis, and which does not pass below the x-axis. We define the height of a Dyck path to be the maximum ordinate of its peaks.

Theorem A.2. [AKOP, Theorem 4.4] *For $\mathfrak{g}$ of type A_n, the number of ideals in $\mathcal{I}^n$ with class of nilpotence K is exactly the same as the number of Dyck paths from $(0,0)$ to $(2n+2,0)$ with height $K+1$.*

Generating function results for Dyck paths translate into the following result for ideals with a given class of nilpotence (cf. the fact quoted in the paragraph before Theorem C.3). Let $U_n(x)$ denote the nth Chebyshev polynomial of the second kind, $U_n(\cos t) = \sin((n+1)t)/\sin t$, or, explicitly,

$$U_n(x) = \sum_{j\ge 0} (-1)^j \binom{n-j}{j} (2x)^{n-2j}.$$

In the statement of the following result, and also later in the paper, we write $\tilde{U}_k$ as short-hand for $U_k\left(1/2\sqrt{x}\right)$.

Theorem A.3. [AKOP, Theorem 4.6] *Let $\mathfrak{g}$ be of type A_n. Let $\alpha_n^{\le}(h)$ denote the number of ideals in $\mathcal{I}^n$ with class of nilpotence at most h. Then*

$$1 + \sum_{n=0}^{\infty} \alpha_n^{\le}(h) x^{n+1} = \frac{\tilde{U}_{h+1}}{\sqrt{x}\,\tilde{U}_{h+2}}. \tag{A.3}$$

A nice implication of this theorem is that not only is the number of Abelian ideals given by a nice compact formula, namely 2^n, but there are also similarly nice formulae for ideals with class of nilpotence at most 2 and 3.

Corollary A.4. [AKOP, Corollary 4.7] *For $\mathfrak{g}$ of type A_n, the number of ideals in $\mathcal{I}^n$ with class of nilpotence at most 2 is the Fibonacci number F_{2n}. The number of ideals in $\mathcal{I}^n$ with class of nilpotence at most 3 is $(3^n+1)/2$.*

Denote by $\alpha_n(h,K)$ the number of such ideals with dimension h and class of nilpotence K and set

$$C_n(q,t) = \sum_{h,K\ge 0} \alpha_n(h,K) t^h q^K. \tag{A.4}$$

This notation reflects the fact that $C_n(1,1)$ is the $(n+1)$st Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$, so that the polynomials $C_n(q,t)$ are (q,t)-analogues of the Catalan numbers[1]. In [AKOP] we proved the following formula for $C_n(q,t)$.

[1]Although these (q,t)-analogues of the Catalan numbers are different from the (q,t)-Catalan

Theorem A.5. [AKOP, Theorem 6.1] *We have*

$$C_n(q,t)=\sum_{K=0}^{n}\left(\sum_{0=i_0<i_1<\dots<i_K<i_{K+1}=n+1}\prod_{j=0}^{K-1}t^{i_{j+1}(i_{j+3}-i_{j+2})}\begin{bmatrix}i_{j+2}-i_j-1\\ i_{j+1}-i_j\end{bmatrix}_t\right)q^K, \tag{A.5}$$

with $i_{K+2}=n+2$. Here, $\left[\begin{smallmatrix}m\\ n\end{smallmatrix}\right]_t$ is the t-binomial coefficient, defined by

$$\begin{bmatrix}m\\ n\end{bmatrix}_t=\begin{cases}\frac{(1-t^m)(1-t^{m-1})\cdots(1-t^{m-n+1})}{(1-t^n)(1-t^{n-1})\cdots(1-t)} & \text{if } m\ge n>0,\\ 1 & \text{if } n=0,\\ 0 & \text{in any other case.}\end{cases}$$

In this paper we provide analogues of Theorems A.1 and A.3 and of Corollary A.4 for $\mathfrak{g}$ of type B, C, and D. We are able to give an analogue of Theorem A.2 only in type C. Finally, again in type C, we also provide an analogue of Theorem A.5, thus obtaining a (q,t)-analogue of the binomial coefficient $\binom{2n}{n}$. In principle, we would also be able to write down analogues of Theorem A.5 in types B and D. However, the corresponding formulas are rather unwieldy, so that we prefer to omit these for the sake of brevity.

Now we state these results. We begin with the type C analogue of Theorem A.1.

Theorem C.1. *Let $\mathfrak{g}$ be of type C_n. Let $\gamma_n(K)$ denote the number of ideals in $\mathcal{I}^n$ with class of nilpotence K. Then*

$$\gamma_n(K)=\begin{cases}\sum_{0<i_1<\dots<i_k<i_{k+1}=n}\prod_{j=1}^{k-1}\binom{i_{j+2}-i_j-1}{i_{j+1}-i_j}\cdot\sum_{\ell=0}^{i_2-i_1-1}\binom{i_1+i_2-1}{\ell} & \text{if } K=2k,\\ \sum_{-i_2<i_1\le 0<i_2<\dots<i_k<i_{k+1}=n}2^{i_1+i_2-1}\prod_{j=1}^{k-1}\binom{i_{j+2}-i_j-1}{i_{j+1}-i_j} & \text{if } K=2k-1,\end{cases} \tag{C.1}$$

where $i_{K+2}=n+1$.

Next we state the type C analogue of Theorem A.3.

Theorem C.2. *Let $\mathfrak{g}$ be of type C_n. Let $\gamma_n^{\le}(h)$ be the number of ideals in $\mathcal{I}^n$ with class of nilpotence at most h. Then the generating function $\sum_{n\ge0}\gamma_n^{\le}(h)x^n$ is given by*

$$\frac{\sum_{i=0}^{\lceil h/2\rceil}\tilde U_{h+1-2i}}{\sqrt{x}\,\tilde U_{h+2}}=\frac{\tilde U_{h+1}+\tilde U_{h-1}+\tilde U_{h-3}+\cdots}{\sqrt{x}\,\tilde U_{h+2}}. \tag{C.2}$$

Theorem C.2 allows to relate the ideals of given class of nilpotence again to path combinatorics. Appealing to the fact (see for example [Kr, Theorem A2 + Fact A3])

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numbers of Garsia and Haiman [GHn], a combinatorial interpretation of the latter (q,t)-Catalan numbers has recently been found by Haglund [H] (the proof [GHd] jointly with Garsia) which is amazingly similar to the combinatorial interpretation of (A.4) of [AKOP] (recalled in Section 2) in terms of two statistics on Ferrers diagrams contained in a staircase. Whereas the first statistics is (basically) identical in both cases (in [AKOP] it is the area of the diagram, respectively in [H] and [GHd] it is the difference of the area of the staircase and the area of the diagram), the difference in the second statistics can be best described by saying that the statistics of [AKOP] (the number of touching points on the diagonal line that bounds the Ferrers diagram; see Figure 1) is a descent-like statistics, whereas Haglund's statistics is the corresponding major-like statistics.

that the generating function $\sum_P x^{\ell(P)/2}$ for paths P with step vectors $(1,1)$ and $(1,-1)$, which start at the origin, never exceed height k (the height being again defined as the maximum ordinate of the peaks of the path), and end at height s ($\ell(P)$ denoting the length of P) is given by

$$\frac{\tilde{U}_{k-s}}{\sqrt{x}\,\tilde{U}_{k+1}},$$

we obtain the following type C analogue of Theorem A.2.

Theorem C.3. *For $\mathfrak{g}$ of type C_n, the number of ideals in $\mathcal{I}^n$ with class of nilpotence K is exactly the same as the number of paths with step vectors $(1,1)$ and $(1,-1)$, which start at the origin, have length $2n$ and height $K+1$, and never pass below the x-axis.*

As we already said in the introduction, it would be desirable to find an explicit bijection between the ideals and the paths in the above theorem.

This lattice path interpretation of the number of ideals with given class of nilpotence, together with the iterated reflection principle formula for paths that stay between two parallel lines (cf. [Mo, Ch. 1, Th. 2]), implies, upon little simplification, an explicit formula for this number. This formula must be preferred over the one in (C.1), as it is much simpler and computationally superior. (An analogous formula in type A has been given in [AKOP, Eq. (4.6)].)

Corollary C.4. *For $\mathfrak{g}$ of type C_n, the number of ideals in $\mathcal{I}^n$ with class of nilpotence at most h is given by*

$$\sum_{s=0}^{\lfloor h/2\rfloor}\sum_{k\in\mathbb{Z}}\frac{1+2s+2k(h+2)}{2n+1}\binom{2n+1}{n-s-k(h+2)}. \tag{C.4}$$

Using Theorem C.2 again, it is easy to deduce the following type C analogue of Corollary A.4. There, F_n denotes again the nth Fibonacci number.

Corollary C.5. *Let $\mathfrak{g}$ be of type C_n. Then for $n\ge 1$ the number of ideals in $\mathcal{I}^n$ with class of nilpotence at most 2 is F_{2n}, and the number of ideals with class of nilpotence at most 3 is $2\cdot 3^{n-1}$.*

Finally, we state the type C analogue of Theorem A.5.

Theorem C.6. *Let $\mathfrak{g}$ be of type C_n. Let $\gamma_n(h,K)$ be the number of ideals in $\mathcal{I}^n$ with dimension h and class of nilpotence K. Then*

$$\sum_{h,K\ge 0}\gamma_n(h,K)t^hq^K=\sum_{k=0}^{n}\sum_{-i_2<i_1<i_2<\dots<i_k<i_{k+1}=n} q^{2k-\chi(i_1\le 0)}\\
\cdot\left(\prod_{j=1}^{k-1}t^{(i_{j+1}+n)(i_{j+3}-i_{j+2})}\begin{bmatrix}i_{j+2}-i_j-1\\ i_{j+1}-i_j\end{bmatrix}_t\right.\\
\left.\cdot\sum_{\ell=0}^{i_2-i_1-1}\begin{bmatrix}i_1+i_2-1\\ \ell\end{bmatrix}_t t^{(i_1+n)(i_3-i_2)-\binom{n-i_2+1}{2}+\binom{\ell+1}{2}}\right), \tag{C.6}$$

where $i_{k+2} = n+1$, $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_t$ is the t-binomial coefficient defined in the statement of Theorem A.5, and χ is the usual truth function, $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise.

Clearly, since in type C the total number of ideals in $\mathcal{I}^n$ is the central binomial coefficient $\binom{2n}{n}$ (see Corollary 4.2), the expression in (C.6) is a (q,t)-analogue of $\binom{2n}{n}$.

Remark. The innermost sum in (C.6) does in fact simplify if $i_1 \le 0$. For, by means of the q-binomial theorem (cf. [GR, Ex. 1.2(vi)]), we have

$$\begin{aligned}\sum_{\ell=0}^{i_2-i_1-1} \begin{bmatrix} i_1+i_2-1 \\ \ell \end{bmatrix}_t t^{\binom{\ell+1}{2}} &= \sum_{\ell=0}^{\infty} \begin{bmatrix} i_1+i_2-1 \\ \ell \end{bmatrix}_t t^{\binom{\ell+1}{2}} \\ &= (1+t)(1+t^2)\cdots(1+t^{i_1+i_2-1}).\end{aligned} \tag{C.7}$$

In view of this, it is now obvious that the special case $t=1$ of Theorem C.6 implies Theorem C.1.

Now we state our results for type B.

Theorem B.1. *Let $\mathfrak{g}$ be of type B_n. Let $\beta_n(K)$ denote the number of ideals in $\mathcal{I}^n$ with class of nilpotence K. Then the generating function $\sum_{n\ge 0}\beta_n(K)x^n$ is given by*

$$\begin{cases} \dfrac{\tilde U_{2k} + \tilde U_k \tilde U_{k+1} \tilde U_{2k-1}}{\sqrt{x}\,\tilde U_{2k}\tilde U_{2k+1}\tilde U_{2k+2}} & \text{if } K = 2k, \\[2ex] \dfrac{\tilde U_{2k} + \tilde U_{k+1}^2 \tilde U_{2k-2} + \tilde U_{k-1}^2 \tilde U_{2k-2} + \tilde U_2}{\tilde U_{2k-1}\tilde U_{2k}\tilde U_{2k+1}} & \text{if } K = 2k-1. \end{cases} \tag{B.1}$$

Analogues of formula (A.1) for type B will be established in the proof of the above theorem in Section 6. An easy consequence of Theorem B.1 is the following type B analogue of Theorem A.3. It can be proved by a straightforward induction on h.

Corollary B.2. *Let $\mathfrak{g}$ be of type B_n. Let $\beta_n^{\le}(h)$ denote the number of ideals in $\mathcal{I}^n$ with class of nilpotence at most h. Then the generating function $\sum_{n\ge 0}\beta_n^{\le}(h)x^n$ is given by*

$$\begin{cases} \dfrac{\sum_{i=1}^{h/2}(2\,i+1)\;\tilde U_{2i-1} + (h+1)\tilde U_{h+1} + \sum_{i=1}^{h/2} 2i\,\tilde U_{2h+3-2i}}{\tilde U_{h+1}\,\tilde U_{h+2}} & \text{if } h \text{ is even}, \\[2ex] \dfrac{\sum_{i=1}^{(h-1)/2}(2\,i+1)\;\tilde U_{2i-1} + (h+1)\tilde U_{h} + \sum_{i=1}^{(h+1)/2} 2i\,\tilde U_{2h+3-2i}}{\tilde U_{h+1}\,\tilde U_{h+2}} & \text{if } h \text{ is odd}. \end{cases} \tag{B.2}$$

The above theorem implies a type B analogue of Corollary A.4.

Corollary B.3. *Let $\mathfrak{g}$ be of type B_n. Then for $n \ge 1$ the number of ideals in $\mathcal{I}^n$ with class of nilpotence at most 2 is $F_{2n} + F_{2n-2} - 2^{n-1}$, and the number of ideals with class of nilpotence at most 3 is $\frac{1}{2}(5\cdot 3^{n-1}+1) - F_{2n-2}$.*

Next we state our results for type D.

Theorem D.1. *Let $\mathfrak g$ be of type D_n. Let $\delta_n(K)$ denote the number of ideals in $\mathcal I^n$ with class of nilpotence K. Then the generating function $\sum_{n\ge0}\delta_n(K)x^n$ is given by $x/(1-x)$ if $K=0$, and otherwise by*

$$\begin{cases} \dfrac{\sqrt{x}\,(2-\tilde U_{2k}+2\tilde U_{2k+2}+3\tilde U_k\tilde U_{k+1}\tilde U_{2k-1})}{\tilde U_{2k}\tilde U_{2k+1}\tilde U_{2k+2}} & \text{if } K=2k,\\ \dfrac{x\,(2\tilde U_{2k+2}+\tilde U_{2k}-3\tilde U_{2k-2}+\tilde U_k\tilde U_{k+1}\tilde U_{2k-1}+4\tilde U_k^2\tilde U_{2k-2}+\tilde U_{k-1}\tilde U_k\tilde U_{2k-3}+2)}{\tilde U_{2k-1}\tilde U_{2k}\tilde U_{2k+1}} & \\ & \text{if } K=2k-1. \end{cases} \tag{D.1}$$

Analogues of formula (A.1) for type D will be established in the proof of the above theorem in Section 7. An easy consequence of Theorem D.1 is the following type D analogue of Theorem A.3. Once again, it can be proved by a straightforward induction on h.

Corollary D.2. *Let $\mathfrak g$ be of type D_n. Let $\delta_n^{\le}(h)$ denote the number of ideals in $\mathcal I^n$ with class of nilpotence at most h. Then the generating function $\sum_{n\ge0}\delta_n^{\le}(h)x^n$ is given by $x/(1-x)$ if $h=0$, by $x(1+2x)/(1-2x)$ if $h=1$, and otherwise by*

$$\begin{cases} x\Big(6\tilde U_1+\sum_{i=1}^{(h-2)/2}(6i+8)\,\tilde U_{2i+1}+(3h+4)\tilde U_{h+1} & \\ \quad+\sum_{i=0}^{(h-2)/2}(6i+5)\,\tilde U_{2h+1-2i}+\tilde U_{2h+3}\Big)\Big/\tilde U_{h+1}\tilde U_{h+2} & \text{if } h \text{ is even},\\ x\Big(6\tilde U_1+\sum_{i=1}^{(h-3)/2}(6i+8)\,\tilde U_{2i+1}+(3h+4)\tilde U_{h} & \\ \quad+\sum_{i=0}^{(h-1)/2}(6i+5)\,\tilde U_{2h+1-2i}+\tilde U_{2h+3}\Big)\Big/\tilde U_{h+1}\tilde U_{h+2} & \text{if } h \text{ is odd}. \end{cases} \tag{D.2}$$

The above theorem implies a type D analogue of Corollary A.4.

Corollary D.3. *Let $\mathfrak g$ be of type D_n. Then for $n\ge2$ the number of ideals in $\mathcal I^n$ with class of nilpotence at most 2 is $5F_{2n-3}-2^{n-2}$, and the number of ideals with class of nilpotence at most 3 is $\frac{13}{2}\cdot3^{n-2}-\frac32+4F_{2n}-7F_{2n-1}$.*

§3 *ad*-NILPOTENT IDEALS IN TYPE A

Throughout this section, $\mathfrak g$ will be of type A_n, i.e., $\mathfrak g$ is the Lie algebra $sl(n+1,\mathbb C)$ of $(n+1)\times(n+1)$ traceless matrices. In the following paragraphs we collect the findings from [AKOP] on how to encode ideals in $\mathcal I^n$ and how to efficiently compute their class of nilpotence.

Recall from the introduction that an ideal $\mathfrak i$ is in $\mathcal I^n$, i.e., is an ad-nilpotent ideal of our fixed Borel subalgebra $\mathfrak b$, if and only if $\mathfrak i$ can be written as $\mathfrak i=\bigoplus_{\alpha\in\Phi}\mathfrak g_\alpha$, where $\Phi\subseteq\Delta^+$ is a dual order ideal in the poset $(\Delta^+,\le)$ of positive roots. This allows us to represent ad-nilpotent ideals conveniently in a geometric fashion, which will be crucial in all subsequent considerations. Clearly, any positive root in A_n can be written as a sum of simple roots. Explicitly, with $\alpha_1,\alpha_2,\dots,\alpha_n$ denoting the simple roots, let us write $\tau_{i,j}=\alpha_i+\dots+\alpha_{n-j+1}$, $1\le i\le n$, $1\le j\le n-i+1$. If we place the roots $\tau_{i,j}$, $j=1,2,\dots,n-i+1$, in the ith row of a diagram, then this defines an arrangement of the positive roots in a staircase fashion. For example, for A_3 we obtain the arrangement

$$\begin{array}{lll} \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1 \\ \alpha_2+\alpha_3 & \alpha_2 & \\ \alpha_3 & & \end{array}$$

Obviously, the above defines an identification of the positive roots with the cells of the staircase diagram $(n, n-1, \dots, 1)$ (for all partition notation we refer the reader to [Ma, Ch. I, Sec. 1]), in which the root $\tau_{i,j}$ is identified with the cell (i, j). Given an ad-nilpotent ideal $\mathfrak{i}$, written as $\mathfrak{i} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, for some collection Φ of positive roots, we can use the above identification to represent $\mathfrak{i}$ as the set of cells that corresponds to the roots in Φ. Since, as we noted above, $\mathfrak{i}$ is a dual order ideal, the set of cells obtained forms a (Ferrers diagram of a) partition. For example, the ideal $\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2+\alpha_3}$ corresponds to the partition $(3, 1, 0)$. This defines a bijection between ad-nilpotent ideals in $sl(n+1, \mathbb{C})$ and subdiagrams of $(n, n-1, \dots, 1)$.

Now, let $m_{i,j}$ be the maximal number m such that the root space $\mathfrak{g}_{\tau_{i,j}}$ occurs in

$$\mathfrak{i}^m := \underbrace{[\cdots[\mathfrak{i},\mathfrak{i}],\dots]}_{m \text{ occurrences of } \mathfrak{i}} .$$

Alternatively, $m_{i,j}$ is the maximal number m such that there is a decomposition of $\tau_{i,j}$ of the form $\tau_{i,j} = \beta_1 + \cdots + \beta_m$ with all β_i's in Φ, the collection of positive roots corresponding to $\mathfrak{i}$. Written succinctly,

$$m_{i,j} = \max\{m : \tau_{i,j} = \beta_1 + \cdots + \beta_m,\ \text{for some } \beta_l \in \Phi,\ l = 1, 2, \dots, m\}. \tag{3.1}$$

By definition, the class of nilpotence of $\mathfrak{i}$ is given by $\max_{i,j} m_{i,j}$. We claim, first, that the numbers $m_{i,j}$ satisfy $m_{i,j} \geq m_{i+1,j}$ and $m_{i,j} \geq m_{i,j+1}$ for all i and j, and, second, that the $m_{i,j}$ (hence, in particular, the class of nilpotence) can be obtained by the following algorithm (see [AKOP, Prop. 3.1]): let λ be the subdiagram of $(n, n-1, \dots, 1)$ that corresponds to $\mathfrak{i}$ according to the identification explained above. Define a filling $(t_{i,j})_{1\leq i\leq n,\, 1\leq j\leq n-i+1}$ of the cells of $(n, n-1, \dots, 1)$ by recursively setting

$$t_{i,j} = \begin{cases} 0 & \text{if } (i,j) \notin \lambda, \\ 1 & \text{if } (i,j) \text{ is a corner cell of } \lambda, \\ \max\limits_{j<k\leq n-i+1} \{t_{i,k} + t_{n-k+2,j}\} & \text{otherwise.} \end{cases} \tag{3.2}$$

It is easy to see that the above rule uniquely defines a filling of the shape $(n, n-1, \dots, 1)$, whose nonzero entries are precisely those corresponding to the cells of λ. E.g., when $n = 4$, the fillings corresponding to $(2, 1, 0, 0)$, $(3, 3, 2, 1)$, $(4, 3, 2, 1)$ are respectively

$$\begin{array}{llll} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & \\ 0 & 0 & & \\ 0 & & & \end{array} \qquad \begin{array}{llll} 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & \\ 2 & 1 & & \\ 1 & & & \end{array} \qquad \begin{array}{llll} 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & \\ 2 & 1 & & \\ 1 & & & \end{array}$$

The claim is that $m_{i,j} = t_{i,j}$ for all i and j.

Let us go through a proof of these two facts that allows to be "recycled" when we shall discuss the computation of class of nilpotence in types B, C, and D. For proving the first claim, suppose that there is a representation of $\tau_{i+1,j}$ as

$$\tau_{i+1,j} = \beta_1 + \cdots + \beta_m, \tag{3.3}$$

with all β_l's in Φ. Then one of the β_l's must be equal to some $\tau_{i+1,k}$, because otherwise the sum on the right-hand side of (3.3), when expanded as a sum of simple roots, would either contain no α_{i+1} or in addition to α_{i+1} also α_i. Without loss of generality let $\beta_1 = \tau_{i+1,k}$. Then

$$\tau_{i,j} = \alpha_i + \tau_{i+1,j} = \alpha_i + \beta_1 + \cdots + \beta_m = \tau_{i,k} + \beta_2 + \cdots + \beta_m.$$

Hence we have $m_{i,j} \ge m_{i+1,j}$. A similar argument proves $m_{i,j} \ge m_{i,j+1}$.

For proving the second claim, we do a reverse induction on $i + j$. The claim is obvious if $i + j = n + 1$. For the induction step, assume that the claim is right for all (i', j') with $i' + j' > i + j$. Now consider the root $\tau_{i,j}$. If $j < k \le n - i + 1$, then we can write it as

$$\tau_{i,j} = \tau_{i,k} + \tau_{n-k+2,j}. \tag{3.4}$$

By the induction hypothesis and (3.1) we know that

$$t_{i,k} = \max\{m : \tau_{i,k} = \beta_1 + \cdots + \beta_m, \text{ for some } \beta_l \in \Phi,\ l = 1, 2, \dots, m\}$$

and

$$t_{n-k+2,j} = \max\{m : \tau_{n-k+2,j} = \beta_1 + \cdots + \beta_m, \text{ for some } \beta_l \in \Phi,\ l = 1, 2, \dots, m\} \tag{3.5}$$

Combined with (3.4), this implies that

$$t_{i,k} + t_{n-k+2,j} \le m_{i,j} \tag{3.6}$$

as long as $t_{i,k}$ and $t_{n-k+2,j}$ are nonzero. But (3.6) is also true if one or both of $t_{i,k}$ and $t_{n-k+2,j}$ should be zero. If both are zero, then there is nothing to prove. If, for instance, $t_{i,k}$ is nonzero, then we have $t_{i,k} = m_{i,k} \le m_{i,j}$, the equality being true by induction hypothesis, the inequality being true because of our first claim.

But $m_{i,j}$ cannot be larger than $t_{i,k} + t_{n-k+2,j}$. For, as we already noted (just replace i by $i - 1$ in the previous argument), in any decomposition

$$\tau_{i,j} = \beta_1 + \cdots + \beta_m, \tag{3.7}$$

with the β_l's being positive roots, one of the β_l's must be equal to some $\tau_{i,k}$. Again, without loss of generality let $\beta_1 = \tau_{i,k}$. Then we have $\beta_2 + \cdots + \beta_m = \tau_{i,j} - \tau_{i,k} = \tau_{n-k+2,j}$. By induction hypothesis we have (3.5), and hence $m - 1 \le t_{n-k+2,j}$. It follows that $m \le 1 + t_{n-k+2,j} \le t_{i,k} + t_{n-k+2,j} \le t_{i,j}$, as required.

Since $m_{i,j} \ge m_{i+1,j}$ and $m_{i,j} \ge m_{i,j+1}$, the same property must be true for the $t_{i,j}$'s. In particular, this implies that the maximum of all entries is entry $t_{1,1}$, so that in fact the class of nilpotence of an ideal $\mathfrak{i}$ is equal to $t_{1,1}$.

If one is only interested to quickly compute the class of nilpotence of some ideal $\mathfrak{i}$ (i.e., just the entry $t_{1,1}$ of the filling constructed by (3.2)), then there is a short-cut through the algorithm (3.2). For a convenient statement of the result, we write, in abuse of notation, $n(\lambda_1, \lambda_2, \dots, \lambda_n)$ for $n(\mathfrak{i})$, given that the partition corresponding to $\mathfrak{i}$ is $(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Proposition 2.1. [AKOP, Prop. 3.2] *Let $\mathfrak{i} \in \mathcal{I}^n$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the corresponding partition. We have $n(0, 0, \dots, 0) = 0$, and otherwise*

$$n(\lambda_1, \lambda_2, \dots, \lambda_n) = n(\lambda_{n+2-\lambda_1}, \dots, \lambda_n) + 1. \tag{3.8}$$

It should be noted that on the left-hand side of (3.8) there appears the class of nilpotence of an ideal in $\mathcal{I}^n$, whereas on the right-hand side there appears the class of nilpotence of an ideal in $\mathcal{I}^{\lambda_1 - 1}$ (with corresponding partition $(\lambda_{n+2-\lambda_1}, \dots, \lambda_n)$). The computation, however, can be carried out completely formally, without reference to ideals, as we now demonstrate by an example.

Example. Let $\mathfrak{i} \in \mathcal{I}^{13}$ be the ideal which corresponds to the partition $(10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0)$. (This is the partition in Figure 1. At this point, all dotted lines should be ignored.) Then, by applying Proposition 2.1 iteratively, we obtain for the class of nilpotence of $\mathfrak{i}$:

$$\begin{aligned} n(\mathfrak{i}) &= n(10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0) \\ &= n(5, 4, 4, 3, 1, 1, 1, 1, 0) + 1 \\ &= n(1, 1, 1, 0) + 2 \\ &= 3. \end{aligned}$$

As is obvious from the example, iterated application of Proposition 2.1 provides a very efficient algorithm for the determination of the class of nilpotence of a given ideal $\mathfrak{i}$.

Since it will be essential subsequently, we wish to point out that this algorithm has a very nice geometric rendering. Let, as before, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the partition corresponding to $\mathfrak{i}$. Consider the Ferrers diagram of λ. As it is contained in the staircase diagram $(n, n-1, \dots, 1)$, it must not cross the antidiagonal line $x + y = n + 1$. We draw a broken ray as follows (see Figure 1, where $n = 13$ and $\lambda = (10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0)$): we start on the vertical edge on the right of cell $(1, \lambda_1)$, and move downward until we touch the antidiagonal $x + y = n + 1$. At the touching point we turn direction from vertical-down to horizontal-left, and move on until we touch a vertical part of the Ferrers diagram. At the touching point we turn direction from horizontal-left to vertical-down. Now the procedure is iterated, until we reach the line $x = 0$. The class of nilpotence of the ideal $\mathfrak{i}$ is equal to the number of touching points on $x + y = n + 1$. In Figure 1, the resulting broken ray is the dotted line outside the Ferrers diagram of $(10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0)$. There are three touching points on $x + y = n + 1 = 14$, in accordance with $n(10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0) = 3$, as we computed just above.

§4 ad-NILPOTENT IDEALS IN TYPES B, C, D

Let $\mathfrak{g}$ be a complex simple Lie algebra of type B_n, C_n, or D_n. In this section we describe a diagrammatic encoding of the positive roots (cf. [Sh, Sec. 2–3]) similar to the one introduced in the previous section for type A. Throughout this section, for any of the three types, the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ denote a basis of simple roots corresponding to the chosen system Δ^+ of positive roots.

We arrange the positive roots in a shifted staircase diagram of shape $(2n-1, 2n-3, \dots, 1)$ for B_n and C_n, and of shape $(2n-2, 2n-4, \dots, 2)$ for D_n as follows.

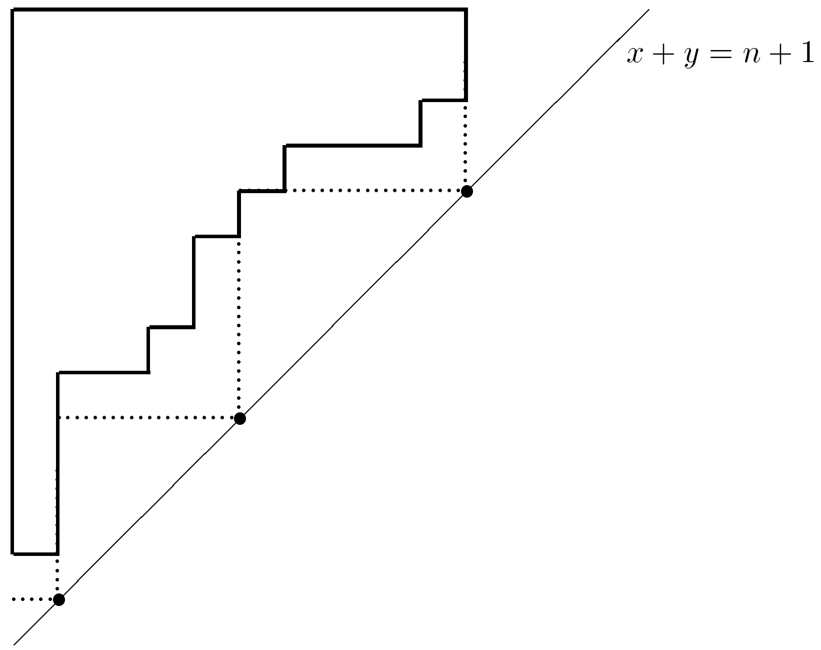


Figure 1

If $\mathfrak{g}$ is of type C_n, then we associate the cell (i,j), $1\le i\le j\le 2n-i$, with the positive root

$$\begin{cases} \alpha_i+\dots+\alpha_{j-1}+2(\alpha_j+\dots+\alpha_{n-1})+\alpha_n & \text{if } j\le n-1,\\ \alpha_i+\dots+\alpha_{2n-j} & \text{if } n\le j\le 2n-i. \end{cases} \tag{4.C}$$

For example, if $n=3$, this defines the following arrangement of positive roots,

$$\begin{array}{ccccc} 2\alpha_1+2\alpha_2+\alpha_3 & \alpha_1+2\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1\\ & 2\alpha_2+\alpha_3 & \alpha_2+\alpha_3 & \alpha_2 & \\ & & \alpha_3 & & \end{array}$$

Likewise, if $\mathfrak{g}$ is of type B_n, then we associate the cell (i,j), $1\le i\le j\le 2n-i$, with the positive root

$$\begin{cases} \alpha_i+\dots+\alpha_j+2(\alpha_{j+1}+\dots+\alpha_n) & \text{if } j\le n-1,\\ \alpha_i+\dots+\alpha_{2n-j} & \text{if } n\le j\le 2n-i. \end{cases} \tag{4.B}$$

For example, if $n=3$, this defines the following arrangement of positive roots,

$$\begin{array}{ccccc} \alpha_1+2\alpha_2+2\alpha_3 & \alpha_1+\alpha_2+2\alpha_3 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1\\ & \alpha_2+2\alpha_3 & \alpha_2+\alpha_3 & \alpha_2 & \\ & & \alpha_3 & & \end{array}$$

If $\mathfrak{g}$ is of type D_n, then we associate the cell (i,j), $1\le i\le j\le 2n-1-i$, with the positive root

$$\begin{cases} \alpha_i+\dots+\alpha_j+2(\alpha_{j+1}+\dots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n & \text{if } j\le n-2,\\ \alpha_i+\dots+\alpha_{n-2}+\alpha_n & \text{if } j=n-1,\\ \alpha_i+\dots+\alpha_{2n-j-1} & \text{if } n\le j\le 2n-1-i. \end{cases} \tag{4.D}$$

For example, if $n = 4$, this defines the following arrangement of positive roots,

$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\alpha_1 + \alpha_2 + \alpha_4$	$\alpha_1 + \alpha_2 + \alpha_3$	$\alpha_1 + \alpha_2$	α_1
	$\alpha_2 + \alpha_3 + \alpha_4$	$\alpha_2 + \alpha_4$	$\alpha_2 + \alpha_3$	α_2	
		α_4	α_3		

Now, for $\mathfrak{g}$ of any of the types B_n, C_n, or D_n, we again associate an ideal $\mathfrak{i} \in \mathcal{I}^n$, represented as before as $\mathfrak{i} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, to a collection of cells, via the above identification of positive roots with cells. The resulting collections are characterized in the proposition below. For convenience, if $A \subseteq (2n-2, 2n-4, \ldots, 2)$ is a collection of cells, we denote by $A^\bullet$ the collection of cells obtained from A by switching columns $n-1$ and n. By a subdiagram of some *shifted* staircase we mean as usual a collection of cells contained in the shifted staircase which forms a *shifted* Ferrers diagram (cf. [Ma, Ch. I, Sec. 1, Ex. 9]).

Proposition 4.1. *If $\mathfrak{g}$ is of type B_n or C_n, then, under the above identification, the ideals in $\mathcal{I}^n$ correspond bijectively to the subdiagrams of $(2n-1, 2n-3, \ldots, 1)$. If $\mathfrak{g}$ is of type D_n, the above identification defines a bijection between the ideals in $\mathcal{I}^n$ and the collection of cells $A \subseteq (2n-2, 2n-4, \ldots, 2)$ such that either A or $A^\bullet$ is a subdiagram of $(2n-2, 2n-4, \ldots, 2)$.*

Corollary 4.2. *The number of ideals in $\mathcal{I}^n$ is given by*

$$|\mathcal{I}^n| = \begin{cases} \binom{2n}{n} & \text{if } \mathfrak{g} \text{ is of type } B_n \text{ or } C_n, \\ \binom{2n}{n} - \binom{2n-2}{n-1} & \text{if } \mathfrak{g} \text{ is of type } D_n. \end{cases}$$

Remark. The previous proposition and corollary are also proved in [CP, Theorem 3.1] and, in equivalent form, in [Sh, Sec. 3]. In particular, Corollary 4.2 follows almost immediately from Proposition 4.1. The most direct argument (which is different from the ones in [CP] and [Sh]) is as follows: for $\mathfrak{g}$ of type B_n or C_n, one maps the shifted subdiagrams of $(2n-1, 2n-3, \ldots, 1)$ to lattice paths which start at the origin, never pass below the x-axis, and consist of $2n$ steps, the steps being up-steps $(1,1)$ and down-steps $(1,-1)$, as before. The correspondence between subdiagrams and paths is best explained with an example at hand. Let $n = 9$ and consider the shifted partition $(16, 13, 11, 8, 7, 5, 3)$, see Figure 4. Rotate the figure by $45°$ in the positive direction and then flip it across a vertical line. Then the zig-zag line which forms the (right) border of the shifted partition becomes such a lattice path (on attaching a few up-steps at the beginning of the path). See Figure 2 for the path corresponding to the partition of Figure 4. (To make a comparison easy, the steps which correspond to thick segments in Figure 4 are also made thick in Figure 2.)

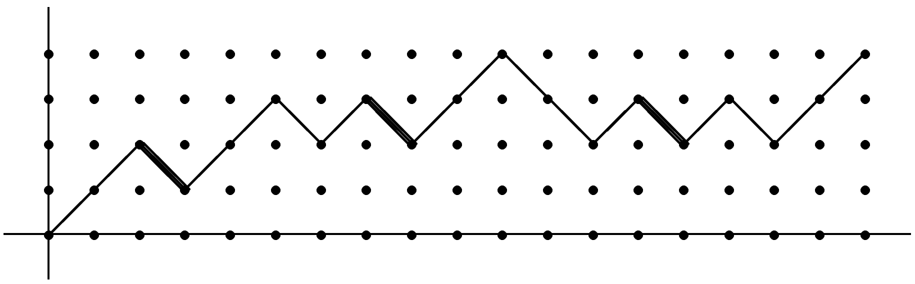


Figure 2

Now one can resort to the well-known result (see e.g. [F, Theorem 1, (4.6) in Ch. III, Sec.4]) that the number of these lattice paths is $\binom{2n}{n}$. The result for type D_n is then an easy consequence. (A very different proof can be found in [CP], where an enumeration result for trapezoidal plane partitions due to Proctor is used.) It can be checked that in all cases the formulas are in accordance with formula (1.1).

We explain now in a rough form how to calculate the class of nilpotence by means of the algorithm used in Section 3 for type A. For convenience, let us write S_{2n-1} for the shifted staircase $(2n-1, 2n-3, \dots, 1)$, S_{2n-2} for the shifted staircase $(2n-2, 2n-4, \dots, 2)$, and T_N for the *ordinary* staircase diagram $(N, N-1, \dots, 1)$. The general procedure will be as follows: we embed the shifted staircase S_N, where $N = 2n-1$ or $N = 2n-2$, respectively, into the ordinary staircase T_N, in such a way that to any collection of cells $A \subseteq S_N$ corresponds uniquely a collection of cells $\tilde{A} \subseteq T_N$. We shall eventually prove in Proposition 4.3 that the class of nilpotence of an ideal $\mathfrak{i} \in \mathcal{I}^n$ can be obtained by applying the algorithm of Section 3 to $\tilde{A}$, respectively to $\widetilde{A^\bullet}$ in the case of type D_n if A is not a diagram, where A is the collection of cells correponding to $\mathfrak{i}$ in the way that was explained earlier.

To explain this in detail, let first $\mathfrak{g}$ be of type C_n. We fill the ordinary staircase T_{2n-1} with the positive roots of C_n by associating the cells of the "upper half" of T_{2n-1} with positive roots as in (4.C), and by associating the cells in the "lower half" of T_{2n-1} in a symmetric fashion in such a way that cell (i, j) gets associated with the same root as cell (j, i). (I.e., the line formed by the cells (i, i), $i = 1, 2, \dots, n$, constitutes a symmetry axis of this arrangement of the positive roots.) Thus, if $n = 3$, this defines the arrangement

$$\begin{array}{lllll}
2\alpha_1+2\alpha_2+\alpha_3 & \alpha_1+2\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1 \\
\alpha_1+2\alpha_2+\alpha_3 & 2\alpha_2+\alpha_3 & \alpha_2+\alpha_3 & \alpha_2 & \\
\alpha_1+\alpha_2+\alpha_3 & \alpha_2+\alpha_3 & \alpha_3 & & \\
\alpha_1+\alpha_2 & \alpha_2 & & & \\
\alpha_1 & & & &
\end{array}$$

Now, given an ideal $\mathfrak{i} \in \mathcal{I}^n$, written as $\mathfrak{i} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, with associated collection of cells $A \subseteq S_{2n-1}$ (which is in fact of the form of a shifted Ferrers diagram), we let $\tilde{A}$ be the collection of all cells in T_{2n-1} that contain a root $\alpha \in \Phi$. Phrased differently, $\tilde{A}$ is the union of A with its mirror image about the symmetry axis. Hence, $\tilde{A}$ forms a self-conjugate (cf. [Ma, Ch. I, Sec. 1]) partition. For example, according to this description, the ideal $\mathfrak{g}_{2\alpha_1+2\alpha_2+\alpha_3} + \mathfrak{g}_{\alpha_1+2\alpha_2+\alpha_3} + \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} + \mathfrak{g}_{2\alpha_2+\alpha_3}$ corresponds to the self-conjugate partition $(3, 2, 1)$.

If $\mathfrak{g}$ is of type B_n, we fill the ordinary staircase T_{2n-1} with elements of the root lattice of B_n by associating the cells of the "upper half" of T_{2n-1} with positive roots as in (4.B), by associating the cell $(i, i-1)$ with the element of the root lattice $2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_n$ (this element is not a root!), $i = 2, 3, \dots, n$, and by associating the cells in the "lower half" of T_{2n-1} in a symmetric fashion in such a way that cell (i, j) gets associated with the same root as cell $(j+1, i-1)$. (I.e., here the line formed by the cells $(i, i-1)$, $i = 2, 3, \dots, n$, constitutes a symmetry axis of this arrangement of elements of the root lattice.) Thus, if $n = 3$, this defines the arrangement

$$\begin{matrix} \alpha_1+2\alpha_2+2\alpha_3 & \alpha_1+\alpha_2+2\alpha_3 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1 \\ 2\alpha_2+2\alpha_3 & \alpha_2+2\alpha_3 & \alpha_2+\alpha_3 & \alpha_2 & \\ \alpha_2+2\alpha_3 & 2\alpha_3 & \alpha_3 & & \\ \alpha_2+\alpha_3 & \alpha_3 & & & \\ \alpha_2 & & & & \end{matrix}$$

Here, given an ideal $\mathfrak{i}\in\mathcal{I}^n$, written as $\mathfrak{i}=\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$, with associated collection of cells $A\subseteq S_{2n-1}$ (which is in fact of the form of a shifted Ferrers diagram), we let $\tilde{A}$ be the collection of all cells in T_{2n-1} that contain a root $\alpha\in\Phi$, together with all cells $(i,i-1)$ that are in the same row as some cell of A. Phrased differently, $\tilde{A}$ is the union of A with its mirror image about the line formed by the cells $(i,i-1)$, $i=2,3,\dots,n$, including the cells on that line which are necessary to "fill the holes". Hence, if we write $\tilde{A}$ as a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2n-1})$, then $(\lambda_2,\dots,\lambda_{2n-1})$ is a self-conjugate partition. Moreover, we have $\lambda_i\neq i-1$ for $i\geq 2$. For example, according to this description, the ideal $\mathfrak{g}_{\alpha_1+2\alpha_2+2\alpha_3}+\mathfrak{g}_{\alpha_1+\alpha_2+2\alpha_3}+\mathfrak{g}_{\alpha_2+2\alpha_3}$ corresponds to the partition $(2,2,1)$.

Finally, if $\mathfrak{g}$ is of type D_n, we fill the ordinary staircase T_{2n-2} with elements of the root lattice of D_n in a similar fashion as in type B. To be precise, we associate the cells of the "upper half" of T_{2n-2} with positive roots as in (4.D), we associate cell $(i,i-1)$ with the element of the root lattice $2\alpha_i+2\alpha_{i+1}+\dots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$ (this element is not a root!), $i=2,3,\dots,n-1$, and we associate the cells in the "lower half" of T_{2n-1} in a symmetric fashion in such a way that cell (i,j) gets associated with the same root as cell $(j+1,i-1)$. (I.e., the line formed by the cells $(i,i-1)$, $i=2,3,\dots,n-1$, constitutes again a symmetry axis of this arrangement of elements of the root lattice.) In particular, we do not associate the cell $(n,n-1)$ with any element of the root lattice. Thus, if $n=4$, this defines the arrangement (there, a cross indicates the cell that is not associated to anything)

$$\begin{matrix} \alpha_1+2\alpha_2+\alpha_3+\alpha_4 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \alpha_1+\alpha_2+\alpha_4 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1 \\ 2\alpha_2+\alpha_3+\alpha_4 & \alpha_2+\alpha_3+\alpha_4 & \alpha_2+\alpha_4 & \alpha_2+\alpha_3 & \alpha_2 & \\ \alpha_2+\alpha_3+\alpha_4 & \alpha_3+\alpha_4 & \alpha_4 & \alpha_3 & & \\ \alpha_2+\alpha_4 & \alpha_4 & \times & & & \\ \alpha_2+\alpha_3 & \alpha_3 & & & & \\ \alpha_2 & & & & & \end{matrix}$$

Given an ideal $\mathfrak{i}\in\mathcal{I}^n$, with associated collection of cells $A\subseteq S_{2n-2}$, we form $\tilde{A}$ in the same way as in type B_n, i.e., $\tilde{A}$ is the union of A with its mirror image about the line formed by the cells $(i,i-1)$, $i=2,3,\dots,n-1$, including the cells on that line which are in the same row as some cell in A. Should A not be a shifted diagram (and, hence, $\tilde{A}$ not be an ordinary diagram), then we shall interchange the $(n-1)$st and the nth column and the nth and $(n+1)$st row, and call the resulting (ordinary) diagram $\widetilde{A^\bullet}$. Whichever of $\tilde{A}$ or $\widetilde{A^\bullet}$ is a diagram, it is contained in T_{2n-2}, and when written as a partition $(\lambda_1,\lambda_2,\dots,\lambda_{2n-2})$, it has the property that $(\lambda_2,\dots,\lambda_{2n-2})$ is a self-conjugate partition and that $\lambda_i\neq i-1$ for $i\geq 2$. For example, according to this description, the set $\widetilde{A^\bullet}$ corresponding to the ideal

$$\mathfrak{g}_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}+\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}+\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_4}+\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3}+\mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4}+\mathfrak{g}_{\alpha_2+\alpha_3}$$

is the partition $(4,3,1,1)$.

After "having set the scene," we come to the crucial result of this section. As promised earlier, it allows to compute the class of nilpotence of ideals in any of the types B_n, C_n, or D_n, by applying the algorithm of Section 3 to a suitable partition.

Proposition 4.3. *Let $\mathfrak{g}$ be of type B_n, C_n, or D_n. Let $\mathfrak{i}$ be in $\mathcal{I}^n$, and let λ be the corresponding ordinary partition according to the case-by-case description given above, i.e., either it stands for $\tilde{A}$ or for $\widetilde{A^\bullet}$, and, in addition, $\lambda = (\lambda_1, \lambda_2, \dots)$ is self-conjugate in the case of type C_n, whereas in types B_n and D_n it is $(\lambda_2, \dots)$ which is self-conjugate. Then $n(\mathfrak{i}) = n(\lambda)$, where $n(\lambda)$ stands for the result of the algorithm of Proposition 2.1.*

Remark. We should clarify what is meant by "the result of the algorithm of Proposition 2.1." For type B_n and type C_n the partition λ is contained in T_{2n-1}. Accordingly we write $\lambda = (\lambda_1, \dots, \lambda_{2n-1})$ (i.e., with $2n-1$ components) and apply (3.8) with n replaced by $2n-1$, and then iterate. For type D_n, however, the partition λ is contained in T_{2n-2}. Thus in this case we write $\lambda = (\lambda_1, \dots, \lambda_{2n-2})$ (i.e., with $2n-2$ components) and apply (3.8) with n replaced by $2n-2$, and then iterate. In the geometric rendering of the algorithm (see Figure 1), the touching points of the broken ray are on the line $x+y=2n$ for type B_n and C_n, and on $x+y=2n-1$ for type D_n.

Proof. Construct the filling $(t_{i,j})$ corresponding to λ according to the algorithm (3.2). Let $\tau_{i,j}$ denote the element of the root lattice associated with cell (i,j).

Now we copy the arguments of Section 3 that show that $t_{i,j}$ is equal to the right-hand side of (3.1). Since (3.4) is also valid for our $\tau_{i,j}$'s (regardless whether we consider B_n, C_n, or D_n!), everything runs through smoothly, with just two modifications. First, the collection of roots Φ (which define the ideal $\mathfrak{i}$) must be replaced by the elements of the root lattice contained in the cells of A. (The reader must recall that in types B_n and D_n this includes some nonroots.) Second, if $i>j$, the argument that in any decomposition (3.3) of $\tau_{i+1,j}$, with the β_l's being positive roots, one of the β_l's must be equal to some $\tau_{i+1,k}$, must now be modified by maintaining that one of the β_l's must be equal to some $\tau_{h,j}$ (because, otherwise, the sum on the right-hand side of (3.3) would either contain no α_{j+1} or in addition to α_{j+1} also α_j). Clearly, if $i=j$, then a decomposition (3.3) of $\tau_{i+1,j}$ must contain some $\tau_{i+1,k}$ as well as some $\tau_{h,j}$. An analogous modification has to be made for the argument following the decomposition (3.7) of $\tau_{i,j}$.

Let us denote the set of the elements of the root lattice contained in the cells of A by Φ'. Since we had to replace Φ by Φ', the conclusion of the above arguments is that $t_{i,j}$ is equal to

$$\max\{m : \tau_{i,j} = \beta_1 + \dots + \beta_m, \text{ for some } \beta_l \in \Phi',\ l = 1,2,\dots,m\}.$$

This expression differs from the right-hand side of (3.1) by a replacement of Φ by Φ'. This indeed makes a difference if the type that we consider is B_n or D_n. However, our goal is actually to prove that it equals exactly the right-hand side of (3.1) as long as $i \neq j+1$, for any of the types B_n, C_n, or D_n. To see that this is true in type B_n, let in a decomposition (3.7) of $\tau_{i,j}$, where all the β_l's are in Φ', one of the β_l's be equal to a nonroot, to $\tau_{r,r-1} = 2\alpha_r + 2\alpha_{r+1} + \dots + 2\alpha_n$ say. Then some other β_l must be necessarily equal to some $\tau_{s,2n-r+1} = \alpha_s + \alpha_{s+1} + \dots + \alpha_{r-1}$. Then we can replace these two elements by $\tau_{r,r}$ and $\tau_{s,2n-r}$, both of which are now roots in Φ.

For, both of which are clearly roots, and in addition $\tau_{s,2n-r} > \tau_{s,2n-r+1}$, whence $\tau_{s,2n-r}$ must belong to Φ since $\tau_{s,2n-r+1}$ does (here we use that Φ is a dual order ideal), and $\tau_{r,r}$ must be in Φ because otherwise we would have $\lambda_r = r-1$ (that is, the cell $(r, r-1)$ would be the rightmost cell in row r). An analogous argument holds in type D_n.

Now that we have shown that $t_{i,j}$ is equal to the right-hand side of (3.1), and since also here $t_{1,1}$ is the maximum of all entries $t_{i,j}$, it follows that $t_{1,1}$ is the class of nilpotence of the ideal $\mathfrak{i}$. Since Proposition 2.1 shows that $t_{1,1}$ can be obtained by repeated application of (3.8), the proof is complete. $\square$

In view of this proposition, we have reduced the problem of counting ideals in $\mathcal{I}^n$ with a given class of nilpotence to the problem of counting certain partitions contained in a staircase with respect to the outcome of the algorithm in Proposition 2.1. This partition enumeration problem is already in a convenient form for being tackled directly in the cases of types B_n and C_n. We shall do that in Sections 6 and 5, respectively. However, in the case of type D_n we have the additional complication that the partition λ is either given by $\tilde{A}$ or by $\widetilde{A^\bullet}$. It means that every partition in which the $(n-1)$st and nth column have the same length is counted once, whereas every partition whose $(n-1)$st and nth column differ in length is actually counted twice. Therefore we have to distinguish between these two cases. So, for a Lie algebra $\mathfrak{g}$ of type D_n, let, as in Theorem D.1, $\delta_n(K)$ denote the number of ideals in $\mathcal{I}^n$ with class of nilpotence K, let $\delta_n^{(1)}(K)$ denote the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2n-2})$ in the staircase T_{2n-2} with the property that $(\lambda_2, \dots, \lambda_{2n-2})$ is self-conjugate, that $\lambda_i \neq i-1$ for $i \geq 2$, and that $n(\lambda) = K$, and let $\delta_n^{(2)}(K)$ denote the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2n-2})$ in the staircase T_{2n-2} that have the property that $(\lambda_2, \dots, \lambda_{2n-2})$ is self-conjugate, that $\lambda_i \neq i-1$ for $i \geq 2$, that the $(n-1)$st and the nth column have the same length, and that $n(\lambda) = K$. Then the above arguments show that

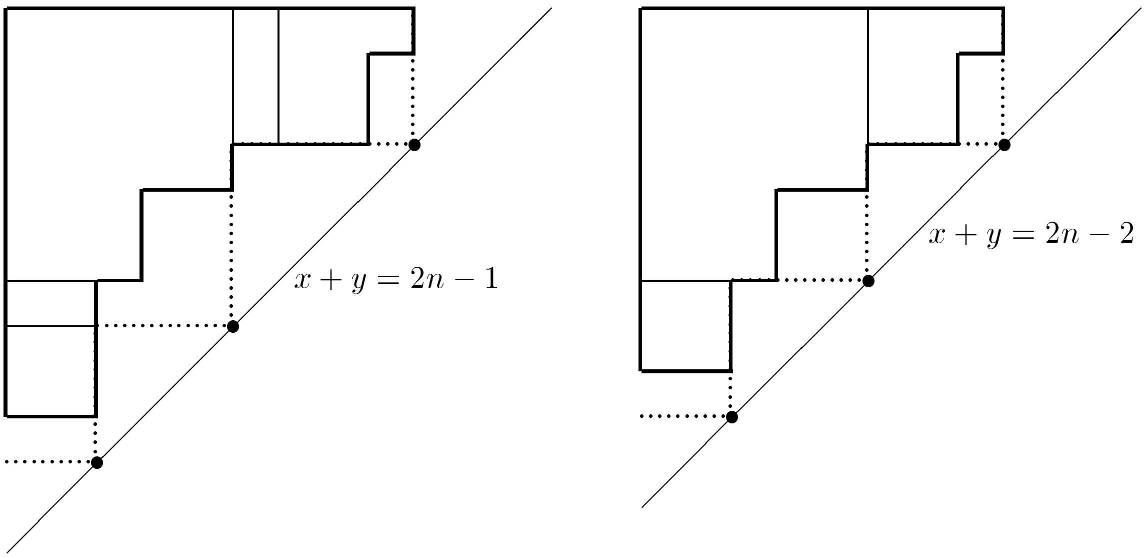
$$\delta_n(K) = 2\delta_n^{(1)}(K) - \delta_n^{(2)}(K). \tag{4.1}$$

In Section 7 we shall show how to compute $\delta_n^{(1)}(K)$. Very conveniently, the computation of $\delta_n^{(2)}(K)$ can be reduced to the computation of the number of ideals with class of nilpotence K in type B_{n-1}.

Proposition 4.4. *For $\mathfrak{g}$ a Lie algebra of type B_n, let, as in Theorem B.1, $\beta_n(K)$ denote the number of ideals in $\mathcal{I}^n$ with class of nilpotence K. Then $\delta_n^{(2)}(K) = \beta_{n-1}(K)$.*

Proof. By Proposition 4.3 we know that $\beta_{n-1}(K)$ is equal to the number of partitions $\mu = (\mu_1, \mu_2, \dots, \mu_{2n-3})$ contained in the staircase T_{2n-3} with the property that $(\mu_2, \dots, \mu_{2n-3})$ is self-conjugate, that $\mu_i \neq i-1$ for $i \geq 2$, and that it satisfies $n(\mu) = K$, where $n(\mu)$ is the result of the algorithm of Proposition 2.1.

Thus it suffices to set up a one-to-one correspondence between the partitions counted by $\delta_n^{(2)}(K)$ and those counted by $\beta_{n-1}(K)$. Such a correspondence is easily set up. Given a partition counted by $\delta_n^{(2)}(K)$ we delete the $(n-1)$st column and the nth row. Clearly, we obtain a partition $\mu = (\mu_1, \mu_2, \dots, \mu_{2n-3})$ contained in T_{2n-3} with the property that $(\mu_2, \dots, \mu_{2n-3})$ is self-conjugate and that $\mu_i \neq i-1$ for $i \geq 2$. It is equally obvious that the mapping can easily be reversed. What



a. A partition counted by $\delta_n^{(2)}(K)$ b. A partition counted by $\beta_{n-1}(K)$

Figure 3

we need in addition is that the result of the algorithm of Proposition 2.1 yields the same value, namely K, for the original partition as well as for the "reduced" partition. This is most obvious from the geometric rendering of the algorithm, see Figure 3, which displays an example with $n = 7$ and $K = 3$. In Part (a) of the figure, the $(n-1)$st column and the nth row are marked, while in Part (b) the places are marked where that column and that row were cut out. When we delete the $(n-1)$st column and the nth row of the original partition then the effect on the broken ray is plainly that a horizontal piece of unit length is cut out (the one that passed exactly under this column) and that a vertical piece of unit length is cut out (the one that passed exactly to the right of that row). Otherwise the ray is completely identical. In particular, the number of touching points with the bounding line must be the same, and this number is exactly equal to the outcome of the algorithm of Proposition 2.1. $\square$

§5 Proofs in type C

Let $\mathfrak{g}$ be of type C_n. We have to prove Theorems C.2 and C.6, upon which the other theorems and corollaries labelled C follow, as we have described in Section 2.

By Proposition 4.3 we know that the number of ideals in $\mathcal{I}^n$ with class of nilpotence K is equal to the number of self-conjugate partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2n-1})$ contained in the staircase $(2n-1, 2n-2, \dots, 1)$ with $n(\lambda) = K$, where $n(\lambda)$ denotes the result of the algorithm of Proposition 2.1.

The strategy which we use to count the latter partitions is based on the following observation: instead of considering the self-conjugate partitions $(\lambda_1, \lambda_2, \dots, \lambda_{2n-1})$, we consider just the "upper halfs," the shifted partitions with row lengths $(\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots)$. (In fact, this is the collection of cells A that contains the roots that define the corresponding ideal; see Section 4.) Draw the shifted partition $(\lambda_1, \lambda_2 - 1, \lambda_2 - 2, \dots)$, see Figures 4 and 5. Start as in the A_{2n-1} algorithm. I.e., begin on the vertical edge on the right of cell $(1, \lambda_1)$, and move downward until we touch the antidiagonal $x + y = 2n$. At the touching point we turn direction from vertical-down to horizontal-left, and move on until we touch a vertical part of the Ferrers diagram. At the touching point we turn direction from horizontal-left to vertical-down. We iterate until we meet the diagonal $x = y$. Let k be the number of touching points on the antidiagonal $x + y = 2n$. Then we claim that if we meet the diagonal $x = y$ while travelling horizontal-left, then the class of nilpotence (of the corresponding ideal) is $2k$, whereas if we meet the diagonal while travelling vertical-downward, then the class of nilpotence is $2k + 1$. For example, if $n = 9$, then for the shifted partition $(16, 13, 11, 8, 7, 5, 3)$ in Figure 4 (corresponding to the self-conjugate partition $(16, 14, 13, 11, 11, 10, 9, 6, 6, 5, 4, 3, 3, 2, 1, 1)$) the class of nilpotence is $2\cdot 3 = 6$, whereas for the partition $(16, 13, 11, 8, 5, 3, 1)$ in Figure 5 (corresponding to the self-conjugate partition $(16, 14, 13, 11, 9, 8, 7, 6, 5, 4, 4, 3, 3, 2, 1, 1)$) the class of nilpotence is $2\cdot 2+1 = 5$. (At this point, the thick segments are without relevance and should therefore be ignored.)

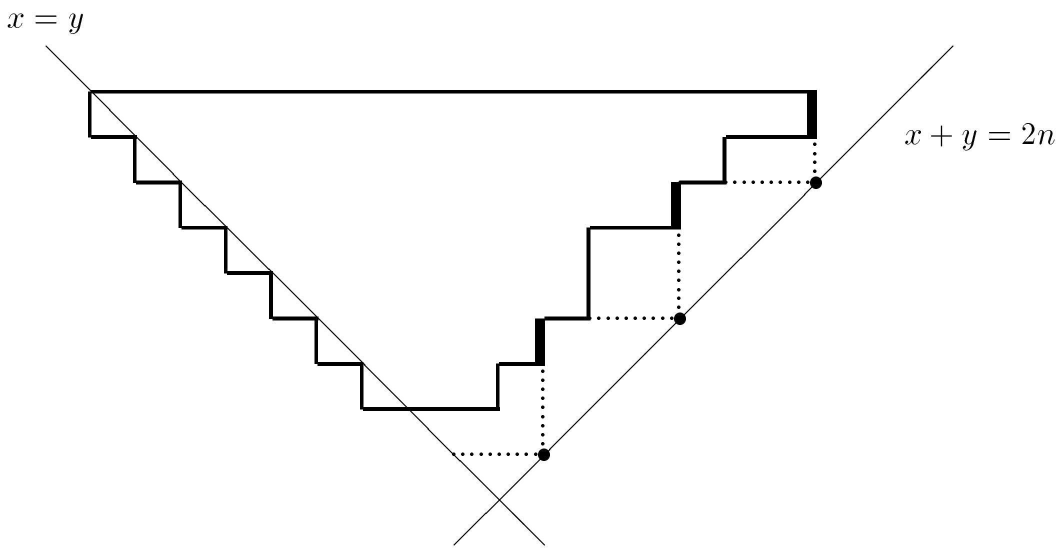


Figure 4

These claims become immediately obvious from the geometrical picture. Consider the self-conjugate partition together with the broken ray that is obtained according to the geometric rendering of the algorithm of Proposition 2.1. Figure 6.a shows the self-conjugate partition $(16, 14, 13, 11, 11, 10, 9, 6, 6, 5, 4, 3, 3, 2, 1, 1)$ (corresponding to the shifted partition in Figure 4) together with the corresponding broken ray. In the figure the ray hits the line $x = y$ while travelling horizontal-left, so that we are in the first of the two possible cases. If we simply reflect the portion of the ray which is below the line $x = y$ about this line (see Figure 6.b), then it is obvious that the touching points below the line $x = y$ get reflected just in between the touching points above $x = y$. Thus, in the first case the total number of touching points of the ray generated by the original self-conjugate partition is exactly twice the number of the touching points above $x = y$, while in the second case it is twice this number plus 1. This proves our claims.

We are now in the position to prove Theorem C.6 (and thereby Theorem C.1, which follows by setting $t = 1$ in Theorem C.6). Subsequently, we shall prove Theorem C.2.

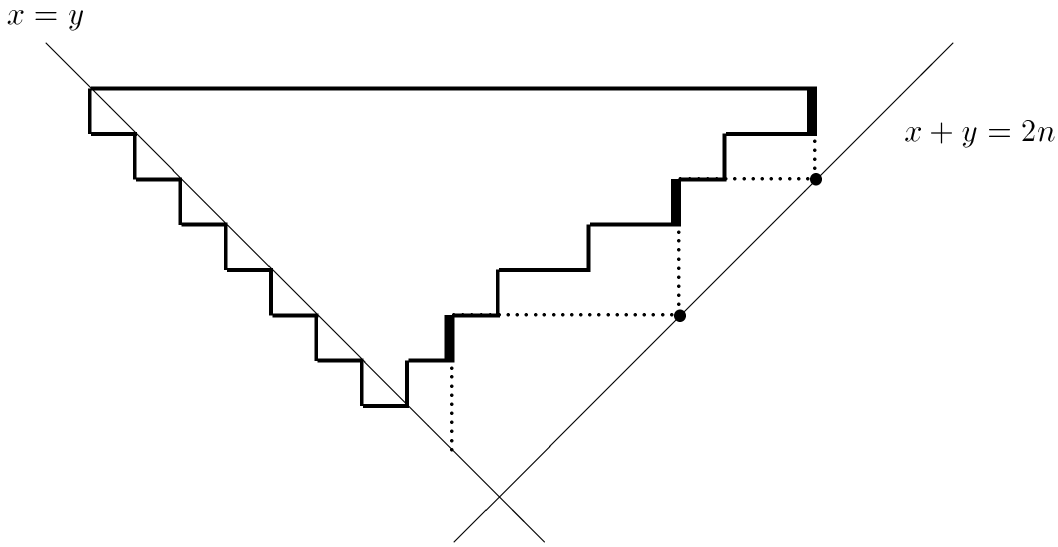
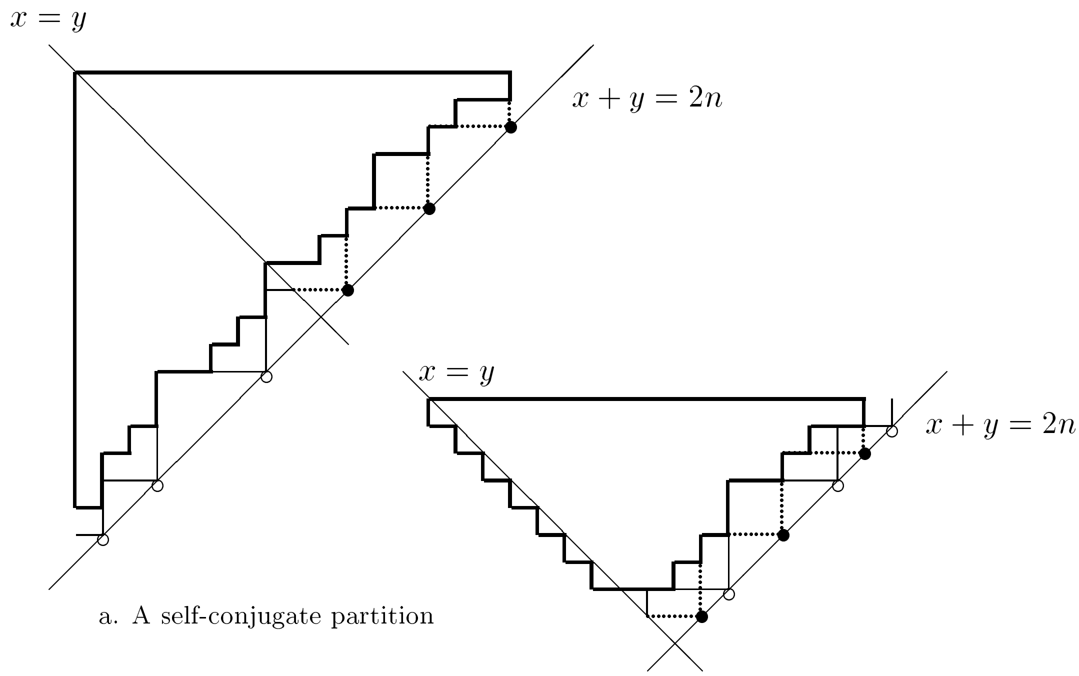


Figure 5



a. A self-conjugate partition

b. One half of the self-conjugate partition

Figure 6

Proof of Theorem C.6. For a fixed K, we aim at computing the generating function $\sum_{h\ge 0}\gamma_n(h,K)t^h$ for ideals $\mathfrak{i}\in\mathcal{I}^n$ with class of nilpotence K. Once this is done, Equation (C.6) will follow immediately. Clearly the dimension of an ideal $\mathfrak{i}$ is equal to the size of the corresponding shifted partition, i.e., to the sum of its parts. Let us denote the size of a partition μ by $|\mu|$. Thus, what we are asking for is to compute the generating function $\sum_\mu t^{|\mu|}$ for shifted partitions μ contained in the shifted staircase S_{2n-1} such that the broken ray construction yields $\lfloor K/2\rfloor$ touching points on $x+y=2n-1$, and $x=y$ is hit while travelling horizontal-left if K is even, respectively vertically-down if K is odd.

Suppose we fix a particular broken ray. How do we obtain all the shifted partitions whose corresponding ray is exactly this fixed one? The answer is immediate once we observe that every time a broken ray hits a vertical part of the shifted partition while travelling horizontal-left, then next the broken ray will continue by a unit step vertically down. These vertical unit steps are shown as thick segments in Figures 4 and 5. In turn, every shifted partition which contains these vertical segments will generate exactly the given broken ray. Let the x-coordinates of these vertical segments be $i_1, i_2, \dots, i_k$, in ascending order. For example, in Figure 4 we have $i_1 = 10$, $i_2 = 13$, and $i_3 = 16$, while in Figure 5 we have $i_1 = 8$, $i_2 = 13$, and $i_3 = 16$. It is easy to see that $i_1, i_2, \dots, i_k$ determine a unique broken ray. If we now use the well-known fact that the generating function $\sum_\nu t^{|\nu|}$, summed over all (ordinary) partitions ν which are contained in an $a \times b$ rectangle, is equal to the t-binomial coefficient $\begin{bmatrix} a+b \\ b \end{bmatrix}_t$ (cf., e.g., [Sta, Prop. 1.3.19]), then if $K = 2k$ we obtain

$$\sum_{h\ge 0} \gamma_n(h,2k)t^h = \sum_{n<i_1<\dots<i_k<i_{k+1}=2n} \prod_{j=1}^{k-1} t^{i_{j+1}(i_{j+3}-i_{j+2})} \begin{bmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{bmatrix}_t \cdot \sum_{\ell=0}^{i_2-i_1-1} \begin{bmatrix} i_1+i_2-2n-1 \\ \ell \end{bmatrix}_t t^{i_1(i_3-i_2)-\binom{2n-i_2+1}{2}+\binom{\ell+1}{2}} \tag{5.1}$$

(where $i_{k+2} = 2n+1$) for the generating function for ideals with class of nilpotence $2k$, and

$$\sum_{h\ge 0} \gamma_n(h,2k-1)t^h = \sum_{2n-i_2<i_1\le n<i_2<\dots<i_k<i_{k+1}=2n} t^{i_1(i_3-i_2)-\binom{2n-i_2+1}{2}} \cdot (1+t)(1+t^2)\cdots(1+t^{i_1+i_2-2n-1}) \prod_{j=1}^{k-1} t^{i_{j+1}(i_{j+3}-i_{j+2})} \begin{bmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{bmatrix}_t \tag{5.2}$$

(where again $i_{k+2} = 2n+1$) for the generating function for ideals with class of nilpotence $2k-1$. Because of (C.7), both formulas can be combined. After having also done the substitution $i_j \to i_j + n$, the result is (C.6). □

Proof of Theorem C.2. We start by first concentrating on the case that h is even, $h = 2k$ say. We want to compute the number of ideals with class of nilpotence at most $2k$. In principle, we could sum up the expressions (5.1) and (5.2) with $t = 1$, but it is more convenient to use an algorithm which determines whether the class of nilpotence is *at most* $2k$ (as opposed to *equal to* $2k$). This algorithm works as follows. Again we are given a shifted partition λ. This time we start at the diagonal $x = y$ (!), at the lowest intersection point with the Ferrers diagram of λ. We move right until we touch the antidiagonal $x + y = 2n$. At the touching point we turn direction from horizontal-right to vertical-up, and move on until we touch a horizontal part of the Ferrers diagram. At the touching point we turn direction from vertical-up to horizontal-right. Etc. Then the class of nilpotence is at most twice the number of touching points on $x + y = 2n$. See Figure 7, where this procedure is applied to the partition of Figure 4.

Let $j_1, j_1+j_2, \dots, j_1+j_2+\dots+j_k$ denote the y-coordinates of the touching points of the broken ray on $x + y = 2n$, in ascending order. For example, in Figure 7 we

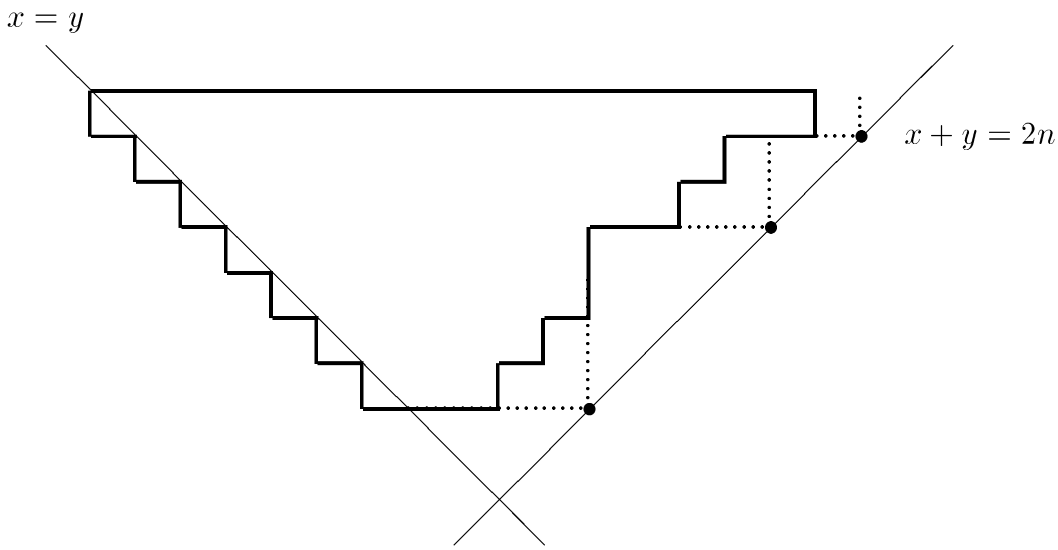


Figure 7

have $j_1=1$, $j_2=2$, and $j_3=4$. In addition, let us write j_{k+1} for $n-j_1-j_2-\dots-j_k$. Then, if we follow the line of argument of the proof of Theorem C.6 (in a slightly modified form), we obtain

$$\sum_{\substack{j_1+j_2+\dots+j_{k+1}=n\\ j_1,j_2,\dots,j_k\ge 1,\, j_{k+1}\ge 0}} \binom{j_1+j_2-1}{j_1}\binom{j_2+j_3-1}{j_2}\cdots\binom{j_{k-1}+j_k-1}{j_{k-1}}\binom{j_k+2j_{k+1}}{j_k}$$

for the number of ideals with class of nilpotence $2k$ or $2k-1$, and therefore, as a moment's thought shows, we obtain

$$\sum_{\substack{j_1+j_2+\dots+j_{k+1}=n\\ j_1,j_2,\dots,j_k,j_{k+1}\ge 0}} \binom{j_1+j_2-1}{j_1}\binom{j_2+j_3-1}{j_2}\cdots\binom{j_{k-1}+j_k-1}{j_{k-1}}\binom{j_k+2j_{k+1}}{j_k}$$

for the number of ideals with class of nilpotence at most $2k$. Now we can easily compute the generating function $\sum_{n\ge 0}\gamma_n^{\le}(2k)x^n$ for the ideals with class of nilpotence at most $2k$. We substitute the previous expression in the generating function, and obtain

$$\sum_{j_1,j_2,\dots,j_k,j_{k+1}\ge 0} x^{j_1+\dots+j_k+j_{k+1}}\binom{j_1+j_2-1}{j_1}\binom{j_2+j_3-1}{j_2}\\ \cdots\binom{j_{k-1}+j_k-1}{j_{k-1}}\binom{j_k+2j_{k+1}}{j_k}.$$

Now we perform the summation over j_1 and obtain $(1/(1-x))^{j_2}$ in the sum. Next the summation over j_2 is performed, etc., thus slowly building up continued fractions of the form of the left-hand side in (5.3) below. If we use the fact that

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{\ddots\, \over 1-x}}}=\frac{\tilde U_h}{\sqrt{x}\,\tilde U_{h+1}}, \tag{5.3}$$

where x occurs h times in the continued fraction (again, $\tilde{U}_k$ is short-hand for $U_k(1/2\sqrt{x})$), then we end up with

$$\sum_{j_{k+1}\ge 0} \frac{1}{\sqrt{x}} \left(\frac{\tilde{U}_k}{\tilde{U}_{k+1}}\right)^{2j_{k+1}+1}.$$

The sum is easily evaluated as it is just a geometric series. This gives

$$\frac{\tilde{U}_k\tilde{U}_{k+1}}{\sqrt{x}\,\left(\tilde{U}_{k+1}^2-\tilde{U}_k^2\right)}.$$

Now, it is easy to check that

$$U_k(x)U_{k+1}(x) = U_{2k+1}(x) + U_{2k-1}(x) + \cdots + U_1(x)$$

and

$$U_{k+1}^2(x) - U_k^2(x) = U_{2k+2}(x).$$

Hence, our generating function equals

$$\frac{\tilde{U}_{2k+1}+\tilde{U}_{2k-1}+\cdots+\tilde{U}_1}{\sqrt{x}\,\tilde{U}_{2k+2}}.$$

This proves Theorem C.2 in the case that h is even.

Next we compute the generating function for ideals with class of nilpotence *equal to* $2k-1$, i.e., $\sum_{n\ge 0}\gamma_n(2k-1)x^n$, where, as before, $\gamma_n(h)$ is the number of ad-nilpotent ideals in $\mathcal{I}^n$ with class of nilpotence equal to h. The case $k=1$ has to be done separately, but this is trivial. (This is already included in the Kostant–Peterson result.) So let us assume $k\ge 2$. If we substitute formula (5.2) with $t=1$ in the generating function, then we obtain

$$\sum_{n\ge 0}\ \sum_{2n-i_2<i_1\le n<i_2<\cdots<i_k<2n} x^n \binom{2n-i_{k-1}-1}{i_k-i_{k-1}} \binom{i_k-i_{k-2}-1}{i_{k-1}-i_{k-2}} \cdots \binom{i_3-i_1-1}{i_2-i_1} 2^{i_1+i_2-2n-1}$$

for our generating function. As before, the sums over $n, i_k, i_{k-1}, \dots, i_3$ are easily computed. On using again that the resulting continued fractions can be expressed in terms of Chebyshev polynomials by means of (5.3), and after having replaced i_1 by i_1+n and i_2 by i_2+n, we obtain

$$\sum_{-i_2<i_1\le 0<i_2} x^{i_2+k-1}\left(\frac{\tilde{U}_1}{x^{k/2-1}\,\tilde{U}_{k-1}}\right)^2 \left(\frac{\tilde{U}_{k-1}}{\sqrt{x}\,\tilde{U}_k}\right)^{i_2-i_1+1} 2^{i_1+i_2-1}.$$

Next we sum over i_1, and subsequently over i_2. In both cases, it is just geometric series that have to be summed, a terminating series when summing over i_1 and two nonterminating series when summing over i_2. The result, after cancellation of factors, is

$$\frac{\tilde U_k}{\left(\tilde U_k - 2\sqrt{x}\,\tilde U_{k-1}\right)\left(\tilde U_k^2 - \tilde U_{k-1}^2\right)}.$$

It is now routine to verify that this is equal to

$$\frac{\tilde U_{2k} + \tilde U_{2k-2} + \cdots + \tilde U_0}{\sqrt{x}\,\tilde U_{2k+1}} - \frac{\tilde U_{2k-1} + \tilde U_{2k-3} + \cdots + \tilde U_1}{\sqrt{x}\,\tilde U_{2k}}.$$

We already know that the second term in this difference is the generating function for ideals with class of nilpotence at most $2k-2$. Hence the first term must be the generating function for ideals with class of nilpotence at most $2k-1$.

At this point Theorem C.2 is completely proved. □

§6 Proofs in type B

We have to prove Theorem B.1, upon which Corollaries B.2 and B.3 follow, as we have described in Section 2.

Proof of Theorem B.1. By Proposition 4.3 we know that $\beta_n(K)$ is equal to the number of partitions $(\lambda_1, \lambda_2, \dots, \lambda_{2n-1})$ contained in the staircase $(2n-1, 2n-2, \dots, 1)$ with the property that $(\lambda_2, \dots, \lambda_{2n-1})$ is self-conjugate, that $\lambda_i \ne i-1$ for $i \ge 2$, and that $n(\lambda) = K$, where $n(\lambda)$ stands for the result of the algorithm of Proposition 2.1. In slight abuse of terminology, we shall refer to $n(\lambda)$ as the "class of nilpotence of λ."

First of all, the cases $K = 0$ and $K = 1$ can be treated directly, the case of $K = 1$ being contained in the Kostant–Peterson result.

For the proof of the theorem for $K \ge 2$, we follow a similar idea as in the proof of Theorems C.6 and C.2 in Section 5. Instead of considering the above (ordinary) partitions $(\lambda_1, \lambda_2, \dots, \lambda_{2n-1})$, we consider again just the "upper halfs," the shifted partitions with row lengths $(\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots)$. (In fact, this is the collection of cells A that contains the roots that define the corresponding ideal; see Section 4.) Again we need an algorithm which, given the shifted partition, allows us to find the class of nilpotence $n(\lambda)$ without having to go back to the ordinary partition λ. The particular construction that we are going to use requires *two* broken rays (as opposed to just one as in Section 5). These two broken rays are constructed as follows: ray 1 starts on the vertical edge on the right of cell $(1, \lambda_1)$ (that is, on the vertical right border of the first row), whereas ray 2 starts on the vertical edge on the right of cell $(2, \lambda_2)$ (that is, on the vertical right border of the second row). Both rays are determined in an algorithmic manner: starting on that edge, we move downward until we touch the antidiagonal $x + y = 2n$. At the touching point we turn direction from vertical-down to horizontal-left, and move on until we touch a vertical part of the Ferrers diagram. At the touching point we turn direction from horizontal-left to vertical-down. Now the procedure is iterated, until we reach the diagonal $x = y - 1$. See Figures 7–14 for typical examples, with ray 1 (starting at the right border of the first row) marked as a thin line, and ray 2 (starting at the right border of the second row) marked as a dotted line.

In order to do the computations, we need to divide the shifted partitions $(\lambda_1, \lambda_2 - 1, \dots)$ into 7 subclasses. To which subclass a particular partition belongs depends on whether the corresponding rays reach $x = y - 1$ while travelling vertical-down or horizontally-left, and whether one ray is above the other or not, as is detailed below. At the same time we shall be able to read off the class of nilpotence from the broken rays.

Below we list the 7 subclasses. It is easy to see that they cover all possibilities. For each subclass we provide a precise characterization, a typical example, and we describe how to read off the class of nilpotence.

Case 1. A partition belongs to subclass 1, if ray 2 reaches $x = y-1$ while travelling horizontal-left, and if either ray 1 reaches $x = y - 1$ while travelling horizontal-left weakly below ray 2, after having touched $x+y = 2n$ one more time than ray 2 (see Figure 8) or ray 1 reaches $x = y - 1$ while travelling vertical-down strictly to the right of ray 2 (see Figure 9; that some edges are thick is irrelevant at the moment).

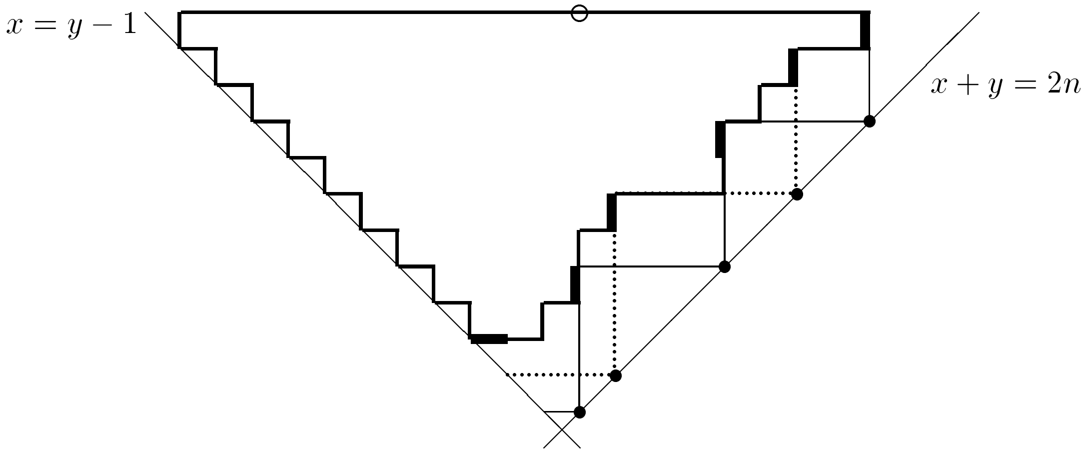


Figure 8

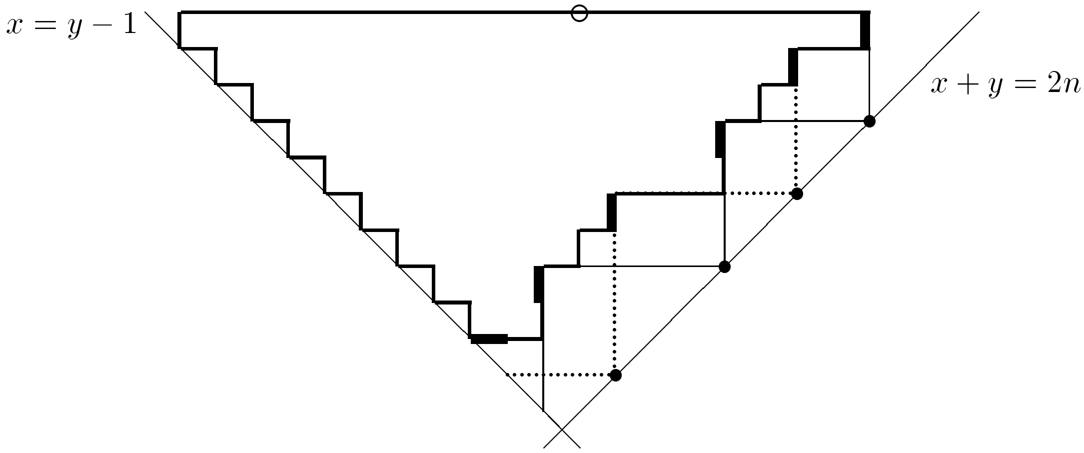


Figure 9

Let k be the number of touching points of ray 2 on the antidiagonal $x+y = 2n$. Then the class of nilpotence of an ideal in this subclass is $2k + 1$.

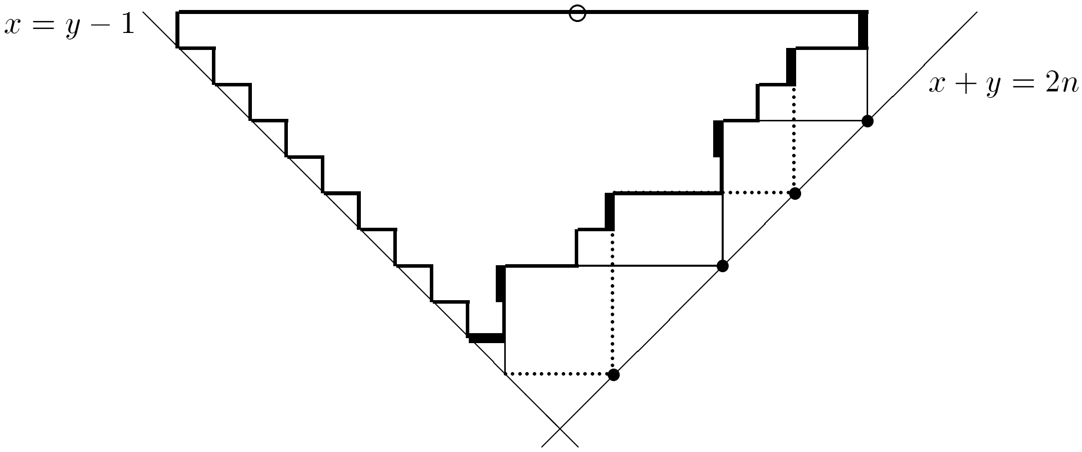


Figure 10

Case 2. A partition belongs to subclass 2, if ray 2 reaches $x = y-1$ while travelling horizontal-left, and if ray 1 reaches $x = y - 1$ while travelling vertical-down weakly to the left of ray 2 (see Figure 10).

Let k be the number of touching points of ray 2 on the antidiagonal $x+y = 2n$. Then the class of nilpotence of an ideal in this subclass is $2k$.

Case 3. A partition belongs to subclass 3, if ray 2 reaches $x = y-1$ while travelling vertical-down, and if ray 1 reaches $x = y - 1$ while travelling horizontal-left strictly above ray 2 (see Figure 11).

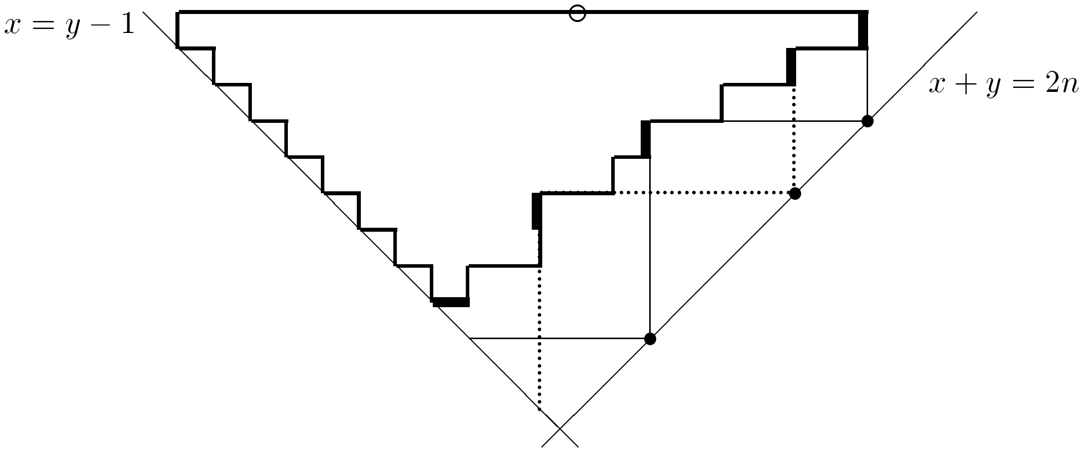


Figure 11

Let k be the number of touching points of ray 1 on the antidiagonal $x+y = 2n$. Then the class of nilpotence of an ideal in this subclass is $2k$.

Case 4. A partition belongs to subclass 4, if ray 2 reaches $x = y-1$ while travelling vertical-down, and if ray 1 reaches $x = y - 1$ while travelling vertical-down weakly to the left of ray 2, after having touched $x + y = 2n$ one more time than ray 2 (see Figure 12).

Let k be the number of touching points of ray 1 on the antidiagonal $x+y = 2n$. Then the class of nilpotence of an ideal in this subclass is $2k$.

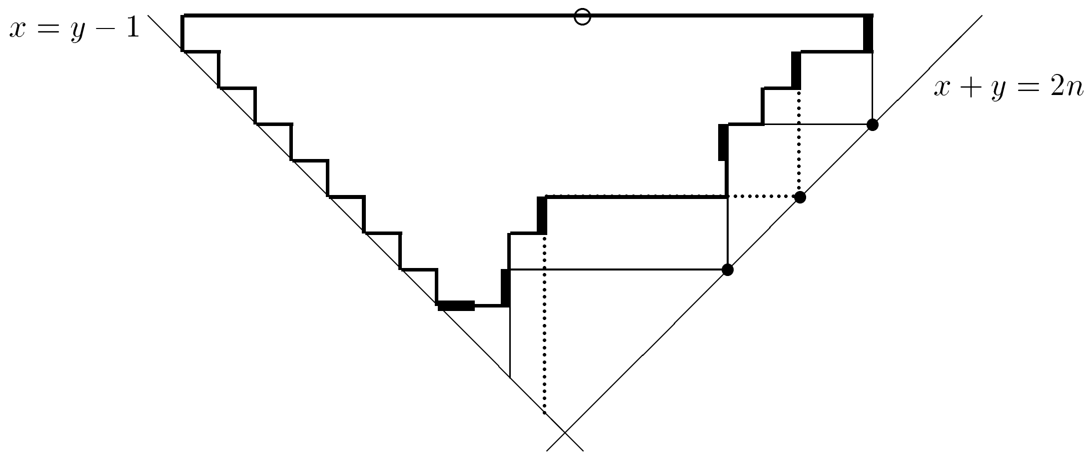


Figure 12

Case 5. A partition belongs to subclass 5, if ray 2 reaches $x = y-1$ while travelling vertical-down, and if ray 1 reaches $x = y-1$ while travelling horizontal-left weakly below ray 2 (see Figure 13).

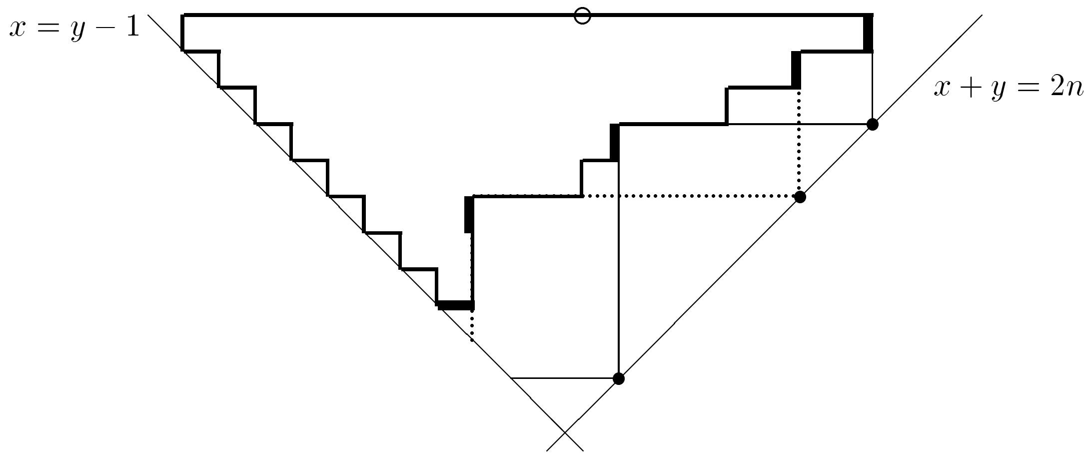


Figure 13

Let k be the number of touching points of ray 1 on the antidiagonal $x + y = 2n$. Then the class of nilpotence of an ideal in this subclass is $2k - 1$.

Case 6. A partition belongs to subclass 6, if ray 2 reaches $x = y-1$ while travelling vertical-down, and if ray 1 reaches $x = y-1$ while travelling vertical-down weakly to the right of ray 2, both rays touching $x + y = 2n$ an equal number of times (see Figure 14).

Let $k-1$ be the number of touching points of ray 1 on the antidiagonal $x+y = 2n$. Then the class of nilpotence of an ideal in this subclass is $2k - 1$.

Case 7. A partition belongs to subclass 7, if ray 2 reaches $x = y-1$ while travelling horizontal-left, and if ray 1 reaches $x = y-1$ while travelling horizontal-left weakly above ray 2, both rays touching $x+y = 2n$ an equal number of times (see Figure 15).

Let k be the number of touching points of ray 2 on the antidiagonal $x + y = 2n$. Then the class of nilpotence of an ideal in this subclass is $2k$.

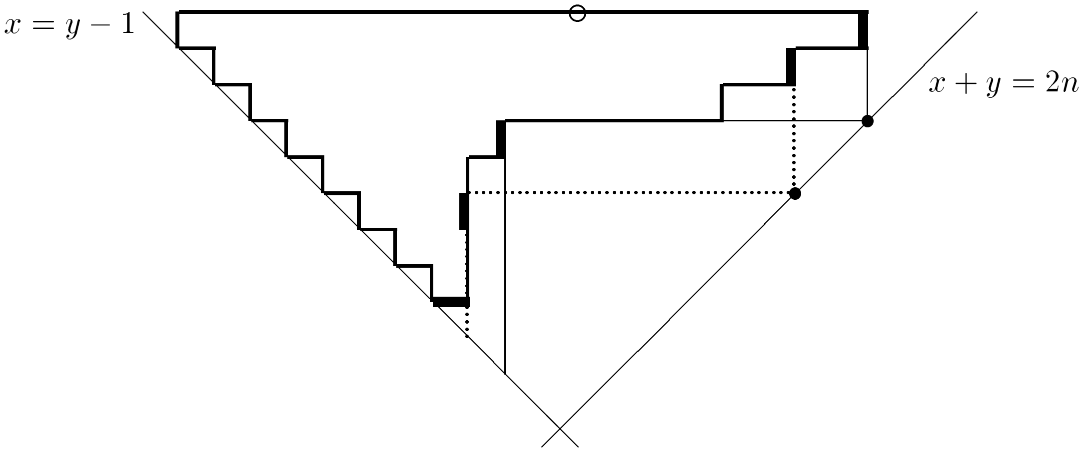


Figure 14

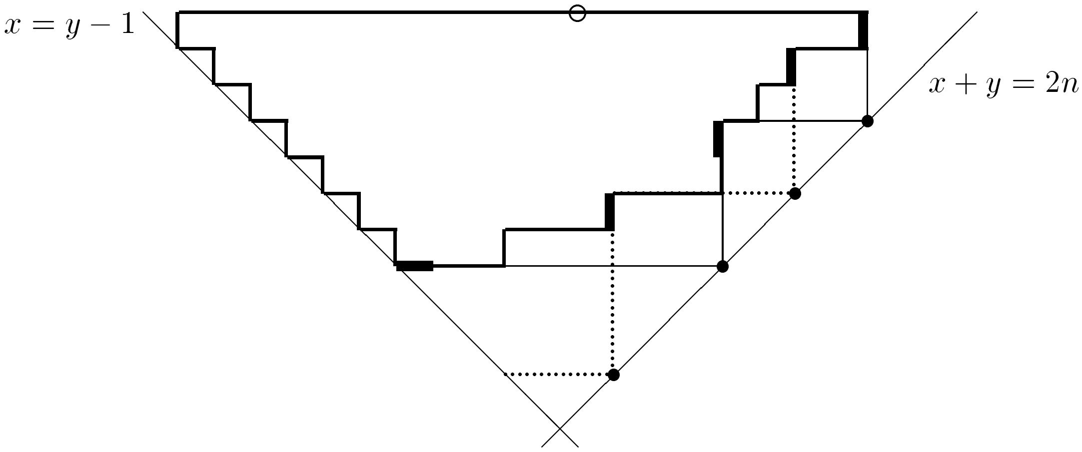


Figure 15

Let $\beta_n^{(i)}(K)$ denote the number of partitions in the ith subclass with class of nilpotence K, $i = 1, 2, \dots, 7$. We will now, for each subclass, compute the corresponding generating function $\sum_{n\ge0} \beta_n^{(i)}(K) x^n$. This is made possible by formulas for $\beta_n^{(i)}(K)$ for each subclass in the spirit of (A.1).

The enumeration in Case 1. For given n, the number $\beta_n^{(1)}(2k+1)$ of partitions in this subclass is given by the multiple summation

$$\sum_{-i_2\le i_1\le i_2\le\dots\le i_{2k+1}\le n-1} \left(\prod_{j=4}^{2k+2} \binom{i_j - i_{j-1} + i_{j-2} - i_{j-3} - 1}{i_{j-2} - i_{j-3}} \right) \sum_{\ell=0}^{i_3-i_2-1} \binom{i_3 + i_1 - 1}{\ell},$$

where, as always in the sequel, $i_{2k+2} = n - 1$. This expression is obtained from the geometrical presentation as in Figures 7 or 8, by denoting the deviation of the first vertical edges on each downward travel along a ray (in Figures 7 and 8 these are the thick vertical edges; the thick horizontal edge marks the downmost and leftmost edge of the shifted Ferrers diagram; it must touch the diagonal $x = y - 1$) from the "reference point" $(n, 0)$ (in Figures 7 and 8 it is marked by a circle) by

$i_{2k+1}, i_{2k}, \dots, i_1$, from right to left. Thus, in Figure 8 we have $i_5 = 8$, $i_4 = 6$, $i_3 = 4$, $i_2 = 1$, $i_1 = 0$, and in Figure 9 we have $i_5 = 8$, $i_4 = 6$, $i_3 = 4$, $i_2 = 1$, $i_1 = -1$. Then arguments very similar to those that we used in the proofs of Theorems C.6 and C.2 in the previous section show that, for fixed $i_{2k+1}, i_{2k}, \dots, i_1$, the number of partitions is equal to the summand in the above sum.

We now compute the generating function $\sum_{n\ge 0} \beta_n^{(1)}(2k+1)x^n$. By definition, this is

$$\sum_{n\ge 0} x^n \sum_{-i_2\le i_1\le i_2\le\dots\le i_{2k+1}\le n-1} \left(\prod_{j=4}^{2k+2} \binom{i_j - i_{j-1} + i_{j-2} - i_{j-3} - 1}{i_{j-2} - i_{j-3}}\right) \cdot \sum_{\ell=0}^{i_3-i_2-1} \binom{i_3+i_1-1}{\ell}.$$

Now we would like to perform the sum over n, then the sum over i_{2k+1}, then over i_{2k}, etc., one after the other. In order to do this conveniently, we have to split the range of the sum into the possibilities

$$-i_2 \le i_1 \le i_2, \quad i_2 + \ell + 1 \le i_3 < \dots < i_{2k+1} < n-1$$

and

$$0 \le i_1 = i_2, \quad i_2 + \ell + 1 \le i_3 = i_4 < i_5 = i_6 < \cdots \\ < i_{2j-1} = i_{2j} < i_{2j+1} < \dots < i_{2k+1} < n-1,$$

for some j between 2 and $k+1$. These two do indeed cover all possibilities which contribute to the sum. For, if we have $i_s = i_{s-1}$ for some $s \ge 4$, then, in order that the summand does not vanish, we must also have $i_{s-2} = i_{s-3}$, etc. The case of s being even is covered by the second possibility above. If s should be odd, then we would also have $i_3 = i_2$. Since then the sum over ℓ is empty, this does not contribute to the sum.

Now, in each of the two cases the sums over n, i_{2k+1}, … , i_4 are easily carried out by the binomial theorem. Thereby, continued fractions as in the left-hand side of (5.3) are slowly built up.

To finish the calculation, we use the identity

$$\sum_{\ell=0}^{i_3-i_2-1} \binom{i_3+i_1-1}{\ell} = \sum_{\ell=0}^{i_3-i_2-1} \binom{i_1+i_2+\ell}{\ell} \cdot \begin{cases} 2^{i_3-i_2-\ell-2} & \text{if } \ell < i_3-i_2-1, \\ 1 & \text{if } \ell = i_3-i_2-1. \end{cases} \tag{6.1}$$

For, now the sums over i_3, ℓ, i_1, and finally i_2, in that order, can easily be carried out, as this amounts again to just summing binomial or even just geometric series. To get rid of the continued fractions, we use Equation (5.3). The result is

$$\frac{\sqrt{x}\,\tilde U_{k+1}\left(\tilde U_k^2 + \tilde U_{k+1}^2\right)}{\tilde U_k^2\,\tilde U_{2k+2}^2\left(\tilde U_k - 2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

The computation for the second possible range is similar. The result is

$$\frac{\sqrt{x}\,\tilde U_{k+1}^2}{\tilde U_{k-j+1}\,\tilde U_{k-j+2}\,\tilde U_k\,\tilde U_{2k+2}\left(\tilde U_k - 2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

In order to obtain the overall generating function for the ideals in this subclass, we have to take the first expression and add to it the sum of the second expressions over j from 2 to $k+1$. Using the identity

$$\frac{1}{U_0(x)U_1(x)}+\frac{1}{U_1(x)U_2(x)}+\dots+\frac{1}{U_s(x)U_{s+1}(x)}=\frac{U_s(x)}{U_{s+1}(x)},$$

we obtain

$$\frac{\sqrt{x}\,\tilde U_{k+1}\left(\tilde U_k^2+\tilde U_{k+1}^2+\tilde U_{k-1}\,\tilde U_{k+1}\,\tilde U_{2k+2}\right)}{\tilde U_k^2\,\tilde U_{2k+2}^2\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

The enumeration in Case 2. By arguments analogous to the ones in Case 1, for given n the number $\beta_n^{(2)}(2k)$ of partitions in this subclass is given by the multiple summation

$$\sum_{\substack{0\le i_2\le\dots\le i_{2k+1}\le n-1\\ -i_3+1\le i_1\le -i_2-1}}\left(\prod_{j=4}^{2k+2}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)2^{i_3+i_1-1}.$$

We now compute the generating function $\sum_{n\ge0}\beta_n^{(2)}(2k)x^n$. We proceed as in Case 1. Here, the analysis is in fact easier. First, there is no sum over ℓ, so that we do not have to use the transformation formula (6.1). Second, both $i_2=i_1$ and $i_3=i_2$ produce vanishing summands, so that we can restrict the summation to

$$-i_3+1\le i_1\le -i_2-1,\quad 0\le i_2<\dots<i_{2k+1}<n-1.$$

As a result, we obtain for the generating function the expression

$$\frac{\sqrt{x}\,\tilde U_{k-1}\,\tilde U_{k+1}}{\tilde U_k^2\,\tilde U_{2k+2}\left(\tilde U_{k+1}-\tilde U_{k-1}\right)\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

The enumeration in Case 3. By arguments analogous to the ones in Case 1, for given n the number $\beta_n^{(3)}(2k)$ of partitions in this subclass is given by the multiple summation

$$\sum_{-i_2\le i_1<0\le i_2\le\dots\le i_{2k}\le n-1}\left(\prod_{j=4}^{2k+1}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)\sum_{\ell=0}^{i_3-i_2-1}\binom{i_3+i_1-1}{\ell}.$$

We now compute the generating function $\sum_{n\ge0}\beta_n^{(3)}(2k)x^n$. We proceed as in Case 1. Since both $i_2=i_1$ and $i_3=i_2$ produce vanishing summands, we can restrict the summation to

$$-i_2\le i_1<0\le i_2<\dots<i_{2k}<n-1.$$

As a result, we obtain for the generating function the expression

$$\frac{\sqrt{x}\,\tilde U_{k-1}\,\tilde U_{k+1}}{\tilde U_k^2\,\tilde U_{2k}\left(\tilde U_{k+1}-\tilde U_{k-1}\right)\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

The enumeration in Case 4. By arguments analogous to the ones in Case 1, for given n the number $\beta_n^{(4)}(2k)$ of partitions in this subclass is given by the multiple summation

$$\sum_{-i_3+1\le i_1\le i_2<0\le i_3\le\dots\le i_{2k+1}\le n-1}\left(\prod_{j=4}^{2k+2}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)2^{i_3+i_1-1}.$$

We now compute the generating function $\sum_{n\ge0}\beta_n^{(4)}(2k)x^n$. We proceed as in Case 1. We split the range of summation into the possibilities

$$-i_3+1\le i_1\le i_2<0\le i_3<\dots<i_{2k+1}<n-1$$

and

$$-i_3+1\le i_1=i_2<0\le i_3=i_4<\dots<i_{2j-1}=i_{2j}<i_{2j+1}<\dots<i_{2k+1}<n-1,$$

for some j between 2 and $k+1$. The contribution to the generating function of the first range is

$$\frac{\sqrt{x}\,\tilde U_{k-1}^2}{\tilde U_k^2\,\tilde U_{2k}\,\left(\tilde U_{k+1}-\tilde U_{k-1}\right)\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)},$$

while the contribution of the second range is

$$\frac{\sqrt{x}\,\tilde U_{k-1}^2}{\tilde U_{k-j+1}\,\tilde U_{k-j+2}\,\tilde U_k\,\tilde U_{2k}\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

Summing the second expression over j from 2 to $k+1$, and adding the result to the first expression gives

$$\frac{\sqrt{x}\,\tilde U_{k-1}^2}{\tilde U_k^2\,\tilde U_{2k}\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}\left(\frac{1}{\tilde U_{k+1}-\tilde U_{k-1}}+\tilde U_{k-1}\right).$$

The enumeration in Case 5. By arguments analogous to the ones in Case 1, for given n the number $\beta_n^{(5)}(2k-1)$ of partitions in this subclass is given by the multiple summation

$$\sum_{\substack{0\le i_2\le\dots\le i_{2k}\le n-1\\-i_3+1\le i_1\le -i_2-1}}\left(\prod_{j=4}^{2k+1}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)2^{i_3+i_1-1}.$$

We now compute the generating function $\sum_{n\ge0}\beta_n^{(5)}(2k-1)x^n$. We proceed as in Case 1. Since both $i_2=i_1$ and $i_3=i_2$ produce vanishing summands, we can restrict the summation to

$$-i_3+1\le i_1\le -i_2-1,\quad 0\le i_2<\dots<i_{2k}<n-1$$

As a result, we obtain for the generating function the expression

$$\frac{\sqrt{x}\,\tilde U_{k-1}}{\tilde U_{2k}^2\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

The enumeration in Case 6. By arguments analogous to the ones in Case 1, for given n the number $\beta_n^{(6)}(2k-1)$ of partitions in this subclass is given by the multiple summation

$$\sum_{-i_3+1\le i_1\le i_2<0\le i_3\le\dots\le i_{2k}\le n-1}\left(\prod_{j=4}^{2k+1}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)2^{i_3+i_1-1}.$$

We now compute the generating function $\sum_{n\ge0}\beta_n^{(6)}(2k-1)x^n$. We proceed as in Case 1. We split the range of summation into the possibilities

$$-i_3+1\le i_1\le i_2<0\le i_3<\dots<i_{2k}<n-1$$

and

$$\begin{aligned}-i_3+1\le i_1=i_2<0\le i_3=i_4<\cdots&\\ <i_{2j-1}=i_{2j}<i_{2j+1}<\dots<i_{2k}<n-1,&\end{aligned}$$

for some j between 2 and k. The contribution to the generating function of the first range is

$$\frac{\sqrt{x}\,\tilde U_{k-1}}{\tilde U_{2k}^2\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)},$$

while the contribution of the second range is

$$\frac{\sqrt{x}\,\tilde U_{k-1}^2}{\tilde U_{k-j}\,\tilde U_{k-j+1}\,\tilde U_k\,\tilde U_{2k}\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

Summing the second expression over j from 2 to k, and adding the result to the first expression gives

$$\frac{\sqrt{x}\,\tilde U_{k-1}}{\tilde U_{2k}^2\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}\left(1+\frac{\tilde U_{k-2}\,\tilde U_{2k}}{\tilde U_k}\right).$$

The enumeration in Case 7. By arguments analogous to the ones in Case 1, for given n the number $\beta_n^{(7)}(2k)$ of partitions in this subclass is given by the multiple summation

$$\sum_{0\le i_1\le i_2\le\dots\le i_{2k}\le n-1}\left(\prod_{j=4}^{2k+1}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)\sum_{\ell=0}^{i_3-i_2-1}\binom{i_3+i_1-1}{\ell}.$$

We now compute the generating function $\sum_{n\ge0}\beta_n^{(7)}(2k)x^n$. We proceed as in Case 1. We split the range of summation into the possibilities

$$0\le i_1\le i_2<\cdots<i_{2k}<n-1$$

and

$$0\le i_1=i_2<i_3=i_4<\cdots<i_{2j-1}=i_{2j}<i_{2j+1}<\cdots<i_{2k}<n-1,$$

for some j between 2 and k. The contribution to the generating function of the first range is

$$\frac{\sqrt{x}\,\tilde U_{k+1}^3}{\tilde U_{k-1}\,\tilde U_k^2\,\tilde U_{2k+2}\left(\tilde U_{k+1}-\tilde U_{k-1}\right)\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)},$$

while the contribution of the second range is

$$\frac{\sqrt{x}\,\tilde U_{k+1}^2}{\tilde U_{k-j}\,\tilde U_{k-j+1}\,\tilde U_k\,\tilde U_{2k+2}\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}.$$

Summing the second expression over j from 2 to k, and adding the result to the first expression gives

$$\frac{\sqrt{x}\,\tilde U_{k+1}^2}{\tilde U_{k-1}\,\tilde U_k^2\,\tilde U_{2k+2}\left(\tilde U_k-2\sqrt{x}\,\tilde U_{k-1}\right)}\left(\frac{\tilde U_{k+1}}{\tilde U_{k+1}-\tilde U_{k-1}}+\tilde U_{k-2}\,\tilde U_k\right).$$

Finally, in order to complete the proof of Theorem B.1, in case that K is even, the expressions of Cases 2, 3, 4 and 7 have to be summed, while in case that K is odd, the expressions of Cases 1 (with k replaced by $k-1$), 5 and 6 have to be summed. Some simplification yields the claimed expressions. □

§7 Proofs in type D

We have to prove Theorem D.1, upon which Corollaries D.2 and D.3 follow, as we have described in Section 2.

Proof of Theorem D.1. According to (4.1) and Proposition 4.4 we have to compute

$$2\sum_{n\ge0}\delta_n^{(1)}(K)x^n-\sum_{n\ge0}\beta_{n-1}(K)x^n. \tag{7.1}$$

By Theorem B.1, we already know an expression for the second sum. It remains to compute $\sum_{n\ge0}\delta_n^{(1)}(K)x^n$, where $\delta_n^{(1)}(K)$ is the number of partitions $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2n-2})$ in the staircase T_{2n-2} with the property that $(\lambda_2,\dots,\lambda_{2n-2})$ is self-conjugate, that $\lambda_i\ne i-1$ for $i\ge2$, and that $n(\lambda)=K$.

We proceed by imitating the arguments used in Case B in the previous section. First, instead of considering the above (ordinary) partitions $(\lambda_1,\lambda_2,\dots,\lambda_{2n-2})$, we consider again just the "upper halfs," the shifted partitions with row lengths

$(\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots)$. We apply again the construction of the two broken rays as in Case B, with the slight modification that it is now the line $x+y=2n-1$ (instead of $x+y=2n$) where the rays get reflected. We divide the shifted partitions that we consider here again into 7 subclasses. The characterization of the 7 subclasses and the description of how to determine the corresponding class of nilpotence are identical to those in Case B, except that, of course, again the line $x+y=2n$ has to be replaced by the line $x+y=2n-1$. We therefore omit to repeat them here and instead refer the reader to Section 6. For writing down formulas for the number of partitions in each subclass we also follow the derivations in Case B. In particular, we choose again $(n,0)$ as the reference point with respect to which deviations of the first vertical edges on each downward travel along a ray are measured.

We now discuss each of the 7 subclasses. Since everything is completely parallel to the computations in Case B, given in the previous section, we can be brief here. For each subclass, we provide the formula in the spirit of (A.1) for the number of shifted partitions in that subclass, and the corresponding generating function.

The enumeration in Case 1: For given n, the number of shifted partitions in this subclass is given by the multiple summation

$$\sum_{-i_2-1\le i_1\le i_2\le\dots\le i_{2k+1}\le n-2}\left(\prod_{j=4}^{2k+2}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)\sum_{\ell=0}^{i_3-i_2-1}\binom{i_3+i_1}{\ell},$$

where $i_{2k+2}=n-1$. For the range

$$-i_2-1\le i_1\le i_2,\quad i_2+\ell+1\le i_3<\dots<i_{2k+1}<n-2$$

we obtain

$$\frac{2\,x\,\tilde U_{k+1}^2}{\tilde U_k\tilde U_{2k+2}^2\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)},$$

for the corresponding generating function, while for the range

$$\begin{aligned}0\le i_1=i_2,\quad i_2+\ell+1\le i_3=i_4<i_5=i_6<\cdots\\ <i_{2j-1}=i_{2j}<i_{2j+1}<\dots<i_{2k+1}<n-2,\end{aligned}$$

for some j between 2 and $k+1$, we obtain

$$\frac{x\,\tilde U_{k+1}}{\tilde U_{k-j+1}\,\tilde U_{k-j+2}\,\tilde U_{2k+2}\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}.$$

The overall generating function for this subclass is

$$\frac{x\,\tilde U_{k+1}\left(2\tilde U_{k+1}+\tilde U_{k-1}\tilde U_{2k+2}\right)}{\tilde U_k\,\tilde U_{2k+2}^2\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}.$$

The enumeration in Case 2: For given n, the number of shifted partitions in this subclass is given by the multiple summation

$$\sum_{\substack{0\le i_2\le\dots\le i_{2k+1}\le n-2\\ -i_3\le i_1\le -i_2-2}}\left(\prod_{j=4}^{2k+2}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)2^{i_3+i_1},$$

where $i_{2k+2}=n-1$. We can restrict the sum to the range

$$-i_3\le i_1\le -i_2-2,\quad 0\le i_2<\dots<i_{2k+1}<n-2.$$

We obtain

$$\frac{x\,\tilde U_{k-1}}{\tilde U_k\,\tilde U_{2k+2}\,\left(\tilde U_{k+1}-\tilde U_{k-1}\right)\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}$$

for the corresponding generating function.

The enumeration in Case 3: For given n, the number of shifted partitions in this subclass is given by the multiple summation

$$\sum_{-i_2-1\le i_1<0\le i_2\le\dots\le i_{2k}\le n-2}\left(\prod_{j=4}^{2k+1}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)\sum_{\ell=0}^{i_3-i_2-1}\binom{i_3+i_1}{\ell},$$

where $i_{2k+1}=n-1$. We can restrict the sum to the range

$$-i_2-1\le i_1<0\le i_2<\dots<i_{2k}<n-2.$$

We obtain

$$\frac{x\,\tilde U_{k+1}}{\tilde U_k\,\tilde U_{2k}\,\left(\tilde U_{k+1}-\tilde U_{k-1}\right)\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}$$

for the corresponding generating function.

The enumeration in Case 4: For given n, the number of shifted partitions in this subclass is given by the multiple summation

$$\sum_{-i_3\le i_1\le i_2<0\le i_3\le\dots\le i_{2k+1}\le n-2}\left(\prod_{j=4}^{2k+2}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)2^{i_3+i_1},$$

where $i_{2k+2}=n-1$. For the range

$$-i_3\le i_1\le i_2<0\le i_3<\dots<i_{2k+1}<n-2$$

we obtain

$$\frac{x\,\tilde U_{k-1}}{\tilde U_k\,\tilde U_{2k}\,\left(\tilde U_{k+1}-\tilde U_{k-1}\right)\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}$$

for the corresponding generating function, while for the range

$$-i_3\le i_1=i_2<0\le i_3=i_4<\dots<i_{2j-1}=i_{2j}<i_{2j+1}<\dots<i_{2k+1}<n-2,$$

for some j between 2 and $k+1$, we obtain

$$\frac{x\,\tilde U_{k-1}}{\tilde U_{k-j+1}\,\tilde U_{k-j+2}\,\tilde U_{2k}\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}.$$

The overall generating function is

$$\frac{x\,\tilde U_{k-1}}{\tilde U_k\,\tilde U_{2k}\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}\left(\frac{1}{\tilde U_{k+1}-\tilde U_{k-1}}+\tilde U_{k-1}\right).$$

The enumeration in Case 5: For given n, the number of shifted partitions in this subclass is given by the multiple summation

$$\sum_{\substack{0\le i_2\le\cdots\le i_{2k}\le n-2\\ -i_3\le i_1\le -i_2-2}}\left(\prod_{j=4}^{2k+1}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)2^{i_3+i_1},$$

where $i_{2k+1}=n-1$. We can restrict the sum to the range

$$-i_3\le i_1\le -i_2-2,\quad 0\le i_2<\cdots<i_{2k}<n-2.$$

We obtain

$$\frac{x\,\tilde U_{k-1}^2}{\tilde U_{2k}^2\,\tilde U_k\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}$$

for the corresponding generating function.

The enumeration in Case 6: For given n, the number of shifted partitions in this subclass is given by the multiple summation

$$\sum_{-i_3\le i_1\le i_2<0\le i_3\le\cdots\le i_{2k}\le n-2}\left(\prod_{j=4}^{2k+1}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)2^{i_3+i_1},$$

where $i_{2k+1}=n-1$. For the range

$$-i_3\le i_1\le i_2<0\le i_3<\cdots<i_{2k}<n-2$$

we obtain

$$\frac{x\,\tilde U_k}{\tilde U_{2k}^2\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}$$

for the corresponding generating function, while for the range

$$\begin{aligned}-i_3+1\le i_1=i_2<0\le i_3=i_4<\cdots&\\ <i_{2j-1}=i_{2j}<i_{2j+1}<\cdots<i_{2k}<n-1,&\end{aligned}$$

for some j between 2 and k, we obtain

$$\frac{x\,\tilde U_{k-1}}{\tilde U_{k-j+1}\,\tilde U_{k-j}\,\tilde U_{2k}\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}.$$

The overall generating function is

$$\frac{x\,\tilde U_k}{\tilde U_{2k}^2\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}\left(1+\frac{\tilde U_{k-2}\tilde U_{2k}}{\tilde U_k}\right).$$

The enumeration in Case 7: For given n, the number of shifted partitions in this subclass is given by the multiple summation

$$\sum_{0\le i_1\le i_2\le\cdots\le i_{2k+1}\le n-2}\left(\prod_{j=4}^{2k+2}\binom{i_j-i_{j-1}+i_{j-2}-i_{j-3}-1}{i_{j-2}-i_{j-3}}\right)\sum_{\ell=0}^{i_3-i_2-1}\binom{i_3+i_1}{\ell},$$

where $i_{2k+2}=n-1$. For the range

$$0\le i_1\le i_2<\cdots<i_{2k}<n-2$$

we obtain

$$\frac{x\,\tilde U_{k+1}^2}{\tilde U_k\,\tilde U_{k-1}\,\tilde U_{2k+2}\,\left(\tilde U_{k+1}-\tilde U_{k-1}\right)\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}$$

for the corresponding generating function, while for the range

$$0\le i_1=i_2<i_3=i_4<\cdots<i_{2j-1}<i_{2j}<i_{2j+1}<\cdots<i_{2k}<n-2,$$

for some j between 2 and k, we obtain

$$\frac{x\,\tilde U_{k+1}}{\tilde U_{k-j+1}\,\tilde U_{k-j}\,\tilde U_{2k+2}\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}.$$

The overall generating function is

$$\frac{x\,\tilde U_{k+1}}{\tilde U_{k-1}\,\tilde U_k\,\tilde U_{2k+2}\,\left(\tilde U_k-2\sqrt{x}\tilde U_{k-1}\right)}\left(\frac{\tilde U_{k+1}}{\tilde U_{k+1}-\tilde U_{k-1}}+\tilde U_{k-2}\tilde U_k\right).$$

Finally, in order to complete the proof of Theorem D.1, we have to combine all our results to obtain the generating function (7.1). In order to obtain the generating function $\sum_{n\ge0}\delta_n^{(1)}(K)x^n$, in case that K is even, the expressions of Cases 2, 3, 4 and 7 have to be summed, while in case that K is odd, the expressions of Cases 1 (with k replaced by $k-1$), 5 and 6 have to be summed. The result is then substituted in (7.1), together with the result of the previous section for $\sum_{n\ge0}\beta_{n-1}(K)x^n$. Some simplification eventually yields the claimed expressions. □

$n(\mathfrak{i})$	E_6	E_7	E_8	F_4	G_2
0	1	1	1	1	1
1	63	127	255	15	3
2	210	662	2200	28	2
3	217	894	3804	21	1
4	150	766	3872	14	0
5	92	576	3372	12	1
6	51	403	2752	5	
7	28	279	2182	4	
8	12	175	1656	2	
9	6	115	1277	2	
10	2	68	955	0	
11	1	44	737	1	
12		23	536		
13		14	412		
14		7	300		
15		4	227		
16		1	157		
17		1	123		
18			81		
19			61		
20			40		
21			30		
22			18		
23			14		
24			7		
25			5		
26			3		
27			2		
28			0		
29			1		
total number	833	4160	25080	105	8

Table 1

§8 The exceptional types

It is not difficult to write down a computer program which determines explicitly the descending central series of a given ideal. The final results are given in Table 1.

Just a few remarks on how to get the data in the table. Clearly we can work at the level of the root system. If $\mathfrak{i} \in \mathcal{I}^n$ is encoded by $\Phi \subseteq \Delta^+$, then $n(\mathfrak{i})$ equals the maximal integer k such that $\theta \in \Phi^k$ (here θ is the highest root of Δ and Φ^k is inductively defined as $\Phi^1 = \Phi$, $\Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta)$. The input for the program (i.e., the Φ's) can be reduced to the determination of the antichains of the root poset. (Indeed, for any finite poset, there is a canonical bijection mapping the antichain $\{a_1, \dots , a_k\}$ to the dual order ideal which is the union of the principal dual order ideals $V_{a_1}, \dots, V_{a_k}$.) In turn, the calculation of the antichains has been done using the Maple program *Coxeter* [Ste]. Note that the total number of ideals in type E_6 is 833, one more than the number given in [Sh, Theorem 3.6]. Our counting agrees with formula (1.1).

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