A DETERMINANT EVALUATION CONJECTURED BY ROLAND BACHER

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Abstract. We evaluate the determinant of a matrix the entries of which are certain binomial sums. This proves a conjecture of Roland Bacher.

Roland Bacher [private communication] made the following conjecture.

Conjecture. *For all non-negative integers n, we have*

$$\det_{0\le i,j\le n-1}\left(\sum_{k\ge0}\binom{2i}{k}\binom{2j}{k}2^k\right)=2^{n^2-n}\prod_{i=0}^{n-1}\frac{i!^2\,(4i)!}{(2i)!^3}.$$

We shall in fact prove the following theorem.

Theorem. *For all non-negative integers n and s with $s\le n$, we have*

$$\det_{0\le i,j\le n-1}\left(\sum_{k\ge0}\binom{2(i+\chi(i\ge s))}{k}\binom{2j}{k}2^k\right)$$
$$=2^{n^2-n}\frac{(4s+1)}{(2n+2s+1)}\binom{n}{s}\frac{(n+s+\frac12)_{n-s}}{(s+\frac12)_{n-s}}\prod_{i=0}^{n-1}\frac{i!^2\,(4i)!}{(2i)!^3},\tag{1}$$

where $\chi(\mathcal{S})=1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S})=0$ otherwise.

Clearly, the above conjecture is the special case $s=n$ of this theorem.

Proof of the theorem. The proof follows the basic set-up in [2], except that we do not make use of recurrences.

The determinants on the left-hand side of (1) are — up to sign — co-factors of the matrix

$$M_n=\left(\sum_{k\ge0}\binom{2i}{k}\binom{2j}{k}2^k\right)_{0\le i,j\le n}.$$

More precisely, they are — again, up to sign — the co-factors where the last column and some row of M_n is omitted. Let C_s denote the determinant of the matrix which arises from M_n by deleting the last column and the s-th row, $s=0,1,\dots,n$. According to basic principles of linear algebra, the C_s's satisfy the obvious equations

$$\sum_{s=0}^{n}(-1)^s(M_n)_{s,j}\cdot X_s=0,\qquad j=0,1,\dots,n-1,\tag{2}$$

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with $X_s = C_s$. The above equations determine the X_s's up to some common multiplicative constant (provided that the matrix M_n is non-singular, which will turn out to be true as a by-product of this proof.). They will be uniquely determined if we fix one of the X_s's to a particular value, say, if we fix X_n to be the determinant of the matrix arising from M_n by deleting the last column and the last row, or, in other words, we fix $X_n = \det M_{n-1}$. In that case, the above system of linear equations forces X_s to be equal to C_s, $s = 0, 1, \dots, n-1$.

In summary, we can turn the above observations into an inductive proof of the Theorem by establishing (2) with $X_s = C_s$, $s = 0, 1, \dots, n$, *and*

$$\sum_{s=0}^{n} (-1)^{s+n} (M_n)_{s,n} \cdot C_s = 2^{n^2+n} \prod_{i=0}^{n} \frac{i!^2\,(4i)!}{(2i)!^3}. \tag{3}$$

(The reader should observe that the right-hand side of (3) is the claimed value of $\det M_n$.)

We now calculate the sum on the left-hand sides of (2) and (3), for $j = 0, 1, \dots, n$. We have

$$\begin{aligned}&\sum_{s=0}^{n} (-1)^{s+j} (M_n)_{s,j} \cdot C_s \\ &= 2^{n^2-n} \prod_{i=0}^{n-1} \frac{i!^2\,(4i)!}{(2i)!^3} \sum_{s=0}^{n} (-1)^{s+j} \sum_{k=0}^{2s} \binom{2s}{k}\binom{2j}{k} 2^k \frac{(4s+1)}{(2n+2s+1)} \binom{n}{s} \frac{(n+s+\frac{1}{2})_{n-s}}{(s+\frac{1}{2})_{n-s}}.\end{aligned}$$

Substitution of the right-hand side in (2) and (3) shows that, what we have to prove is

$$\begin{aligned}\sum_{s=0}^{n} (-1)^{s+j} \sum_{k=0}^{2s} \binom{2s}{k}\binom{2j}{k} 2^k \frac{(4s+1)}{(2n+2s+1)} \binom{n}{s} \frac{(n+s+\frac{1}{2})_{n-s}}{(s+\frac{1}{2})_{n-s}}& \\ = \delta_{j,n} 2^{2n} \frac{n!^2\,(4n)!}{(2n)!^3}&.\end{aligned} \tag{4}$$

where $\delta_{j,n}$ is the usual Kronecker delta.

In order to proceed, we rewrite the sum over k has a contour integral,

$$\sum_{k=0}^{2s} \binom{2s}{k}\binom{2j}{k} 2^k = \frac{1}{2\pi i} \int_{\mathcal{C}} (1+z)^{2s} (z+2)^{2j} \frac{dz}{z^{2j+1}},$$

where $\mathcal{C}$ is a small contour encircling the origin once in positive direction. We substitute this on the left-hand side of (4), and we write the sum over s in hypergeometric notation. Thus, we obtain

$$(-1)^j \frac{(n+\frac{1}{2})_n}{(2n+1)\,(\frac{1}{2})_n} \frac{1}{2\pi i} \int_{\mathcal{C}} (z+2)^{2j} {}_3F_2\left[\begin{matrix}\frac{5}{4}, \frac{1}{2}, -n \\ \frac{1}{4}, n+\frac{3}{2}\end{matrix}; (z+1)^2\right] \frac{dz}{z^{2j+1}} \tag{5}$$

for the left-hand side of (4). We now apply the contiguous relation

$${}_3F_2\left[\begin{matrix}a, b, c \\ d, e\end{matrix}; x\right] = {}_3F_2\left[\begin{matrix}a-1, b, c \\ d, e\end{matrix}; x\right] + \frac{xbc}{de} {}_3F_2\left[\begin{matrix}a, b+1, c+1 \\ d+1, e+1\end{matrix}; x\right]$$

with $a = 5/4$, $b = 1/2$, $c = -n$, $d = 1/4$, $e = n + \frac{3}{2}$, $x = (z+1)^2$. This leads to the expression

$$(-1)^j \frac{(n+\frac{1}{2})_n}{(2n+1)\,(\frac{1}{2})_n} \frac{1}{2\pi i} \int_C (z+2)^{2j} \left({}_2F_1\left[\begin{matrix}\frac{1}{2}, -n\\ n+\frac{3}{2}\end{matrix}; (z+1)^2\right] \right.$$
$$\left. - \frac{4n(z+1)^2}{2n+3} {}_2F_1\left[\begin{matrix}\frac{3}{2}, -n+1\\ n+\frac{5}{2}\end{matrix}; (z+1)^2\right] \right) \frac{dz}{z^{2j+1}}$$

Next, we apply the transformation formula (see [1, (1.8.10), terminating form])

$${}_2F_1\left[\begin{matrix}a, -N\\ c\end{matrix}; z\right] = \frac{(c-a)_N}{(c)_N} {}_2F_1\left[\begin{matrix}a, -N\\ 1+a-c-N\end{matrix}; 1-z\right],$$

where N is a nonnegative integer, to both ${}_2F_1$-series. This yields the expression

$$(-1)^j \frac{(n+1)_n}{(4n+1)\,(\frac{1}{2})_n} \frac{1}{2\pi i} \int_C (z+2)^{2j} \left({}_2F_1\left[\begin{matrix}\frac{1}{2}, -n\\ -2n\end{matrix}; -z(z+2)\right] \right.$$
$$\left. - (1+z(z+2)) {}_2F_1\left[\begin{matrix}\frac{3}{2}, -n+1\\ -2n+1\end{matrix}; -z(z+2)\right] \right) \frac{dz}{z^{2j+1}},$$

or, if we expand the hypergeometric series and evaluate the contour integrals,

$$(-1)^j \frac{4^n}{(4n+1)} \left(\sum_{\ell=0}^{n} (-1)^\ell \frac{(\frac{1}{2})_\ell\,(-n)_\ell}{(-2n)_\ell\,\ell!} \binom{2j+\ell}{2j-\ell} 2^{2\ell} \right.$$
$$\left. - \sum_{\ell=0}^{n-1} (-1)^\ell \frac{(\frac{3}{2})_\ell\,(-n+1)_\ell}{(-2n+1)_\ell\,\ell!} \binom{2j+\ell}{2j-\ell} 2^{2\ell} - \sum_{\ell=0}^{n-1} (-1)^\ell \frac{(\frac{3}{2})_\ell\,(-n+1)_\ell}{(-2n+1)_\ell\,\ell!} \binom{2j+\ell+1}{2j-\ell-1} 2^{2\ell+2} \right).$$

We replace ℓ by $\ell - 1$ in the last sum, and then we put everything together into one sum over ℓ. After some simplification, we arrive at

$$(-1)^{j+1} 4^n \sum_{\ell=1}^{n} (-1)^\ell \frac{(\frac{3}{2})_{\ell-1}\,(-n+1)_{\ell-1}}{(1-2n)_\ell\,(\ell-1)!} \binom{2j+\ell}{2j-\ell} 2^{2\ell-2}.$$

We write the sum in hypergeometric notation and obtain

$$(-1)^{j+1} 4^n \frac{2j(2j+1)}{2(2n-1)} {}_3F_2\left[\begin{matrix}-2j+1, 2j+2, -n+1\\ 2, -2n+2\end{matrix}; 1\right]. \tag{6}$$

This ${}_3F_2$-series can be evaluated by means of Watson's ${}_3F_2$-summation formula (see [1, (2.3.3.13); Appendix (III.23)])

$${}_3F_2\left[\begin{matrix}a, b, c\\ \frac{a}{2}+\frac{b}{2}+\frac{1}{2}, 2c\end{matrix}; 1\right] = \frac{\Gamma(\frac{1}{2})\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})\,\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2})}{\Gamma(\frac{a}{2}+\frac{1}{2})\,\Gamma(\frac{b}{2}+\frac{1}{2})\,\Gamma(c-\frac{a}{2}+\frac{1}{2})\,\Gamma(c-\frac{b}{2}+\frac{1}{2})}. \tag{7}$$

Care must however be taken when applying this formula to our series since direct application of the formula leads to singularities. Instead of (6), let us consider

$$(-1)^{j+1} 4^n \frac{2j(2j+1)}{2(2n-1)} \lim_{\varepsilon\to 0} {}_3F_2\left[\begin{matrix}-2j-2\varepsilon+1, 2j+2\varepsilon+2, -n+\varepsilon+1\\ 2, -2n+\varepsilon+2\end{matrix}; 1\right].$$

Now we may apply (7), which transforms the above expression into

$$(-1)^{j+1}4^n\frac{2j(2j+1)}{2(2n-1)}\\ \times\lim_{\varepsilon\to0}\frac{\Gamma(\frac{1}{2})\,\Gamma(-n+\varepsilon+\frac{3}{2})\,\Gamma(-n+\varepsilon)}{\Gamma(-j-\varepsilon+1)\,\Gamma(j+\varepsilon+\frac{3}{2})\,\Gamma(-n+j+2\varepsilon+1)\,\Gamma(-n-j+\frac{1}{2})}. \quad (8)$$

Here, the critical terms are $\Gamma(-n+\varepsilon)$, $\Gamma(-j-\varepsilon+1)$, $\Gamma(-n+j+2\varepsilon+1)$, which tend to infinity for $\varepsilon\to0$, the last one only if $j<n$. Consequently, if $j<n$, the limit in (8) obviously vanishes, whence (4) is indeed true in this case.

It remains to consider the case $j=n$. Since we have

$$\lim_{\varepsilon\to0}\frac{\Gamma(-n+\varepsilon)}{\Gamma(-n-\varepsilon+1)}=\frac{1}{n},$$

the expression (8) reduces to

$$(-1)^{n+1}4^n\frac{(2n+1)}{(2n-1)}\frac{\Gamma(\frac{1}{2})\,\Gamma(-n+\frac{3}{2})}{\Gamma(n+\frac{3}{2})\,\Gamma(-2n+\frac{1}{2})}.$$

Further rewriting converts this into

$$4^n\frac{n!^2\,(4n)!}{(2n)!^3},$$

which is again in accordance with (4). This completes the proof of the theorem. $\square$

Remark. Roland Bacher has a similar conjecture for the determinant

$$\det_{0\le i,j\le n-1}\left(\sum_{k\ge0}\binom{2i}{k}\binom{2j}{k}4^k\right).$$

To find a proof appears to be harder than in the case of the conjecture proved here.

References

[1] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.
[2] D. Zeilberger, *The holonomic Ansatz II. Automatic discovery(!) and proof(!!) of holonomic determinant evaluations*, Ann. Combin. **11** (2007), 241–247.

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