Determinants of (generalised) Catalan numbers

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Abstract. We show that recent determinant evaluations involving Catalan numbers and generalisations thereof have most convenient explanations by combining the Lindström–Gessel–Viennot theorem on non-intersecting lattice paths with a simple determinant lemma from [*Manuscripta Math.* **69** (1990), 173–202]. This approach leads also naturally to extensions and generalisations.

1. Introduction. Determinants of matrices containing combinatorial numbers have always been of big attraction to many researchers. The combinatorial numbers which are in the centre of the present paper are the *Catalan numbers* C_n, defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

and extensions thereof. (The reader is referred to [43, Exercise 6.19] for extensive information on these numbers.) Numerous papers exist in the literature featuring determinants of matrices the entries of which contain Catalan numbers or generalisations thereof, see [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 22, 23, 24, 32, 33, 35, 37, 38, 39, 40, 41, 42, 44, 45] (and this list is for sure incomplete). The contexts in which they appear are also widespread, ranging from lattice path enumeration (cf. [2, 3, 5, 6, 7, 8, 12, 13, 23, 18, 32, 35, 37, 45]), plane partition and tableaux counting (cf. [2, 11, 16, 22]), continued fractions and orthogonal polynomials (cf. [13, 15, 40, 44, 45]), statistical physics (cf. [7, 8, 32, 37]), up to commutative algebra and algebraic geometry (cf. [24]).

One of the most popular themes in this context is *Hankel determinants* of Catalan numbers, that is, determinants of the form $\det_{0\le i,j\le n-1}(a_{i+j})$, where the sequence $(a_i)_{i\ge0}$ involves Catalan numbers. (Cf. [30, Sec. 2.7] and [31, Sec. 5.4] for general information

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on the evaluation of Hankel determinants.) To be more specific, Hankel determinant evaluations such as

$$\det_{0\le i,j\le n-1}(C_{i+j}) = 1, \tag{1.1}$$

$$\det_{0\le i,j\le n-1}(C_{i+j+1}) = 1, \tag{1.2}$$

or

$$\det_{0\le i,j\le n-1}(C_{i+j+2}) = n+1 \tag{1.3}$$

have been addressed numerous times in the literature. More recently, it has been observed in [33] and proved in [15] that

$$\det_{0\le i,j\le n-1}(C_{i+j}+C_{i+j+1}) = F_{2n}, \tag{1.4}$$

where F_m denotes the m-th *Fibonacci number*; that is, $F_0 = F_1 = 1$ and $F_m = F_{m-1}+F_{m-2}$ for $m \ge 2$.

Let k be a fixed positive integer. Taking the *generalised Catalan number* $C_{n,k}$ from [27], given by

$$C_{n,k} = \frac{n-(k-1)\lfloor\frac{n}{k-1}\rfloor+1}{n+\lfloor\frac{n}{k-1}\rfloor+1}\binom{n+\lfloor\frac{n}{k-1}\rfloor+1}{n+1}, \tag{1.5}$$

the determinant evaluation in (1.4) has been generalised in [10] to

$$\det_{0\le i,j\le n-1}(C_{(k-1)i+j,k}+C_{(k-1)(i+1)+j,k}) = \sum_{s=0}^{n}\binom{(k-1)s+n}{n-s} \tag{1.6}$$

and

$$\det_{0\le i,j\le n-1}(C_{(k-1)i+j,k}+C_{(k-1)i+j+1,k}) = \sum_{s=0}^{n}\binom{\lfloor\frac{s}{k-1}\rfloor+n}{n-s}. \tag{1.7}$$

Many different methods have been employed to prove the formulae (1.1)–(1.4), (1.6), and (1.7), among which are direct determinant manipulations, non-intersecting lattice paths, orthogonal polynomials, and LU factorisation. However, in the author's opinion, none of the published approaches explains in a satisfactory and uniform way why these identities exist. This is what we aim to do in this article. We show that the above determinant evaluations are actually special cases of families of more general determinant evaluations, which result from a combination of a simple determinant lemma from [29] (see Lemma 1) and the main theorem on non-intersecting lattice paths (see Theorem 2). In particular, Theorem 3, originally due to Gessel and Viennot [22], shows that one can in fact evaluate the determinant $\det_{0\le i,j\le n-1}(C_{\alpha_i+j})$ in closed form, where $\alpha_0,\alpha_1,\dots,\alpha_{n-1}$ are arbitrary non-negative numbers, thus largely generalising (1.1)–(1.3). Together with the simple determinant expansion in Lemma 4, this explains as well formula (1.4). Moreover, it also enables us to embed (1.4) in a much larger family of determinant evaluations, see Corollary 5.

Starting point for our "explanation" of (1.6) and (1.7) is the (well-known) observation that the generalised Catalan numbers in (1.5) count lattice paths which stay below a slanted line (see (4.1)). Then, altogether, the main theorem on non-intersecting lattice paths, combinatorial arguments involving paths, and the aforementioned determinant lemma, lead in fact again to much more general determinant evaluations, which we present in Theorem 6 (which generalises Theorem 3, and, thus, identities (1.1)–(1.3)), Corollary 7, and Theorem 9, respectively.

The final section, Section 5, is provided here for the sake of completeness. There, we consider generalised Catalan numbers which are different from those in (1.5). We briefly survey the deep results from [18, 23] which are known on Hankel determinants involving these numbers.

We emphasise that we do not address *weighted* generalisations in this paper. Sometimes these can also be approached by the method of non-intersecting lattice paths (cf. [13, 32, 37, 45] for examples), but often more refined methods are needed (cf. [7, 8, 12, 14, 32, 37]).

2. Preliminaries. In this section we present the two auxiliary results on which everything else in this paper is based: a general determinant lemma and the Lindström–Gessel–Viennot theorem on non-intersecting lattice paths.

We begin with the determinant lemma, which is taken from [29, Lemma 2.2] (see also [30, Lemma 3]). There are many ways to prove it. The "easiest" is by condensation (once one takes the determinant identity by Desnanot and Jacobi which underlies the condensation method for granted; cf. [30, Sec. 2.3]). For alternative proofs see [9, Theorem 2.9], [29], and [30, App. B].

Lemma 1. *Let $X_0,\dots,X_{n-1}$, $A_1,\dots,A_{n-1}$, and $B_1,\dots,B_{n-1}$ be indeterminates. Then there holds*

$$\det_{0\le i,j\le n-1}\Big((X_i+A_{n-1})(X_i+A_{n-2})\cdots(X_i+A_{j+1})(X_i+B_j)(X_i+B_{j-1})\cdots(X_i+B_1)\Big)$$
$$=\prod_{0\le i<j\le n-1}(X_i-X_j)\prod_{1\le i\le j\le n-1}(B_i-A_j).$$

The other auxiliary result that we need is a determinant formula for the enumeration of non-intersecting lattice paths, originally due to Lindström [34, Lemma 1], rediscovered by Gessel and Viennot [22] (and special cases in [20, Sec. 5.3] and in [28, 26]; for a more detailed historical account see Footnote 6 in [32]). Since, as is the case in most applications, we do not need the theorem in its most general form, we state the special case that serves our purposes. The lattice paths to which we want to apply the theorem are lattice paths in the integer plane consisting of horizontal and vertical unit steps in the positive direction. We make the convention for the rest of the paper that this is what we mean whenever we speak of "lattice paths," unless it is specified otherwise. A family $(P_0,P_1,\dots,P_{n-1})$ of lattice paths P_i, $i=0,1,\dots,n-1$, is called *non-intersecting* if no two paths in the family have a point in common. Examples can be found in Figure 1–5. With the above terminology, we have the following theorem.

Theorem 2. *Let $A_0,A_1,\dots,A_{n-1}$ and $E_0,E_1,\dots,E_{n-1}$ be lattice points in the plane integer lattice such that for $i<j$ and $k<l$ any lattice path from A_i to E_l has a*

common point with any lattice path from A_j to E_k. Then the number of all families $(P_0, P_1, \dots, P_{n-1})$ of non-intersecting lattice paths, P_i running from A_i to E_i, $i = 0, 1, \dots, n-1$, is given by

$$\det_{0\le i,j\le n-1} \big(P(A_j \to E_i)\big),$$

where $P(A \to E)$ denotes the number of all lattice paths from A to E.

3. Determinants of Catalan numbers. In this section we present evaluations of determinants in which the entries involve Catalan numbers. These determinant evaluations generalise Eqs. (1.1)–(1.4). Our first theorem is a common generalisation of Eqs. (1.1)–(1.3). It is originally due to Gessel and Viennot [22, paragraph between Theorems 21 and 22], who gave essentially the proof that we present below.

Theorem 3. *Let n be a positive integer and $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ non-negative integers. Then*

$$\det_{0\le i,j\le n-1} \left(C_{\alpha_i+j}\right) = \prod_{0\le i<j\le n-1} (\alpha_j - \alpha_i) \prod_{i=0}^{n-1} \frac{(i+n)!\,(2\alpha_i)!}{(2i)!\,\alpha_i!\,(\alpha_i+n)!}.$$

Proof. We use the observation $C_m = (-1)^m 2^{2m+1}\binom{1/2}{m+1}$ to rewrite the determinant in question as

$$(-1)^{\binom{n}{2}+\sum_{i=0}^{n-1}\alpha_i} 2^{n^2+2\sum_{i=0}^{n-1}\alpha_i} \det_{0\le i,j\le n-1}\left(\binom{1/2}{\alpha_i+j+1}\right). \tag{3.1}$$

Then, by taking some factors out of the i-th row of the matrix of which we want to compute the determinant, $i = 0, 1, \dots, n-1$, we obtain

$$\begin{aligned}\det_{0\le i,j\le n-1} \left(C_{\alpha_i+j}\right) &= (-1)^{\sum_{i=0}^{n-1}\alpha_i} 2^{n^2+2\sum_{i=0}^{n-1}\alpha_i} \prod_{i=0}^{n-1} \frac{\left(\frac{1}{2}-\alpha_i\right)_{\alpha_i+1}}{(\alpha_i+n)!}\\ &\quad\times \det_{0\le i,j\le n-1}\Bigg(\left(\alpha_i+j-\frac{1}{2}\right)\left(\alpha_i+j-\frac{3}{2}\right)\cdots\left(\alpha_i+\frac{1}{2}\right)\\ &\qquad\qquad\cdot(\alpha_i+j+2)(\alpha_i+j+3)\cdots(\alpha_i+n)\Bigg).\end{aligned}$$

Now Lemma 1 can be applied with $X_i = \alpha_i$, $A_j = j+1$, and $B_j = j - \frac{1}{2}$. After some manipulation, one arrives at the claimed result. $\square$

Remarks. (a) Clearly, the determinant evaluations (1.1)–(1.3) are all special cases of Theorem 3.

(b) From a conceptual point of view, the "actual" theorem is the determinant evaluation for the determinant in (3.1). In fact, there holds the determinant evaluation (see [30, Theorem 26, (3.12)])

$$\det_{0\le i,j\le n-1}\left(\binom{A}{\alpha_i+j}\right) = \frac{\prod_{0\le i<j\le n-1}(\alpha_i-\alpha_j)}{\prod_{i=0}^{n-1}(\alpha_i+n-1)!} \frac{\prod_{i=0}^{n-1}(A+i)!}{\prod_{i=0}^{n-1}(A-\alpha_i)!}.$$

(The cited theorem in [30] is even a q-analogue of the above identity.) This identity is not restricted to integral A and α_i: if one interprets the appearing factorials and binomials as suitable expressions involving gamma functions (cf. [25, §5.5, (5.96), (5.100)]), then it continues to hold for real or complex A and α_i as long as there do not appear any singularities.

(c) Gessel and Viennot used Theorem 3 to derive tableaux enumeration results, see [22, Theorems 22 and 23].

(d) The most elegant proof of the special cases (1.1)–(1.3) of Theorem 3 is by using non-intersecting lattice paths. For (1.1) and (1.2), these (one figure) proofs appear as side results already in [45]. For (1.3), a corresponding proof is explained in [5], as well as a non-intersecting lattice path proof for the "next" case, the evaluation of the Hankel determinant $\det_{0\le i,j\le n-1}\left(C_{i+j+3}\right)$. On the other hand, it seems unlikely that the full statement of Theorem 3 can be explained combinatorially.

(e) An ubiquitous determinant is the Hankel determinant $\det_{0\le i,j\le n-1}\left(C_{i+j+\beta}\right)$, where β is some fixed positive integer, which, by Theorem 3 with $\alpha_i = i+\beta$, $i=0,1,\dots,n-1$, has a closed form product evaluation. By Theorem 2 and the standard combinatorial interpretation of the Catalan number C_m as the number of lattice paths from the origin to (m,m) which never pass above the diagonal $x=y$, this determinant counts families $(P_0,P_1,\dots,P_{n-1})$ of non-intersecting lattice paths, where P_i runs from $(-i,-i)$ to $(i+\beta, i+\beta)$ and does not pass over the diagonal $x=y$, $i=0,1,\dots,n-1$. In [16], Desainte–Catherine and Viennot showed that this counting problem is equivalent to the problem of counting tableaux with a bounded number of columns, all rows being of even length. Desainte–Catherine and Viennot solve the counting problem by evaluating the determinant by means of the quotient-difference algorithm (see also [46]). A weighted version of the tableaux counting problem of Desainte–Catherine and Viennot was solved by Désarménien [17, Théorème 1.2]. The determinant and its associated counting problem arise also in the context of the determination of the multiplicity of Pfaffian rings, see Theorem 2 and the accompanying remarks in [24].

It is a simple observation that, given a determinant in which each entry is the sum of two expressions,

$$\det_{0\le i,j\le n-1}(a_{i,j}+b_{i,j})$$

say, one can use linearity of the determinant in the rows to expand this determinant into the sum

$$\sum_{S\subseteq\{0,1,\dots,n-1\}} \det_{0\le i,j\le n-1}\left(c_{i,j}^{(S)}\right), \tag{3.2}$$

where $c_{i,j}^{(S)} = a_{i,j}$ if $i\in S$ and $c_{i,j}^{(S)} = b_{i,j}$ otherwise. This sum consists of 2^n terms and, normally, will therefore not be very useful. Should it happen, however, that the $(i+1)$-st row of the matrix $(a_{i,j})_{0\le i,j\le n-1}$ and the i-th row of the matrix $(b_{i,j})_{0\le i,j\le n-1}$ agree for all i, $i=0,1,\dots,n-2$, then most terms in the sum (3.2) would vanish because, for a given set S, there will always be two identical rows in the determinant $\det_{0\le i,j\le n-1}\left(c_{i,j}^{(S)}\right)$ if there is an $s\in\{0,1,\dots,n-1\}$ with $s\notin S$ and $s+1\in S$. We summarise this last observation in the following lemma.

Lemma 4. *For all positive integers n we have*

$$\det_{0\le i,j\le n-1}\left(a_{i,j}+a_{i+1,j}\right)=\sum_{s=0}^{n}\det_{0\le i,j\le n-1}\left(a_{i+\chi(i\ge s),j}\right),$$

where $\chi(\mathcal{S})=1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S})=0$ otherwise.

By combining Theorem 3 and the above lemma, we obtain the following corollary.

Corollary 5. *Let n be a positive integer and $\alpha_0,\alpha_1,\dots,\alpha_n$ non-negative integers. Then*

$$\det_{0\le i,j\le n-1}\left(C_{\alpha_i+j}+C_{\alpha_{i+1}+j}\right)=\prod_{0\le i<j\le n}(\alpha_j-\alpha_i)\prod_{i=0}^{n-1}\frac{(i+n)!}{(2i)!}\prod_{i=0}^{n}\frac{(2\alpha_i)!}{\alpha_i!\,(\alpha_i+n)!}$$
$$\times\sum_{s=0}^{n}\frac{\alpha_s!\,(\alpha_s+n)!}{(2\alpha_s)!\prod_{j=0}^{s-1}(\alpha_s-\alpha_j)\prod_{j=s+1}^{n}(\alpha_j-\alpha_s)}.$$

Remarks. (a) It can be checked that for $\alpha_i=i$, $i=0,1,\dots,n$, Corollary 5 reduces to

$$\det_{0\le i,j\le n-1}\left(C_{i+j}+C_{i+j+1}\right)=\sum_{s=0}^{n}\binom{n+s}{n-s}=\sum_{s=0}^{n}\binom{2n-s}{s}.$$

In view of the well-known formula

$$F_m=\sum_{s=0}^{m}\binom{m-s}{s}$$

for Fibonacci numbers F_m, this immediately implies (1.4).

(b) Proofs of (1.4) by using non-intersecting lattice paths can be found in [5] and [13, Sec. 3].

4. Determinants of generalised Catalan numbers, I. The purpose of this section is to show how the determinant evaluation in Lemma 1 and Theorem 2 on non-intersecting lattice paths lead to proofs of (1.6) and (1.7), and, in fact, of generalisations thereof.

Let μ be a positive integer. We denote by $P\big((0,0)\to(c,d)\mid x\ge\mu y\big)$ the number of lattice paths starting at the origin and ending at (c,d), which never pass above the line $x=\mu y$. Then it is well-known (see [36, Theorem 3]) that

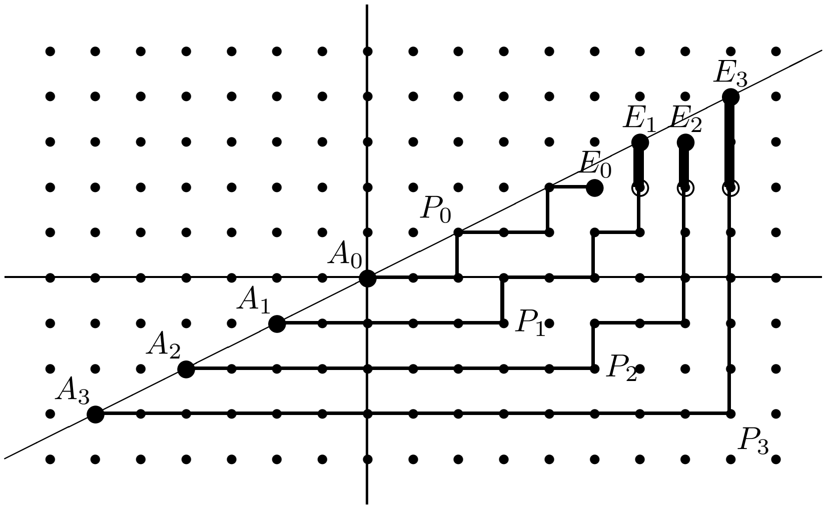
$$P\big((0,0)\to(c,d)\mid x\ge\mu y\big)=\frac{c-\mu d+1}{c+d+1}\binom{c+d+1}{d}. \tag{4.1}$$

The generalised Catalan numbers in (1.5) are a special case: by (4.1), the number $C_{n,k}$ is equal to the number of lattice paths starting at the origin and ending at $\left(n,\lfloor\frac{n}{k-1}\rfloor\right)$, which never pass above the line $x=(k-1)y$.

Theorem 6. *Let n and k be positive integers and $\beta, \alpha_0, \alpha_1, \dots, \alpha_{n-1}$ non-negative integers with $0 \le \beta \le k-1$. Then*

$$\det_{0\le i,j\le n-1} \left(C_{(k-1)\alpha_i+j+\beta,k} \right) = \prod_{0\le i<j\le n-1} (\alpha_j - \alpha_i) \prod_{i=0}^{n-1} \frac{((k-1)i+\beta+n)!\,(k\alpha_i+\beta)!}{(ki+\beta)!\,\alpha_i!\,((k-1)\alpha_i+\beta+n)!}.$$

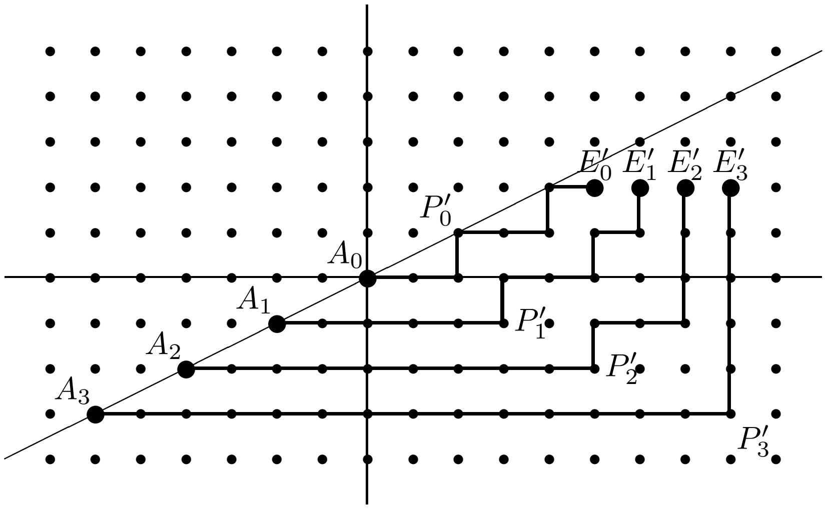
Proof. By (4.1) and Theorem 2, we can interpret the determinant in the theorem as the number of families $(P_0, P_1, \dots, P_{n-1})$ of non-intersecting lattice paths, where P_i runs from $(-(k-1)\alpha_i, -\alpha_i)$ to $\left(i+\beta, \lfloor \frac{i+\beta}{k-1} \rfloor\right)$ and does not pass over the line $x = (k-1)y$, $i = 0, 1, \dots, n-1$.



A family of non-intersecting lattice paths

Figure 1

If we would directly apply Theorem 2 together with the formula on the right-hand side of (4.1) to the above problem of counting non-intersecting lattice paths, then we would have to deal with a determinant which is not easy to handle directly. However, we may use combinatorics to simplify the determinant. (The following simplifications could also be achieved by row and column manipulations of the determinant. However, they become much simpler and much more transparent in terms of non-intersecting lattice paths.) The simplification is best explained with an example at hand. Let us consider the case $k = 3$, $n = 4$, $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\beta = 5$. Then the starting and end points $A_0, \dots, A_3, E_0, \dots, E_3$ are as shown in Figure 1, which at the same time shows an example of a family of non-intersecting lattice paths with these starting and end points. (That portions of some paths are indicated by thick lines and some points are circled should be ignored for the moment.) Since the path family is non-intersecting, and since the paths $P_1, P_2, \dots, P_{n-1}$ must stay (weakly) below the line $x = (k-1)y$ and must avoid E_0, they must pass to the right of E_0. Hence, they must all end with vertical steps above the height at which we find E_0, that is, above $y=\lfloor \frac{\beta}{k-1}\rfloor$. These vertical steps (in our example in Figure 1 they are indicated by thick lines) can therefore be deleted without changing the enumeration problem. Thus, we may equivalently count families $P_0', P_1', \dots, P_{n-1}'$ of non-intersecting lattice paths, where P_i' runs from $A_i=(-(k-1)\alpha_i,-\alpha_i)$ to $E_i'=\left(i+\beta,\lfloor \frac{\beta}{k-1}\rfloor\right)$ and does not pass over the line $x=(k-1)y$, $i=0,1,\dots,n-1$. The modified family of paths which corresponds to our example in Figure 1 is shown in Figure 2.



The non-intersecting lattice paths of Figure 1 without forced vertical steps

Figure 2

If we now use Theorem 2 and (4.1), then we obtain

$$\det_{0\le i,j\le n-1}\left(C_{(k-1)\alpha_i+j+\beta,k}\right)=\det_{0\le i,j\le n-1}\left(\frac{j+\beta+1}{k\alpha_i+j+\beta+1}\binom{k\alpha_i+j+\beta+1}{\alpha_i}\right). \tag{4.2}$$

By taking some factors out of the i-th row of the matrix of which the determinant on the right-hand side is taken, $i=0,1,\dots,n-1$, we obtain

$$\begin{aligned}\det_{0\le i,j\le n-1}\left(C_{(k-1)\alpha_i+j+\beta,k}\right)&=\frac{(\beta+n)!}{\beta!}\prod_{i=0}^{n-1}\frac{(k\alpha_i+\beta)!}{\alpha_i!\,((k-1)\alpha_i+\beta+n)!}\\&\quad\times\det_{0\le i,j\le n-1}\Big((k\alpha_i+\beta+1)(k\alpha_i+\beta+2)\cdots(k\alpha_i+\beta+j)\\&\qquad\cdot((k-1)\alpha_i+\beta+j+2)((k-1)\alpha_i+\beta+j+3)\cdots((k-1)\alpha_i+\beta+n)\Big)\\&=k^{\binom n2}(k-1)^{\binom n2}\frac{(\beta+n)!}{\beta!}\prod_{i=0}^{n-1}\frac{(k\alpha_i+\beta)!}{\alpha_i!\,((k-1)\alpha_i+\beta+n)!}\\&\quad\times\det_{0\le i,j\le n-1}\left(\left(\alpha_i+\frac{\beta+1}{k}\right)\left(\alpha_i+\frac{\beta+2}{k}\right)\cdots\left(\alpha_i+\frac{\beta+j}{k}\right)\right.\\&\qquad\left.\cdot\left(\alpha_i+\frac{\beta+j+2}{k-1}\right)\left(\alpha_i+\frac{\beta+j+3}{k-1}\right)\cdots\left(\alpha_i+\frac{\beta+n}{k-1}\right)\right).\end{aligned}$$

Now Lemma 1 can be applied with $X_i = \alpha_i$, $A_j = (\beta+j+1)/(k-1)$, and $B_j = (\beta+j)/k$. After some manipulation, one arrives at the claimed result. $\square$

REMARK. Again, from a conceptual point of view, the "actual" theorem is the determinant evaluation for the determinant on the right-hand side of (4.2). In an equivalent form, this is

$$\det_{0\le i,j\le n-1}\left(\binom{k\alpha_i+j+\beta}{\alpha_i-1}\right) = \frac{\beta!}{(\beta+n)!}\prod_{0\le i<j\le n-1}(\alpha_j-\alpha_i)\prod_{i=0}^{n-1}\frac{((k-1)i+\beta+n)!\,(k\alpha_i+\beta)!}{(ki+\beta)!\,(\alpha_i-1)!\,((k-1)\alpha_i+\beta+n)!}. \tag{4.3}$$

By observing that

$$\binom{k\alpha_i+j+\beta}{\alpha_i-1} = \binom{k\alpha_i+j+\beta}{(k-1)\alpha_i+j+\beta+1} = (-1)^{(k-1)\alpha_i+j+\beta-1}\binom{-\alpha_i}{(k-1)\alpha_i+j+\beta-1},$$

one sees that the determinant evaluation (4.3) is equivalent to [30, Theorem 26, (3.13)]. In particular, for the validity of (4.3), the restriction on β in the statement of Theorem 6 is not necessary. Moreover, the identity (4.3) continues to hold for real or complex β and α_i if one interprets the appearing factorials and binomials as suitable expressions involving gamma functions (cf. [25, §5.5, (5.96), (5.100)]), as long as there do not appear any singularities.

By combining Theorem 6 and Lemma 4, we obtain the following corollary.

Corollary 7. *Let n and k be a positive integers and $\beta,\alpha_0,\alpha_1,\dots,\alpha_n$ non-negative integers with $0\le\beta\le k-1$. Then*

$$\det_{0\le i,j\le n-1}\left(C_{(k-1)\alpha_i+j+\beta,k}+C_{(k-1)\alpha_{i+1}+j+\beta,k}\right) = \prod_{0\le i<j\le n}(\alpha_j-\alpha_i)\prod_{i=0}^{n-1}\frac{((k-1)i+\beta+n)!}{(ki+\beta)!}\prod_{i=0}^{n}\frac{(k\alpha_i+\beta)!}{\alpha_i!\,((k-1)\alpha_i+\beta+n)!} \times\sum_{s=0}^{n}\frac{\alpha_s!\,((k-1)\alpha_s+\beta+n)!}{(k\alpha_s+\beta)!\prod_{j=0}^{s-1}(\alpha_s-\alpha_j)\prod_{j=s+1}^{n}(\alpha_j-\alpha_s)}.$$

REMARK. It can be checked that for $\alpha_i=i$, $i=0,1,\dots,n$, Corollary 7 reduces to (1.6).

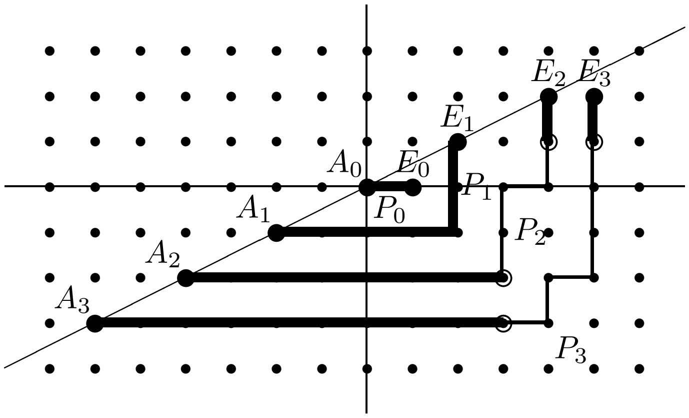
Finally, we address the determinant evaluation (1.7). We shall actually consider the more general determinant

$$\det_{0\le i,j\le n-1}\left(C_{(k-1)i+j+\beta,k}+C_{(k-1)i+j+\beta+1,k}\right), \tag{4.4}$$

where $\beta\le k-1$. Clearly, we can again apply Lemma 4 (with the roles of rows and columns interchanged) to obtain

$$\det_{0\le i,j\le n-1}\left(C_{(k-1)i+j+\beta,k}+C_{(k-1)i+j+\beta+1,k}\right) = \sum_{s=0}^{n}\det_{0\le i,j\le n-1}\left(C_{(k-1)i+j+\chi(j\ge s)+\beta,k}\right). \tag{4.5}$$

By Theorem 2, for any fixed s, the determinant on the right-hand side has a combinatorial interpretation in terms of non-intersecting lattice paths in a manner completely analogous to the one in the proof of Theorem 6: it counts families $(P_0, P_1, \dots, P_{n-1})$ of non-intersecting lattice paths, where P_i runs from $(-(k-1)i, -i)$ to $\big(i + \chi(i \ge s) + \beta, \lfloor \frac{i+\chi(i\ge s)+\beta}{k-1} \rfloor\big)$ and does not pass over the line $x = (k-1)y$, $i = 0, 1, \dots, n-1$. Figure 3 shows an example with $n = 4$, $k = 3$, $\beta = 1$ and $s = 2$.

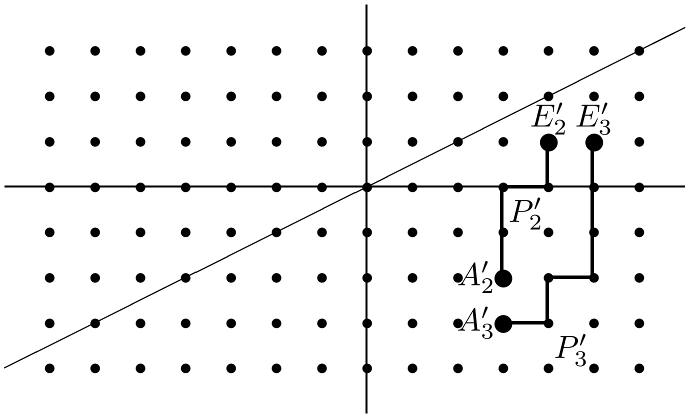


Another family of non-intersecting lattice paths

Figure 3

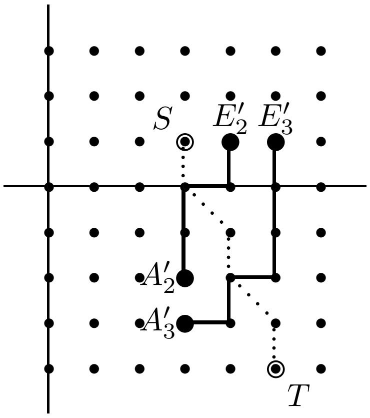
Again, if the paths want to stay below the line $x = (k-1)y$ (being allowed to touch it) and to be non-intersecting, then there are forced path portions. These come in three different flavours. First, for $0 \le i < s$, the path P_i must start with $ki + \beta$ horizontal steps, followed by as many vertical steps as necessary to reach the end point E_i. (See the paths P_0 and P_1 in Figure 3.) Second, for $s \le i \le n-1$, the path P_i must start with $(k-1)i + s + \beta$ horizontal steps. (See the thick portions of P_2 and P_3 below the x-axis in Figure 3.) Third, the paths $P_s, P_{s+1}, \dots, P_{n-1}$ must all end with vertical steps above the height at which we find E_s, that is, above $y = \lfloor \frac{s+\beta+1}{k-1} \rfloor$. Moreover, if E_s should lie on the line $x = (k-1)y$ (that is, if $k-1$ divides $s+\beta+1$), then all of $P_s, P_{s+1}, \dots, P_{n-1}$ must have an additional vertical step from height $\frac{s+\beta+1}{k-1} - 1$ to $\frac{s+\beta+1}{k-1}$. (See the thick vertical parts of P_2 and P_3 in Figure 3. In the example, we have indeed that $E_s = E_2$ lies on $x = (k-1)y = 2y$.) The last observation can also be succinctly rephrased by saying that the paths $P_s, P_{s+1}, \dots, P_{n-1}$ must all end with vertical steps above $y = \lfloor \frac{s+\beta}{k-1} \rfloor$.

We may delete the forced portions without changing the enumeration problem. (See Figure 4 for the resulting path family after the forced portions have been removed in Figure 3.) The boundary $x = (k-1)y$ may also be removed without changing the enumeration problem. (See Figure 5 for the result in case of our running example from Figures 3 and 4. The dotted path should be ignored at the moment.) In that manner, we see that the determinant on the right-hand side of (4.5) is equal to the number of families $(P'_s, P'_{s+1}, \dots, P'_{n-1})$ of non-intersecting lattice paths, where P'_i runs from $(s+\beta, -i)$ to $\Big(i + \beta + 1, \lfloor \frac{s+\beta}{k-1} \rfloor\Big)$, $i = s, s+1, \dots, n-1$.



The non-intersecting lattice paths of Figure 3 after removing forced path portions

Figure 4



The dual path for the non-intersecting lattice paths of Figure 4

Figure 5

A more general enumeration problem can actually be solved in closed form. We formulate the corresponding result in the proposition below.

Proposition 8. *Let a, b, c and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ be integers with $a \le \alpha_0 \le \alpha_1 \le \cdots \le \alpha_{n-1}$ and $b \le c$. Then the number of families $(P_0, P_1, \ldots, P_{n-1})$ of non-intersecting lattice paths, the path P_i running from $(a, b-i)$ to (α_i, c), $i = 0, 1, \ldots, n-1$, is given by*

$$\prod_{0 \le i < j \le n-1} (\alpha_j - \alpha_i) \prod_{i=0}^{n-1} \frac{(\alpha_i + c - a - b)!}{(\alpha_i - a)!\,(c - b + i)!}.$$

Proof. By Theorem 2, the number in question is equal to the determinant

$$\det_{0 \le i,j \le n-1} \left(\binom{\alpha_j - a + c - b + i}{c - b + i} \right).$$

By taking some factors out of the j-th column of the matrix of which we want to compute the determinant, $j = 0, 1, \dots, n-1$, we obtain the equivalent expression

$$\prod_{j=0}^{n-1} \frac{(\alpha_j + c - a - b)!}{(\alpha_j - a)!\,(c - b + j)!} \\ \times \det_{0 \le i,j \le n-1} \big((\alpha_j + c - a - b + 1)(\alpha_j + c - a - b + 2) \cdots (\alpha_j + c - a - b + i) \big).$$

This determinant is of the form $\det_{0 \le i,j \le n-1} \big(p_i(\alpha_j) \big)$, where $p_i(x)$ is a polynomial in x of degree i with leading coefficient 1, $i = 0, 1, \dots, n-1$. Such a determinant can be easily reduced to the Vandermonde determinant $\det_{0 \le i,j \le n-1} \big(\alpha_j^i \big)$ by using elementary column operations (see also [30, Proposition 1]). The evaluation of the Vandermonde determinant being well-known, the assertion of our proposition follows immediately. $\square$

We now apply Proposition 8 to obtain the announced generalisation of (1.7).

Theorem 9. *Let n and k be positive integers and β a non-negative integer with $0 \le \beta \le k-1$. Then*

$$\det_{0 \le i,j \le n-1} \big(C_{(k-1)i+j+\beta,k} + C_{(k-1)i+j+\beta+1,k} \big) = \sum_{s=0}^{n} \binom{\lfloor \frac{s+\beta}{k-1} \rfloor + n}{n - s}. \tag{4.6}$$

Proof. Coming back to the enumeration problem which is illustrated in Figure 5, we apply Proposition 8 with n replaced by $n - s$, $a = s + \beta$, $b = -s$, $c = \lfloor \frac{s+\beta}{k-1} \rfloor$, and $\alpha_i = s+i+\beta+1$, $i = 0, 1, \dots, n-s-1$. The claimed result follows upon little simplification. $\square$

Remarks. (a) Clearly, Eq. (1.7) is the special case $\beta = 0$ of Theorem 9.

(b) It should be clear that, from a conceptual point of view, Proposition 8 represents the actual key result which is behind Theorem 9. On the other hand, if we are just interested in proving Theorem 9, then we do not need the determinant evaluation in the proof of Proposition 8. Namely, for Theorem 9, we only need the special case of Proposition 8 in which the end points of the paths P_i are successive lattice points on a horizontal line, and the first end point is by one unit to the right of the vertical line on which we find the starting points (see Figure 5). The reader is referred to the proof of Theorem 9 for the exact choices of the starting and end points.

There is an elegant argument using the concept of *dual paths*, an idea which is again due to Gessel and Viennot [21, Sec. 4], to see that the number of families of non-intersecting lattice paths with starting and end points as described above is a binomial coefficient. In order to explain the idea, we let S be the lattice point one unit to the left of the first end point, that is, $S = (s + \beta, \lfloor \frac{s+\beta}{k-1} \rfloor)$, and we let T be the lattice point exactly below the last end point, the height of which is by one unit lower than the height of the last starting point, that is, $T = (n + \beta, -n)$. See Figure 5. We now connect S with T by moving vertically downwards, unless we hit one of the existing paths. If the latter happens, then we continue by a diagonal step $(1,-1)$, etc. The "dual" path which results for our example in Figure 5 is indicated by dotted line segments.

It is easy to see that families of non-intersecting lattice paths connecting the starting and end points as above are in bijection with all paths from S to T consisting of vertical down steps $(0,-1)$ and diagonal down steps $(1,-1)$. However, each such path from S to T has exactly $s+\lfloor\frac{s+\beta}{k-1}\rfloor$ vertical steps and $n-s$ diagonal steps, the order of which can be chosen freely. Thus, there are $\binom{\lfloor\frac{s+\beta}{k-1}\rfloor+n}{n-s}$ such paths and, therefore, as many families of non-intersecting lattice paths with the starting and end points as above, in accordance with (4.6).

For a slightly different lattice path proof of (1.5) see [6].

5. Determinants of generalised Catalan numbers, II. Let again k be a fixed positive integer. Rewriting the Catalan number C_n in the form

$$C_n=\frac{1}{2n+1}\binom{2n+1}{n+1}=\frac{1}{2n+1}\binom{2n+1}{n}, \tag{5.1}$$

another way to construct "generalised" Catalan numbers is by considering numbers of the form

$$\frac{l}{kn+l}\binom{kn+l}{n}.$$

Tamm [44] has studied Hankel determinants involving such numbers to considerable extent. It seems that, beyond $k=3$, one cannot expect any closed form results (see also [19]). On the other hand, for $k=3$ there exist several beautiful results, with many variations, due to Eğecioğlu, Redmond and Ryavec [18] (for some of them) and Gessel and Xin [23] (for most of them). As the proof methods show, these results lie on a much deeper level than their counterparts in Theorems 3 or 6. In the theorem below, we present the determinant evaluations involving generalised Catalan numbers of the form (5.1). For a complete list see [31, Theorem 31].

Theorem 10. *For any positive integer n, there hold*

$$\det_{0\le i,j\le n-1}\left(\frac{1}{3i+3j+1}\binom{3i+3j+1}{i+j}\right)=\prod_{i=0}^{n-1}\frac{(\frac{2}{3})_i\,(\frac{1}{6})_i\,(\frac{4}{3})_i\,(\frac{5}{6})_i}{(\frac{1}{2})_{2i}\,(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.2}$$

$$\det_{0\le i,j\le n-1}\left(\frac{1}{3i+3j+4}\binom{3i+3j+4}{i+j+1}\right)=\prod_{i=0}^{n-1}\frac{(\frac{4}{3})_i\,(\frac{5}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.3}$$

$$\det_{0\le i,j\le n-1}\left(\frac{1}{3i+3j+2}\binom{3i+3j+2}{i+j+1}\right)=\prod_{i=0}^{n-1}\frac{(\frac{4}{3})_i\,(\frac{5}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.4}$$

$$\det_{0\le i,j\le n-1}\left(\frac{1}{3i+3j+5}\binom{3i+3j+5}{i+j+2}\right)=\prod_{i=0}^{n}\frac{(\frac{2}{3})_i\,(\frac{1}{6})_i\,(\frac{4}{3})_i\,(\frac{5}{6})_i}{(\frac{1}{2})_{2i}\,(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.5}$$

$$\det_{0\le i,j\le n-1}\left(\frac{2}{3i+3j+1}\binom{3i+3j+1}{i+j+1}\right)=\prod_{i=0}^{n}\frac{(\frac{2}{3})_i\,(\frac{1}{6})_i\,(\frac{4}{3})_i\,(\frac{5}{6})_i}{(\frac{1}{2})_{2i}\,(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.6}$$

$$\det_{0\le i,j\le n-1}\left(\frac{2}{3i+3j+4}\binom{3i+3j+4}{i+j+2}\right)=\prod_{i=0}^{n}\frac{(\frac{4}{3})_i\,(\frac{5}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.7}$$

where $(\alpha)_i:=\alpha(\alpha+1)\cdots(\alpha+i-1)$ *is the usual Pochhammer symbol. Let* $a_0=-2$ *and* $a_m=\frac{1}{3m+1}\binom{3m+1}{m}$ *for* $m\ge 1$*. Then*

$$\det_{0\le i,j\le n-1}(a_{i+j})=\prod_{i=0}^{n-1}(-2)\frac{(\frac{1}{3})_i\,(-\frac{1}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{1}{2})_{2i}\,(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.8}$$

Let $b_0=10$ *and* $b_m=\frac{2}{3m+2}\binom{3m+2}{m}$ *for* $m\ge 1$*. Then*

$$\det_{0\le i,j\le n-1}(b_{i+j})=\prod_{i=0}^{n-1}10\frac{(\frac{2}{3})_i\,(\frac{1}{6})_i\,(\frac{7}{3})_i\,(\frac{11}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.9}$$

Furthermore,

$$\det_{0\le i,j\le n-1}\left(\frac{2}{3i+3j+5}\binom{3i+3j+5}{i+j+1}\right)=\prod_{i=0}^{n}\frac{(\frac{4}{3})_i\,(\frac{5}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.10}$$

Let $c_0=\frac{7}{2}$ *and* $c_m=\frac{2}{3m+1}\binom{3m+1}{m+1}$ *for* $m\ge 1$*. Then*

$$\det_{0\le i,j\le n-1}(c_{i+j})=\prod_{i=0}^{n}\frac{(\frac{4}{3})_i\,(\frac{5}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.11}$$

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