SUPERCONGRUENCES SATISFIED BY COEFFICIENTS OF ${}_2F_1$ HYPERGEOMETRIC SERIES

HENG HUAT CHAN, ARISTIDES KONTOGEORGIS, CHRISTIAN KRATTENTHALER AND ROBERT OSBURN

Dedicated to Paulo Ribenboim on the occasion of his 80th birthday

Abstract. Recently, Chan, Cooper and Sica conjectured two congruences for coefficients of classical ${}_2F_1$ hypergeometric series which also arise from power series expansions of modular forms in terms of modular functions. We prove these two congruences using combinatorial properties of the coefficients.

1. Introduction

The sequence

$$\alpha_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

introduced by R. Apéry [1] in his proof of the irrationality of $\zeta(3)$, has many interesting arithmetical properties. For example, F. Beukers [3, p. 276] showed that α_n arises from the power series expansion of a modular form of weight 2 in terms of a modular function.[1] More precisely, if $q = e^{2\pi i\tau}$ with $\operatorname{Im}\tau > 0$,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

$$Z(\tau) = \frac{\big(\eta(2\tau)\eta(3\tau)\big)^7}{\big(\eta(\tau)\eta(6\tau)\big)^5} \quad \text{and} \quad X(\tau) = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12},$$

then

$$Z(\tau) = \sum_{n=0}^{\infty} \alpha_n X^n(\tau). \tag{1.1}$$

Other properties of α_n were soon discovered by S. Chowla, J. Cowles and M. Cowles [8]. They showed that for all primes $p > 3$,

$$\alpha_p \equiv \alpha_1 \pmod{p^3}.$$

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[1] Beukers gave the modular form in terms of Lambert series. The product form can be found in [12].

Subsequently, I. M. Gessel [9] showed that, for all positive integers n and primes $p > 3$,

$$\alpha_{np} \equiv \alpha_n \pmod{p^3}. \tag{1.2}$$

Recently, an analogue of Apéry numbers was found. The corresponding sequence is formed by the Domb numbers [5], defined by

$$\beta_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}.$$

It can be shown (see [5, (4.14)]) that if

$$\mathcal{Z}(\tau) = \frac{\big(\eta(\tau)\eta(3\tau)\big)^4}{\big(\eta(2\tau)\eta(6\tau)\big)^2} \quad \text{and} \quad \mathcal{X}(\tau) = \left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6,$$

then

$$\mathcal{Z}(\tau) = \sum_{n=0}^{\infty} \beta_n \mathcal{X}^n(\tau). \tag{1.3}$$

In [7], H. H. Chan, S. Cooper and F. Sica showed, using Gessel's idea, that

$$\beta_{np} \equiv \beta_n \pmod{p^3}. \tag{1.4}$$

The similarities between (1.1) and (1.3), as well as between (1.2) and (1.4), indicated that perhaps sequences arising from power series expansions of modular forms of weight 2 in terms of modular functions may have properties similar to (1.2) and (1.4). Motivated by this idea, Chan, Cooper and Sica constructed seven sequences a_n from η-quotients, analogues of theta functions and various modular functions, and they conjectured that, under certain conditions on the primes p, these seven sequences satisfy congruences of the type

$$a_{np} \equiv a_n \pmod{p^r}, \tag{1.5}$$

with $r = 1$, 2, or 3. Unfortunately, these conjectures do not follow immediately from Gessel's method, and therefore new methods have to be devised. The purpose of this note is to give an elementary approach to proving two of these conjectures, which feature the value $r = 2$.

Theorem 1.1. *Let* $(a)_n = (a)(a+1)(a+2)\cdots(a+n-1)$.
(a) *For* $p \equiv 1 \pmod 4$ *and*

$$s_n = 64^n \frac{\left(\frac{1}{4}\right)_n^2}{(1)_n^2},$$

we have

$$s_{np} \equiv s_n \pmod{p^2}. \tag{1.6}$$

(b) *For* $p \equiv 1 \pmod 6$ *and*

$$t_n = 108^n \frac{\left(\frac{1}{6}\right)_n \left(\frac{1}{3}\right)_n}{(1)_n^2},$$

we have

$$t_{np} \equiv t_n \pmod{p^2}. \tag{1.7}$$

The proof of (1.6) will be given in Sections 2 to 4. The proof of (1.7) will be given in Section 5. Some parts of the proof of (1.7) will only be sketched as they are similar to that of (1.6).

We conclude this introduction by indicating the analogues of (1.1) and (1.3).

Let

$$Z_2 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+n^2} \quad \text{and} \quad X_2 = \frac{\eta^{12}(2\tau)}{Z_2^6}.$$

Then the s_n's are obtained from the expansion

$$Z_2 = \sum_{n=0}^{\infty} s_n X_2^n.$$

Incidentally, the coefficients s_n can be obtained from the coefficients $\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n /(1)_n^2$ studied by S. Ramanujan via a special case of Kummer's transformation

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x\right) = \frac{1}{\sqrt[4]{1-x}}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; \frac{x}{x-1}\right),$$

where ${}_2F_1(a, b; c; z)$ is the classical Gaußian hypergeometric series.

Let

$$Z_3 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \quad \text{and} \quad X_3 = \frac{\eta^6(\tau)\eta^6(\tau)}{Z_3^6}.$$

Then the t_n's are obtained from the expansion

$$Z_3 = \sum_{n=0}^{\infty} t_n X_3^n.$$

The series associated with the coefficients t_n were studied in [4] and [6], and these coefficients are related to the coefficients $\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n /(1)_n^2$ studied by Ramanujan and the Borweins by means of the transformation formula

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right) = {}_2F_1\left(\frac{1}{3}, \frac{1}{6}; 1; 4x(1-x)\right).$$

We remark here that, using (3.4), it is immediate (see (3.3) and (5.2)) that, if $u_n = 64^n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n /(1)_n^2$ and $v_n = 27^n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n /(1)_n^2$, then

$$u_p \equiv u_1 \pmod{p^2} \quad \text{and} \quad v_p \equiv v_1 \pmod{p^2}.$$

Although it is not clear how one can deduce the corresponding congruences for s_p and t_p from congruences satisfied by u_p and v_p using the ${}_2F_1$ transformation formulas, our proof of Theorem 1.1 is clearly motivated by these relations.

2. A Lemma for the proof of (1.6)

In this section, we establish a simple lemma which is interesting in its own right.

Lemma 2.1. *For positive integer n and prime $p \equiv 1 \pmod 4$,*

$$\left(\frac{3}{4}\right)_p \equiv 3\left(\frac{1}{4}\right)_p \pmod{p^3}. \tag{2.1}$$

Proof. By isolating the terms involving multiples of p on both sides of (2.1), we find that it suffices to prove the congruence

$$\prod_{k=0}^{\frac{3p-7}{4}}\left(\frac{3}{4}+k\right)\prod_{k=\frac{3p+1}{4}}^{p-1}\left(\frac{3}{4}+k\right) \equiv \prod_{k=0}^{\frac{p-5}{4}}\left(\frac{1}{4}+k\right)\prod_{k=\frac{p+3}{4}}^{p-1}\left(\frac{1}{4}+k\right) \pmod{p^2}. \tag{2.2}$$

Let the product on the left-hand side be $L(p)$ and the product on the right-hand side be $R(p)$. We group some of the terms in $L(p)$ in pairs as follows:

$$\left(\frac{3}{4}+\frac{3p-3}{4}-k\right)\left(\frac{3}{4}+\frac{3p-3}{4}+k\right)$$

for

$$1 \le k \le \frac{p-1}{4}.$$

We then conclude that

$$L(p) \equiv \prod_{k=1}^{\frac{p-1}{4}}(-k^2)\prod_{k=0}^{\frac{p-3}{2}}\left(\frac{3}{4}+k\right) \pmod{p^2}.$$

Similarly, for

$$1 \le k \le \frac{p-1}{4},$$

we perform the following pairing of some of the terms in the product in $R(p)$:

$$\left(\frac{1}{4}+\frac{p-1}{4}-k\right)\left(\frac{1}{4}+\frac{p-1}{4}+k\right).$$

Hence we have

$$R(p) \equiv \prod_{k=1}^{\frac{p-1}{4}}(-k^2)\prod_{k=\frac{p+1}{2}}^{p-1}\left(\frac{1}{4}+k\right) \pmod{p^2}.$$

It now remains to verify that

$$\prod_{k=0}^{\frac{p-3}{2}}\left(\frac{3}{4}+k\right)\equiv\prod_{k=\frac{p+1}{2}}^{p-1}\left(\frac{1}{4}+k\right)\pmod{p^2}. \tag{2.3}$$

Denoting the left-hand side of (2.3) by $l(p)$ and the right-hand side by $r(p)$, we observe that we can write $l(p)$ and $r(p)$ as

$$\begin{aligned} l(p) &= \prod_{k=0}^{\frac{p-5}{4}}\left(\frac{3}{4}+\frac{p-5}{4}-k\right)\left(\frac{3}{4}+\frac{p-1}{4}+k\right) \\ &\equiv \prod_{k=0}^{\frac{p-5}{4}}\left(-\frac{1}{4}-k-k^2\right)\pmod{p^2} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} r(p) &= \prod_{k=0}^{\frac{p-5}{4}}\left(\frac{1}{4}+\frac{p+1}{2}+\frac{p-5}{4}-k\right)\left(\frac{1}{4}+\frac{p+1}{2}+\frac{p-1}{4}+k\right) \\ &\equiv \prod_{k=0}^{\frac{p-5}{4}}\left(-\frac{1}{4}-k-k^2\right)\pmod{p^2}, \end{aligned} \tag{2.5}$$

which implies (2.3). This completes the proof of (2.2). □

As a consequence, we have the following congruence.

Corollary 2.2. *Let p be a prime such that $p\equiv 1 \pmod 4$. Then*

$$\prod_{\substack{k=0\\ k\neq\frac{3p-3}{4}}}^{p-1}(3+4k)\equiv\prod_{\substack{k=0\\ k\neq\frac{p-1}{4}}}^{p-1}(1+4k)\pmod{p^2}. \tag{2.6}$$

3. Simple properties of s_n and the congruence (1.6) for $n=1$

We first observe that

$$s_n=\frac{\left(\frac{1}{4}\right)_n^2 64^n}{(n!)^2}=\frac{4^n}{(n!)^2}\prod_{i=0}^{n-1}(1+4i)^2. \tag{3.1}$$

Lemma 3.1. *If p is a prime satisfying $p\equiv 1 \pmod 4$, then*

$$s_p\equiv s_1\pmod{p^2}. \tag{3.2}$$

Proof. From (3.1), we find that

$$s_p=\frac{4^p}{(p!)^2}\prod_{i=0}^{p-1}(1+4i)^2.$$

Observe that

$$s_p = \frac{4^p}{((p-1)!)^2} \prod_{\substack{i=0 \\ i \neq \frac{p-1}{4}}}^{p-1} (1+4i)^2.$$

By (2.6), we find that

$$\begin{aligned} s_p &\equiv \frac{4^p}{((p-1)!)^2} \prod_{\substack{i=0 \\ i \neq \frac{p-1}{4}}}^{p-1} (1+4i) \prod_{\substack{k=0 \\ k \neq \frac{3p-3}{4}}}^{p-1} (3+4k) \pmod{p^2} \\ &\equiv \frac{1}{3} \frac{4^p}{(p!)^2} \prod_{i=0}^{p-1} (1+4i) \prod_{i=0}^{p-1} (3+4i) \pmod{p^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} s_p &\equiv \frac{1}{3} \frac{4^p}{(p!)^2} \prod_{i=0}^{p-1} (1+4i)(3+4i) \pmod{p^2} \\ &\equiv \frac{1}{3} \frac{4^p}{(p!)^2} \prod_{i=0}^{p-1} \frac{(1+4i)(3+4i)(2+4i)(4+4i)}{2^2(1+2i)(2+2i)} \pmod{p^2} \\ &\equiv \frac{1}{3} \binom{4p}{2p} \binom{2p}{p} \pmod{p^2}. \end{aligned} \tag{3.3}$$

It is known that (see [11], respectively [2, Theorem 4]), for positive integers a and b, with $a \geq b$, and primes $p > 3$,

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p^3}. \tag{3.4}$$

Using (3.4) in the last expression in (3.3), we conclude that

$$s_p \equiv \frac{1}{3} \binom{4p}{2p} \binom{2p}{p} \equiv \frac{1}{3} \binom{4}{2} \binom{2}{1} \equiv 4 \pmod{p^2}.$$

□

We end this section with a simple observation. Let

$$F(n) = 4^{p-1} \prod_{\substack{j=0 \\ j \neq \frac{p-1}{4}}}^{p-1} (1+4j+4np)^2 \prod_{i=0}^{p-2} \frac{1}{(1+i+np)^2}. \tag{3.5}$$

From (3.2), we have the following congruence for $F(0)$.

Corollary 3.2. *Let p be a prime with $p \equiv 1 \pmod 4$. Then*

$$F(0) \equiv 1 \pmod{p^2}. \tag{3.6}$$

4. Completion of the proof of (1.6)

Lemma 4.1. *Let $F(n)$ be defined as in* (3.5) *and suppose $p \equiv 1 \pmod 4$. Then $F(n) \pmod{p^2}$ is independent of n.*

Proof. We first consider the denominator of $F(n)$. We have

$$\prod_{i=0}^{p-2} \frac{1}{(1+i+np)^2} = \prod_{k=1}^{(p-1)/2} \frac{1}{(np+k)^2((n+1)p-k)^2}$$
$$\equiv \prod_{k=1}^{(p-1)/2} \frac{1}{k^2(p-k)^2} \pmod{p^2}.$$

Next, we split the numerator of $F(n)$ into two parts, namely,

$$\prod_{\substack{j=0\\ j\neq \frac{p-1}{4}}}^{p-1} (1+4j+4np)^2 = A(n)B(n),$$

where

$$\begin{aligned} A(n) &= \prod_{j=1}^{(p-1)/4} \left(1+4\left(\frac{p-1}{4}-j\right)+4np\right)^2 \\ &\qquad \times \left(1+4\left(\frac{p-1}{4}+j\right)+4np\right)^2 \\ &\equiv \prod_{j=1}^{(p-1)/4} 16^2 j^4 \pmod{p^2} \end{aligned}$$

and

$$\begin{aligned} B(n) &= \prod_{k=1}^{(p-1)/4} \left(4np+2p+3+4\left(\frac{p-1}{4}-k\right)\right)^2 \\ &\qquad \times \left(4np+2p+3+4\left(\frac{p-1}{4}+k-1\right)\right)^2 \\ &= \prod_{k=1}^{(p-1)/4} \left(4np+3p-(4k-2)\right)^2 \left(4np+3p+(4k-2)\right)^2 \\ &\equiv \prod_{k=1}^{(p-1)/4} \left(-4+16k-16k^2\right) \pmod{p^2}. \end{aligned}$$

The above computations show that both $A(n) \pmod{p^2}$ and $B(n) \pmod{p^2}$ are independent of n. Hence, $F(n) \pmod{p^2}$ is independent of n. □

Using (3.6), we arrive at the following conclusion.

Corollary 4.2. *For all positive integers n and $p \equiv 1 \pmod 4$, we have*

$$F(n) \equiv F(0) \equiv 1 \pmod{p^2}.$$

Completion of the proof of (1.6). Our aim is to show that

$$s_{np} \equiv s_n \pmod{p^2}$$

for all positive integers n and primes $p \equiv 1 \pmod 4$. We shall accomplish this by an induction on n.

From (3.1), we find that

$$s_{n+1} = 4\left(\frac{1+4n}{1+n}\right)^2 s_n. \tag{4.1}$$

Therefore

$$s_{n+k} = 4^k \prod_{i=0}^{k-1} \left(\frac{1+4(i+n)}{1+n+i}\right)^2 s_n.$$

In particular,

$$s_{n+p} = 4^p \prod_{i=0}^{p-1} \left(\frac{1+4(i+n)}{1+n+i}\right)^2 s_n. \tag{4.2}$$

Now, for the induction hypothesis, suppose that

$$s_{np} \equiv s_n \pmod{p^2}. \tag{4.3}$$

By (4.2), we find that

$$\begin{aligned} s_{(n+1)p} = s_{np+p} &= s_{np} 4^p \prod_{i=0}^{p-1} \left(\frac{1+4(i+np)}{1+i+np}\right)^2 \\ &\equiv s_n 4^p \prod_{i=0}^{p-1} \left(\frac{1+4(i+np)}{1+i+np}\right)^2 \pmod{p^2}, \end{aligned}$$

where we used (4.3) in the last congruence. We observe that, if

$$4^p \prod_{i=0}^{p-1} \left(\frac{1+4(i+np)}{1+i+np}\right)^2 \equiv 4\left(\frac{1+4n}{1+n}\right)^2 \pmod{p^2}, \tag{4.4}$$

then we would have

$$s_{(n+1)p} \equiv s_n 4\left(\frac{1+4n}{1+n}\right)^2 \equiv s_{n+1} \pmod{p^2},$$

by (4.1). But the congruence (4.4) is exactly the congruence in Corollary 4.2. This completes our proof of (1.6). □

5. A Lemma for the proof of (1.7)

Lemma 5.1. *Let $p = 6q + 1$ be a prime. Then*

$$4^p \left(\frac{1}{6}\right)_p \equiv \left(\frac{2}{3}\right)_p \pmod{p^3}.$$

Proof. We want to reduce the congruence to one that we can manage. Clearing denominators and dividing the terms which are multiples of p on both sides, we see that we need to prove that

$$\begin{aligned}&2^{6q} 1 \cdot 7 \cdots (6q-5)(6q+7) \cdots (36q+1)\\ &\qquad \equiv 2 \cdot 5 \cdots (12q-1)(12q+5) \cdots (18q+2) \pmod{p^2}.\end{aligned}$$

We next match the terms $6q + 1 - 6k$ to $6q + 1 + 6k$ for $1 \leq k \leq q$ and simplify the left-hand side to

$$2^{6q} \prod_{k=1}^{q} (6q+1-6k)(6q+1+6k) \cdot M(q) \equiv 2^{8q} \cdot 3^{2q} \prod_{k=1}^{q} (-k^2) \cdot M(q) \pmod{p^2},$$

where

$$M(q) = \prod_{k=1}^{4q} (12q + 1 + 6k).$$

But $M(q)$ can also be expressed as

$$M(q) = \prod_{k=1}^{2q} (24q+4-(6k-3))(24q+4+6k-3) \equiv 3^{4q} \prod_{k=1}^{2q} (2k-1)^2 \pmod{p^2}.$$

Hence the left-hand side is

$$2^{8q} \cdot 3^{6q} \prod_{k=1}^{q} (-k^2) \prod_{k=1}^{2q} (2k-1)^2.$$

Similarly, the right-hand side can be expressed as

$$\prod_{k=1}^{2q} (12q + 2 - 3k)(12q + 2 + 3k) \cdot N(q) \equiv 3^{4q} \prod_{k=1}^{2q} k^2 \cdot N(q) \pmod{p^2},$$

where $N(q)$ is given by

$$\begin{aligned}N(q) &= \prod_{k=1}^{q} \left(3q + \frac{1}{2} - \frac{6k-3}{2}\right) \left(3q + \frac{1}{2} + \frac{6k-3}{2}\right)\\ &\equiv \left(\frac{3}{2}\right)^{2q} \prod_{k=1}^{q} (-(2k-1)^2) \pmod{p^2}.\end{aligned}$$

Simplifying both sides, we observe that we need to prove that

$$2^{10q} \prod_{k=q+1}^{2q} (2k-1)^2 \equiv \prod_{k=q+1}^{2q} k^2 \pmod{p^2}.$$

We rewrite both sides, so that the above congruence turns out to be equivalent to

$$2^{10q}\prod_{k=q+1}^{2q}(p-(2k-1))(p+(2k-1)) \equiv \prod_{k=q+1}^{2q}(p-k)(p+k) \pmod{p^2}.$$

This leads to

$$2^{11q}\prod_{k=q+1}^{2q}(p-(2k-1)) \equiv \prod_{k=q+1}^{2q}(p+k) \pmod{p^2},$$

since

$$\prod_{k=q+1}^{2q}(p+(2k-1)) = 2^q\prod_{k=q+1}^{2q}(p-k).$$

Now rewriting

$$\prod_{k=q+1}^{2q}(p-(2k-1)) = 2^q\prod_{k=q+1}^{2q}(3q-k+1),$$

we see that we must show that

$$\begin{aligned}2^{12q}&(q+1)(q+2)\cdots(2q)\\ &\equiv (q+1)(q+2)\cdots(2q)\,(1+p\,(H_{2q}-H_q)) \pmod{p^2},\end{aligned}$$

where

$$H_n=\sum_{k=1}^{n}\frac{1}{k}.$$

Equivalently, we need to verify that

$$\frac{2^{6q}-1}{p}\cdot 2 \equiv H_{2q}-H_q \pmod{p}.$$

But it is known (see [10, Theorem 132]) that

$$\frac{2^{p-1}-1}{p} \equiv H_{6q}-\frac{H_{3q}}{2} \pmod{p}.$$

Since

$$H_{6q}\equiv 0 \pmod{p},$$

it suffices to show that

$$-H_{3q}+H_q-H_{2q}\equiv 0 \pmod{p}.$$

Observe that

$$H_{3q}=1+\frac{1}{2}+\cdots+\frac{1}{q}+H_{2q}-H_q+\frac{1}{2q+1}+\cdots+\frac{1}{3q}.$$

Now, for $1\le i\le q$, we pair the terms in the sums at both ends as follows:

$$\frac{1}{i}+\frac{1}{3q+1-i}=\frac{3q+1}{i(3q+1-i)}\equiv\frac{1}{2i(3q+1-i)}\equiv\frac{1}{i}-\frac{2}{2i-1} \pmod{p}.$$

Hence, we deduce that

$$H_{3q} \equiv H_q - 2\left(H_{2q} - \frac{H_q}{2}\right) + H_{2q} - H_q \equiv -H_{2q} + H_q \pmod{p},$$

which completes the proof of the lemma. □

We are now ready to show that if

$$t_n = 108^n \frac{\left(\frac{1}{6}\right)_n \left(\frac{1}{3}\right)_n}{(1)_n^2}$$

then

$$t_p \equiv t_1 \pmod{p^2} \tag{5.1}$$

for all primes $p \equiv 1 \pmod 6$. By Lemma 5.1,

$$t_p \equiv 27^p \frac{\left(\frac{2}{3}\right)_p \left(\frac{1}{3}\right)_p}{(1)_p^2} \pmod{p^2}. \tag{5.2}$$

But the last expression can be written as

$$\binom{3p}{p}\binom{2p}{p} \equiv 6 \equiv t_1 \pmod{p^2},$$

by using (3.4). This completes the proof of (5.1).

The proof of (1.7) for $n > 1$ is similar to the proof of (1.6). We will simply list the corresponding identities that are needed in the proof. These are:

(i) The sequence t_n satisfies

$$t_{n+1} = 6\frac{(1+6n)(1+3n)}{(1+n)^2} t_n$$

and

$$t_{n+p} = t_n 6^p \prod_{i=0}^{p-1} \frac{(1+6n+6i)(1+3n+3i)}{(1+n+i)^2}.$$

(ii) The expression

$$G(n) = 6^{p-1} \prod_{\substack{j=0 \\ j \neq \frac{p-1}{6}}}^{p-1} (1+6j+6np) \prod_{\substack{j=0 \\ j \neq \frac{p-1}{3}}}^{p-1} (1+3j+3np) \prod_{i=0}^{p-2} \frac{1}{(1+i+np)^2}$$

is independent of n modulo p^2, and

$$G(n) \equiv G(0) \equiv 1 \pmod{p^2}.$$

The proofs of (i) and (ii) are similar to those presented in Section 4.

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Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543; Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111, Bonn, Germany

E-mail address: matchh@nus.edu.sg

Department of Mathematics, University of the Aegean, 83200 Karlovassi, Samos, Greece

E-mail address: aristides.kontogeorgis@gmail.com

Fakultät für Mathematik, Universität Wien, Nordbergstrasze 15, A-1090 Vienna, Austria

WWW: http://www.mat.univie.ac.at/~kratt

School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland
E-mail address: robert.osburn@ucd.ie