ADVANCED DETERMINANT CALCULUS: A COMPLEMENT

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Abstract. This is a complement to my previous article "Advanced Determinant Calculus" (*Séminaire Lotharingien Combin.* **42** (1999), Article B42q, 67 pp.). In the present article, I share with the reader my experience of applying the methods described in the previous article in order to solve a particular problem from number theory (G. Almkvist, J. Petersson and the author, *Experiment. Math.* **12** (2003), 441–456). Moreover, I add a list of determinant evaluations which I consider as interesting, which have been found since the appearance of the previous article, or which I failed to mention there, including several conjectures and open problems.

1. Introduction

In the previous article [109], I described several methods to evaluate determinants, and I provided a long list of known determinant evaluations. The present article is meant as a complement to [109]. Its purpose is three-fold: first, I want to shed light on the problem of evaluating determinants from a slightly different angle, by sharing with the reader my experience of applying the methods from [109] in order to solve a particular problem from number theory (see Sections 3 and 4); second, I shall address the question why it is apparently in the first case combinatorialists (such as myself) who are so interested in determinant evaluations and get so easily excited about them (see Section 2); and, finally third, I add a list of determinant evaluations, which I consider as interesting, which have been found since the appearance of [109], or which I

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failed to mention in the list given in Section 3 of [109] (see Section 5), including several conjectures and open problems.

2. Enumerative combinatorics, nice formulae, and determinants

Why are combinatorialists so fascinated by determinant evaluations?

A simplistic answer to this question goes as follows. Clearly, binomial coefficients $\binom{n}{k}$ or Stirling numbers (of the second kind) $S(n,k)$ are basic objects in (enumerative) combinatorics; after all they count the subsets of cardinality k of a set with n elements, respectively the ways of partitioning such a set of n elements into k pairwise disjoint non-empty subsets. Thus, if one sees an identity such as[1]

$$\det_{1\le i,j\le n}\left(\binom{a+b}{a-i+j}\right)=\prod_{i=1}^{n}\frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!},\tag{2.1}$$

or[2]

$$\det_{1\le i,j\le n}\left(S(i+j,i)\right)=\prod_{i=1}^{n} i^i\tag{2.2}$$

(and there are many more of that kind; see [109] and Section 5), there is an obvious excitement that one cannot escape.

Although this is indeed an explanation which applies in many cases, there is also an answer on a more substantial level, which brings us to the reason why *I* like (and need) determinant evaluations.

The *favourite question* for an enumerative combinatorialist (such as myself) is

How many $\langle\dots\rangle$ are there?

Here, $\langle\dots\rangle$ can be permutations with certain properties, certain partitions, certain paths, certain trees, etc. The *favourite theorem* then is:

Theorem 1. *The number of $\langle\dots\rangle$ of size n is equal to*

$$NICE(n).$$

I have already explained the meaning of $\langle\dots\rangle$. What does $NICE(n)$ stand for? Typical examples for $NICE(n)$ are formulae such as

$$\frac{1}{n+1}\binom{2n}{n}\tag{2.3}$$

(*Catalan numbers*; cf. [178, Ex. 6.19]) or

$$\prod_{i=0}^{n-1}\frac{(3i+1)!}{(n+i)!}\tag{2.4}$$

---

[1]For more information on this determinant see Theorems 2 and 4 in this section and [109, Sections 2.2, 2.3 and 2.5].

[2]This determinant evaluation follows easily from the matrix factorisation

$$\left(S(i+j,i)\right)_{1\le i,j\le n}=\left((-1)^k k^i/(k!\,(i-k)!)\right)_{1\le i,k\le n}\cdot\left(k^j\right)_{1\le k,j\le n},$$

application of [109, Theorem 26, (3.14)] to the first determinant, and application of the Vandermonde determinant evaluation to the second.

(the number of $n \times n$ alternating sign matrices and several other combinatorial objects; cf. [28]). Let us be more precise.

"Definition". *The symbol $NICE(n)$ is a formula of the type*

$$\xi^n \cdot \mathrm{Rat}(n) \cdot \prod_{i=1}^{k} \frac{(a_i n + b_i)!}{(c_i n + d_i)!}, \tag{2.5}$$

where $\mathrm{Rat}(n)$ *is a rational function in* n*, and where* $a_i, c_i \in \mathbb{Z}$ *for* $i = 1, 2, \dots, k$*,* $\mathbb{Z}$ *denoting the set of integers. The parameters* b_i, c_i, ξ *can be arbitrary real or complex numbers.* (*If necessary,* $(a_i n + b_i)!$ *has to be interpreted as* $\Gamma(a_i n + b_i + 1)$*, where* $\Gamma(x)$ *is the Euler gamma function, and similarly for* $(c_i n + d_i)!$.)

Clearly, the formulae (2.3) and (2.4) fit this "Definition".[3]

If one is working on a particular problem, how can one recognise that one is looking at a sequence of numbers given by $NICE(n)$? The key observation is that, if we factorise $(an + b)!$ into its prime factors, where a and b are integers, then, as n runs through the positive integers, the numbers $(an + b)!$ explode quickly, whereas the prime factors occurring in the factorisation will grow only moderately, more precisely, they will grow roughly linearly. Thus, if we encounter a sequence the prime factorisation of which has this property, we can be sure that there is a formula $NICE(n)$ for this sequence. Even better, as I explain in Appendix A of [109], the program `Rate`[4] will (normally[5]) be able to guess the formula.

To illustrate this, let us look at a particular example. Let us suppose that the first few values of our sequence are the following:

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440,\\ 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420.$$

The prime factorisation of the second-to-last number is (we are using *Mathematica* here)

```
In[1]:= FactorInteger[477638700]
```

[3] The writing $NICE(n)$ is borrowed from Doron Zeilberger [193, Recitation III]. The technical term for a formula of the type (2.5) is *"hypergeometric term"*, see [144, Sec. 3.2], whereas, most often, the colloquial terms *"closed form"* or *"nice formula"* are used for it, see [193, Recitation II]. More recently, some authors call sequences given by formulae of that type sequences of "round" numbers, see [117, Sec. 6].

[4] `Rate` is available from `http://igd.univ-lyon1.fr/~kratt`. It is based on a rather simple algorithm which involves rational interpolation. In contrast to what I read, with great surprise, in [46], the explanations of how `Rate` works in Appendix A of [109] can be read and understood without any knowledge about determinants and, in particular, without any knowledge of the fifty or so pages that precede Appendix A in [109].

[5] `Rate` will *always* be able to guess a formula of the type (2.5) if there are enough initial terms of the sequence available. However, there is a larger class of sequences which have the property that the size of the primes in the prime factorisation of the terms of the sequence grows only slowly with n. These are sequences given by formulae containing "Abelian" factors, such as n^n. Unfortunately, `Rate` does not know how to handle such factors. Recently, Rubey [158] proposed an algorithm for covering Abelian factors as well. His implementation `Guess` is written in *Axiom* and is available at `http://www.mat.univie.ac.at/~rubey/martin.html`.

```
Out[1]= {{2, 2}, {3, 1}, {5, 2}, {7, 1}, {11, 1}, {23, 1}, {29, 1}, {31, 1}}
```

whereas the prime factorisations of the next-to-last and the last number in this sequence are

```
In[2]:= FactorInteger[1767263190]

Out[2]= {{2, 1}, {3, 1}, {5, 1}, {7, 1}, {11, 1}, {23, 1}, {29, 1}, {31, 1},

>    {37, 1}}

In[3]:= FactorInteger[6564120420]

Out[3]= {{2, 2}, {3, 1}, {5, 1}, {11, 1}, {13, 1}, {23, 1}, {29, 1}, {31, 1},

>    {37, 1}}
```

(To decipher this for the reader unfamiliar with *Mathematica*: the prime factorisation of the last number is $2^2 3^1 5^1 11^1 13^1 23^1 29^1 31^1 37^1$.) One observes, first of all, that the occurring prime factors are rather small in comparison to the numbers of which they are factors, and, second, that the size of the prime factors grows only very slowly (from 31 to 37). Thus, we *can be sure* that there is a "nice" formula $NICE(n)$ for this sequence. Indeed, `Rate` needs only the first five members of the sequence to come up with a guess for $NICE(n)$:

```
In[4]:= <<rate.m

In[5]:= Rate[1,2,5,14,42]

              i0       1
            4    Gamma[- + i0]
                       2
Out[5]= ----------------------
        Sqrt[Pi] Gamma[2 + i0]
```

As the reader will have guessed, `Rate` uses the parameter i_0 instead of n. In fact, the formula is a fancy way to write $\frac{1}{i_0+1}\binom{2i_0}{i_0}$, that is, we were looking at the sequence of Catalan numbers (2.3).

To see the sharp contrast, here are the first few terms of another sequence:

$$1, 2, 9, 272, 589185.$$

(Also these are combinatorial numbers. They count the perfect matchings of the n-dimensional hypercube; cf. [146, Problem 19].) Let us factorise the last two numbers:

```
In[6]:= FactorInteger[272]

Out[6]= {{2, 4}, {17, 1}}

In[7]:= FactorInteger[589185]

Out[7]= {{3, 2}, {5, 1}, {13093, 1}}
```

The presence of the big prime factor 13093 in the last factorisation is a sure sign that we cannot expect a formula $NICE(n)$ as described in the "Definition" for this sequence of numbers. (There may well be a simple formula of a different kind. It is not very likely, though. In any case, such a formula has not been found up to this date.)

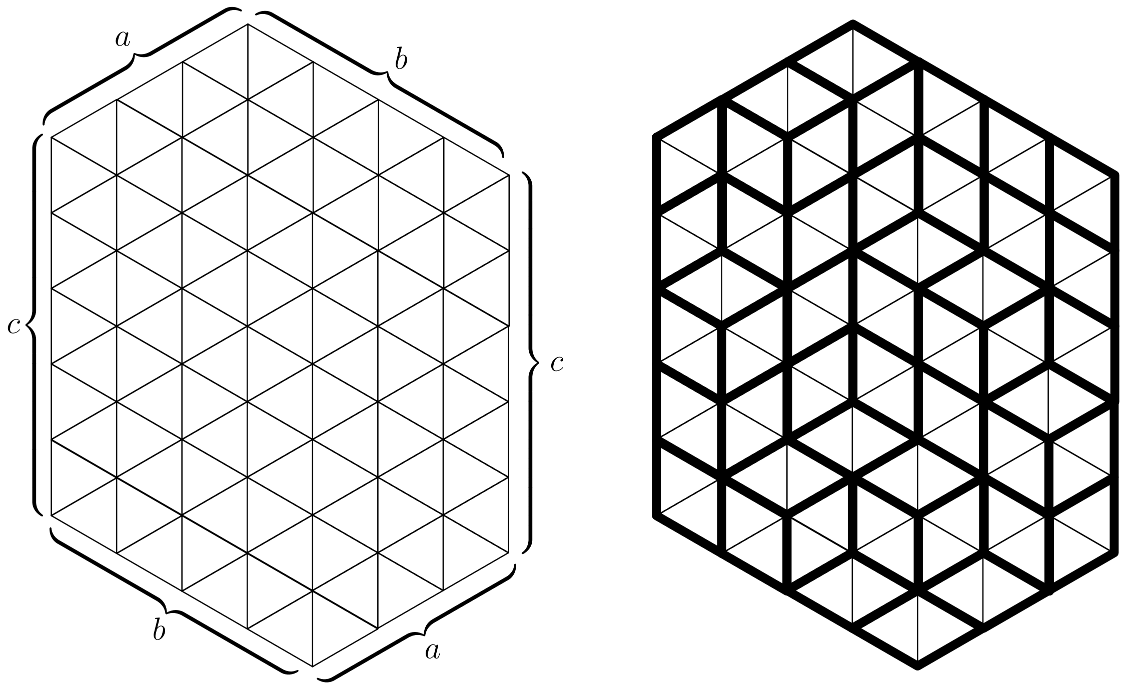
Now, that I have sufficiently explained all the ingredients in the "prototype theorem" Theorem 1, I can explain why theorems of this form are so attractive (at least to me): the objects (i.e., the permutations, partitions, paths, trees, etc.) that it deals with are usually very simple to explain, the statement is very simple and can be understood by anybody, the result $NICE(n)$ has a very elegant form, and yet, very often it is not easy at all to give a proof (not to mention a true *explanation* why such an elegant result occurs.)

Here are two examples. They concern *rhombus tilings*, by which I mean tilings of a region by rhombi with side lengths 1 and angles of 60° and 120°. The first one is a one century old theorem due to MacMahon [131, Sec. 429, $q \to 1$; proof in Sec. 494].[6]

Theorem 2. *The number of rhombus tilings of a hexagon with side lengths* a, b, c, a, b, c *whose angles are* 120° (*see Figure 1.a for an example of such a hexagon, and Figure 1.b for an example of a rhombus tiling*) *is equal to*

$$\prod_{i=1}^{c} \frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!}. \tag{2.6}$$

□



a. A hexagon with sides a, b, c, a, b, c, where $a = 3$, $b = 4$, $c = 5$

b. A rhombus tiling of a hexagon with sides a, b, c, a, b, c

FIGURE 1

[6]To be correct, MacMahon did not know anything about rhombus tilings, they did not exist in enumerative combinatorics at the time. The objects that he considered were *plane partitions*. However, there is a very simple bijection between plane partitions contained in an $a \times b \times c$ box and rhombus tilings of a hexagon with side lengths a, b, c, a, b, c, as explained for example in [51].

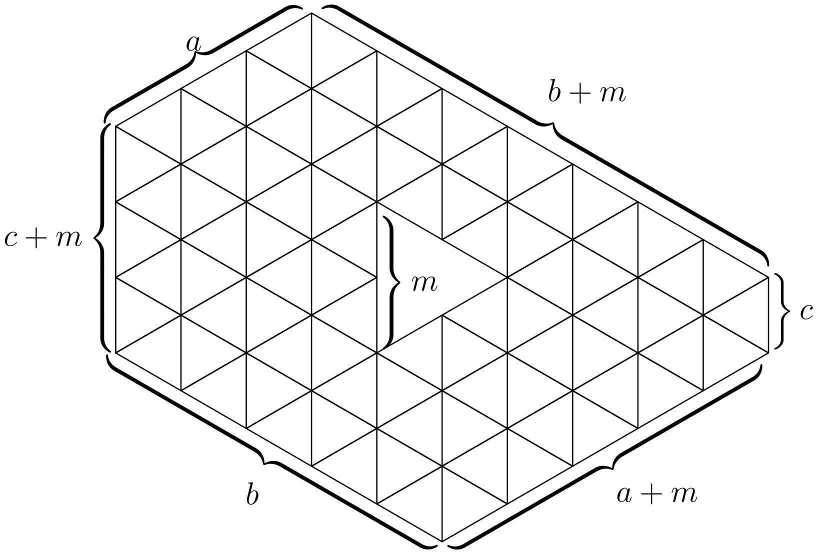


FIGURE 2. A hexagon with triangular hole

The second one is more recent, and is due to Ciucu, Eisenkölbl, Zare and the author [40, Theorem 1].

Theorem 3. *If a, b, c have the same parity, then the number of lozenge tilings of a hexagon with side lengths $a, b+m, c, a+m, b, c+m$, with an equilateral triangle of side length m removed from its centre (see Figure 2 for an example) is given by*

$$\frac{\mathrm{H}(a+m)\,\mathrm{H}(b+m)\,\mathrm{H}(c+m)\,\mathrm{H}(a+b+c+m)}{\mathrm{H}(a+b+m)\,\mathrm{H}(a+c+m)\,\mathrm{H}(b+c+m)}\frac{\mathrm{H}(m+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(m+\lfloor\frac{a+b+c}{2}\rfloor)}{\mathrm{H}(\frac{a+b}{2}+m)\,\mathrm{H}(\frac{a+c}{2}+m)\,\mathrm{H}(\frac{b+c}{2}+m)}$$
$$\times\frac{\mathrm{H}(\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{c}{2}\rfloor)}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{c}{2}\rfloor)}$$
$$\times\frac{\mathrm{H}(\frac{m}{2})^2\,\mathrm{H}(\frac{a+b+m}{2})^2\,\mathrm{H}(\frac{a+c+m}{2})^2\,\mathrm{H}(\frac{b+c+m}{2})^2}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a+b+c}{2}\rfloor)\,\mathrm{H}(\frac{a+b}{2})\,\mathrm{H}(\frac{a+c}{2})\,\mathrm{H}(\frac{b+c}{2})},\tag{2.7}$$

where

$$\mathrm{H}(n):=\begin{cases}\prod_{k=0}^{n-1}\Gamma(k+1) & \text{for } n \text{ an integer,}\\ \prod_{k=0}^{n-\frac{1}{2}}\Gamma(k+\frac{1}{2}) & \text{for } n \text{ a half-integer.}\end{cases}\tag{2.8}$$

□

(There is a similar theorem if the parities of a, b, c should not be the same, see [40, Theorem 2]. Together, the two theorems generalise MacMahon's Theorem 2.[7])

The reader should notice that the right-hand side of (2.6) is indeed of the form $NICE(a)$, while the right-hand side of (2.7) is of the form $NICE(m/2)$.

Where is the connexion to determinants? As it turns out, these two theorems are in fact *determinant evaluation theorems*. More precisely, Theorem 2 is equivalent to the following theorem.

[7]Bijective proofs of Theorem 2 which "explain" the "nice" formula are known [106, 108]. I do not ask for a bijective proof of Theorem 3 because I consider the task of finding one as daunting.

Theorem 4.

$$\det_{1\le i,j\le c}\left(\binom{a+b}{a-i+j}\right)=\prod_{i=1}^{c}\frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!}. \tag{2.9}$$

□

(The reader should notice that this is exactly (2.1) with n replaced by c.) On the other hand, Theorem 3 is equivalent to the theorem below.[8]

Theorem 5. *If m is even, the determinant*

$$\det_{1\le i,j\le a+m}\begin{pmatrix}\dbinom{b+c+m}{b-i+j} & 1\le i\le a\\ \dbinom{\frac{b+c}{2}}{\frac{b+a}{2}-i+j} & a+1\le i\le a+m\end{pmatrix} \tag{2.10}$$

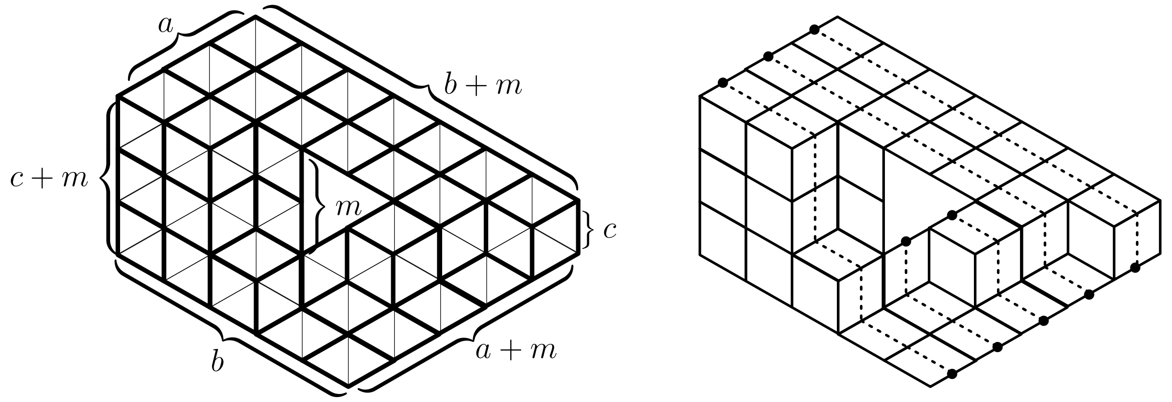
is equal to (2.7). □

The link between rhombus tilings (and equivalent objects such as plane partitions, semistandard tableaux, etc.) and determinants which explains the above two equivalence statements is *non-intersecting lattice paths*.[9] The latter are families of paths in a lattice with the property that no two paths in the family have a point in common. Indeed, rhombus tilings are (usually) in bijection with families of non-intersecting paths in the integer lattice $\mathbb{Z}^2$ which consist of unit horizontal and vertical steps. (Figure 3 illustrates the bijection for the rhombus tilings which appear in Theorem 3 in an example. In that bijection, all horizontal steps of the paths are in the positive direction, and all vertical steps are in the negative direction. See the explanations that accompany [40, Figure 8] for a detailed description. Since, as I explained, Theorem 3 essentially is a generalisation of Theorem 2, this gives also an idea for the bijection for the rhombus tilings which appear in the latter theorem. For other instances of bijections between rhombus tilings and non-intersecting lattice paths see [41, 43, 44, 55, 56, 57, 61, 112, 141].). In the case that the starting points and the end points of the lattice paths are fixed, the following many-author-theorem applies.[10]

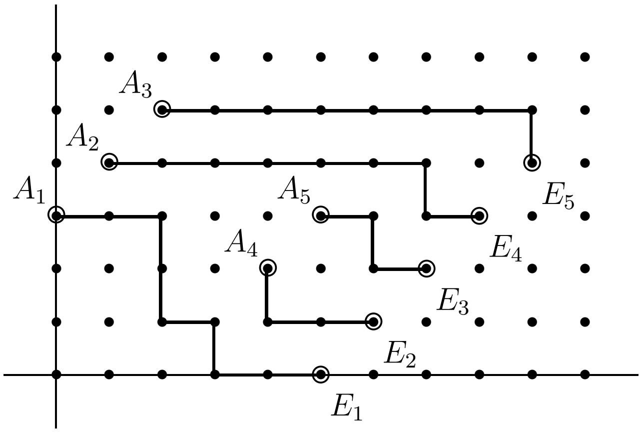
[8]To be correct, this is a little bit oversimplified. The truth is that equivalence holds only if m is even. An additional argument is necessary for proving the result for the case that m is odd. We refer the reader who is interested in these details to [40, Sec. 2].

[9]There exists in fact a second link between rhombus tilings and determinants which is not less interesting or less important. It is a well-known fact that rhombus tilings are in bijection with perfect matchings of certain hexagonal graphs. (See for example [116, Figures 13 and 14].) In view of this fact, this second link is given by Kasteleyn's theorem [98] saying that the number of perfect matchings of a planar graph is given by the Pfaffian of a slight perturbation of the adjacency matrix of the graph. See [116] for an exposition of Kasteleyn's result, including historical notes, and for adaptations taking symmetries of the graph into account.

[10]This result was discovered and rediscovered several times. In a probabilistic form, it occurs for the first time in work by Karlin and McGregor [96, 97]. In matroid theory, it is discovered in its discrete form by Lindström [126, Lemma 1]. Then, in the 1980s the theorem is rediscovered at about the same time in three different communities, not knowing from each other at the time: in statistical physics by Fisher [62, Sec. 5.3] in order to apply it to the analysis of vicious walkers as a model of wetting and melting, in combinatorial chemistry by John and Sachs [91] and Gronau, Just, Schade, Scheffler and Wojciechowski [78] in order to compute Pauling's bond order in benzenoid hydrocarbon molecules, and in enumerative combinatorics by Gessel and Viennot [73, 74] in order to count tableaux and plane



a. A lozenge tiling of the cored hexagon in Figure 2 b. The corresponding path family



c. The path family made orthogonal

FIGURE 3

Theorem 6 (KARLIN–MCGREGOR, LINDSTRÖM, GESSEL–VIENNOT, FISHER, JOHN–SACHS, GRONAU–JUST–SCHADE–SCHEFFLER–WOJCIECHOWSKI). *Let $A_1, A_2, \dots, A_n$ and $E_1, E_2, \dots, E_n$ be lattice points such that for $i < j$ and $k < l$ any lattice path between A_i and E_l has a common point with any lattice path between A_j and E_k. Then the number of all families $(P_1, P_2, \dots, P_n)$ of non-intersecting lattice paths, P_i running from A_i to E_i, $i = 1, 2, \dots, n$, is given by*

$$\det_{1\le i,j\le n} \big(P(A_j \to E_i)\big),$$

where $P(A \to E)$ denotes the number of all lattice paths from A to E. □

It goes beyond the scope of this article to include the proof of this theorem here. However, I cannot help telling that it is an extremely beautiful and simple proof that

partitions. Since only Lindström, and then Gessel and Viennot state the result in its most general form (not reproduced here), I call this theorem most often the "Lindström–Gessel–Viennot theorem." It must be also mentioned that the so-called "Slater determinant" in quantum mechanics (cf. [170] and [171, Ch. 11]) may qualify as an "ancestor" of the Lindström–Gessel–Viennot determinant.

every mathematician should have seen once, even if (s)he does not have any use for it in her/his own research. I refer the reader to [73, 74, 179].

Now the origin of the determinants becomes evident. In particular, since, for rhombus tilings, we have to deal with lattice paths in the integer lattice consisting of unit horizontal and vertical steps, and since the number of such lattice paths which connect two lattice points is given by a binomial coefficient, we see that the enumeration of rhombus tilings must be a rich source for binomial determinants. This is indeed the case, and there are several instances in which such determinants can be evaluated in the form $NICE(.)$ (see [38, 40, 41, 43, 44, 55, 56, 57, 61, 71, 109] and Section 5). Often the evaluation part is highly non-trivial.

The evaluation of the determinant (2.9) is not very difficult (see [109, Sections 2.2, 2.3, 2.5] for 3 different ways to evaluate it). On the other hand, the evaluation of the determinant (2.10) requires some effort (see [40, Sec. 7]).

To conclude this section, I state another determinant evaluation, to which I shall come back later. Its origin lies as well in the enumeration of rhombus tilings and plane partitions (see [105, Theorem 10] and [42, Theorem 2.1]).

Theorem 7. *For any complex numbers x and y there holds*

$$\det_{0\le i,j\le n-1}\left(\frac{(x+y+i+j-1)!}{(x+2i-j)!\,(y+2j-i)!}\right)\\ =\prod_{i=0}^{n-1}\frac{i!\,(x+y+i-1)!\,(2x+y+2i)_i\,(x+2y+2i)_i}{(x+2i)!\,(y+2i)!},\qquad(2.11)$$

where the shifted factorials *or* Pochhammer symbols $(a)_k$ *are defined by* $(a)_k:=a(a+1)\cdots(a+k-1)$, $k\ge1$, *and* $(a)_0:=1$. *(In this formula, a factorial* $m!$ *has to be interpreted as* $\Gamma(m+1)$ *if* m *is not a non-negative integer.)*

3. A determinant from number theory

However, determinants do not only arise in combinatorics, they also arise in other fields. In this section, I want to present a determinant which arose in number theory, explain in some detail its origin, and then outline the steps which led to its evaluation, thereby giving the reader an opportunity to look "behind the scenes" while one tries to make the determinant evaluation methods described in [109] work.

The story begins with the following two series expansions for π. The first one is due to Bill Gosper [77],

$$\pi=\sum_{n=0}^{\infty}\frac{50n-6}{\binom{3n}{n}2^n},\qquad(3.1)$$

and was used by Fabrice Bellard [20, file `pi1.c`] to find an algorithm for computing the n-th decimal of π without computing the earlier ones, thus improving an earlier algorithm due to Simon Plouffe [145]. The second one,

$$\pi=\frac{1}{740025}\left(\sum_{n=1}^{\infty}\frac{3P(n)}{\binom{7n}{2n}2^{n-1}}-20379280\right),\qquad(3.2)$$

where

$$P(n) = -885673181n^5 + 3125347237n^4 - 2942969225n^3 + 1031962795n^2 - 196882274n + 10996648,$$

is due to Fabrice Bellard [20], and was used by him in his world record setting computation of the 1000 billionth *binary* digit of π, being based on the algorithm in [19].

Going beyond that, my co-authors from [5], Gert Almkvist and Joakim Petersson, asked themselves the following question:

Are there more expansions of the type

$$\pi = \sum_{n=0}^{\infty} \frac{S(n)}{\binom{mn}{pn} a^n},$$

where $S(n)$ is some polynomial in n (depending on m, p, a)?

How can one go about to get some intuition about this question? One chooses some specific m, p, a, goes to the computer, computes

$$p(k) = \sum_{n=0}^{\infty} \frac{n^k}{\binom{mn}{pn} a^n}$$

to many, many digits for $k = 0, 1, 2, \dots$, puts

$$\pi, p(0), p(1), p(2), \dots$$

into the LLL-algorithm (which comes, for example, with the *Maple* computer algebra package), and one waits whether the algorithm comes up with an integral linear combination of $\pi, p(0), p(1), p(2), \dots$.[11] Indeed, Table 1 shows the parameter values, where the LLL-algorithm gave a result.

For example, it found

$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{16n}{8n} 16^n},$$

where

$$r = 3^6 5^3 7^2 11^2 13^2$$

and

$$S(n) = -869897157255 - 3524219363487888n + 112466777263118189n^2 - 1242789726208374386n^3 + 6693196178751930680n^4 - 19768094496651298112n^5 + 32808347163463348736n^6 - 28892659596072587264n^7 + 10530503748472012800n^8,$$

[11]For readers unfamiliar with the LLL-algorithm: in this particular application, it takes as an input rational numbers $r_1, r_2, \dots, r_m$ (which, in our case, will be the numbers 1 and the rational approximations of π, $p(0)$, $p(1)$, ... which we computed), and, if successful, outputs *small* integers $c_1, c_2, \dots, c_m$ such that $c_1r_1 + c_2r_2 + \dots + c_mr_m$ is *very small.* Thus, if r_i was a good approximation for the real number x_i, $i = 1, 2, \dots, m$, one can expect that actually $c_1x_1 + c_2x_2 + \dots + c_mx_m = 0$. See [123, Sec. 1, in particular the last paragraph] and [45, Ch. 2] for the description of and more information on this important algorithm. In particular, also here, the output of the algorithm (if there is) is just a (very guided) *guess.* Thus, a proof is still needed, although the probability that the guess is wrong is infinitesimal. As a matter of fact, it is very likely that Bellard had no proof of his formula (3.2) ...

m	p	a	$\deg(S)$	
3	1	2	1	(Gosper)
7	2	2	5	(Bellard)
8	4	-4	4	
10	4	4	8	
12	4	-4	8	
16	8	16	8	
24	12	-64	12	
32	16	256	16	
40	20	-4^5	20	
48	24	4^6	24	
56	28	-4^7	28	
64	32	4^8	32	
72	36	-4^9	36	
80	40	4^{10}	40	

TABLE 1

and

$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{32n}{16n} 256^n},$$

where

$$r = 2^3 3^{10} 5^6 7^3 11^1 13^2 17^2 19^2 23^2 29^2 31^2$$

and

$$\begin{aligned} S(n) = &- 206211188475634747908570928087 5\\ &+ 1505491740302839023753569717261882091900n\\ &- 11240114940408765821383938671621197529197 5n^2\\ &+ 325788165194268289181855772622584067411000 2n^3\\ &- 5167730951089063050060789859946303626796128 0n^4\\ &+ 5173379779873548193227869095411790431485227 20n^5\\ &- 3526396494329560718758086392841258152390245120n^6\\ &+ 17114576623599516622750121611007480594379936358 4n^7\\ &- 60739416613228219940886539658145904402068029440n^8\\ &+ 159935882563435860391195903248596461569183580160n^9\\ &- 313951952615028230229958218839819183812205608960n^{10}\\ &+ 457341091673257198565533286493831205566468325376n^{11}\\ &- 486846784774707448105420279985074159657397780480n^{12}\\ &+ 367314505118245777241612044490633887668208926720n^{13}\\ &- 185647326591648164598342857319777582801297080320n^{14} \end{aligned}$$

$$+\,56224688035707015687999128994324690418467340288n^{15}$$
$$-\,76872557788165577860739777951493604086120448 00n^{16}.$$

Of course, there could be many more.

If one looks more closely at Table 1, then, if one disregards the first, second and fourth line, one cannot escape to notice a pattern: *apparently, for each $k=1,2,\dots$, there is a formula*

$$\pi=\sum_{n=0}^{\infty}\frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}},$$

where $S_k(n)$ is some polynomial in n of degree $4k$.

In order to make progress on this observation, we have to first see how one can prove such an identity, once it is found. In fact, this is not difficult at all. To illustrate the idea, let us go through a proof of Gosper's identity (3.1).

The beta integral evaluation (cf. [12, Theorem 1.1.4]) gives

$$\frac{1}{\binom{3n}{n}}=(3n+1)\int_0^1 x^{2n}(1-x)^n dx.$$

Hence the right hand side of the formula will be

$$\int_0^1\sum_{n=0}^{\infty}(50n-6)(3n+1)\left(\frac{x^2(1-x)}{2}\right)^n dx.$$

We have

$$\sum_{n=0}^{\infty}(50n-6)(3n+1)y^n=\frac{2(56y^2+97y-3)}{(1-y)^3}. \tag{3.3}$$

Thus, if substituted, we obtain

$$\begin{aligned}RHS&=8\int_0^1\frac{28x^6-56x^5+28x^4-97x^3+97x^2-6}{(x^3-x^2+2)^3}dx\\&=\left[\frac{4x(x-1)(x^3-28x^2+9x+8)}{(x^3-x^2+2)^2}+4\arctan(x-1)\right]_0^1=\pi.\end{aligned} \tag{3.4}$$

(Clearly, both (3.3) and (3.4) are routine calculations, and therefore we did not do it by hand, but let them be worked out by *Maple*.)

Now let us fix $k\ge 1$. We apply the same procedure to $\sum_{n=0}^{\infty}S_k(n)/\binom{8kn}{4kn}(-4)^{kn}$, where $S_k(n)$ is (hopefully) some (unknown) polynomial in n. The beta integral evaluation gives

$$\frac{1}{\binom{8kn}{4kn}}=(8kn+1)\int_0^1 x^{4kn}(1-x)^{4kn}dx.$$

Hence, if $S_k(n)$ should have degree d in n,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}}&=\int_0^1\sum_{n=0}^{\infty}(8kn+1)S_k(n)\left(\frac{x^{4k}(1-x)^{4k}}{(-4)^k}\right)^n dx\\&=\int_0^1\frac{P_k(x)}{\left(x^{4k}(1-x)^{4k}-(-4)^k\right)^{d+2}}\,dx,\end{aligned} \tag{3.5}$$

where $P_k(x)$ is some polynomial in x. For convenience, let us write P as a short-hand for P_k. Let $Q(x) := x^{4k}(1-x)^{4k} - (-4)^k$. Now we make the wild assumption that

$$\int \frac{P(x)}{Q(x)^{d+2}}\,dx = \frac{R(x)}{Q(x)^{d+1}} + 2\arctan(x) + 2\arctan(x-1),$$

for some polynomial $R(x)$ with $R(0) = R(1) = 0$. Then the original sum would indeed be equal to π. The last equality is equivalent to

$$\frac{P}{Q^{d+2}} = \frac{R'}{Q^{d+1}} - (d+1)\frac{Q'R}{Q^{d+2}} + 2\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right),$$

or

$$QR' - (d+1)Q'R = P - 2Q^{d+2}\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right).$$

In our examples, we observed that

$$R(x) = (2x-1)\check{R}\big(x(1-x)\big)$$

for a polynomial $\check{R}$. So, let us make the substitution

$$t = x(1-x).$$

Then, after some simplification, the above differential equation becomes

$$-(1-4t)Q\frac{d\check{R}}{dt} + (2Q + 4k(4k+1)(1-4t)t^{4k-1})\check{R} - P + 2(3-2t)\frac{Q^{4k+2}}{t^2-2t+2} = 0, \quad (3.6)$$

where $Q(t) = t^{4k} - (-4)^k$.

Now, writing $N(k) = 4k(4k+1)$, we make the Ansatz

$$\check{R}(t) = \sum_{j=1}^{N(k)-1} a(j)t^j,$$

$$S_k(n) = \sum_{j=0}^{4k} a(N(k)+j)n^j.$$

(The reader should recall that $S_k(n)$ defines $P_k(t) = P(t)$ through (3.5).) Comparing coefficients of powers of t on both sides of (3.6), we get a system of $N(k) + 4k$ linear equations for the unknowns $a(1), a(2), \dots, a(N(k)+4k)$.

Hence: *If the determinant of this system of linear equations is non-zero, then there does indeed exist a representation*

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^n}.$$

To see whether we could indeed hope for the determinant to be non-zero, we went again to the computer and looked at the values of the determinant in some small instances. (Obviously, we do not want to do this by hand, since for $k = 1$ the matrix is already a 24×24 matrix!) So, let us program the matrix. (We shall see the mathematical definition of the matrix in just a moment, see (3.8).[12])

[12]To tell the truth, this is the form of the matrix after some simplifications have already been carried out. (In particular, we are looking at a matrix which is slightly smaller than the original one.) See [5, beginning of Section 4] for these details. There, the matrix in (3.8) is called M'''.

```
In[8]:= a[k_,j_]:=Module[{Var=j/(4k)},
                    (-1)^(Var-1)*8k(4k+1)(-4)^(k*(Var+1))*
                    Product[4k*l-1,{l,1,4k-Var}]*Product[4k*l+1,{l,1,Var-1}]
                    ]
```

```
In[9]:= A[k_, i_, j_] := Module[{Var},
        Var = {Floor[(i - 2)/(4*k - 1)],
            Floor[(j - 1)/(4*k)], Mod[i - 2, 4*k - 1],
            Mod[j - 1, 4*k]};
        If[i == 1,
          If[Mod[j, 4*k] === 0, a[k, j], 0],
          If[Var[[1]] - Var[[2]] == 0,
            Switch[Var[[3]] - Var[[4]], 0, f1[k, Var[[3]]+1, j], -1,
              f0[k, Var[[3]]+1, j], _, 0],
            If[Var[[1]] - Var[[2]] == 1,
              Switch[Var[[3]] - Var[[4]], 0, g1[k, Var[[3]]+1, j], -1,
                g0[k, Var[[3]]+1, j], _, 0], 0]]]]
In[10]:= A[k_] := Table[A[k, i, j], {i, 1, 16*k^2}, {j, 1, 16*k^2}]
In[11]:= f0[k_, t_, j_] := j*(-4)^k;
         f1[k_, t_, j_] := -(2 + 4*j)*(-4)^k;
         g0[k_, t_, j_] := (4*k*(4*k + 1) - j);
         g1[k_, t_, j_] := (-4*4*k*(4*k + 1) + 2 + 4*j)
```

We shall not try to digest this at this point. Let us accept the program as a black box, and let us compute the determinant for $k = 2$.

```
In[12]:= Det[A[2]]
Out[12]= -601576375580370166777074138698518196031142518971568946712\
>      220413667478103830277423172597130645906407512102309266227981 4\
>      0151955456000000000
```

Magnificent! This is certainly *not* zero. However, what are we going to do with this gigantic number? Remembering our discussion about "nice" numbers and "nice" formulae in the preceding section, let us factorise it in its prime factors.

```
In[13]:= FactorInteger[%]
Out[13]= {{-1, 1}, {2, 325}, {3, 39}, {5, 11}, {7, 11}, {11, 3}, {13, 2}}
```

I would say that this is sensational: a number with 139 digits, and the biggest prime factor is 13! As a matter of fact, this is not just a rare exception. Table 2 shows the factorisations of the first five determinants. (We could not go further because of the exploding size of the matrix of which the determinant is taken.)

k	$\det(A(k))$
1	$2^{59}3^{5}5^{6}7^{1}$
2	$-2^{325}3^{39}5^{11}7^{11}11^{3}13^{2}$
3	$2^{772}3^{146}5^{28}7^{17}11^{17}13^{18}17^{4}19^{3}23^{1}$
4	$-2^{1913}3^{111}5^{58}7^{38}11^{21}13^{22}17^{24}19^{7}23^{5}29^{2}31^{1}$
5	$2^{2932}3^{202}5^{306}7^{69}11^{29}13^{27}17^{28}19^{29}23^{9}29^{6}31^{5}37^{2}$

TABLE 2

Thus, these experimental results *make us sure* that there must be a "nice" formula for the determinant. Indeed, we prove in [5] that[13]

$$\det(A(k)) = (-1)^{k-1} 2^{16k^3+20k^2+6k} k^{8k^2+2k} (4k+1)!^{4k} \prod_{j=1}^{4k} \frac{(2j)!}{j!^2}. \tag{3.7}$$

Hence the desired theorem follows.

Theorem 8. *For all $k \geq 1$ there is a formula*

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}},$$

where $S_k(n)$ is a polynomial in n of degree $4k$ with rational coefficients. The polynomial $S_k(n)$ can be found by solving the previously described system of linear equations.

I must admit that we were extremely lucky that it was indeed possible to *evaluate* the determinant *explicitly*. To recall, "all" we needed to prove our theorem (Theorem 8) was to show that the determinant was *non-zero*. To be honest, I would not have the slightest idea how to do this here without finding the exact value of the determinant.

Now, after all this somewhat "dry" discussion, let me present the determinant. We had to determine the determinant of the $16k^2 \times 16k^2$ matrix

$$\begin{pmatrix}
0\dots0* & 0\dots0* & 0\dots0* & \dots & \dots & \dots & 0\dots0* \\
F_1 & 0 & 0 & \dots & \dots & \dots & 0 \\
G_1 & F_2 & 0 & \dots & \dots & \dots & 0 \\
0 & G_2 & F_3 & & & & \vdots \\
0 & 0 & G_3 & & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & F_{4k-1} & 0 \\
\vdots & & & 0 & 0 & G_{4k-1} & F_{4k} \\
0 & \dots & \dots & \dots & 0 & 0 & G_{4k}
\end{pmatrix}, \tag{3.8}$$

[13]Strictly speaking, this is not a formula $NICE(k)$ according to my "Definition" in the preceding section, because of the presence of the "Abelian" factors k^{8k^2+2k} and $(4k+1)!^{4k}$, see Footnote 5. Nevertheless, the reader will certainly admit that this is a *nice* and *closed* formula.

where the ℓ-th non-zero entry in the first row (these are marked by $*$) is

$$(-1)^{\ell-1}(-4)^{(\ell+1)k}8k(4k+1)\left(\prod_{i=1}^{4k-\ell}(4ik-1)\right)\left(\prod_{i=1}^{\ell-1}(4ik+1)\right),$$

and where each block F_t and G_t is a $(4k-1)\times(4k)$ matrix (that is, these are *rectangular* blocks!) with non-zero entries only on its (two) main diagonals,

$$F_t=\begin{pmatrix} f_1(4(t-1)k+1) & f_0(4(t-1)k+2) & 0 & \dots & & & \\ 0 & f_1(4(t-1)k+2) & f_0(4(t-1)k+3) & 0 & \dots & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & 0 & f_1(4tk-2) & f_0(4tk-1) & 0 \\ & & & 0 & f_1(4tk-1) & f_0(4tk) \end{pmatrix}$$

and

$$G_t=\begin{pmatrix} g_1(4(t-1)k+1) & g_0(4(t-1)k+2) & 0 & \dots & & & \\ 0 & g_1(4(t-1)k+2) & g_0(4(t-1)k+3) & 0 & \dots & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & 0 & g_1(4tk-2) & g_0(4tk-1) & 0 \\ & & & 0 & g_1(4tk-1) & g_0(4tk) \end{pmatrix}.$$

We have almost worked our way through the definition of the determinant. The only missing piece is the definition of the functions f_0, f_1, g_0, g_1 in the blocks F_t and G_t. Here it is:

$$\begin{aligned} f_0(j)&=j(-4)^k,\\ f_1(j)&=-(4j+2)(-4)^k,\\ g_0(j)&=(N(k)-j),\\ g_1(j)&=-(4N(k)-4j-2), \end{aligned} \tag{3.9}$$

where, as before, we write $N(k)=4k(4k+1)$ for short.

4. The evaluation of the determinant

I now describe how the determinant of (3.8) was evaluated by applying to it the methods described in [109]. To make this section as self-contained as possible, for each of them I briefly recall how it works before putting it into action.

"Method" 0: Do row and column operations until the determinant reduces to something manageable.

In fact, at a first glance, this does not look too bad. Our matrix (3.8), of which we want to compute the determinant and show that it is non-zero, is a very sparse matrix. Moreover, it looks almost like a two-diagonal matrix. It seems that one should be able to do a few row and column manipulations and thus reduce the matrix to a matrix of a simpler form of which we can evaluate the determinant.

Well, we tried that. Unfortunately, the above impression is deceiving. First of all, the diagonals of the blocks do not really fit together to form diagonals which run from one end of the matrix to the other. Second, there remains still the first row which does not fit the pattern of the rest of the matrix. So, whatever we did, we ended up nowhere. Maybe we should try something more sophisticated …

Method 1 [109, Sec. 2.6]: *LU-factorisation.* Suppose we are given a family of matrices $A(1), A(2), A(3), \dots$ of which we want to compute the determinants. Suppose further that we can write

$$A(k) \cdot U(k) = L(k),$$

where $U(k)$ is an upper triangular matrix with 1s on the diagonal, and where $L(k)$ is a lower triangular matrix. Then, clearly,

$$\det(A(k)) = \text{product of the diagonal entries of } L(k).$$

But how do we find $U(k)$ and $L(k)$? We go to the computer, crank out $U(k)$ and $L(k)$ for $k = 1, 2, 3, \dots$, until we are able to make a guess. Afterwards we prove the guess by proving the corresponding identities.

Well, we programmed that, we stared at the output on the computer screen, but we could not make any sense of it.

Method 2 [109, Sec. 2.3]: *Condensation.* This is based on a determinant formula due to Jacobi (see [28, Ch. 4] and [100, Sec. 3]). Let A be an $n \times n$ matrix. Let $A^{j_1,j_2,\dots,j_\ell}_{i_1,i_2,\dots,i_\ell}$ denote the submatrix of A in which rows $i_1, i_2, \dots, i_\ell$ and columns $j_1, j_2, \dots, j_\ell$ are omitted. Then there holds

$$\det A \cdot \det A^{1,n}_{1,n} = \det A^1_1 \cdot \det A^n_n - \det A^n_1 \cdot \det A^1_n. \tag{4.1}$$

If we consider a family of matrices $A(1), A(2), \dots$, and if all the consecutive minors of $A(n)$ belong to the same family, then this allows one to give an inductive proof of a conjectured determinant evaluation for $A(n)$.

Let me illustrate this by reproducing Amdeberhan's condensation proof [8] of (2.11). Let $M_n(x, y)$ denote the determinant in (2.11). Then we have

$$\begin{aligned}
\big(M_n(x,y)\big)^n_n &= M_{n-1}(x,y),\\
\big(M_n(x,y)\big)^1_1 &= M_{n-1}(x+1,y+1),\\
\big(M_n(x,y)\big)^1_n &= M_{n-1}(x-1,y+2),\\
\big(M_n(x,y)\big)^n_1 &= M_{n-1}(x+2,y-1),\\
\big(M_n(x,y)\big)^{1,n}_{1,n} &= M_{n-2}(x+1,y+1).
\end{aligned} \tag{4.2}$$

Thus, we know that Equation (4.1) is satisfied with A replaced by $M_n(x, y)$, where the minors appearing in (4.1) are given by (4.2). This can be interpreted as a recurrence for the sequence $\big(M_n(x, y)\big)_{n\ge0}$. Indeed, given $M_0(x, y)$ and $M_1(x, y)$, the equation (4.1) determines $M_n(x, y)$ uniquely for all $n \ge 0$ (given that $M_n(x, y)$ never vanishes). Thus, since the right-hand side of (2.11) is indeed never zero, for the proof of (2.11) it suffices to check (2.11) for $n = 0$ and $n = 1$, and that the right-hand side of (2.11) also satisfies (4.1), all of which is a routine task.

Now, a short glance at the definition of our matrix (3.8) will convince us quickly that application of this method to it is rather hopeless. For example, omission of the first row already brings us outside of our family of matrices. So, also this method is not much help to solve our problem, which is really a pity, because it is the most painless of all ...

Method 3 [109, Sec. 2.4]: *Identification of factors.* In order to sketch the idea, let us quickly go through a (standard) proof of the *Vandermonde determinant evaluation*,

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\prod_{1\le i<j\le n}(X_j-X_i). \tag{4.3}$$

Proof. If $X_{i_1}=X_{i_2}$ with $i_1\ne i_2$, then the Vandermonde determinant (4.3) certainly vanishes because in that case two rows of the determinant are identical. Hence, $(X_{i_1}-X_{i_2})$ divides the determinant as a polynomial in the X_i's. But that means that the complete product $\prod_{1\le i<j\le n}(X_j-X_i)$ (which is exactly the right-hand side of (4.3)) must divide the determinant.

On the other hand, the determinant is a polynomial in the X_i's of degree at most $\binom n2$. Combined with the previous observation, this implies that the determinant equals the right-hand side product times, possibly, some constant. To compute the constant, compare coefficients of $X_1^0X_2^1\cdots X_n^{n-1}$ on both sides of (4.3). This completes the proof of (4.3). $\square$

At this point, let us extract the essence of this proof. The basic steps are:

(S1) *Identification of factors*
(S2) *Determination of degree bound*
(S3) *Computation of the multiplicative constant.*

As I report in [109], this turns out to be an extremely powerful method which has numerous applications. To give an idea of the flavour of the method, I show a few steps when it is applied to the determinant in (2.11) (ignoring the fact that we have already found a very simple proof of its evaluation; see [105, proof of Theorem 10] for the complete proof using the "identification of factors" method).

To get started, we have to transform the assertion (2.11) into an assertion about polynomials. This is easily done, we just have to factor

$$(x+y+i-1)!/(x+2i)!/(y+2n-i-2)!$$

out of the i-th row of the determinant. If we subsequently cancel common factors on both sides of (2.11), we arrive at the equivalent assertion

$$\det_{0\le i,j\le n-1}\Big((x+y+i)_j\,(x+2i-j+1)_j\,(y+2j-i+1)_{2n-2j-2}\Big)$$
$$=\prod_{i=0}^{n-1}\Big(i!\,(y+2i+1)_{n-i-1}\,(2x+y+2i)_i\,(x+2y+2i)_i\Big), \tag{4.4}$$

where, as before, $(\alpha)_k$ is the standard notation for shifted factorials (Pochhammer symbols) explained in the statement of Theorem 7.

In order to apply the same idea as in the above evaluation of the Vandermonde determinant, as a first step we have to show that the right-hand side of (4.4) divides the determinant on the left-hand side as a polynomial in x and y. For example, we would have to prove that $(x+2y+2n-2)$ (actually, $(x+2y+2n-2)^{\lfloor (n+1)/3\rfloor}$, we will come to that in a moment) divides the determinant. Equivalently, if we set $x=-2y-2n+2$ in the determinant, then it should vanish. How could we prove that? Well, if it vanishes then there must be a linear combination of the columns, or of the rows, that vanishes. Equivalently, for $x=-2y-2n+2$ we find a vector in the kernel of the matrix in

(4.4), respectively of its transpose. More generally (and this addresses the fact that we actually want to prove that $(x+2y+2n-2)^{\lfloor (n+1)/3\rfloor}$ divides the determinant):

> *For proving that $(x+2y+2n-2)^E$ divides the determinant, we find E linear independent vectors in the kernel.*

(For a formal justification that this does indeed suffice, see Section 2 of [107], and in particular the Lemma in that section.)

Okay, how is this done in practice? You go to your computer, crank out these vectors in the kernel, for $n=1,2,3,\dots$, and try to make a guess what they are in general. To see how this works, let us do it in our example. First of all, we program the kernel of the matrix in (4.4) with $x=-2y-2n+2$ (again, we are using *Mathematica* here).[14]

```
In[14]:= p=Pochhammer;
         m[i_,j_,n_]:=p[x+y+i, j]*p[y+2*j+1-i, 2*n-2*j-2]*p[x-j+1+2i,j];
         V[n_]:=(x=-2y-2n+2;
                Var=Sum[c[j]*Table[m[i,j,n],{i,0,n-1}],{j,0,n-1}];
                Var=Solve[Var==Table[0,{n}],Table[c[i],{i,0,n-1}]];
                Factor[Table[c[i],{i,0,n-1}]/.Var])
```

What the computer gives is the following:

```
In[15]:= V[2]

Out[15]= {{-2 c[1], c[1]}}

In[16]:= V[3]

Out[16]= {{-2 c[2], -c[2], c[2]}}

In[17]:= V[4]

Out[17]= {{-2 c[3], -3 c[3], 0, c[3]}}

In[18]:= V[5]

Out[18]= {{-2 c[4], -5 c[4], -2 c[3] - c[4], c[3], c[4]}}

In[19]:= V[6]

Out[19]= {{-2 c[5], -7 c[5], -2 (c[4] + 2 c[5]), -c[4], c[4], c[5]}}

In[20]:= V[7]

Out[20]= {{-2 c[6], -9 c[6], -2 c[5] - 9 c[6], -3 c[5] - c[6], 0,
>     c[5], c[6]}}
```

At this point, the computations become somewhat slow. So we should help our computer. Indeed, on the basis of what we have obtained so far, it is "obvious" that, somewhat unexpectedly, y does not appear in the result. Therefore we simply set y equal to some random number, and then the computer can go much further without any effort.

```
In[21]:= y=101
```

[14]In the program, `V[n]` represents the kernel, which is clearly a vector space. In the computer output, it is given in parametric form, the parameters being the `c[i]`'s.

```
In[22]:= V[8]

Out[22]= {{-2 c[7], -11 c[7], -2 (c[6] + 8 c[7]), -5 (c[6] + c[7]),

>     -2 c[5] - c[6], c[5], c[6], c[7]}}
In[23]:= V[9]

Out[23]= {{-2 c[8], -13 c[8], -2 c[7] - 25 c[8], -7 (c[7] + 2 c[8]),

>     -2 c[6] - 4 c[7] - c[8], -c[6], c[6], c[7], c[8]}}
In[24]:= V[10]

Out[24]= {{-2 c[9], -15 c[9], -2 (c[8] + 18 c[9]), -3 (3 c[8] + 10 c[9]),

>     -2 c[7] - 9 c[8] - 6 c[9], -3 c[7] - c[8], 0, c[7], c[8], c[9]}}
In[25]:= V[11]

Out[25]= {{-2 c[10], -17 c[10], -2 c[9] - 49 c[10], -11 (c[9] + 5 c[10]),

>     -2 (c[8] + 8 c[9] + 10 c[10]), -5 c[8] - 5 c[9] - c[10],

>     -2 c[7] - c[8], c[7], c[8], c[9], c[10]}}
```

Let us extract some information out of these data. For convenience, we write M_n for the matrix in (4.4) in the sequel. For example, by just looking at the coefficients of $c[n-1]$ appearing in $V[n]$, we extract that

the vector $(-2,1)$ is in the kernel of M_2,
the vector $(-2,-1,1)$ is in the kernel of M_3,
the vector $(-2,-3,0,1)$ is in the kernel of M_4,
the vector $(-2,-5,-1,0,1)$ is in the kernel of M_5,
the vector $(-2,-7,-4,0,0,1)$ is in the kernel of M_6,
the vector $(-2,-9,-9,-1,0,0,1)$ is in the kernel of M_7,
the vector $(-2,-11,-16,-5,0,0,0,1)$ is in the kernel of M_8,
the vector $(-2,-13,-25,-14,-1,0,0,0,1)$ is in the kernel of M_9,
the vector $(-2,-15,-36,-30,-6,0,0,0,0,1)$ is in the kernel of M_{10},
the vector $(-2,-17,-49,-55,-20,-1,0,0,0,0,1)$ is in the kernel of M_{11}.

Okay, now we have to make sense out of this. Our vectors in the kernel have the following structure: first, there are some negative numbers, then follow a few zeroes, and finally there is a trailing 1. I believe that we do not have any problem to guess what the zeroeth[15] or the first coordinate of our vector is. Since the second coordinates are always negatives of squares, there is also no problem there. What about the third coordinates? Starting with the vector for M_7, these are $-1,-5,-14,-30,-55,\dots$. I guess, rather than thinking hard, we should consult `Rate` (see Footnote 4):

[15]The indexing convention in the matrix in (4.4) of which the determinant is taken is that rows and columns are indexed by $0,1,\dots,n-1$. We keep this convention here.

```
In[26]:= Rate[-1,-5,-14,-30,-55]

          -(i0 (1 + i0) (1 + 2 i0))
Out[26]= {-------------------------}
                      6
```

After replacing `i0` by $n-6$ (as we should), this becomes $-(n-6)(n-5)(2n-11)/6$. An interesting feature of this formula is that it also works well for $n=6$ and $n=5$. Equipped with this experience, we let `Rate` work out the fourth coordinate:

```
In[27]:= Rate[0,0,0,-1,-6,-20]

                              2
          -((-3 + i0) (-2 + i0)  (-1 + i0))
Out[27]= {---------------------------------}
                         12
```

After replacement of `i0` by $n-5$, this is $-(n-8)(n-7)^2(n-6)/12$. Let us summarise our results so far: the first five coordinates of our vector in the kernel of M_n are

$$-2,\ -(2n-5),\ -\frac{(n-4)(2n-8)}{2},\ -\frac{(n-6)(n-5)(2n-11)}{6},\\ -\frac{(n-8)(n-7)(n-6)(2n-14)}{12}.$$

I would say, there is a clear pattern emerging: the s-th coordinate is equal to

$$-\frac{(n-2s)_{s-1}\,(2n-3s-2)}{s!}=-\frac{(2n-3s-2)}{(n-s-1)}\frac{(n-2s)_s}{s!}.$$

Denoting the above expression by $f(n,s)$, the vector

$$(f(n,0),f(n,1),\dots,f(n,n-2),1)$$

is apparently in the kernel of M_n for $n\ge 2$. To prove this, we have to show that

$$\sum_{s=0}^{n-2}\frac{(2n-3s-2)}{(n-s-1)}\frac{(n-2s)_s}{(s)!}\\ \cdot(-y-2n+i+2)_s\,(-2y-2n+2i-s+3)_s\,(y+2s-i+1)_{2n-2s-2}\\ =(-y-2n+i+2)_{n-1}\,(-2y-3n+2i+4)_{n-1}.$$

In [105] it was argued that this identity follows from a certain hypergeometric identity due to Singh [169]. However, for just having *some* proof of this identity, this careful literature search was not necessary. In fact, nowadays, *once you write down a binomial or hypergeometric identity, it is already proved!* One simply puts the binomial/hypergeometric sum into the *Gosper–Zeilberger algorithm* (see [144, 193, 194, 195]), which outputs a recurrence for it, and then the only task is to verify that the (conjectured) right-hand side also satisfies the same recurrence, and to check the identity for sufficiently many initial values (which one has already done anyway while producing the conjecture).[16]

[16]As you may have suspected, this is again a little bit oversimplified. But not much. The Gosper–Zeilberger algorithm applies *always* to hypergeometric sums, and there are only very few

As I mentioned earlier, actually we need more vectors in the kernel. However, this is not difficult. Take a closer look, and you will see that the pattern persists (set $c[n-1]=0$ in the vector for $V[n]$, etc.). It will take you no time to work out a full-fledged conjecture for $\lfloor (n+1)/3 \rfloor$ linear independent vectors in the kernel of M_n.

I do not want to go through Steps (S2) and (S3), that is, the degree calculation and the computation of the constant. As it turns out, to do this conveniently you need to introduce more variables in the determinant in (4.4). Once you do this, everything works out very smoothly. I refer the reader to [105] for these details.

Now, let us come back to our determinant, the determinant of (3.8), and apply "identification of factors" to it.[17] To begin with, here is bad news: "identification of factors" crucially requires the existence of indeterminates. But, where are they in (3.8)? If we look at the definition of the matrix (3.8), which, in the end, depends on the auxiliary functions f_0, f_1, g_0, g_1 defined in (3.9), then we see that there are no indeterminates at all. Everything is (integral) numbers. So, to get even started, we need to introduce indeterminates in a way such that the more general determinant would still factor "nicely." We do not have much guidance. Maybe, since we already made the abbreviation $N(k) = 4k(4k+1)$, we should replace $N(k)$ by X? Okay, let us try this, that is, let us put

$$\begin{aligned} f_0(j) &= j(-4)^k,\\ f_1(j) &= -(4j+2)(-4)^k,\\ g_0(j) &= (X-j),\\ g_1(j) &= -(4X-4j-2) \end{aligned} \tag{4.5}$$

instead of (3.9). Let us compute the new determinant for $k=2$. We program the new functions f_0, f_1, g_0, g_1,

```
In[28]:= f0[k_, t_, j_] := j*(-4)^k;
         f1[k_, t_, j_] := -(2 + 4*j)*(-4)^k;
         g0[k_, t_, j_] := (X - j);
         g1[k_, t_, j_] := (-4*X + 2 + 4*j)
```

we enter the new determinant for $k=2$,

```
In[29]:= Factor[Det[A[2]]]
```

binomial sums where it does not apply. (For the sake of completeness, I mention that there are also several algorithms available to deal with multi-sums, see [34, 190]. These do, however, rather quickly challenge the resources of today's computers.) *Maple* implementations written by Doron Zeilberger are available from `http://www.math.rutgers.edu/~zeilberg`, those written by Frédéric Chyzak are available from `http://algo.inria.fr/chyzak/mgfun.html`, *Mathematica* implementations written by Peter Paule, Axel Riese, Markus Schorn, Kurt Wegschaider are available from `http://www.risc.uni-linz.ac.at/research/combinat/risc/software`.

[17]What I describe in the sequel is, except for very few excursions that ended up in a dead end, and which are therefore omitted here, the way how the determinant evaluation was found.

and, after a waiting time of more than 15 minutes,[18] we obtain

```
Out[29]= -140639960847488232315491052598657851591836968104151763б\

                                                                    2
>      117837623599720038400000000 (-64 + X) (-48 + X) (-40 + X)

                 3           4           5           6   7
>      (-32 + X)   (-24 + X)   (-16 + X)   (-8 + X)   X

                                                                    2
>      (9653078694297600 - 916000657637376 X + 36130368757760 X

                        3                4              5             6
>       - 758218948608 X   + 8928558848 X   - 55938432 X   + 145673 X )
```

Not bad. There are many factors which are linear in X. (This is what we were after.) However, the irreducible polynomial of degree 6 gives us some headache. (The degrees of the irreducible part of the polynomial will grow quickly with k.) How are we going to guess what this factor could be, and, even more daunting, even if we should be able to come up with a guess, how would we go about to prove it?

So, maybe we should modify our choice of how to introduce indeterminates into the matrix. In fact, we overlooked something: maybe, in a hidden manner, the variable X is also there at other places in (3.9), that is, when X is specialised to $N(k) = 4k(4k+1)$ at these places it becomes invisible. More specifically, maybe we should insert the difference $X - 4k(4k+1)$ in the definitions of f_0 and f_1 (which would disappear for $X = 4k(4k+1)$). So, maybe we should try:

$$\begin{aligned}
f_0(j) &= (4k(4k+1) - X + j)(-4)^k,\\
f_1(j) &= -(16k(4k+1) - 4X + 2 + 4j)(-4)^k,\\
g_0(j) &= (X - j),\\
g_1(j) &= -(4X - 4j - 2),
\end{aligned}$$

Okay, let us modify our computer program accordingly,

[18]which I use to explain why our computer needs so long to calculate this determinant of a very sparse matrix of size $16 \cdot 2^2 = 64$: isn't it true that, nowadays, determinants of matrices with several hundreds of rows and columns can be calculated without the slightest difficulty (particularly if they are very sparse)? Well, we should not forget that this is true for determinants of matrices with *numerical* entries. However, our matrix (3.8) with the modified definitions (4.5) of f_0, f_1, g_0, g_1 has now entries which are *polynomials* in X. Hence, when our computer algebra program applies (internally) some elimination algorithm to compute the determinant, huge rational expressions will slowly build up and will slow down the calculations (and, at times, will make our computer crash ...). As I learn from Dave Saunders, *Maple* and *Mathematica* do currently in fact not use the best known algorithms for dealing with determinants of matrices with polynomial entries. (This may have to do with the fact that the developers try to optimise the algorithms for numerical determinants in the first case.) It is known how to avoid the expression swell and compute polynomial matrix determinants in time about mn^3, where n is the dimension of the matrix and m is the bit length of the determinant (roughly, in univariate case, m is degree times maximum coefficient length).

```
In[30]:= f0[k_, t_, j_] := (4*k*(4*k + 1) - X + j)*(-4)^k;
         f1[k_, t_, j_] := -(4*4*k*(4*k + 1) - 4*X + 2 + 4*j)*(-4)^k;
         g0[k_, t_, j_] := (X - j);
         g1[k_, t_, j_] := (-4*X + 2 + 4*j)
```

and let us compute the new determinant for $k = 2$:

```
In[31]:= Factor[Det[A[2]]]
```

This makes us wait for another 15 minutes, after which we are rewarded with:

```
Out[31]= -29677797539762467990136980979441210445413476349407084 1\

>      11553651961247547703174722717904176349374398811662525586 32\

>      6166741975040000000000 (-141 + 2 X) (-139 + 2 X) (-137 + 2 X)

>      (-135 + 2 X) (-133 + 2 X) (-131 + 2 X) (-129 + 2 X)
```

Excellent! There is no big irreducible polynomial anymore. Everything is linear factors in X. But, wait, there is still a problem: in the end (recall Step (S2)!) we will have to compare the degrees of the determinant and of the right-hand side as polynomials in X. If we expand the determinant according to its definition, then the conclusion is that the degree of the determinant is bounded above by $16k^2 - 1$, which, for $k = 2$ is equal to 31. The right-hand side polynomial however which we computed above has degree 7. This is a big gap!

I skip some other things (ending up in dead ends …) that we tried at this point. Altogether they pointed to the fact that, apparently, *one* indeterminate is not sufficient. Perhaps it is a good idea to "diversify" the variable X, that is, to make two variables, X_1 and X_2, out of X:

$$
\begin{aligned}
f_0(j) &= (4k(4k+1) - X_2 + j)(-4)^k,\\
f_1(j) &= -(16k(4k+1) - 4X_1 + 2 + 4j)(-4)^k,\\
g_0(j) &= (X_2 - j),\\
g_1(j) &= -(4X_1 - 4j - 2).
\end{aligned}
$$

We program this,

```
In[32]:= f0[k_, t_, j_] := (4*k*(4*k + 1) - X[2] + j)*(-4)^k;
         f1[k_, t_, j_] := -(4*4*k*(4*k + 1) - 4*X[1] + 2 + 4*j)*(-4)^k;
         g0[k_, t_, j_] := (X[2] - j);
         g1[k_, t_, j_] := (-4*X[1] + 2 + 4*j)
```

and, in order to avoid overstraining our computer, compute this time the new determinant for $k = 1$:

```
In[33]:= Factor[Det[A[1]]]
```

After some minutes there appears

```
Out[33]= 324259173170675712000 0 (-37 + 2 X[1]) (-35 + 2 X[1])

                                                   3                      2
>       (-33 + 2 X[1]) (1 + 2 X[1] - 2 X[2])  (3 + 2 X[1] - 2 X[2])
```

```
>       (5 + 2 X[1] - 2 X[2])
```

on the computer screen. On the positive side: the determinant still factors completely into linear factors, something which we could not expect a priori. Moreover, the degree (in X_1 and X_2) has increased, it is now equal to 9 although we were only computing the determinant for $k = 1$. However, a gap remains, the degree should be $16k^2 - 1 = 15$ if $k = 1$.

Thus, it may be wise to introduce another genuine variable, Y. For example, we may think of simply homogenising the definitions of f_0, f_1, g_0, g_1:

$$\begin{aligned}
f_0(j) &= (4k(4k+1)Y - X_2 + jY)(-4)^k,\\
f_1(j) &= -(16k(4k+1)Y - 4X_1 + (2+4j)Y)(-4)^k,\\
g_0(j) &= (X_2 - jY),\\
g_1(j) &= -(4X_1 - (4j+2)Y).
\end{aligned}$$

We program this,

```
In[34]:= f0[k_, t_, j_] := (4*k*(4*k + 1)*Y - X[2] + j*Y)*(-4)^k;
         f1[k_, t_, j_] := -(4*4*k*(4*k + 1)*Y - 4*X[1] +
                                   (2 + 4*j)*Y)*(-4)^k;
         g0[k_, t_, j_] := (X[2] - j*Y);
         g1[k_, t_, j_] := (-4*X[1] + (2 + 4*j)*Y)
In[35]:= Factor[Det[A[1]]]
```

we wait for some more minutes, and we obtain

```
                                          6
Out[35]= -3242591731706757120000 Y  (33 Y - 2 X[1]) (35 Y - 2 X[1])

                                                    3                        2
>       (37 Y - 2 X[1]) (Y + 2 X[1] - 2 X[2])  (3 Y + 2 X[1] - 2 X[2])

>       (5 Y + 2 X[1] - 2 X[2])
```

Great! The degree in X_1, X_2, Y is 15, as it should be!

At this point, one becomes greedy. The more variables we have, the easier will be the proof. We "diversify" the variables X_1, X_2, Y, that is, we make them $X_{1,t}, X_{2,t}, Y_t$ if they appear in the blocks F_t or G_t, respectively, $t = 1, 2, \dots, 4k$ (cf. (3.8) and the *Mathematica* code for the precise meaning of this definition):

$$\begin{aligned}
f_0(j) &= (4k(4k+1)Y_t - X_{2,t} + jY_t)(-4)^k,\\
f_1(j) &= -(16k(4k+1)Y_t - 4X_{1,t} + (2+4j)Y_t)(-4)^k,\\
g_0(j) &= (X_{2,t} - jY_t),\\
g_1(j) &= -(4X_{1,t} - (4j+2)Y_t).
\end{aligned} \tag{4.6}$$

Now there are so many variables so that there is no way to do the factorisation of the new determinant for $k = 1$ on the computer unless one plays tricks (which we did).[19] But let us pretend that we are able to do it:

[19]See Footnote 18 for the explanation of the complexity problem. "Playing tricks" would mean to compute the determinant for various special choices of the variables $X_{1,t}, X_{2,t}, Y_t$, and then reconstruct

```
In[36]:= f0[k_, t_, j_] := (4*k*(4*k + 1)*Y[t] - X[2, t]
                                     + j*Y[t])*(-4)^k;
         f1[k_, t_, j_] := -(4*4*k (4*k + 1)*Y[t] - 4*X[1, t] +
                                 (2 + 4*j)*Y[t])*(-4)^k;
         g0[k_, t_, j_] := (X[2, t] - j*Y[t]);
         g1[k_, t_, j_] := (-4*X[1, t] + (2 + 4*j)*Y[t])
In[37]:= Factor[Det[A[1]]]
Out[37]= 32425917317067571200000 (2 X[1, 1] - 33 Y[1]) Y[1]
>       (2 X[1, 1] - 2 X[2, 1] + Y[1]) (2 X[1, 2] - 35 Y[2]) Y[2]
>       (2 X[1, 2] - 2 X[2, 2] + Y[2]) (-2 X[2, 2] Y[1] + 2 X[1, 1]
>        Y[2] + 3 Y[1] Y[2]) (2 X[1, 3] - 37 Y[3]) Y[3] (2 X[1, 3] -
>        2 X[2, 3] + Y[3]) (-2 X[2, 3] Y[1] + 2 X[1, 1] Y[3] +
>        5 Y[1] Y[3]) (-2 X[2, 3] Y[2] + 2 X[1, 2] Y[3] + 3 Y[2] Y[3])
```

By staring a little bit at this result (and the one that we computed for $k = 2$), we extracted that, apparently, we have

$$\det A^X = (-1)^{k-1} 4^{2k(4k^2+7k+2)} k^{2k(4k+1)} \prod_{i=1}^{4k} (i+1)_{4k-i+1} \\ \times \prod_{a=1}^{4k-1} \big(2X_{1,a} - (32k^2 + 2a - 1)Y_a\big) \\ \times \prod_{1\le a\le b\le 4k-1} (2X_{2,b}Y_a - 2X_{1,a}Y_b - (2b - 2a + 1)Y_aY_b), \quad (4.7)$$

where A^X denotes the new general matrix given through (3.8) and (4.6), and where, as before, $(\alpha)_k$ is the standard notation for shifted factorials (Pochhammer symbols)

---

the general result by interpolation. This is possible because we know an a priori degree bound (namely 15) for the polynomial. However, this would become infeasible for $k = 3$, for example. "Playing tricks" then would mean to be content with an "almost sure" guess, the latter being based on features of the (unknown) general result that are already visible in the earlier results, and on calculations done for special values of the variables. For example, if we encounter determinants $\det M(k)$, where the $M(k)$'s are some square matrices, $k = 1, 2, \dots$, and the results for $k = 1, 2, \dots, k_0 - 1$ show that $x - y$ must be a factor of $\det M(k)$ to some power, then one would specialise y to some value that would make $x - y$ distinct from any other linear factors containing x, and, supposing that $y = 17$ is such a choice, compute $\det M(k_0)$ with $y = 17$. The exact power of $x - y$ in the unspecialised determinant $\det M(k_0)$ can then be read off from the exponent of $x - 17$ in the specialised one. If it should happen that it is also infeasible to calculate $\det M(k_0)$ with x still unspecialised, then there is still a way out. In that case, one specialises y *and* x, in such a way that $x - y$ would be a prime p that one expects not to occur as a prime factor in any other factor of the determinant $\det M(k_0)$. The exact power of $x - y$ in the unspecialised determinant $\det M(k_0)$ can then be read off from the exponent of p in the prime factorisation of the specialised determinant. See Subsections 5.7 and 5.8, and in particular Footnote 27 for further instances where this trick was applied.

explained in the statement of Theorem 7. The special case that we need in the end to prove our Theorem 8 is $X_{1,t} = X_{2,t} = N(k)$ and $Y_t = 1$.

Now we are in business. Here is the *Sketch of the proof of* (4.7):

Re (S1): For each factor of the (conjectured) result (4.7), we find a linear combination of the rows which vanishes if the factor vanishes. (In other terms: if the indeterminates in the matrix are specialised so that a particular factor vanishes, we find a vector in the kernel of the transpose of the specialised matrix.) For example, to explain the factor $(2X_{1,1} - (32k^2+1)Y_1)$, we found:

If $X_{1,1} = \frac{32k^2+1}{2} Y_1$, then

$$\frac{2(X_{2,4k-1} - (N(k)-1)Y_{4k-1})}{(-4)^{k(4k+1)+1}(16k^2+1)\prod_{\ell=1}^{4k-1}(4\ell k+1)} \cdot (\text{row } 0 \text{ of } A^X)$$
$$+ \sum_{s=0}^{4k} \sum_{t=0}^{4k-2} \left(\frac{(-1)^{s(k-1)}2^t}{4^{sk}} \prod_{\ell=0}^{s-1} \frac{4k-1+4\ell k}{16k^2+1-4\ell k} \prod_{\ell=4k-t}^{4k-1} \frac{2X_{1,\ell} - (32k^2+2\ell-1)Y_\ell}{X_{2,\ell-1} - (16k^2+\ell-1)Y_{\ell-1}} \right)$$
$$\cdot (\text{row } (16k^2 - (4k-1)s - t - 1) \text{ of } A^X) = 0, \quad (4.8)$$

as is easy to verify. (Since the coefficients of the various rows in (4.8) are rational functions in the indeterminates $X_{1,t}, X_{2,t}, Y_t$, they are rather easy to work out from computer data. One does not even need `Rate` ...)

Re (S2): The total degree in the $X_{1,t}$'s, $X_{2,t}$'s, Y_t's of the product on the right-hand side of (4.7) is $16k^2 - 1$. As we already remarked earlier, the degree of the determinant is at most $16k^2 - 1$. Hence, the determinant is equal to the product times, possibly, a constant.

Re (S3): For the evaluation of the constant, we compare coefficients of

$$X_{1,1}^{4k} X_{1,2}^{4k-1} \cdots X_{1,4k-1}^{2} Y_1^1 Y_2^2 \cdots Y_{4k-1}^{4k-1}.$$

After some reflection, it turns out that the constant is equal to a determinant of the same form, that is, of the form (3.8), but with auxiliary functions

$$\begin{aligned} f_0(j) &= (N(k)+j)(-4)^k, \\ f_1(j) &= 4(-4)^k, \\ g_0(j) &= -j, \\ g_1(j) &= -4. \end{aligned} \quad (4.9)$$

What a set-back! It seems that we are in the same situation as at the very beginning. We started with the determinant of the matrix (3.8) with auxiliary functions (3.9), and we ended up with the same type of determinant, with auxiliary functions (4.9). There is little hope though: the functions in (4.9) are somewhat simpler as those in (3.9). Nevertheless, we have to play the same game again; that is, if we want to apply the method of identification of factors, then we have to introduce indeterminates. Skipping the experimental part, we came up with

$$\begin{aligned} f_0(j) &= (Z_t + j)(-4)^k, \\ f_1(j) &= 4(-4)^k X_t, \\ g_0(j) &= -j, \\ g_1(j) &= -4X_t, \end{aligned}$$

where t has the same meaning as before in (4.6). Denoting the new matrix by A^Z, computer calculations suggested that apparently

$$\det A^Z = (-1)^{k-1} 2^{16k^3+20k^2+14k-1} k^{4k} (4k+1)! \prod_{a=1}^{4k-1} \left(X_a^{4k+1-a} \prod_{b=0}^{a-1} (Z_a - 4bk) \right). \tag{4.10}$$

The special case that we need is $Z_t = N(k)$ and $X_t = 1$.

So, we apply again the method of identification of factors. Everything runs smoothly (except that the details of the verification of the factors are somewhat more unpleasant here). When we come finally to the point that we want to determine the constant, it turns out that the constant is equal to — no surprise anymore — the determinant of a matrix of the form (3.8) with auxiliary functions

$$\begin{aligned} f_0(j) &= (-4)^k, \\ f_1(j) &= 4(-4)^k, \\ g_0(j) &= 0, \\ g_1(j) &= -4. \end{aligned}$$

Now, is this good or bad news? In other words, while painfully working through the steps of "identification of factors," will we forever continue producing new determinants of the form (3.8), which we must again handle by the same method? To give it away: this is indeed *very good* news. The function $g_0(j)$ vanishes identically! It makes it possible that now Method 0 (= do some row and column manipulations) works. (See [5] for the details.) We are — finally — done with the proof of (4.7), and, since the right-hand side *does not* vanish for $X_{1,t} = X_{2,t} = N(k)$ and $Y_t = 1$, with the proof of Theorem 8! $\square$

5. More determinant evaluations

This section complements the list of known determinant evaluations given in Section 3 of [109]. I list here several determinant evaluations which I believe are interesting or attractive (and, in the ideal case, both), that have appeared since [109], or that I failed to mention in [109]. I also include several conjectures and open problems, some of them old, some of them new. As in [109], each evaluation is accompanied by some remarks providing information on the context in which it arose. Again, the selection of determinant evaluations presented reflects totally my taste, which must be blamed in the case of any shortcomings. The order of presentation follows loosely the order of presentation of determinants in [109].

5.1. More basic determinant evaluations. I begin with two determinant evaluations belonging to the category "standard determinants" (see Section 2.1 in [109]). They are among those which I missed to state in [109]. The reminder for inclusion here is the paper [9]. There, Amdeberhan and Zeilberger propose an *automated approach* towards determinant evaluations via the condensation method (see "Method 2" in Section 4). They provide a list of examples which can be obtained in that way. As they remark at the end of the paper, all of these are special cases of Lemma 5 in [109], with the exception of three, namely Eqs. (8)–(10) in [9]. In their turn, two of them, namely (8) and (9), are special cases of the following evaluation. (For (10), see Lemma 11 below.)

Lemma 9. *Let $P(Z)$ be a polynomial in Z of degree $n-1$ with leading coefficient L. Then*

$$\det_{1\le i,j\le n}\left(P(X_i+Y_j)\right)=L^n\prod_{i=1}^{n}\binom{n-1}{i}\prod_{1\le i<j\le n}(X_i-X_j)(Y_j-Y_i). \tag{5.1}$$

□

This lemma is easily proved along the lines of the standard proof of the Vandermonde determinant evaluation which we recalled in Section 4 (see the proof of (4.3)) or by condensation.

A multiplicative version of Lemma 9 is the following.

Lemma 10. *Let $P(Z)=p_{n-1}Z^{n-1}+p_{n-2}Z^{n-2}+\dots+p_0$. Then*

$$\det_{1\le i,j\le n}\left(P(X_iY_j)\right)=\prod_{i=0}^{n-1}p_i\prod_{1\le i<j\le n}(X_i-X_j)(Y_i-Y_j). \tag{5.2}$$

□

On the other hand, identity (10) from [9] can be generalised to the following Cauchy-type determinant evaluation. As all the identities from [9], it can also be proved by the condensation method.

Lemma 11. *Let $a_0,a_1,\dots,a_{n-1}$, $c_0,c_1,\dots,c_{n-1}$, b, x and y be indeterminates. Then, for any positive integer n, there holds*

$$\begin{aligned}\det_{0\le i,j\le n-1}&\left(\frac{(x+a_i+c_j)(y+bi+c_j)}{(x+a_i+bi+c_j)}\right)\\&=b^{n-1}\,(n-1)!\left(\binom{n}{2}b+(n-1)x+y+\sum_{i=1}^{n-1}a_i+\sum_{i=0}^{n-1}c_i\right)\\&\times\frac{\displaystyle\prod_{0\le i<j\le n-1}(c_j-c_i)\prod_{i=1}^{n-1}(y-x-a_i)\prod_{1\le i<j\le n-1}((j-i)b-a_i+a_j)}{\displaystyle\prod_{i=1}^{n-1}\prod_{j=0}^{n-1}(x+a_i+bi+c_j)}.\end{aligned} \tag{5.3}$$

□

Speaking of Cauchy-type determinant evaluations, this brings us to a whole family of such evaluations which were instrumental in Kuperberg's recent advance [117] on the enumeration of (symmetry classes of) *alternating sign matrices*. The reason that determinants, and also Pfaffians, play an important role in this context is due to Propp's discovery (described for the first time in [59, Sec. 7] and exploited in [115, 117]) that alternating sign matrices are in bijection with *configurations in the six vertex model*, and due to determinant and Pfaffian formulae due to Izergin [88] and Kuperberg [117] for certain multivariable partition functions of the six vertex model under various boundary conditions. In many cases, this leads to determinants which are, or are similar to, *Cauchy's evaluation of the double alternant* (see [136, vol. III, p. 311] and (5.5) below) or *Schur's Pfaffian version* [166, pp. 226/227] of it (see (5.7) below).

Let me recall that the *Pfaffian* $\operatorname{Pf}(A)$ of a skew-symmetric $(2n)\times(2n)$ matrix A is defined by

$$\operatorname{Pf}(A)=\sum_{\pi}(-1)^{c(\pi)}\prod_{(ij)\in\pi}A_{ij}, \tag{5.4}$$

where the sum is over all perfect matchings π of the complete graph on $2n$ vertices, where $c(\pi)$ is the *crossing number* of π, and where the product is over all edges (ij), $i<j$, in the matching π (see e.g. [179, Sec. 2]). What links Pfaffians so closely to determinants is (aside from similarity of definitions) the fact that the Pfaffian of a skew-symmetric matrix is, up to sign, the square root of its determinant. That is, $\det(A)=\operatorname{Pf}(A)^2$ for any skew-symmetric $(2n)\times(2n)$ matrix A (cf. [179, Prop. 2.2]). See the corresponding remarks and additional references in [109, Sec. 2.8].

The following three theorems present the relevant evaluations. They are Theorems 15–17 from [117]. All of them are proved using identification of factors (see "Method 3" in Section 4). The results in Theorem 12 contain whole sets of indeterminates, whereas the results in Theorems 13 and 14 only have two indeterminates p and q, respectively three indeterminates p, q and r, in them. Identity (5.8) is originally due to Laksov, Lascoux and Thorup [119] and Stembridge [179], independently. The reader must be warned that the statements in [117, Theorems 15–17] are often blurred by typos.

Theorem 12. *Let $x_1,x_2,\dots$ and $y_1,y_2,\dots$ be indeterminates. Then, for any positive integer n, there hold*

$$\det_{1\le i,j\le n}\left(\frac{1}{x_i+y_j}\right)=\frac{\prod\limits_{1\le i<j\le n}(x_i-x_j)(y_i-y_j)}{\prod\limits_{1\le i,j\le n}(x_i+y_j)}, \tag{5.5}$$

$$\det_{1\le i,j\le n}\left(\frac{1}{x_i+y_j}-\frac{1}{1+x_iy_j}\right)=\frac{\prod\limits_{1\le i<j\le n}(1-x_ix_j)(1-y_iy_j)(x_j-x_i)(y_j-y_i)}{\prod\limits_{1\le i,j\le n}(x_i+y_j)(1+x_iy_j)}\times\prod_{i=1}^{n}(1-x_i)(1-y_i), \tag{5.6}$$

$$\operatorname*{Pf}_{1\le i,j\le 2n}\left(\frac{x_i-x_j}{x_i+x_j}\right)=\prod_{1\le i<j\le 2n}\frac{x_i-x_j}{x_i+x_j}. \tag{5.7}$$

$$\operatorname*{Pf}_{1\le i,j\le 2n}\left(\frac{x_i-x_j}{1-x_ix_j}\right)=\prod_{1\le i<j\le 2n}\frac{x_i-x_j}{1-x_ix_j}. \tag{5.8}$$

□

Theorem 13. *Let p and q be indeterminates. Then, for any positive integer n, there hold*

$$\det_{1\le i,j\le n}\left(\frac{q^{n+j-i}-q^{-(n+j-i)}}{p^{n+j-i}-p^{-(n+j-i)}}\right)=\frac{\prod\limits_{1\le i\ne j\le n}(p^{j-i}-p^{-(j-i)})\prod\limits_{1\le i,j\le n}(qp^{j-i}-q^{-1}p^{-(j-i)})}{\prod\limits_{1\le i,j\le n}(p^{n+j-i}-p^{-(n+j-i)})}, \tag{5.9}$$

$$\det_{1\le i,j\le n}\left(\frac{q^{j-i}+q^{-(j-i)}}{p^{j-i}+p^{-(j-i)}}\right)=(-1)^{\binom n2}\frac{2^n\prod\limits_{\substack{1\le i\ne j\le n\\ 2\mid j-i}}(p^{j-i}-p^{-j+i})\prod\limits_{\substack{1\le i,j\le n\\ 2\nmid j-i}}(qp^{j-i}-q^{-1}p^{-j+i})}{\prod\limits_{1\le i,j\le n}(p^{j-i}+p^{-j+i})}, \tag{5.10}$$

$$\det_{1\le i,i\le n}\left(\frac{q^{n+j+i}-q^{-(n+j+i)}}{p^{n+j+i}-p^{-(n+j+i)}}-\frac{q^{n+j-i}-q^{-(n+j-i)}}{p^{n+j-i}-p^{-(n+j-i)}}\right)$$
$$=\frac{\prod\limits_{1\le i<j\le 2n}(p^{j-i}-p^{-(j-i)})\prod\limits_{\substack{1\le i,j\le 2n+1\\ 2\mid j}}(qp^{j-i}-q^{-1}p^{-(j-i)})}{\prod\limits_{1\le i,j\le n}(p^{n+j-i}-p^{-(n+j-i)})(p^{n+j+i}-p^{-(n+j+i)})}, \tag{5.11}$$

$$\det_{1\le i,j\le n}\left(\frac{q^{j+i}+q^{-(j+i)}}{p^{j+i}+p^{-(j+i)}}-\frac{q^{j-i}+q^{-j+i}}{p^{j-i}+p^{-(j-i)}}\right)$$
$$=(-1)^{\binom n2}\frac{2^n\prod\limits_{1\le i<j\le n}(p^{2(j-i)}-p^{-2(j-i)})^2\prod\limits_{\substack{1\le i,j\le 2n+1\\ 2\nmid i,\,2\mid j}}(qp^{j-i}-q^{-1}p^{-(j-i)})}{\prod\limits_{1\le i,j\le n}(p^{j-i}+p^{-(j-i)})(p^{j+i}+p^{-(j+i)})}. \tag{5.12}$$

□

Theorem 14. *Let p, q, and r be indeterminates. Then, for any positive integer n, there hold*

$$\underset{1\le i,j\le 2n}{\operatorname{Pf}}\left(\frac{(q^{j-i}-q^{-(j-i)})(r^{j-i}-r^{-(j-i)})}{(p^{j-i}-p^{-(j-i)})}\right)\\
=\frac{\prod\limits_{1\le i<j\le n}(p^{j-i}-p^{-(j-i)})^2\prod\limits_{1\le i,j\le n}(qp^{j-i}-q^{-1}p^{-(j-i)})(rp^{j-i}-r^{-1}p^{-(j-i)})}{\prod\limits_{1\le i,j\le n}(p^{n+j-i}-p^{-(n+j-i)})} \tag{5.13}$$

$$\underset{1\le i,j\le 2n}{\operatorname{Pf}}\Bigg((p^{j+i}-p^{-(j+i)})(p^{j-i}-p^{-(j-i)})\left(\frac{q^{j+i}-q^{-(j+i)}}{p^{j+i}-p^{-(j+i)}}-\frac{q^{j-i}-q^{-(j-i)}}{p^{j-i}-p^{-(j-i)}}\right)\\
\cdot\left(\frac{r^{j+i}-r^{-(j+i)}}{p^{j+i}-p^{-(j+i)}}-\frac{r^{j-i}-r^{-(j-i)}}{p^{j-i}-p^{-(j-i)}}\right)\Bigg)\\
=\frac{\prod\limits_{1\le i<j\le 2n}(p^{j-i}-p^{-(j-i)})\prod\limits_{\substack{1\le i,j\le 2n+1\\ 2|j}}(qp^{j-i}-q^{-1}p^{-(j-i)})(rp^{j-i}-r^{-1}p^{-(j-i)})}{\prod\limits_{1\le i<j\le 2n}(p^{j+i}-p^{-(j+i)})}. \tag{5.14}$$

□

Subsequent to Kuperberg's work, Okada [139] related Kuperberg's determinants and Pfaffians to characters of classical groups, by coming up with rather complex, but still beautiful determinant identities. In particular, this allowed him to settle one more of the conjectured enumeration formulae on symmetry classes of alternating sign matrices. Generalising even further, Ishikawa, Okada, Tagawa and Zeng [85] have found more such determinant identities. Putting them into the framework of certain special representations of the symmetric group, Lascoux [122] has clarified the mechanism which gives rise to these identities.

The next six determinant lemmas are corollaries of *elliptic determinant evaluations* due to Rosengren and Schlosser [156]. (The latter will be addressed later in Subsection 5.11.) They partly extend the fundamental determinant lemmas in [109, Sec. 2.2]. For the statements of the lemmas, we need the notion of a *norm* of a polynomial $a_0+a_1z+\cdots+a_kz^k$, which we define to be the reciprocal of the product of its roots, or, more explicitly, as $(-1)^k a_k/a_0$.

If we specialise $p=0$ in Lemma 68, (5.122), then we obtain a determinant identity which generalises at the same time the Vandermonde determinant evaluation, Lemma 9 and Lemma 10.

Lemma 15. *Let $P_1,P_2,\dots,P_n$ be polynomials of degree n and norm t, given by*

$$P_j(x)=(-1)^n t a_{j,0}x^n+\sum_{k=0}^{n-1}a_{j,k}x^k.$$

Then

$$\det_{1\le i,j\le n}\left(P_j(x_i)\right)=(1-tx_1\cdots x_n)\left(\prod_{1\le i<j\le n}(x_j-x_i)\right)\det_{\substack{1\le i\le n\\0\le j\le n-1}}(a_{i,j}). \tag{5.15}$$

□

Further determinant identities which generalise other *Weyl denominator formulae* (cf. [109, Lemma 2]) could be obtained from the special case $p=0$ of the other determinant evaluations in Lemma 68.

A generalisation of Lemma 6 from [109] in the same spirit can be obtained by setting $p=0$ in Theorem 75. It is given as Corollary 5.1 in [156].

Lemma 16. *Let $x_1,\dots,x_n$, $a_1,\dots,a_n$, and t be indeterminates. For each $j=1,\dots,n$, let P_j be a polynomial of degree j and norm $ta_1\cdots a_j$. Then there holds*

$$\det_{1\le i,j\le n}\left(P_j(x_i)\prod_{k=j+1}^{n}(1-a_kx_i)\right)\\ =\frac{1-ta_1\cdots a_nx_1\cdots x_n}{1-t}\prod_{i=1}^{n}P_i(1/a_i)\prod_{1\le i<j\le n}a_j(x_j-x_i). \tag{5.16}$$

□

We continue with a consequence of Theorem 76 (see Corollary 5.3 in [156]). The special case $P_{j-1}(x)=1$, $j=1,\dots,n$, is Lemma A.1 of [162], which was needed in order to obtain an A_n *matrix inversion* that played a crucial role in the derivation of *multiple basic hypergeometric series identities.* A slight generalisation was given in [164, Lemma A.1].

Lemma 17. *Let $x_1,\dots,x_n$ and b be indeterminates. For each $j=1,\dots,n$, let $P_{j-1}(x)$ be a polynomial in x of degree at most $j-1$ with constant term 1, and let $Q(x)=(1-y_1x)\cdots(1-y_{n+1}x)$. Then there holds*

$$Q(b)\det_{1\le i,j\le n}\left(x_i^{n+1-j}P_{j-1}(x_i)-b^{n+1-j}P_{j-1}(b)\frac{Q(x_i)}{Q(b)}\right)\\ =(1-bx_1\cdots x_ny_1\cdots y_{n+1})\prod_{i=1}^{n}(x_i-b)\prod_{1\le i<j\le n}(x_i-x_j). \tag{5.17}$$

□

Pairing the (i,j)-entry in the determinant in (5.16) with itself, but with x_i replaced by $1/x_i$, one can construct another determinant which evaluates in closed form. The result given below is Corollary 5.5 in [156]. It is the special case $p=0$ of Theorem 77.

Lemma 18. *Let $x_1,\dots,x_n$, $a_1,\dots,a_n$, and $c_1,\dots,c_{n+2}$ be indeterminates. For each $j=1,\dots,n$, let P_j be a polynomial of degree j with norm $(c_1\cdots c_{n+2}a_{j+1}\cdots a_n)^{-1}$.*

Then there holds

$$\det_{1\le i,j\le n}\Bigg(x_i^{-n-1}\prod_{k=1}^{n+2}(1-c_kx_i)\,P_j(x_i)\prod_{k=j+1}^{n}(1-a_kx_i)\\
-x_i^{n+1}\prod_{k=1}^{n+2}(1-c_kx_i^{-1})\,P_j(x_i^{-1})\prod_{k=j+1}^{n}(1-a_kx_i^{-1})\Bigg)\\
=\frac{a_1\cdots a_n}{x_1\cdots x_n\,(1-c_1\cdots c_{n+2}a_1\cdots a_n)}\prod_{i=1}^{n}P_i(1/a_i)\\
\times\prod_{1\le i<j\le n+2}(1-c_ic_j)\prod_{i=1}^{n}(1-x_i^2)\prod_{1\le i<j\le n}a_j(x_i-x_j)(1-1/x_ix_j). \quad (5.18)$$

□

It is worthwhile to state the limit case $c_{n+2}\to\infty$ of this lemma separately, in which case the norm constraint on the polynomials P_j drops out, but, in return, the degree of P_j gets lowered by one (see [156, Cor. 5.8]).

Lemma 19. *Let $x_1,\dots,x_n$, $a_2,\dots,a_n$, and $c_1,\dots,c_{n+1}$ be indeterminates. For each $j=1,\dots,n$, let P_{j-1} be a polynomial of degree at most $j-1$. Then there holds*

$$\det_{1\le i,j\le n}\Bigg(x_i^{-n}\prod_{k=1}^{n+1}(1-c_kx_i)\,P_{j-1}(x_i)\prod_{k=j+1}^{n}(1-a_kx_i)\\
-x_i^{n}\prod_{k=1}^{n+1}(1-c_kx_i^{-1})\,P_{j-1}(x_i^{-1})\prod_{k=j+1}^{n}(1-a_kx_i^{-1})\Bigg)\\
=\prod_{i=1}^{n}P_{i-1}(1/a_i)\prod_{1\le i<j\le n+1}(1-c_ic_j)\\
\times\prod_{i=1}^{n}x_i^{-1}(1-x_i^2)\prod_{1\le i<j\le n}a_j(x_i-x_j)(1-1/x_ix_j). \quad (5.19)$$

□

Dividing both sides of (5.19) by $\prod_{i=2}^{n}a_i^{i-1}$ and then letting a_i tend to ∞, $i=2,3,\dots,n$, we arrive at the determinant evaluation below (see [156, Cor. 5.11]). Its special case $P_{j-1}(x)=1$, $j=1,\dots,n$, is Lemma A.11 of [162], needed there in order to obtain a C_n *matrix inversion*, which was later applied in [163] to derive *multiple q-Abel and q-Rothe summations*.

Lemma 20. *Let $x_1, \dots, x_n$, and $c_1, \dots, c_{n+1}$ be indeterminates. For each $j = 1, \dots, n$, let P_{j-1} be a polynomial of degree at most $j-1$. Then there holds*

$$\det_{1\le i,j\le n}\left(x_i^{-j}\prod_{k=1}^{n+1}(1-c_kx_i)\,P_{j-1}(x_i) - x_i^{j}\prod_{k=1}^{n+1}(1-c_kx_i^{-1})\,P_{j-1}(x_i^{-1})\right)$$
$$= \prod_{i=1}^{n} P_{i-1}(0) \prod_{1\le i<j\le n+1}(1-c_ic_j) \prod_{i=1}^{n} x_i^{-1}(1-x_i^2) \prod_{1\le i<j\le n}(x_j-x_i)(1-1/x_ix_j). \tag{5.20}$$

□

It is an attractive feature of this determinant identity that it contains, at the same time, the *Weyl denominator formulae* for the classical root systems B_n, C_n and D_n as special cases (cf. [109, Lemma 2]). This is seen by setting $P_j(x) = 1$ for all j, $c_1 = c_2 = \dots = c_{n-1} = 0$, and then $c_n = 0$, $c_{n+1} = -1$ for the type B_n case, $c_n = c_{n+1} = 0$ for the type C_n case, and $c_n = 1$, $c_{n+1} = -1$ for the type D_n case, respectively.

A determinant which is of completely different type, but which also belongs to the category of basic determinant evaluations, is the determinant of a matrix where only two (circular) diagonals are filled with non-zero elements. It was applied with advantage in [80] to evaluate *Scott-type permanents.*

Lemma 21. *Let n and r be positive integers, $r \le n$, and $x_1, x_2, \dots, x_n$, $y_1, y_2, \dots, y_n$ be indeterminates. Then, with $d = \gcd(r, n)$, we have*

$$\det\begin{pmatrix}
x_1 & 0 & \dots & 0 & y_{n-r+1} & 0 & & \\
0 & x_2 & 0 & & 0 & y_{n-r+2} & 0 & \\
 & & \ddots & & & & \ddots & 0\\
0 & & & & & & 0 & y_n\\
y_1 & 0 & & & & & & \\
0 & y_2 & 0 & & & & & \\
 & 0 & \ddots & 0 & & & \ddots & 0\\
 & & 0 & y_{n-r} & 0 & & 0 & x_n
\end{pmatrix}$$
$$= \prod_{i=1}^{d}\left(\prod_{j=1}^{n/d} x_{i+(j-1)d} - (-1)^{n/d}\prod_{j=1}^{n/d} y_{i+(j-1)d}\right). \tag{5.21}$$

(*I.e., in the matrix there are only nonzero entries along two diagonals, one of which is a broken diagonal.*) □

A further basic determinant evaluation which I missed to state in [109] is the evaluation of the determinant of a *skew circulant matrix* attributed to Scott [167] in [136, p. 356]. It was in fact recently used by Fulmek in [70] to find a closed form formula for the number of *non-intersecting lattice paths with equally spaced starting and end points living on a cylinder*, improving on earlier results by Forrester [67] on the *vicious walker model* in *statistical mechanics*, see [70, Lemma 9].

Theorem 22. *Let n by a fixed positive integer, and let $a_0, a_1, \dots, a_{n-1}$ be indeterminates. Then*

$$\det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ -a_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ -a_{n-2} & -a_{n-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \hdotsfor{6} \\ -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & a_0 \end{pmatrix}$$
$$= \prod_{i=0}^{n-1} (a_0 + \omega^{2i+1} a_1 + \omega^{2(2i+1)} a_2 + \dots + \omega^{(n-1)(2i+1)} a_{n-1}), \tag{5.22}$$

where ω is a primitive $(2n)$-th root of unity. □

5.2. More confluent determinants. Here I continue the discussion from the beginning of Section 3 in [109, Theorems 20–24]. There I presented determinant evaluations of matrices which, essentially, consist of several vertical strips, each of which is formed by taking a certain column vector and gluing it together with its derivative, its second derivative, etc., respectively by a similar construction where the derivative is replaced by a difference or q-difference operator.

Since most of this subsection will be under the influence of the so-called *"q-disease"*,[20] we shall need the standard q-notations $(a;q)_k$, denoting the *q-shifted factorial* and being given by $(a;q)_0 := 1$ and

$$(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1})$$

if k is a positive integer, as well as $\begin{bmatrix}\alpha\\k\end{bmatrix}_q$, denoting the *q-binomial coefficient* and being defined by $\begin{bmatrix}\alpha\\k\end{bmatrix}_q = 0$ if $k<0$, $\begin{bmatrix}\alpha\\0\end{bmatrix}_q = 1$, and

$$\begin{bmatrix}\alpha\\k\end{bmatrix}_q := \frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

if k is a positive integer. Clearly we have $\lim_{q\to 1} \begin{bmatrix}\alpha\\k\end{bmatrix}_q = \binom{\alpha}{k}$.

[20]The distinctive symptom of this disease is to invariably raise the question "Is there also a q-analogue?" My epidemiological research on MathSciNet revealed that, while basically non-existent during the 1970s, this disease slowly spread during the 1980s, and then had a sharp increase around 1990, when it jumped from about 20 papers per year published with the word "q-analog∗" in it to over 80 in 1995, and since then it has been roughly stable at 60–70 papers per year. In its simplest form, somebody who is infected by this disease takes a combinatorial identity, and replaces every occurrence of a positive integer n by its *"q-analogue"* $1+q+q^2+\dots+q^{n-1}$, inserts some powers of q at the appropriate places, and hopes that the result of these manipulations would be again an identity, thus constituting a "q-analogue" of the original equation. I refer the reader to the bible [72] for a rich source of q-identities, and for the right way to look at (most) combinatorial q-identities. In another form, given a certain set of objects of which one knows the exact number, one defines a *statistics* stat on these objects and now tries to evaluate $\sum_{O \text{ an object}} q^{\mathrm{stat}(O)}$. For a very instructive text following these lines see [63], with emphasis on the objects being permutations. There is also an important third form of the disease in which one works in the ring of polynomials in variables $x, y, \dots$ with coefficients being rational functions in q, and in which some pairs of variables satisfy commutation relations of the type $xy = qxy$. The study of such polynomial rings and algebras is often motivated by *quantum groups* and *quantum algebras*. The reader may want to consult [104] to learn more about this direction. While my description did not make this clear, the three described forms of the q-disease are indeed strongly inter-related.

The first result that I present is a q-extension of the evaluation of the *confluent alternant* due to Schendel [161] (cf. [109, paragraph before Theorem 20]). In fact, Theorem 23 of [109] already provided a q-extension of (a generalisation of) Schendel's formula. However, in [93, Theorem 1], Johnson found a different q-extension. The theorem below is a slight generalisation of it. (The theorem below reduces to Johnson's theorem if one puts $C = 0$. For $q = 1$, the theorem below and [109, Theorem 23] become equivalent. To go from one determinant to the other in this special case, one would have to take a certain factor out of each column.)

Theorem 23. *Let n be a non-negative integer, and let $A_m(X)$ denote the $n\times m$ matrix*

$$\left(\begin{bmatrix} C+i\\ i-j\end{bmatrix}_q (X;q)_{i-j}\right)_{0\le i\le n-1,\,0\le j\le m-1}.$$

Given a composition of n, $n = m_1 + \dots + m_\ell$, there holds

$$\det_{n\times n}\left(A_{m_1}(X_1)\,A_{m_2}(X_2)\dots A_{m_\ell}(X_\ell)\right)$$
$$= q^{\sum_{1\le i<j<k\le\ell} m_i m_j m_k} \prod_{1\le i<j\le\ell}\prod_{g=1}^{m_i}\prod_{h=1}^{m_j}\frac{(q^{h-1}X_i - q^{g-1}X_j)(1-q^{C+g+h-1+\sum_{r=1}^{i-1}m_r})}{(1-q^{g+h-1+\sum_{r=1}^{i-1}m_r})}. \qquad (5.23)$$

□

In [93, Theorem 2], Johnson provides as well a confluent q-extension of the evaluation of Cauchy's double alternant (5.5). Already the case $q = 1$ seems to not have appeared in the literature earlier. Here, I was not able to introduce an additional parameter (as, for example, the C in Theorem 23).

Theorem 24. *Let n be a non-negative integer, and let $A_m(X)$ denote the $n\times m$ matrix*

$$\left(\frac{1}{(Y_i-X)(Y_i-qX)(Y_i-q^2X)\cdots(Y_i-q^{j-1}X)}\right)_{1\le i\le n,\,1\le j\le m}.$$

Given a composition of n, $n = m_1 + \dots + m_\ell$, there holds

$$\det_{n\times n}\left(A_{m_1}(X_1)\,A_{m_2}(X_2)\dots A_{m_\ell}(X_\ell)\right)$$
$$= \frac{\left(\prod_{1\le i<j\le n}(Y_i-Y_j)\right)\left(\prod_{1\le i<j\le\ell}\prod_{g=1}^{m_i}\prod_{h=1}^{m_j}(q^{h-1}X_j - q^{g-1}X_i)\right)}{\prod_{i=1}^{n}\prod_{j=1}^{\ell}(Y_i-X_j)(Y_i-qX_j)(Y_i-q^2X_j)\cdots(Y_i-q^{m_j-1}X_j)}. \qquad (5.24)$$

□

A surprising mixture between the confluent alternant and a confluent double alternant appears in [39, Theorem A.1]. Ciucu used it there in order to prove a *Coulomb gas law* and a *superposition principle* for the joint correlation of certain collections of holes for the *rhombus tiling model on the triangular lattice*. (His main result is in fact based on an even more general, and more complex, determinant evaluation, see [39, Theorem 8.1].)

Theorem 25. *Let $s_1, s_2, \dots, s_m \ge 1$ and $t_1, t_2, \dots, t_n \ge 1$ be integers. Write $S = \sum_{i=1}^m s_i$, $T = \sum_{j=1}^n t_j$, and assume $S \ge T$. Let $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_n$ be indeterminates. Define N to be the $S \times S$ matrix*

$$N = \begin{bmatrix} A & B \end{bmatrix} \tag{5.25}$$

whose blocks are given by

$$A = \left(\begin{array}{cccc c cccc}
\frac{\binom{0}{0}}{y_1-x_1} & \frac{\binom{1}{0}}{(y_1-x_1)^2} & \cdots & \frac{\binom{t_1-1}{0}}{(y_1-x_1)^{t_1}} & & \frac{\binom{0}{0}}{y_n-x_1} & \frac{\binom{1}{0}}{(y_n-x_1)^2} & \cdots & \frac{\binom{t_n-1}{0}}{(y_n-x_1)^{t_n}} \\
\frac{\binom{1}{1}}{(y_1-x_1)^2} & \frac{\binom{2}{1}}{(y_1-x_1)^3} & \cdots & \frac{\binom{t_1}{1}}{(y_1-x_1)^{t_1+1}} & \cdots & \frac{\binom{1}{1}}{(y_n-x_1)^2} & \frac{\binom{2}{1}}{(y_n-x_1)^3} & \cdots & \frac{\binom{t_n}{1}}{(y_n-x_1)^{t_n+1}} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\binom{s_1-1}{s_1-1}}{(y_1-x_1)^{s_1}} & \frac{\binom{s_1}{s_1-1}}{(y_1-x_1)^{s_1+1}} & \cdots & \frac{\binom{s_1+t_1-2}{s_1-1}}{(y_1-x_1)^{s_1+t_1-1}} & & \frac{\binom{s_1-1}{s_1-1}}{(y_n-x_1)^{s_1}} & \frac{\binom{s_1}{s_1-1}}{(y_n-x_1)^{s_1+1}} & \cdots & \frac{\binom{s_1+t_n-2}{s_1-1}}{(y_n-x_1)^{s_1+t_n-1}} \\
 & & \vdots & & & & & \vdots & \\
\frac{\binom{0}{0}}{y_1-x_m} & \frac{\binom{1}{0}}{(y_1-x_m)^2} & \cdots & \frac{\binom{t_1-1}{0}}{(y_1-x_m)^{t_1}} & & \frac{\binom{0}{0}}{y_n-x_m} & \frac{\binom{1}{0}}{(y_n-x_m)^2} & \cdots & \frac{\binom{t_n-1}{0}}{(y_n-x_m)^{t_n}} \\
\frac{\binom{1}{1}}{(y_1-x_m)^2} & \frac{\binom{2}{1}}{(y_1-x_m)^3} & \cdots & \frac{\binom{t_1}{1}}{(y_1-x_m)^{t_1+1}} & \cdots & \frac{\binom{1}{1}}{(y_n-x_m)^2} & \frac{\binom{2}{1}}{(y_n-x_m)^3} & \cdots & \frac{\binom{t_n}{1}}{(y_n-x_m)^{t_n+1}} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\binom{s_m-1}{s_m-1}}{(y_1-x_m)^{s_m}} & \frac{\binom{s_m}{s_m-1}}{(y_1-x_m)^{s_m+1}} & \cdots & \frac{\binom{s_m+t_1-2}{s_m-1}}{(y_1-x_m)^{s_m+t_1-1}} & & \frac{\binom{s_m-1}{s_m-1}}{(y_n-x_m)^{s_m}} & \frac{\binom{s_m}{s_m-1}}{(y_n-x_m)^{s_m+1}} & \cdots & \frac{\binom{s_m+t_n-2}{s_m-1}}{(y_n-x_m)^{s_m+t_n-1}}
\end{array}\right) \tag{5.26}$$

and

$$B = \left(\begin{array}{cccc}
\binom{0}{0}x_1^0 & \binom{1}{0}x_1 & \cdots & \binom{S-T-1}{0}x_1^{S-T-1} \\
\binom{0}{1}x_1^{-1} & \binom{1}{1}x_1^0 & \cdots & \binom{S-T-1}{1}x_1^{S-T-2} \\
\vdots & \vdots & & \vdots \\
\binom{0}{s_1-1}x_1^{1-s_1} & \binom{1}{s_1-1}x_1^{2-s_1} & \cdots & \binom{S-T-1}{s_1-1}x_1^{S-T-s_1} \\
 & & \vdots & \\
\binom{0}{0}x_m^0 & \binom{1}{0}x_m & \cdots & \binom{S-T-1}{0}x_m^{S-T-1} \\
\binom{0}{1}x_m^{-1} & \binom{1}{1}x_m^0 & \cdots & \binom{S-T-1}{1}x_m^{S-T-2} \\
\vdots & \vdots & & \vdots \\
\binom{0}{s_m-1}x_m^{1-s_m} & \binom{1}{s_m-1}x_m^{2-s_m} & \cdots & \binom{S-T-1}{s_m-1}x_m^{S-T-s_m}
\end{array}\right). \tag{5.27}$$

Then we have

$$\det N = \frac{\prod_{1\le i<j\le m}(x_j - x_i)^{s_is_j} \prod_{1\le i<j\le n}(y_i - y_j)^{t_it_j}}{\prod_{i=1}^m \prod_{j=1}^n (y_j - x_i)^{s_it_j}}. \tag{5.28}$$

□

This theorem generalises at the same time numerous previously obtained determinant evaluations. It reduces of course to Cauchy's double alternant when $m = n$ and $s_1 = s_2 = \cdots = s_m = t_1 = t_2 = \cdots = t_n = 1$. (In that case, the submatrix B is empty.) It reduces to the confluent alternant for $t_1 = t_2 = \cdots = t_n = 0$ (i.e., in the case where the submatrix A is empty). The case $m = rn$, $s_1 = s_2 = \cdots = s_m = 1$, $t_1 = t_2 = \cdots = t_n = r$ is stated as an exercise in [137, Ex. 42, p. 360]. Finally, a mixture of the double alternant and the Vandermonde determinant appeared already in [80, Theorem (Cauchy+)] where it was used to evaluate *Scott-type permanents*. This mixture turns out to be the special case $s_1 = s_2 = \cdots = s_m = t_1 = t_2 = \cdots = t_n = 1$ (but not necessarily $m = n$) of Theorem 25.

If $S = T$ (i.e., in the case where the submatrix B is empty), Theorem 25 provides the evaluation of a confluent double alternant which is different from the one in Theorem 24 for $C = 0$ and $q = 1$. While, for the general form of Theorem 25, I was not able to find a q-analogue, I was able to find one for this special case, that is, for the case where B is empty. In view of the fact that there are also q-analogues for the other extreme case where the submatrix A is empty (namely Theorem 23 and [109, Theorem 23]), I still suspect that a q-analogue of the general form of Theorem 25 should exist.

Theorem 26. *Let $s_1, s_2, \dots, s_m \ge 1$ and $t_1, t_2, \dots, t_n \ge 1$ be integers such that $s_1 + s_2 + \cdots + s_m = t_1 + t_2 + \cdots + t_n$. Let $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_n$ be indeterminates. Let A be the matrix*

$$A = \begin{pmatrix}
\frac{\begin{bmatrix}0\\0\end{bmatrix}_q}{\langle x_1,y_1\rangle^{1}} & \frac{\begin{bmatrix}1\\0\end{bmatrix}_q}{\langle x_1,y_1\rangle^{2}} & \cdots & \frac{\begin{bmatrix}t_1-1\\0\end{bmatrix}_q}{\langle x_1,y_1\rangle^{t_1}} & & \frac{\begin{bmatrix}0\\0\end{bmatrix}_q}{\langle x_1,y_n\rangle^{1}} & \frac{\begin{bmatrix}1\\0\end{bmatrix}_q}{\langle x_1,y_n\rangle^{2}} & \cdots & \frac{\begin{bmatrix}t_n-1\\0\end{bmatrix}_q}{\langle x_1,y_n\rangle^{t_n}} \\
\frac{\begin{bmatrix}1\\1\end{bmatrix}_q}{\langle x_1,y_1\rangle^{2}} & \frac{\begin{bmatrix}2\\1\end{bmatrix}_q}{\langle x_1,y_1\rangle^{3}} & \cdots & \frac{\begin{bmatrix}t_1\\1\end{bmatrix}_q}{\langle x_1,y_1\rangle^{t_1+1}} & \cdots & \frac{\begin{bmatrix}1\\1\end{bmatrix}_q}{\langle x_1,y_n\rangle^{2}} & \frac{\begin{bmatrix}2\\1\end{bmatrix}_q}{\langle x_1,y_n\rangle^{3}} & \cdots & \frac{\begin{bmatrix}t_n\\1\end{bmatrix}_q}{\langle x_1,y_n\rangle^{t_n+1}} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\begin{bmatrix}s_1-1\\s_1-1\end{bmatrix}_q}{\langle x_1,y_1\rangle^{s_1}} & \frac{\begin{bmatrix}s_1\\s_1-1\end{bmatrix}_q}{\langle x_1,y_1\rangle^{s_1+1}} & \cdots & \frac{\begin{bmatrix}s_1+t_1-2\\s_1-1\end{bmatrix}_q}{\langle x_1,y_1\rangle^{s_1+t_1-1}} & & \frac{\begin{bmatrix}s_1-1\\s_1-1\end{bmatrix}_q}{\langle x_1,y_n\rangle^{s_1}} & \frac{\begin{bmatrix}s_1\\s_1-1\end{bmatrix}_q}{\langle x_1,y_n\rangle^{s_1+1}} & \cdots & \frac{\begin{bmatrix}s_1+t_n-2\\s_1-1\end{bmatrix}_q}{\langle x_1,y_n\rangle^{s_1+t_n-1}} \\
 & & \vdots & & & & & \vdots & \\
\frac{\begin{bmatrix}0\\0\end{bmatrix}_q}{\langle x_m,y_1\rangle^{1}} & \frac{\begin{bmatrix}1\\0\end{bmatrix}_q}{\langle x_m,y_1\rangle^{2}} & \cdots & \frac{\begin{bmatrix}t_1-1\\0\end{bmatrix}_q}{\langle x_m,y_1\rangle^{t_1}} & & \frac{\begin{bmatrix}0\\0\end{bmatrix}_q}{\langle x_m,y_n\rangle^{1}} & \frac{\begin{bmatrix}1\\0\end{bmatrix}_q}{\langle x_m,y_n\rangle^{2}} & \cdots & \frac{\begin{bmatrix}t_n-1\\0\end{bmatrix}_q}{\langle x_m,y_n\rangle^{t_n}} \\
\frac{\begin{bmatrix}1\\1\end{bmatrix}_q}{\langle x_m,y_1\rangle^{2}} & \frac{\begin{bmatrix}2\\1\end{bmatrix}_q}{\langle x_m,y_1\rangle^{3}} & \cdots & \frac{\begin{bmatrix}t_1\\1\end{bmatrix}_q}{\langle x_m,y_1\rangle^{t_1+1}} & \cdots & \frac{\begin{bmatrix}1\\1\end{bmatrix}_q}{\langle x_m,y_n\rangle^{2}} & \frac{\begin{bmatrix}2\\1\end{bmatrix}_q}{\langle x_m,y_n\rangle^{3}} & \cdots & \frac{\begin{bmatrix}t_n\\1\end{bmatrix}_q}{\langle x_m,y_n\rangle^{t_n+1}} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\begin{bmatrix}s_m-1\\s_m-1\end{bmatrix}_q}{\langle x_m,y_1\rangle^{s_m}} & \frac{\begin{bmatrix}s_m\\s_m-1\end{bmatrix}_q}{\langle x_m,y_1\rangle^{s_m+1}} & \cdots & \frac{\begin{bmatrix}s_m+t_1-2\\s_m-1\end{bmatrix}_q}{\langle x_m,y_1\rangle^{s_m+t_1-1}} & & \frac{\begin{bmatrix}s_m-1\\s_m-1\end{bmatrix}_q}{\langle x_m,y_n\rangle^{s_m}} & \frac{\begin{bmatrix}s_m\\s_m-1\end{bmatrix}_q}{\langle x_m,y_n\rangle^{s_m+1}} & \cdots & \frac{\begin{bmatrix}s_m+t_n-2\\s_m-1\end{bmatrix}_q}{\langle x_m,y_n\rangle^{s_m+t_n-1}}
\end{pmatrix}, \tag{5.29}$$

where $\langle x, y\rangle^e := (y - x)(qy - x)(q^2y - x)\cdots(q^{e-1}y - x)$. Then we have

$$\det A = q^{\frac{1}{6}\sum_{i=1}^{m}(s_i-1)s_i(2s_i-1)} \left(\prod_{1\le i<j\le m} \prod_{g=1}^{s_i} \prod_{h=1}^{s_j} (q^{g-1}x_j - q^{h-1}x_i) \right)$$
$$\times \left(\prod_{1\le i<j\le n} \prod_{g=1}^{t_i} \prod_{h=1}^{t_j} (q^{g-1}y_i - q^{h-1}y_j) \right) \left(\prod_{j=1}^{m} \prod_{i=1}^{n} \prod_{g=1}^{t_i} \prod_{h=1}^{s_j} \frac{1}{(q^{g+h-2}y_i - x_j)} \right). \quad (5.30)$$

□

5.3. More determinants containing derivatives and compositions of series. Inspired by formulae of Mina [135], Kedlaya [99] and Strehl and Wilf [180] for determinants of matrices the entries of which being given by (coefficients of) *multiple derivatives* and *compositions of formal power series* (see also [109, Lemma 16]), Chu embedded all these in a larger context in the remarkable systematic study [33]. He shows that, at the heart of these formulae, there is the *Faà di Bruno formula*[21] for multiple derivatives of a composition of two formal power series. Using it, he derives the following determinant reduction formulae [33, Theorems 4.1 and 4.2] for determinants of matrices containing multiple derivatives of compositions of formal power series.

Theorem 27. *Let $f(x)$ and $\phi_k(x)$ and $w_k(x)$, $k = 0, 1, \dots, n$, be formal power series in x with coefficients in a commutative ring. Then*

$$\det_{0\le i,j,\le n} \left(\frac{d^j}{dx^j} \Big(w_j(x)\phi_i\big(f(x)\big) \Big) \right) = \big(f'(x)\big)^{\binom{n+1}{2}} \left(\prod_{k=0}^{n} w_k(x) \right) \det_{0\le i,j,\le n} \left(\phi_i^{(j)}\big(f(x)\big) \right), \quad (5.31)$$

where $\phi^{(j)}(x)$ is short for $\frac{d^j}{dx^j}\phi(x)$. If, in addition, $w_k(x)$ is a polynomial of degree at most k, $k = 1, 2, \dots, n$, then

$$\det_{1\le i,j,\le n} \left(\frac{d^j}{dx^j} \Big(w_j(x)\phi_i\big(f(x)\big) \Big) \right) = \big(f'(x)\big)^{\binom{n+1}{2}} \left(\prod_{k=1}^{n} w_k(x) \right) \det_{1\le i,j,\le n} \left(\phi_i^{(j)}\big(f(x)\big) \right). \quad (5.32)$$

□

Specialising the series $\phi_i(x)$ so that the determinants on the right-hand sides of (5.31) or (5.32) can be evaluated, he obtains numerous nice corollaries. Possible choices are $\phi_i(x) = \exp(y_i x)$, $\phi_i(x) = \log(1 + y_i x)$, $\phi_i(x) = x^{y_i}$, or $\phi_i(x) = (a_i + b_i x)/(c_i + d_i x)$. See [33, Cor. 4.3 and 4.4] for the corresponding results.

Further reduction formulae and determinant evaluations from [33] address determinants of matrices formed out of coefficients of iterated compositions of formal power

[21] As one can read in [92], "Faà di Bruno was neither the first to state the formula that bears his name nor the first to prove it." In Section 4 of that article, the author tries to trace back the roots of the formula. It is apparently impossible to find the author of the formula with certainty. In his book [15, p. 312], Arbogast describes a recursive rule how, from the top term, to generate all other terms in the formula. However, the explicit formula is never written down. (I am not able to verify the conclusions in [48]. It seems to me that the author mixes the knowledge that we have today with what is really written in [15].) The formula appears explicitly in Lacroix's book [118, p. 629], but Lacroix's precise sources remain unknown. I refer the reader to [92, Sec. 4] and [48] for more detailed remarks on the history of the formula.

series. In order to have a convenient notation, let us write $f^{\langle n\rangle}(x)$ for the n-fold composition of f with itself,

$$f^{\langle n\rangle}(x)=f(f(\cdots(f(x)))),$$

with n occurrences of f on the right-hand side. Chu shows [33, Sec. 1.4] that it is possible to extend this n-fold composition to values of n other than non-negative integers. This given, Theorems 4.6 and 4.7 from [33] read as follows.

Theorem 28. *Let $f(x)=x+\sum_{m=2}^{\infty}f_mx^m$, $g(x)=\sum_{m=1}^{\infty}g_mx^m$ and $w_k(x)$, $k=1,2,\dots,n$, be formal power series with coefficients in some commutative ring. Then*

$$\det_{1\le i,j\le n}\left([x^j]w_j(x)f^{\langle y_i\rangle}(g(x))\right)=f_2^{\binom n2}g_1^{\binom {n+1}2}\left(\prod_{k=1}^{n}w_k(0)\right)\left(\prod_{1\le i<j\le n}(y_j-y_i)\right), \tag{5.33}$$

where $[x^j]h(x)$ denotes the coefficient of x^j in the series $h(x)$. If, in addition, $w_n(0)=0$, then

$$\det_{1\le i,j\le n}\left([x^{j+1}]w_j(x)f^{\langle y_i\rangle}(g(x))\right)\\=f_2^{\binom n2}g_1^{\binom {n+1}2}\left(\prod_{1\le i<j\le n}(y_j-y_i)\right)\det_{1\le i,j\le n}\left([x^{1+j-i}]w_j(x)\right). \tag{5.34}$$

Furthermore, we have

$$\det_{1\le i,j\le n}\left([x^{j+1}]f^{\langle y_i\rangle}(x)\right)=f_2^{\binom {n+1}2}\left(\prod_{k=1}^{n}y_k\right)\left(\prod_{1\le i<j\le n}(y_j-y_i)\right). \tag{5.35}$$

□

5.4. More on Hankel determinant evaluations. Section 2.7 of [109] was devoted to *Hankel determinants*. There, I tried to convince the reader that, whenever you think that a certain Hankel determinant evaluates nicely, then the explanation will be (sometimes more sometimes less) hidden in the theory of *continued fractions* and *orthogonal polynomials*. In retrospect, it seems that the success of this try was mixed. Since readers are always right, this has to be blamed entirely on myself, and, indeed, the purpose of the present subsection is to rectify some shortcomings from then.

Roughly speaking, I explained in [109] that, given a Hankel determinant

$$\det_{0\le i,j\le n-1}(\mu_{i+j}), \tag{5.36}$$

to find its evaluation one should expand the generating function of the sequence of coefficients $(\mu_k)_{k\ge0}$ in terms of a continued fraction, respectively find the sequence of orthogonal polynomials $(p_n(x))_{n\ge0}$ with moments μ_k, $k=0,1,\dots$, and then the value of the Hankel determinant (5.36) can be read off the coefficients of the continued fraction, respectively from the recursion coefficients of the orthogonal polynomials. What I missed to state is that the knowledge of the orthogonal polynomials makes it also possible to find the value of the Hankel determinants which start with μ_1 and μ_2, respectively (instead of μ_0). In the theorem below I summarise the results that were already discussed in [109] (for which classical references are [186, Theorem 51.1] or [185, Cor. 6, (19), on p. IV-17; Proposition 1, (7), on p. V-5]), and I add the two missing ones.

Theorem 29. *Let $(\mu_k)_{k\ge0}$ be a sequence with generating function $\sum_{k=0}^{\infty}\mu_k x^k$ written in the form*

$$\sum_{k=0}^{\infty}\mu_k x^k=\cfrac{\mu_0}{1+a_0x-\cfrac{b_1x^2}{1+a_1x-\cfrac{b_2x^2}{1+a_2x-\cdots}}}\,. \tag{5.37}$$

Then

$$\det_{0\le i,j\le n-1}(\mu_{i+j})=\mu_0^n b_1^{n-1}b_2^{n-2}\cdots b_{n-2}^2b_{n-1}. \tag{5.38}$$

Let $(p_n(x))_{n\ge0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree n, which is orthogonal with respect to some functional L, that is, $L(p_m(x)p_n(x))=\delta_{m,n}c_n$, where the c_n's are some non-zero constants and $\delta_{m,n}$ is the Kronecker delta. Let

$$p_{n+1}(x)=(a_n+x)p_n(x)-b_np_{n-1}(x) \tag{5.39}$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function $\sum_{k=0}^{\infty}\mu_kx^k$ for the moments $\mu_k=L(x^k)$ satisfies (5.37) *with the a_i's and b_i's being the coefficients in the three-term recurrence* (5.39). *In particular, the Hankel determinant evaluation* (5.38) *holds, with the b_i's from the three-term recurrence* (5.39).

If $(q_n)_{n\ge0}$ is the sequence recursively defined by $q_0=1$, $q_1=-a_0$, and

$$q_{n+1}=-a_nq_n-b_nq_{n-1},$$

then in the situation above we also have

$$\det_{0\le i,j\le n-1}(\mu_{i+j+1})=\mu_0^n b_1^{n-1}b_2^{n-2}\cdots b_{n-2}^2b_{n-1}q_n \tag{5.40}$$

and

$$\det_{0\le i,j\le n-1}(\mu_{i+j+2})=\mu_0^n b_1^{n-1}b_2^{n-2}\cdots b_{n-2}^2b_{n-1}\sum_{k=0}^{n}q_k^2b_{k+1}\cdots b_{n-1}b_n. \tag{5.41}$$

□

I did not find a reference for (5.40) and (5.41). These two identities follow however easily from Viennot's combinatorial model [185] for orthogonal polynomials and Hankel determinants of moments. More precisely, in this theory the moments μ_k are certain generating functions for *Motzkin paths*, and, due to Theorem 6, the Hankel determinants $\det_{0\le i,j\le n-1}(\mu_{i+j+m})$ are generating functions for families $(P_1,P_2,\dots,P_n)$ of non-intersecting Motzkin paths, P_i running from $(-i,0)$ to $(j+m,0)$. In the case $m=0$, it is explained in [185, Ch. IV] how to find the corresponding Hankel determinant evaluation (5.38) using this combinatorial model. The idea is that in that case there is a unique family of non-intersecting Motzkin paths, and its weight gives the right-hand side of (5.38). If $m=1$ or $m=2$ one can argue similarly. The paths are uniquely determined with the exception of their portions in the strip $0\le x\le m$. The various possibilities that one has there then yield the right-hand sides of (5.40) and (5.41).

Since there are so many explicit families of orthogonal polynomials, and, hence, so many ways to apply the above theorem, I listed only a few standard Hankel determinant evaluations explicitly in [109]. I did append a long list of references and sketched in which ways these give rise to more Hankel determinant evaluations. Apparently, these remarks were at times too cryptic, in particular concerning the theme *"orthogonal polynomials as moments."* This is treated systematically in the two papers [86, 87] by Ismail and Stanton. There it is shown that certain classical polynomials $(r_n(x))_{n\ge0}$, such as, for example, the *Laguerre polynomials*, the *Meixner polynomials*, or the *Al-Salam–Chihara polynomials* (but there are others as well, see [86, 87]), are *moments* of other families of classical orthogonal polynomials. Thus, application of Theorem 29 with $\mu_n = r_n(x)$ immediately tells that the evaluations of the corresponding Hankel determinants

$$\det_{0\le i,j\le n-1}\big(r_{i+j}(x)\big) \tag{5.42}$$

(and also the higher ones in (5.40) and (5.41)) are known. In particular, the explicit forms can be extracted from the coefficients of the three-term recursions for these other families of orthogonal polynomials. Thus, whenever you encounter a determinant of the form (5.42), you must check whether $(r_n(x))_{n\ge0}$ is a family of orthogonal polynomials (which, as I explained in [109], one does by consulting the compendium [102] of hypergeometric orthogonal polynomials compiled by Koekoek and Swarttouw), and if the answer is "yes", you will find the solution of your determinant evaluation through the results in [86, 87] by applying Theorem 29.

While Theorem 29 describes in detail the connexion between Hankel determinants and the continued fractions of the type (5.37), which are often called *J-fractions* (which is short for *Jacobi continued fractions*), I missed to tell in [109] that there is also a close relation between Hankel determinants and so-called *S-fractions* (which is short for *Stieltjes continued fractions*). I try to remedy this by the theorem below (cf. for example [94, Theorem 7.2], where S-fractions are called *regular C-fractions*). In principle, since S-fractions are special cases of J-fractions (5.37) in which the coefficients a_i are all zero, the corresponding result for the Hankel determinants is in fact implied by Theorem 29. Nevertheless, it is useful to state it separately. I am not able to give a reference for (5.46), but, again, it is not too difficult to derive it from Viennot's combinatorial model [185] for orthogonal polynomials and moments that was mentioned above.

Theorem 30. *Let $(\mu_k)_{k\ge0}$ be a sequence with generating function $\sum_{k=0}^{\infty}\mu_k x^k$ written in the form*

$$\sum_{k=0}^{\infty}\mu_k x^k = \cfrac{\mu_0}{1+\cfrac{a_1x}{1+\cfrac{a_2x}{1+\cdots}}}\ . \tag{5.43}$$

Then

$$\det_{0\le i,j\le n-1}(\mu_{i+j}) = \mu_0^n(a_1a_2)^{n-1}(a_3a_4)^{n-2}\cdots(a_{2n-5}a_{2n-4})^2(a_{2n-3}a_{2n-2}), \tag{5.44}$$

$$\det_{0\le i,j\le n-1}(\mu_{i+j+1}) = (-1)^n\mu_0^na_1^n(a_2a_3)^{n-1}(a_4a_5)^{n-2}\cdots(a_{2n-4}a_{2n-3})^2(a_{2n-2}a_{2n-1}), \tag{5.45}$$

and

$$\det_{0\le i,j\le n-1}(\mu_{i+j+2})=\mu_0^n a_1^n (a_2a_3)^{n-1}(a_4a_5)^{n-2}\cdots(a_{2n-4}a_{2n-3})^2(a_{2n-2}a_{2n-1})\\ \times \sum_{0\le i_1-1<i_2-2<\cdots<i_n-n\le n} a_{i_1}a_{i_2}\cdots a_{i_n}. \tag{5.46}$$

□

Using this theorem, Tamm [181, Theorem 3.1] observed that from *Gauß' continued fraction for the ratio of two contiguous* $_2F_1$*-series* one can deduce several interesting binomial Hankel determinant evaluations, some of them had already been found earlier by Eğecioğlu, Redmond and Ryavec [53, Theorem 4] while working on *polynomial Riemann hypotheses*. Gessel and Xin [75] undertook a systematic analysis of this approach, and they arrived at a set of 18 Hankel determinant evaluations, which I list as (5.48)–(5.65) in the theorem below. They are preceded by the Hankel determinant evaluation (5.47), which appears only in [53, Theorem 4].

Theorem 31. *For any positive integer n, there hold*

$$\det_{0\le i,j\le n-1}\left(\binom{3i+3j+2}{i+j}\right)=\prod_{i=0}^{n-1}\frac{(\frac43)_i\,(\frac56)_i\,(\frac53)_i\,(\frac76)_i}{(\frac32)_{2i}\,(\frac52)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.47}$$

$$\det_{0\le i,j\le n-1}\left(\frac{1}{3i+3j+1}\binom{3i+3j+1}{i+j}\right)=\prod_{i=0}^{n-1}\frac{(\frac23)_i\,(\frac16)_i\,(\frac43)_i\,(\frac56)_i}{(\frac12)_{2i}\,(\frac32)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.48}$$

$$\det_{0\le i,j\le n-1}\left(\frac{1}{3i+3j+4}\binom{3i+3j+4}{i+j+1}\right)=\prod_{i=0}^{n-1}\frac{(\frac43)_i\,(\frac56)_i\,(\frac53)_i\,(\frac76)_i}{(\frac32)_{2i}\,(\frac52)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.49}$$

$$\det_{0\le i,j\le n-1}\left(\frac{1}{3i+3j+2}\binom{3i+3j+2}{i+j+1}\right)=\prod_{i=0}^{n-1}\frac{(\frac43)_i\,(\frac56)_i\,(\frac53)_i\,(\frac76)_i}{(\frac32)_{2i}\,(\frac52)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.50}$$

$$\det_{0\le i,j\le n-1}\left(\frac{1}{3i+3j+5}\binom{3i+3j+5}{i+j+2}\right)=\prod_{i=0}^{n}\frac{(\frac23)_i\,(\frac16)_i\,(\frac43)_i\,(\frac56)_i}{(\frac12)_{2i}\,(\frac32)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.51}$$

$$\det_{0\le i,j\le n-1}\left(\frac{2}{3i+3j+1}\binom{3i+3j+1}{i+j+1}\right)=\prod_{i=0}^{n}\frac{(\frac23)_i\,(\frac16)_i\,(\frac43)_i\,(\frac56)_i}{(\frac12)_{2i}\,(\frac32)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.52}$$

$$\det_{0\le i,j\le n-1}\left(\frac{2}{3i+3j+4}\binom{3i+3j+4}{i+j+2}\right)=\prod_{i=0}^{n}\frac{(\frac43)_i\,(\frac56)_i\,(\frac53)_i\,(\frac76)_i}{(\frac32)_{2i}\,(\frac52)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.53}$$

$$\det_{0\le i,j\le n-1}\left(\frac{2}{(3i+3j+1)(3i+3j+2)}\binom{3i+3j+2}{i+j+1}\right)=\prod_{i=0}^{n-1}2\frac{(\frac53)_i\,(\frac16)_i\,(\frac73)_i\,(\frac56)_i}{(\frac32)_{2i}\,(\frac52)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.54}$$

$$\det_{0\le i,j\le n-1}\left(\frac{2}{(3i+3j+4)(3i+3j+5)}\binom{3i+3j+5}{i+j+2}\right)\\ =(-1)^n\prod_{i=1}^{n}\frac{(\frac53)_i\,(\frac16)_i\,(\frac43)_i\,(-\frac16)_i}{(\frac12)_{2i}\,(\frac32)_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.55}$$

$$\det_{0\le i,j\le n-1}\left(\frac{(9i+9j+5)}{(3i+3j+1)(3i+3j+2)}\binom{3i+3j+2}{i+j+1}\right)=\prod_{i=0}^{n-1}5\frac{(\frac{2}{3})_i\,(\frac{7}{6})_i\,(\frac{4}{3})_i\,(\frac{11}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}, \tag{5.56}$$

$$\det_{0\le i,j\le n-1}\left(\frac{(9i+9j+14)}{(3i+3j+4)(3i+3j+5)}\binom{3i+3j+5}{i+j+2}\right)=\prod_{i=1}^{n}2\frac{(\frac{2}{3})_i\,(\frac{7}{6})_i\,(\frac{1}{3})_i\,(\frac{5}{6})_i}{(\frac{1}{2})_{2i}\,(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.57}$$

Let $a_0=-2$ *and* $a_m=\frac{1}{3m+1}\binom{3m+1}{m}$ *for* $m\ge 1$. *Then*

$$\det_{0\le i,j\le n-1}(a_{i+j})=\prod_{i=0}^{n-1}(-2)\frac{(\frac{1}{3})_i\,(-\frac{1}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{1}{2})_{2i}\,(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.58}$$

Let $b_0=10$ *and* $b_m=\frac{2}{3m+2}\binom{3m+2}{m}$ *for* $m\ge 1$. *Then*

$$\det_{0\le i,j\le n-1}(b_{i+j})=\prod_{i=0}^{n-1}10\frac{(\frac{2}{3})_i\,(\frac{1}{6})_i\,(\frac{7}{3})_i\,(\frac{11}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.59}$$

Furthermore,

$$\det_{0\le i,j\le n-1}\left(\frac{2}{3i+3j+5}\binom{3i+3j+5}{i+j+1}\right)=\prod_{i=0}^{n}\frac{(\frac{4}{3})_i\,(\frac{5}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.60}$$

Let $c_0=\frac{7}{2}$ *and* $c_m=\frac{2}{3m+1}\binom{3m+1}{m+1}$ *for* $m\ge 1$. *Then*

$$\det_{0\le i,j\le n-1}(c_{i+j})=\prod_{i=0}^{n}\frac{(\frac{4}{3})_i\,(\frac{5}{6})_i\,(\frac{5}{3})_i\,(\frac{7}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.61}$$

Let $d_0=-5$ *and* $d_m=\frac{8}{(3m+1)(3m+2)}\binom{3m+2}{m}$ *for* $m\ge 1$. *Then*

$$\det_{0\le i,j\le n-1}(d_{i+j})=\prod_{i=0}^{n-1}(-5)\frac{(\frac{4}{3})_i\,(-\frac{1}{6})_i\,(\frac{8}{3})_i\,(\frac{7}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.62}$$

Furthermore,

$$\det_{0\le i,j\le n-1}\left(\frac{8}{(3i+3j+4)(3i+3j+5)}\binom{3i+3j+5}{i+j+1}\right)=\prod_{i=0}^{n-1}2\frac{(\frac{7}{3})_i\,(\frac{5}{6})_i\,(\frac{8}{3})_i\,(\frac{7}{6})_i}{(\frac{5}{2})_{2i}\,(\frac{7}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.63}$$

Let $e_0=14$ *and* $e_m=\frac{2(9m+5)}{(3m+1)(3m+2)}\binom{3m+2}{m}$ *for* $m\ge 1$. *Then*

$$\det_{0\le i,j\le n-1}(e_{i+j})=\prod_{i=0}^{n-1}14\frac{(\frac{1}{3})_i\,(\frac{5}{6})_i\,(\frac{5}{3})_i\,(\frac{13}{6})_i}{(\frac{3}{2})_{2i}\,(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.64}$$

Furthermore,

$$\det_{0\le i,j\le n-1}\left(\frac{2(9i+9j+14)}{(3i+3j+4)(3i+3j+5)}\binom{3i+3j+5}{i+j+1}\right)=\prod_{i=1}^{n}2\frac{(\frac{2}{3})_i\,(\frac{7}{6})_i\,(\frac{1}{3})_i\,(\frac{5}{6})_i}{(\frac{1}{2})_{2i}\,(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}. \tag{5.65}$$

□

Some of the numbers appearing on the right-hand sides of the formulae in this theorem have combinatorial significance, although no intrinsic explanation is known why this is the case. More precisely, the numbers on the right-hand sides of (5.48), (5.51) and (5.52) count *cyclically symmetric transpose-complementary plane partitions* (cf. [132] and [28]), whereas those on the right-hand sides of (5.47), (5.49), (5.50) and (5.53) count *vertically symmetric alternating sign matrices* (cf. [117] and [28]).

In [53, Theorem 4], Eğecioğlu, Redmond and Ryavec prove also the following common generalisation of (5.49) and (5.50). (The first identity is the special case $x = 0$, while the second is the special case $x = 1$ of the following theorem.)

Theorem 32. *For $m \ge 0$, let $s_m(x) = \sum_{k=0}^{m} \frac{k+1}{m+1}\binom{3m-k+1}{m-k} x^k$. Then, for any positive integer n, there holds*

$$\det_{0\le i,j\le n-1}\left(s_{i+j}(x)\right) = \prod_{i=0}^{n-1} \frac{\left(\frac{4}{3}\right)_i \left(\frac{5}{6}\right)_i \left(\frac{5}{3}\right)_i \left(\frac{7}{6}\right)_i}{\left(\frac{3}{2}\right)_{2i} \left(\frac{5}{2}\right)_{2i}} \left(\frac{27}{4}\right)^{2i}. \tag{5.66}$$

□

As I mentioned above, in [109] I only stated a few special Hankel determinant evaluations explicitly, because there are too many ways to apply Theorems 29 and 30. I realise, however, that I should have stated the evaluation of the *Hankel determinant of Catalan numbers* there. I make this up now by doing this in the theorem below. I did not do it then because orthogonal polynomials are not needed for its evaluation (the orthogonal polynomials which are tied to Catalan numbers as moments are *Chebyshev polynomials*, but, via Theorems 29 and 30, one would only cover the cases $m = 0, 1, 2$ in the theorem below). In fact, the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ can be alternatively written as $C_n = (-1)^n 2^{2n+1}\binom{1/2}{n+1}$, and therefore the Hankel determinant evaluation below follows from [109, Theorem 26, (3.12)]. This latter observation shows that even a more general determinant, namely $\det_{0\le i,j\le n-1}(C_{\lambda_i+j})$, can be evaluated in closed form. For historical remarks on this ubiquitous determinant see [76, paragraph before the Appendix].

Theorem 33.

$$\det_{0\le i,j\le n-1}\left(C_{m+i+j}\right) = \prod_{1\le i\le j\le m-1} \frac{2n+i+j}{i+j}. \tag{5.67}$$

□

As in [109], let me conclude the part on Hankel determinants by pointing the reader to further papers containing interesting results on them, high-lighting sometimes the point of view of orthogonal polynomials that I explained above, sometimes a combinatorial point of view. The first point of view is put forward in [191] (see [110] for the solution of the conjectures in that paper) in order to present Hankel determinant evaluations of matrices with *hypergeometric ${}_2F_1$-series* as entries. The orthogonal polynomials approach is also used in [49] to show that a certain Hankel determinant defined by *Catalan numbers* evaluates to *Fibonacci numbers*. In [14], one finds Hankel determinant evaluations involving generalisations of the *Bernoulli numbers*. The combinatorial point of view dominates in [4, 35, 36, 37, 54], where Hankel determinants involving *q-Catalan numbers, q-Stirling numbers*, and *q-Fibonacci numbers* are considered.

A very interesting new direction, which seems to have much potential, is opened up by Luque and Thibon in [127]. They show that *Selberg-type integrals* can be evaluated by means of *Hankel hyperdeterminants*, and they prove many hyperdeterminant generalisations of classical Hankel determinant evaluations.

At last, (but certainly not least!), I want to draw the reader's attention to Lascoux's "unorthodox"[22] approach to Hankel determinants and orthogonal polynomials through symmetric functions which he presents in detail in [121, Ch. 4, 5, 8]. In particular, Theorem 29, Eq. (5.38) are the contents of Theorem 8.3.1 in [121] (see also the end of Section 5.3 there), and Theorem 30, Eqs. (5.44) and (5.45) are the contents of Theorem 4.2.1 in [121]. The usefulness of this symmetric function approach is, for example, demonstrated in [82, 83] in order to evaluate Hankel determinants of matrices the entries of which are *Rogers–Szegő*, respectively *Meixner polynomials*.

5.5. More binomial determinants. A vast part of Section 3 in [109] is occupied by binomial determinants. As I mentioned in Section 2 of the present article, an extremely rich source for binomial determinants is rhombus tiling enumeration. I want to present here some which did not already appear in [109].

To begin with, I want to remind the reader of an old problem posed by Andrews in [11, p. 105]. The determinant in this problem is a variation of a determinant which enumerates *cyclically symmetric plane partitions* and *descending plane partitions*, which was evaluated by Andrews in [10] (see also [109, Theorem 32]; the latter determinant arises from the one in (5.68) by replacing $j+1$ by j in the bottom of the binomial coefficient).

Problem 34. *Evaluate the determinant*

$$D_1(n) := \det_{0\le i,j\le n-1}\left(\delta_{ij}+\binom{\mu+i+j}{j+1}\right), \tag{5.68}$$

where δ_{ij} is the Kronecker delta. In particular, show that

$$\frac{D_1(2n)}{D_1(2n-1)} = (-1)^{\binom{n-1}{2}}\frac{2^n\,\left(\frac{\mu}{2}+n\right)_{\lceil n/2\rceil}\,\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}}{(n)_n\,\left(-\frac{\mu}{2}-2n+\frac{3}{2}\right)_{\lceil (n-2)/2\rceil}}. \tag{5.69}$$

[22] It could easily be that it is the "modern" treatment of the theory which must be labelled with the attribute "unorthodox." As Lascoux documents in [121], in his treatment *he* follows the tradition of great masters such as Cauchy, Jacobi or Wroński ...

The determinants $D_1(n)$ are rather intriguing. Here are the first few values:

$$\begin{aligned}
D_1(1) &= \mu+1,\\
D_1(2) &= (\mu+1)(\mu+2),\\
D_1(3) &= \frac{1}{12}(\mu+1)(\mu+2)(\mu+3)(\mu+14),\\
D_1(4) &= \frac{1}{72}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+9)(\mu+14),\\
D_1(5) &= \frac{1}{8640}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+9)\\
&\qquad\qquad \times (3432+722\mu+45\mu^2+\mu^3),\\
D_1(6) &= \frac{1}{518400}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+6)(\mu+8)(\mu+13)\\
&\qquad\qquad \times (\mu+15)(3432+722\mu+45\mu^2+\mu^3),\\
D_1(7) &= \frac{1}{870912000}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+6)(\mu+7)(\mu+8)^2\\
&\qquad\qquad \times (\mu+13)(\mu+15)^2(\mu+34)(\mu^3+47\mu^2+954\mu+5928),\\
D_1(8) &= \frac{1}{731566080000}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+6)(\mu+7)(\mu+8)^3\\
&\qquad\qquad \times (\mu+10)(\mu+15)^2(\mu+17)(\mu+19)\\
&\qquad\qquad \times (\mu+21)(\mu+34)(\mu^3+47\mu^2+954\mu+5928).
\end{aligned}$$

"So," these determinants factor almost completely, there is only a relatively small (in degree) irreducible factor which is not linear. (For example, this factor is of degree 6 for $D_1(9)$ and $D_1(10)$, and of degree 7 for $D_1(11)$ and $D_1(12)$.) Moreover, this "bigger" factor is always the same for $D_1(2n-1)$ and $D_1(2n)$. Not only that, the quotient which is predicted in (5.69) is at the same time a building block in the result of the evaluation of the determinant which enumerates the cyclically symmetric and descending plane partitions (see [11]). All this begs for an explanation in terms of a factorisation of the matrix of which the determinant is taken from. In fact, for the plane partition matrix there is such a factorisation, due to Mills, Robbins and Rumsey [132, Theorem 5] (see also [109, Theorem 36]). The question is whether there is a similar one for the matrix in (5.68).

Inspired by this conjecture and by the variations in [40, Theorems 11–13] (see [109, Theorem 35]) on Andrews' original determinant evaluation in [10], Guoce Xin (private communication) observed that, if we change the sign in front of the Kronecker delta in (5.68), then the resulting determinant factors completely into linear factors.

Conjecture 35. *Let μ be an indeterminate and n be a non-negative integer. The determinant*

$$\det_{0\le i,j\le n-1}\left(-\delta_{ij}+\binom{\mu+i+j}{j+1}\right) \tag{5.70}$$

is equal to

$$(-1)^{n/2}2^{n(n+2)/4}\frac{\left(\frac{\mu}{2}\right)_{n/2}}{\left(\frac{n}{2}\right)!}\left(\prod_{i=0}^{(n-2)/2}\frac{i!^2}{(2i)!^2}\right)$$
$$\times\left(\prod_{i=0}^{\lfloor (n-4)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{5}{2}\right)_{(n-4i-2)/2}^2\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+3\right)_{(n-4i-4)/2}^2\right) \tag{5.71}$$

if n is even, and it is equal to

$$(-1)^{(n-1)/2}2^{(n+3)(n+1)/4}\left(\frac{\mu-1}{2}\right)_{(n+1)/2}\left(\prod_{i=0}^{(n-1)/2}\frac{i!\,(i+1)!}{(2i)!\,(2i+2)!}\right)$$
$$\times\left(\prod_{i=0}^{\lfloor (n-3)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{5}{2}\right)_{(n-4i-3)/2}^2\left(-\frac{\mu}{2}-\frac{3n}{2}+\frac{3}{2}+3i\right)_{(n-4i-1)/2}^2\right) \tag{5.72}$$

if n is odd.

In fact, it seems that also the "next" determinant, the determinant where one replaces $j+1$ at the bottom of the binomial coefficient in (5.70) by $j+2$ factors completely when n is odd. (It does not when n is even, though.)

Conjecture 36. *Let μ be an indeterminate. For any odd non-negative integer n there holds*

$$\det_{0\le i,j\le n-1}\left(-\delta_{ij}+\binom{\mu+i+j}{j+2}\right)$$
$$=(-1)^{(n-1)/2}2^{(n-1)(n+5)/4}(\mu+1)\frac{\left(\frac{\mu}{2}-1\right)_{(n+1)/2}}{\left(\frac{n+1}{2}\right)!}\left(\prod_{i=0}^{(n-1)/2}\frac{i!^2}{(2i)!^2}\left(\frac{\mu}{2}+3i+\frac{3}{2}\right)_{(n-4i-1)/2}^2\right)$$
$$\times\left(\prod_{i=0}^{\lfloor (n-3)/4\rfloor}\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+\frac{5}{2}\right)_{(n-4i-3)/2}^2\right). \tag{5.73}$$

For the combinatorialist I add that all the determinants in Problem 34 and Conjectures 35 and 36 count certain rhombus tilings, as do the original determinants in [10, 11, 40].

Alain Lascoux (private communication) did not understand why we should stop here, and he hinted at a parametric family of determinant evaluations into which the case of odd n of Conjecture 35 is embedded as a special case.

Conjecture 37. *Let μ be an indeterminate. For any odd non-negative integers n and r there holds*

$$\det_{0\le i,j\le n-1}\left(-\delta_{i,j+r-1}+\binom{\mu+i+j}{j+r}\right)$$
$$=(-1)^{(n-r)/2}2^{(n^2+6n-2nr+r^2-4r+2)/4}\left(\prod_{i=0}^{r-2}i!\right)\left(\prod_{i=0}^{(r-3)/2}\frac{(n-2i-2)!^2}{\left(\frac{n-2i-3}{2}\right)!^2\,(n+2i)!\,(n+2i+2)!}\right)$$
$$\times(\mu-r)\left(\frac{\mu+1}{2}\right)_{(n-r)/2}\left(\prod_{i=1}^{r-1}(m-r+i)_{n+r-2i+1}\right)\left(\prod_{i=0}^{(n-1)/2}\frac{i!\,(i+1)!}{(2i)!\,(2i+2)!}\right)$$
$$\times\left(\prod_{i=0}^{\lfloor (n-r-2)/4\rfloor}\left(\frac{\mu}{2}+3i+r+\frac{3}{2}\right)^2_{(n-4i-r-2)/2}\left(-\frac{\mu}{2}-\frac{3n}{2}+\frac{r}{2}+3i+1\right)^2_{(n-4i-r)/2}\right). \tag{5.74}$$

The next binomial determinant that I want to mention is, strictly speaking, not a determinant but a Pfaffian (see (5.4) for the definition). While doing (-1)-*enumeration of sef-complementary plane partitions*, Eisenkölbl [57] encountered an, I admit, complicated looking Pfaffian,

$$\operatorname*{Pf}_{1\le i,j\le a}\big(M(m_1,m_2,n_1,n_2,a,b)\big), \tag{5.75}$$

where a is even and b is odd, and where

$$M_{ij}(m_1,m_2,n_1,n_2,a,b)$$
$$=\sum_{l=1}^{\lfloor (a+b-1)/4)\rfloor}(-1)^{i+j}\left(\binom{n_1}{(b-1)/2+\lfloor (i-1)/2\rfloor-l+1}\binom{m_1}{-a/2+\lfloor (j-1)/2\rfloor+l}\right.$$
$$\left.-\binom{n_1}{(b-1)/2+\lfloor (j-1)/2\rfloor-l+1}\binom{m_1}{-a/2+\lfloor (i-1)/2\rfloor+l}\right)$$
$$+\sum_{l=1}^{\lceil (a+b-1)/4\rceil}\left(\binom{n_2}{(b-1)/2+\lfloor i/2\rfloor-l+1}\binom{m_2}{-a/2+\lfloor j/2\rfloor+l-1}\right.$$
$$\left.-\binom{n_2}{(b-1)/2+\lfloor j/2\rfloor-l+1}\binom{m_2}{-a/2+\lfloor i/2\rfloor+l-1}\right).$$

Remarkably however, experimentally this Pfaffian, first of all, factors completely into factors which are linear in the variables m_1,m_2,n_1,n_2, but not only that, there seems to be complete separation, that is, each linear factor contains only one of m_1,m_2,n_1,n_2. One has the impression that this phenomenon should have an explanation in a factorisation of the matrix in (5.75). However, the task of finding one does not seem to be an easy one in view of the "entangledness" of the parameters in the sums of the matrix entries.

Problem 38.[23] *Find and prove the closed form evaluation of the Pfaffian in* (5.75).

[23]Theresia Eisenkölbl has recently solved this problem in [58].

Our next determinants can be considered as *shuffles* of two binomial determinants. Let us first consider

$$\det_{1\le i,j\le a+m}\begin{pmatrix}\dbinom{b+c+m}{b-i+j} & 1\le i\le a\\ \dbinom{\frac{b+c}{2}}{\frac{b+a}{2}-i+j+\varepsilon} & a+1\le i\le a+m\end{pmatrix}. \tag{5.76}$$

In fact, if $\varepsilon=0$, and if a,b,c all have the same parity, then this is exactly the determinant in (2.10), the evaluation of which proves Theorem 3, as we explained in Section 2. If $\varepsilon=1/2$ and a has parity different from that of b and c, then the corresponding determinant was also evaluated in [40], and this evaluation implied the companion result to Theorem 3 that we mentioned immediately after the statement of the theorem. In the last section of [40], it is reported that, apparently, there are also nice closed forms for the determinant in (5.76) for $\varepsilon=1$ and $\varepsilon=3/2$, both of which imply as well enumeration theorems for *rhombus tilings of a hexagon with an equilateral triangle removed from its interior* (see Conjectures 1 and 2 in [40]). We reproduce the conjecture for $\varepsilon=1$ here, the one for $\varepsilon=3/2$ is very similar in form.

Conjecture 39. *Let a,b,c,m be non-negative integers, a,b,c having the same parity. Then for $\varepsilon=1$ the determinant in* (5.76) *is equal to*

$$\begin{aligned}\frac{1}{4}&\frac{\mathrm{H}(a+m)\,\mathrm{H}(b+m)\,\mathrm{H}(c+m)\,\mathrm{H}(a+b+c+m)}{\mathrm{H}(a+b+m)\,\mathrm{H}(a+c+m)\,\mathrm{H}(b+c+m)}\\
&\qquad\times\frac{\mathrm{H}(m+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(m+\lfloor\frac{a+b+c}{2}\rfloor)}{\mathrm{H}(\frac{a+b}{2}+m+1)\,\mathrm{H}(\frac{a+c}{2}+m-1)\,\mathrm{H}(\frac{b+c}{2}+m)}\\
&\times\frac{\mathrm{H}(\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{c}{2}\rfloor)}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{c}{2}\rfloor)}\\
\times&\frac{\mathrm{H}(\frac{m}{2})^2\,\mathrm{H}(\frac{a+b+m}{2})^2\,\mathrm{H}(\frac{a+c+m}{2})^2\,\mathrm{H}(\frac{b+c+m}{2})^2}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a+b+c}{2}\rfloor)\,\mathrm{H}(\frac{a+b}{2}-1)\,\mathrm{H}(\frac{a+c}{2}+1)\,\mathrm{H}(\frac{b+c}{2})}P_1(a,b,c,m),\end{aligned} \tag{5.77}$$

where $P_1(a,b,c,m)$ is the polynomial given by

$$P_1(a,b,c,m)=\begin{cases}(a+b)(a+c)+2am & \textit{if } a \textit{ is even},\\ (a+b)(a+c)+2(a+b+c+m)m & \textit{if } a \textit{ is odd},\end{cases}$$

and where $\mathrm{H}(n)$ *is the hyperfactorial defined in* (2.8).

Two other examples of determinants in which the upper part is given by one binomial matrix, while the lower part is given by a different one, arose in [43, Conjectures A.1 and A.2]. Again, both of them seem to factor completely into linear factors, and both of them imply enumeration results for *rhombus tilings of a certain V-shaped region*. The right-hand sides of the (conjectured) results are the weirdest "closed" forms in enumeration that I am aware of.[24] We state just the first of the two conjectures, the other is very similar.

[24]No non-trivial simplifications seem to be possible.

Conjecture 40. *Let x, y, m be non-negative integers. Then the determinant*

$$\det_{1\le i,j\le m+y}\left(\begin{cases}\binom{x+i}{x-i+j} & i=1,\dots,m\\ \binom{x+2m-i+1}{m+y-2i+j+1} & i=m+1,\dots,m+y\end{cases}\right) \tag{5.78}$$

is equal to

$$\prod_{i=1}^{m}\frac{(x+i)!}{(x-i+m+y+1)!\,(2i-1)!}\prod_{i=m+1}^{m+y}\frac{(x+2m-i+1)!}{(2m+2y-2i+1)!\,(m+x-y+i-1)!}$$
$$\times 2^{\binom{m}{2}+\binom{y}{2}}\prod_{i=1}^{m-1}i!\prod_{i=1}^{y-1}i!\prod_{i\ge0}(x+i+\tfrac32)_{m-2i-1}\prod_{i\ge0}(x-y+\tfrac52+3i)_{\lfloor\frac{3y}{2}-\frac{9i}{2}\rfloor-2}$$
$$\times\prod_{i\ge0}(x+\tfrac{3m}{2}-y+\lceil\tfrac{3i}{2}\rceil+\tfrac32)_{3\lceil\frac{y}{2}\rceil-\lceil\frac{9i}{2}\rceil-2}\prod_{i\ge0}(x+\tfrac{3m}{2}-y+\lfloor\tfrac{3i}{2}\rfloor+2)_{3\lfloor\frac{y}{2}\rfloor-\lfloor\frac{9i}{2}\rfloor-1}$$
$$\times\prod_{i\ge0}(x+m-\lfloor\tfrac{y}{2}\rfloor+i+1)_{2\lfloor\frac{y}{2}\rfloor-m-2i}\prod_{i\ge0}(x+\lfloor\tfrac{y}{2}\rfloor+i+2)_{m-2\lfloor\frac{y}{2}\rfloor-2i-2}$$
$$\times\frac{\prod\limits_{i=0}^{y}(x-y+3i+1)_{m+2y-4i}\prod\limits_{i=0}^{\lceil\frac{y}{2}\rceil-1}(x+m-y+i+1)_{3y-m-4i}}{\prod\limits_{i\ge0}(x+\frac{m}{2}-\frac{y}{2}+i+1)_{y-2i}\,(x+\frac{m}{2}-\frac{y}{2}+i+\frac32)_{y-2i-1}}$$
$$\times\frac{\prod\limits_{i=0}^{y}(x+i+2)_{2m-2i-1}}{(x+y+2)_{m-y-1}\,(m+x-y+1)_{m+y}}. \tag{5.79}$$

Here, shifted factorials occur with positive as well as with negative indices. The convention with respect to which these have to be interpreted is

$$(\alpha)_k:=\begin{cases}\alpha(\alpha+1)\cdots(\alpha+k-1) & \text{if } k>0,\\ 1 & \text{if } k=0,\\ 1/(\alpha-1)(\alpha-2)\cdots(\alpha+k) & \text{if } k<0.\end{cases}$$

All products $\prod_{i\ge0}(f(i))_{g(i)}$ in (5.79) have to be interpreted as the products over all $i\ge0$ for which $g(i)\ge0$.

For further conjectures of determinants of shuffles of two binomial matrices I refer the reader to Conjectures 1–3 in Section 4 of [71]. All of them imply also enumeration results for rhombus tilings of hexagons. This time, these would be results about the number of *rhombus tilings of a symmetric hexagon with some fixed rhombi on the symmetry axis.*

5.6. Determinants of matrices with recursive entries. Binomial coefficients $\binom{i+j}{i}$ satisfy the basic recurrence of the Pascal triangle,

$$p_{i,j}=p_{i,j-1}+p_{i-1,j}. \tag{5.80}$$

We have seen many determinants of matrices with entries containing binomial coefficients in the preceding subsection and in [109, Sec. 3]. In [17], Bacher reports an experimental study of determinants of matrices $(p_{i,j})_{0\le i,j\le n-1}$, where the coefficients $p_{i,j}$ satisfy the recurrence (5.80) (and sometimes more general recurrences), but where the initial conditions for $p_{i,0}$ and $p_{0,i}$, $i\ge 0$, are different from the ones for binomial coefficients. He makes many interesting observations. The most intriguing one says that these determinants satisfy also a linear recurrence (albeit a much longer one). It is intriguing because it points towards the possibility of *automatising determinant evaluations*[25], something that several authors (cf. e.g. [9, 109, 143]) have been aiming at (albeit, with only limited success up to now). The conjecture (and, in fact, a generalisation thereof) has been proved by Petkovšek and Zakrajšek in [192]. Still, there remains a large gap to fill until computers will replace humans doing determinant evaluations.

The paper [17] contained as well several pretty conjectures on closed form evaluations of special cases of such determinants. These were subsequently proved in [111]. We state three of them in the following three theorems. The first two are proved in [111] by working out the LU-factorisation (see "Method 1" in Section 4) for the matrices of which the determinant is computed. The third one is derived by simple row and column operations.

Theorem 41. *Let $(a_{i,j})_{i,j\ge 0}$ be the sequence given by the recurrence*

$$a_{i,j}=a_{i-1,j}+a_{i,j-1}+x\,a_{i-1,j-1},\qquad i,j\ge 1,$$

and the initial conditions $a_{i,0}=\rho^i$ and $a_{0,i}=\sigma^i$, $i\ge 0$. Then

$$\det_{0\le i,j\le n-1}(a_{i,j})=(1+x)^{\binom{n-1}{2}}(x+\rho+\sigma-\rho\sigma)^{n-1}. \tag{5.81}$$

□

Theorem 42. *Let $(a_{i,j})_{i,j\ge 0}$ be the sequence given by the recurrence*

$$a_{i,j}=a_{i-1,j}+a_{i,j-1}+x\,a_{i-1,j-1},\qquad i,j\ge 1,$$

and the initial conditions $a_{i,i}=0$, $i\ge 0$, $a_{i,0}=\rho^{i-1}$ and $a_{0,i}=-\rho^{i-1}$, $i\ge 1$. Then

$$\det_{0\le i,j\le 2n-1}(a_{i,j})=(1+x)^{2(n-1)^2}(x+\rho)^{2n-2}. \tag{5.82}$$

□

Theorem 43. *Let $(a_{i,j})_{i,j\ge 0}$ be the sequence given by the recurrence*

$$a_{i,j}=a_{i-1,j}+a_{i,j-1}+x\,a_{i-1,j-1},\qquad i,j\ge 1,$$

and the initial conditions $a_{i,0}=i$ and $a_{0,i}=-i$, $i\ge 0$. Then

$$\det_{0\le i,j\le 2n-1}(a_{i,j})=(1+x)^{2n(n-1)}. \tag{5.83}$$

□

Certainly, the proofs in [111] are not very illuminating. Neuwirth [138] has looked more carefully into the structure of recursive sequences of the type as those in Theorems 41–43. Even more generally, he looks at sequences $(f_{i,j})_{i,j\ge 0}$ satisfying the recurrence relation

$$f_{i,j}=c_jf_{i-1,j}+d_jf_{i,j-1}+e_jf_{i-1,j-1},\quad i,j\ge 1, \tag{5.84}$$

[25] The reader should recall that the successful automatisation [144, 188, 190, 194, 195] of the evaluation of binomial and hypergeometric sums is fundamentally based on producing recurrences by the computer.

for some given sequences $(c_j)_{j\ge1}$, $(d_j)_{j\ge1}$, $(e_j)_{j\ge1}$. He approaches the problem by finding appropriate *matrix decompositions* for the (infinite) matrix $(f_{i,j})_{i,j\ge0}$. In two special cases, he is able to apply his decomposition results to work out the LU-factorisation of the matrix $(f_{i,j})_{i,j\ge0}$ explicitly, which then yields an elegant determinant evaluation in both of these cases. Neuwirth's first result [138, Theorem 5] addresses the case where the initial values $f_{0,j}$ satisfy a first order recurrence determined by the coefficients d_j from (5.84). It generalises Theorem 41. There is no restriction on the initial values $f_{i,0}$ for $i\ge1$.

Theorem 44. *Let $(c_j)_{j\ge1}$, $(d_j)_{j\ge1}$ and $(e_j)_{j\ge1}$ be given sequences, and let $(f_{i,j})_{i,j\ge0}$ be the doubly indexed sequence given by the recurrence* (5.84) *and the initial conditions $f_{0,0}=1$ and $f_{0,j}=d_jf_{0,j-1}$, $j\ge1$. Then*

$$\det_{0\le i,j\le n-1}(f_{i,j})=\prod_{0\le i<j\le n-1}(e_{i+1}+c_jd_{i+1}). \tag{5.85}$$

□

Neuwirth's second result [138, Theorem 6] also generalises Theorem 41, but in a different way. This time, the initial values $f_{0,j}$, $j\ge1$, are free, whereas the initial values $f_{i,0}$ satisfy a first order recurrence determined by the coefficients c_j from the recurrence (5.84). Below, we state its most attractive special case, in which all the c_j's are identical.

Theorem 45. *Let $(d_j)_{j\ge1}$ and $(e_j)_{j\ge1}$ be given sequences, and let $(f_{i,j})_{i,j\ge0}$ be the doubly indexed sequence given by the recurrence* (5.84) *with $c_j=c$ for all j and the initial conditions $f_{0,0}=1$ and $f_{i,0}=cf_{i-1,0}$, $i\ge1$. Then*

$$\det_{0\le i,j\le n-1}(f_{i,j})=\prod_{i=1}^{n-1}(cd_i+e_i)^{n-i}. \tag{5.86}$$

□

5.7. Determinants for signed permutations. The next class of determinants that we consider are determinants of matrices in which rows and columns are indexed by *elements of reflection groups* (the latter being groups generated by reflections of hyperplanes in real n-dimensional space; see [84] for more information on these groups, and, more generally, on Coxeter groups). The prototypical example of a reflection group is the *symmetric group* $\mathfrak{S}_n$ of permutations of an n-element set. In [109], there appeared two determinant evaluations associated to the symmetric group, see Theorems 55 and 56 in [109]. They concerned evaluations of determinants of the type

$$\det_{\sigma,\pi\in\mathfrak{S}_n}\left(q^{\operatorname{stat}(\sigma\pi^{-1})}\right), \tag{5.87}$$

due to Varchenko, Zagier, and Thibon, respectively, in which stat is the statistic *"number of inversions,"* respectively *"major index."* We know that in many fields of mathematics there exist certain diseases which are typical for that field. Algebraic combinatorics is no exception. Here, I am *not* talking of the earlier mentioned "q-disease" (see Footnote 20; although, due to the presence of q we might also count it as a case of q-disease), but of the disease which manifests itself by the question "And what about the other types?"[26] So let us ask this question, that is, are there theorems similar to the two theorems which we mentioned above for other reflection groups?

So, first of all we need analogues of the statistics "number of inversions" and "major index" for other reflection groups. Indeed, these are available in the literature. The analogue of "number of inversions" is the so-called *length* of an element in a Coxeter group (see [84] for the definition). As a matter of fact, a closed form evaluation of the determinant (5.87), where $\mathfrak{S}_n$ is replaced by any finite reflection group, and where stat is the length, is known (and was already implicitly mentioned in [109]). This result is due to Varchenko [184, Theorem (1.1), where $a(H)$ is specialised to q]. His result is actually much more general, as it is valid for real hyperplane arrangements in which each hyperplane is assigned a different weight. I will not state it here explicitly because I do not want to go through the definitions and notations which would be necessary for doing that.

So, what about analogues of the "major index" for other reflection groups? These are also available, and there are in fact several of them. The first person to introduce a major index for reflection groups other than the symmetric groups was Reiner in [150]. He proposed a major index for the *hyperoctahedral group* B_n, which arose naturally in his study of *P-partitions for signed posets.* The elements of B_n are often called *signed permutations*, and they are all elements of the form $\pi_1\pi_2\dots\pi_n$, where $\pi_i \in \{\pm1, \pm2, \dots, \pm n\}$, $i = 1, 2, \dots, n$, and where $|\pi_1||\pi_2|\dots|\pi_n|$ is a permutation in $\mathfrak{S}_n$. To define their multiplicative structure, it is most convenient to view $\pi = \pi_1\pi_2\dots\pi_n$ as a linear operator on $\mathbb{R}^n$ acting by permutation and sign changes of the co-ordinates. To be precise, the action is given by $\pi(e_i) = (\operatorname{sgn}\pi_i)e_{|\pi_i|}$, where e_i is the i-th standard basis vector in $\mathbb{R}^n$, $i = 1, 2, \dots, n$. The multiplication of two signed permutations is then simply the composition of the corresponding linear operators.

The major index $\operatorname{maj}_B \pi$ of an element $\pi \in B_n$ which Reiner defined is, as in the symmetric group case, the sum of all positions of *descents* in π. (There is a natural notion of "descent" for any Coxeter group.) Concretely, it is

$$\operatorname{maj}_B \pi := \chi(\pi_n < 0) + \sum_{i=1}^{n-1} i \cdot \chi(\pi_i >_B \pi_{i+1}),$$

where we impose the order $1 <_B 2 <_B \cdots <_B n <_B -n <_B \cdots <_B -2 <_B -1$ on our ground set $\{\pm1, \pm2, \dots, \pm n\}$, and where $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise.

[26]In order to give a reader who is not acquainted with the language and theory of reflection groups an idea what this question is referring to, I mention that all finite reflection groups have been classified, each having been assigned a certain "type." So, usually one proves something for the symmetric group $\mathfrak{S}_n$, which, according to this classification, has type A_{n-1}, and then somebody (which could be oneself) will ask the question "Can you also do this for the other types?", meaning whether or not there exists an analogous result for the other finite reflection groups.

There is overwhelming computational evidence[27] that the "major-determinant" for B_n, i.e., the determinant (5.87) with stat $= \operatorname{maj}_B$ and with $\mathfrak{S}_n$ replaced by B_n, factors completely into cyclotomic polynomials.

Conjecture 46. *For any positive integer n, we have*

$$\det_{\sigma,\pi\in B_n}\left(q^{\operatorname{maj}_B(\sigma\pi^{-1})}\right)=\prod_{i=1}^{n}(1-q^{2i})^{2^{n-1}n!/i}\prod_{i=2}^{n}(1-q^{i})^{2^{n}n!(i-1)/i}. \tag{5.88}$$

A different major index for B_n was proposed by Adin and Roichman in [3]. It arises there naturally in a combinatorial study of *polynomial algebras which are diagonally invariant under B_n*. (In fact, more generally, *wreath products* of the form $C_m\wr\mathfrak{S}_n$, where C_m is the cyclic group of order m, and their diagonal actions on polynomial algebras are studied in [3]. These groups are also sometimes called *generalised reflection groups.* In this context, B_n is the special case $C_2\wr\mathfrak{S}_n$.) If we write $\operatorname{neg}\pi$ for the number of i for which π_i is negative, then the *flag-major index* fmaj of Adin and Roichman is defined by

$$\operatorname{fmaj}\pi:=2\operatorname{maj}_A\pi+\operatorname{neg}\pi, \tag{5.89}$$

where maj_A is the "ordinary" major index due to MacMahon,

$$\operatorname{maj}_A\pi:=\sum_{i=1}^{n-1}i\cdot\chi(\pi_i>\pi_{i+1}).$$

If one now goes to the computer and calculates the determinant on the left-hand side of (5.88) with maj_B replaced by fmaj for $n=1,2,3,4,5$ (see Footnote 27 for the precise

[27]The reader may wonder what this computational evidence could actually be. After all, we are talking about a determinant of a matrix of size $2^n n!$. More concretely, for $n=1,2,3,4,5$ these are matrices of size 2, 8, 48, 384, 3840, respectively. While *Maple* or *Mathematica* have no problem to compute these determinants for $n=1$ and $n=2$, it takes already considerable time to do the computation for $n=3$, and it is, of course, completely hopeless to let them compute the one for $n=4$, a determinant of a matrix of size 384 which has polynomial entries (cf. Footnote 18). However, the results for $n=1,2,3$ already "show" that the determinant will factor completely into factors of the form $1-q^i$, $i=1,2,\dots,2n$. One starts to expect the same to be true for higher n. To get a formula for $n=4$, one would then apply the tricks explained in Footnote 19. That is, one specialises q to 4, at which value the first 8 cyclotomic polynomials (in fact, even more) are clearly distinguishable by their prime factorisations, and one computes the determinant. The exponents of the various factors $1-q^i$ can then be extracted from the exponents of the prime factors in the prime factorisation of the determinant with $q=4$. Unfortunately, the data collected for $n=1,2,3,4$ do not suffice to come up with a guess, and, on the other hand, *Maple* and *Mathematica* will certainly be incapable to compute a determinant of a matrix of size 3840 (which, just to store it on the disk, occupies already 10 megabytes ...). So then, what did I mean when I said that the conjecture is based on data including $n=5$? This turned out to become a "test case" for *LinBox*, a C++ template library for exact high-performance linear algebra [52], which is freely available under `http://linalg.org`. To be honest, I was helped by Dave Saunders and Zhendong Wan (two of the developers) who applied *LinBox* to do rank and Smith normal form computations for the specialised matrix with respect to various prime powers (each of which taking several hours). The specific computational approach that worked here is quite recent (thus, it came just in time for our purpose), and is documented in [160]. The results of the computations made it possible to come up with a "sure" prediction for the exponents with which the various prime factors occur in the prime factorisation of the specialised determinant. As in the case $n=4$, the exponents of the various factors $1-q^i$, $i=1,2,\dots,2n$ can then easily be extracted. (The guesses were subsequently also tested with special values of q other than $q=4$.)

meaning of "calculating the determinant for $n = 1, 2, 3, 4, 5$"), then again the results factor completely into cyclotomic factors. Even more generally, it seems that one can treat the two parts on the right-hand side of (5.89), that is "major index" and "number of negative letters," separately.

Conjecture 47. *For any positive integer n, we have*

$$\det_{\sigma,\pi\in B_n}\left(q^{\operatorname{maj}_A(\sigma\pi^{-1})}p^{\operatorname{neg}(\sigma\pi^{-1})}\right)=\prod_{i=1}^{n}(1-p^{2i})^{2^{n-1}n!/i}\prod_{i=2}^{n}(1-q^{i})^{2^n n!(i-1)/i}. \tag{5.90}$$

I should remark that Adin and Roichman have shown in [3] that the statistics fmaj is equidistributed with the statistics length on B_n. However, even in the case where we just look at the flag-major determinant (that is, the case where $q = p^2$ in Conjecture 47), this does not seem to help. (Neither length nor flag-major index satisfy a simple law with respect to multiplication of signed permutations.) In fact, from the data one sees that the flag-major determinants are different from the length determinants (that is, the determinants (5.87), where $\mathfrak{S}_n$ is replaced by B_n and stat is flag-major, respectively length).

Initially, I had my program wrong, and, instead of taking the (ordinary) major index maj_A of the signed permutation $\pi = \pi_1\pi_2\dots\pi_n$ in (5.89), I computed taking the major index of the *absolute value* of π. This absolute value is obtained by forgetting all signs of the letters of π, that is, writing $|\pi|$ for the absolute value of π, $|\pi| = |\pi_1|\,|\pi_2|\,\dots\,|\pi_n|$. Curiously, it seems that also this "wrong" determinant factors nicely. (Again, the evidence for this conjecture is based on data which were obtained in the way described in Footnote 27.)

Conjecture 48. *For any positive integer n, we have*

$$\det_{\sigma,\pi\in B_n}\left(q^{\operatorname{maj}_A|\sigma\pi^{-1}|}p^{\operatorname{neg}(\sigma\pi^{-1})}\right)=(1-p^2)^{2^{n-1}n\cdot n!}\prod_{i=2}^{n}(1-q^{i})^{2^n n!(i-1)/i}. \tag{5.91}$$

Since, as I indicated earlier, Adin and Roichman actually define a flag-major index for wreath products $C_m\wr\mathfrak{S}_n$, a question that suggests itself is whether or not we can expect closed product formulae for the corresponding determinants. Clearly, since we are now dealing with determinants of the size $m^n n!$, computer computations will exhaust our computer's resources even faster if $m > 2$. The calculations that I was able to do suggest strongly that there is indeed an extension of the statement in Conjecture 47 to the case of arbitrary m, if one uses the definition of major index and the "negative" statistics for $C_m\wr\mathfrak{S}_n$ as in [3]. (See [3, Section 3] for the definition of the major index. The sum on the right-hand side of [3, (3.1)] must be taken as the extension of the "negative" statistics neg to $C_m\wr\mathfrak{S}_n$.)

Problem 49. *Find and prove the closed form evaluation of*

$$\det_{\sigma,\pi\in C_m\wr\mathfrak{S}_n}\left(q^{\operatorname{maj}(\sigma\pi^{-1})}p^{\operatorname{neg}(\sigma\pi^{-1})}\right), \tag{5.92}$$

where maj *and* neg *are the extensions to $C_m\wr\mathfrak{S}_n$ of the statistics* maj_A *and* neg *in Conjecture* 47*, as described in the paragaph above.*

Together with Brenti, Adin and Roichman proposed another major statistics for signed permutations in [1]. They call it the *negative major index*, denoted nmaj, and it is defined as the sum of the ordinary major index and the sum of the absolute values of the negative letters, that is,

$$\operatorname{nmaj}\pi := \operatorname{maj}_A \pi + \operatorname{sneg}\pi,$$

where $\operatorname{sneg}\pi := -\sum_{i=1}^{n} \chi(\pi_i < 0)\pi_i$. Also for this statistics, the corresponding determinant seems to factor nicely. In fact, it seems that one can again treat the two components of the definition of the statistics, that is, "major index" and "sum of negative letters," separately. (Once more, the evidence for this conjecture is based on data which were obtained in the way described in Footnote 27.)

Conjecture 50. *For any positive integer n, we have*

$$\det_{\sigma,\pi\in B_n}\left(q^{\operatorname{maj}_A(\sigma\pi^{-1})}p^{\operatorname{sneg}(\sigma\pi^{-1})}\right) = \prod_{i=1}^{n}(1-p^{2i^2})^{2^{n-1}n!/i}\prod_{i=2}^{n}(1-q^i)^{2^n n!(i-1)/i}. \tag{5.93}$$

If one compares the (conjectured) result with the (conjectured) one for the "flag-major determinant" in Conjecture 47, then one notices the somewhat mind-boggling fact that one obtains the right-hand side of (5.93) from the one of (5.90) by simply replacing (in the factored form of the latter) $1-p^{2i}$ by $1-p^{2i^2}$, everything else, the exponents, the "q-part", is identical. It is difficult to imagine an intrinsic explanation why this should be the case.

Since Thibon's proof of the evaluation of the determinant (5.87) with stat being the (ordinary) major index for permutations (see [109, Appendix C]) involved the *descent algebra* of the symmetric group, viewed in terms of *non-commutative symmetric functions*, one might speculate that to prove Conjectures 46–48 and 50 it may be necessary to work with B_n versions of descent algebras (which exist, see [175]) and non-commutative symmetric functions (which also exist, see [32]).

Adriano Garsia points out that all the determinants in (5.87), (5.88), (5.90)–(5.93) are special instances of *group determinants*. (See the excellent survey article [120] for information on group determinants.) The main theorem on group determinants, due to Frobenius, says that a general group determinant factorises into irreducible factors, each of which corresponding to an irreducible representation of the group, and the exact exponent to which the irreducible factor is raised is the degree of the corresponding irreducible representation. In view of this, Garsia poses the following problem, a solution of which would refine the above conjectures, Problem 49, and the earlier mentioned results of Varchenko, Zagier, and Thibon.

Problem 51. *For each of the above special group determinants, determine the closed formula for the value of the irreducible factor corresponding to a fixed irreducible representation of the involved group ($\mathfrak{S}_n$, B_n, $C_m \wr \mathfrak{S}_n$, respectively).*

It seems that a solution to this problem has not even been worked out for the determinant which is the subject of the results of Varchenko and Zagier, that is, for the determinant (5.87) with stat being the number of inversions.

For further work on statistics for (generalised) reflection groups (thus providing further prospective candidates for forming interesting determinants), I refer the reader to [1, 2, 18, 22, 23, 26, 27, 64, 65, 66, 79, 149]. I must report that, somewhat disappointingly, it seems that the various major indices proposed for the group D_n of even signed permutations (see [26, 27, 150]) apparently do not give rise to determinants in the same way as above that have nice product formulae. This remark seems to also apply to determinants formed in an anlogous way by using the various statistics proposed for the alternating group in [148].

5.8. More poset and lattice determinants. Continuing the discussion of determinants which arise under the influence of the above-mentioned "reflection group disease," we turn our attention to two miraculous determinants which were among the last things Rodica Simion was able to look at. Some of her considerations in this direction are reported in [165].

The first of the two is a determinant of a matrix the rows and columns of which are indexed by *type B non-crossing partitions.* This determinant is inspired by the evaluation of an analogous one for *ordinary* non-crossing partitions (that is, in "reflection group language," *type A non-crossing partitions*), due to Dahab [50] and Tutte [183] (see [109, Theorem 57, (3.69)]). Recall (see [177, Ch. 1 and 3] for more information) that a (*set*) *partition* of a set S is a collection $\{B_1, B_2, \dots, B_k\}$ of pairwise disjoint non-empty subsets of S such that their union is equal to S. The subsets B_i are also called *blocks* of the partition. One partially orders partitions by refinement. With respect to this partial order, the partitions form a lattice. We write $\pi \vee_A \gamma$ (the A stands for the fact that, in "reflection group language", we are looking at "type A partitions") for the join of π and γ in this lattice. Roughly speaking, the join of π and γ is formed by considering altogether all the blocks of π and γ. Subsequently, whenever we find two blocks which have a non-empty intersection, we merge them into a bigger block, and we keep doing this until all the (merged) blocks are pairwise disjoint.

If $S = \{1, 2, \dots, n\}$, we call a partition *non-crossing* if for any $i < j < k < l$ the elements i and k are in the same block at the same time as the elements j and l are in the same block only if these two blocks are the same. (I refer the reader to [168] for a survey on non-crossing partitions.)

Reiner [151] introduced non-crossing partitions in type B. Partitions of type B_n are (ordinary) partitions of $\{1, 2, \dots, n, -1, -2, \dots, -n\}$ with the property that whenever B is a block then so is $-B := \{-b : b \in B\}$, and that there is at most one block B with $B = -B$. A block B with $B = -B$, if present, is called the *zero-block* of the partition. We denote the set of all type B_n partitions by Π_n^B, the number of zero blocks of a partition π by $\operatorname{zbk}\pi$, and we write $\operatorname{nzbk}\pi$ for half of the number of the non-zero blocks. Type B_n non-crossing partitions are a subset of type B_n partitions. Imposing the order $1 < 2 < \cdots < n < -1 < -2 < \cdots < -n$ on our ground-set, the definition of type B_n non-crossing partitions is identical with the one for type A non-crossing partitions, that is, given this order on the ground-set, a B_n partition is called *non-crossing* if for any $i < j < k < l$ the elements i and k are in the same block at the same time as the elements j and l are in the same block only if these two blocks are the same. We write NC_n^B for the set of all B_n non-crossing partitions.

The determinant defined by type B_n non-crossing partitions that Simion tried to evaluate was the one in (5.94) below.[28] The use of the "type A" join $\vee_A$ for two

[28] In fact, instead of $\vee_A$, the "ordinary" join, she used the join in the type B_n partition lattice Π_n^B. However, the number of *non-zero* blocks will be the same regardless of whether we take the join of two type B_n non-crossing partitions with respect to "ordinary" join or with respect to "type B_n" join. This is in contrast to the numbers of zero blocks, which can differ largely. (To be more precise, one way to form the "type B_n" join is to first form the "ordinary" join, and then merge all zero blocks

type B non-crossing partitions in (5.94) may seem strange. However, this is certainly a well-defined operation. The result may neither be a type B partition nor a non-crossing one, it will just be an ordinary partition of the ground-set $\{1,2,\dots,n,-1,-2,\dots,-n\}$. We extend the notion of "zero block" and "non-zero block" to these objects in the obvious way. Simion observed that, as in the case of the analogous type A_n partition determinant due to Dahab and Tutte, it factors apparently completely into factors which are Chebyshev polynomials. Based on some additional numerical calculations,[29] I propose the following conjecture.

Conjecture 52. *For any positive integer n, we have*

$$\det_{\pi,\gamma\in \mathrm{NC}^B_n}\left(q^{\mathrm{nzbk}(\pi\vee_A\gamma)}\right)=\prod_{i=1}^n\left(\frac{U_{3i-1}(\sqrt{q}/2)}{U_{i-1}(\sqrt{q}/2)}\right)^{\binom{2n}{n-i}},\tag{5.94}$$

where $U_m(x):=\sum_{j\ge0}(-1)^j\binom{m-j}{j}(2x)^{m-2j}$ is the m-th Chebyshev polynomial of the second kind*.*

If proved, this would solve Problem 1 in [165]. It would also solve Problem 2 from [165], because $U_{m-1}(\sqrt{q}/2)$ is, up to multiplication by a power of q, equal to the product $\prod_{j\mid m}f_j(q)$, where the polynomials $f_j(q)$ are the ones of [165]. A simple computation then shows that, when the right-hand side product of (5.94) is expressed in terms of the $f_j(q)$'s, one obtains

$$\prod_{k=1}^n f_{3k}(q)^{e_{n,k}},\tag{5.95}$$

where

$$e_{n,k}=\sum_{\substack{\ell=1\\ \ell\not\equiv0\ (\mathrm{mod}\ 3)}}^{\lfloor n/k\rfloor}\binom{2n}{n-\ell k}.$$

This agrees with the data in [165] and with the further ones I have computed (see Footnote 29).

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into one big block.) The reason that I insist on using $\vee_A$ is that this is crucial for the more general Conjecture 53. To tell the truth, the discovery of the latter conjecture is due to a programming error on my behalf (that is, originally I aimed to program the "type B" join, but it happened to be the "ordinary" join ...).

[29]Evidently, more than five years later, thanks to technical progress since then, one can go much farther when doing computer calculations. The evidence for Conjecture 52 which I have is based on, similar to the conjectures and calculations on determinants for signed permutations in Subsection 5.7 (see Footnote 27), the exact form of the determinants for $n=1,2,3,4$, which were already computed by Simion, and, essentially, the exact form of the determinants for $n=5$ and 6. By "essentially" I mean, as earlier, that I computed the determinant for many special values of q, which then let me make a guess on the basis of comparison of the prime factors in the factorised results with the prime factors of the candidate factors, that is the irreducible factors of the Chebyshev polynomials. Finally, for guessing the general form of the exponents, the available data were not sufficient for `Rate` (see Footnote 4). So I consulted the fabulous *On-Line Encyclopedia of Integer Sequences* (`http://www.research.att.com/~njas/sequences/Seis.html`), originally created by Neil Sloane and Simon Plouffe [173], and since many years continuously further developed by Sloane and his team [172]. An appropriate selection from the results turned up by the Encyclopedia then led to the exponents on the right-hand side of (5.94).

Even more seems to be true. The following conjecture predicts the evaluation of the more general determinant where we also keep track of the zero blocks.

Conjecture 53. *For any positive integer n, we have*

$$\det_{\pi,\gamma\in \mathrm{NC}_n^B}\left(q^{\mathrm{nzbk}(\pi\vee_A\gamma)}z^{\mathrm{zbk}(\pi\vee_A\gamma)}\right)=z^{\frac{1}{2}\binom{2n}{n}}\prod_{i=1}^{n}\Big(2T_{2i}(\sqrt{q}/2)+2-z\Big)^{\binom{2n}{n-i}},\tag{5.96}$$

where $T_m(x):=\frac{1}{2}\sum_{j\ge0}(-1)^j\frac{m}{m-j}\binom{m-j}{j}(2x)^{m-2j}$ *is the m-th* Chebyshev polynomial of the first kind.

Again, the conjecture is supported by extensive numerical calculations. It is not too difficult to show, by using some identities for Chebyshev polynomials, that Conjecture 53 implies Conjecture 52.

The other determinant which Simion looked at (cf. [165, Problem 9ff]), was the B_n analogue of a determinant of a matrix the rows and columns of which are indexed by *non-crossing matchings*, due to Lickorish [124], and evaluated by Ko and Smolinsky [101] and independently by Di Francesco [68] (see [109, Theorem 58]). As we may regard (ordinary) non-crossing matchings as partitions all the blocks of which consist of two elements, we *define* a B_n *non-crossing matching* to be a B_n non-crossing partition all the blocks of which consist of two elements. We shall be concerned with B_{2n} non-crossing matchings, which we denote by NCmatch$(2n)$. With this notation, the following seems to be true.

Conjecture 54. *For any positive integer n, we have*

$$\det_{\pi,\gamma\in \mathrm{NCmatch}(2n)}\left(q^{\mathrm{nzbk}(\pi\vee_A\gamma)}z^{\mathrm{zbk}(\pi\vee_A\gamma)}\right)=\prod_{i=1}^{n}\Big(2T_{2i}(q/2)+2-z^2\Big)^{\binom{2n}{n-i}}.\tag{5.97}$$

The reader should notice the remarkable fact that, in the case that Conjectures 53 and 54 are true, the right-hand side of (5.97) is, up to a power of z, equal to the right-hand side of (5.96) with q replaced by q^2 and z replaced by z^2. An intrinsic explanation why this should be the case is not known. An analogous relation between the determinants of Tutte and of Lickorish, respectively, was observed, and proved, in [47]. Also here, no intrinsic explanation is known.

The reader is referred to [165] for further open problems related to the determinants in Conjectures 52–54. Finally, it may also be worthwhile to look at determinants defined using D_n non-crossing partitions and non-crossing matchings, see [16] and [151] for two possible definitions of those.

The reader may have wondered why in Conjectures 52 and 53 we considered determinants defined by type B_n non-crossing partitions, which form in fact a lattice, but used the extraneous type A join[30] in the definition of the determinant, instead of the join which is intrinsic to the lattice of type B_n non-crossing partitions. In particular, what would happen if we would make the latter choice? As it turns out, for that situation there exists an elegant general theorem due to Lindström [125], which I missed to state in [109]. I refer to [177, Ch. 3] for the explanation of the poset terminology used in the statement.

[30]and not even the one in the type A *non-crossing* partition lattice!

Theorem 55. *Let L be a finite meet semilattice, R be a commutative ring, and $f : L \times L \to R$ be an incidence function, that is, $f(x,y) = 0$ unless $x \wedge y = x$. Then*

$$\det_{x,y\in L}\big(f(x\wedge y,x)\big) = \prod_{y\in L}\left(\sum_{x\in L}\mu(x,y)f(x,y)\right), \tag{5.98}$$

where μ is the Möbius function of L. □

Clearly, this does indeed answer our question, we just have to specialise $f(x,y) = h(x)$ for $x \wedge y = x$, where h is some function from L to R. The fact that the above theorem talks about meets instead of joins is of course no problem because this is just a matter of convention.

Having an answer in such a great generality, one is tempted to pose the problem of finding a general theorem that would encompass the above-mentioned determinant evaluations due to Tutte, Dahab, Ko and Smolinsky, Di Francesco, as well as Conjectures 52 and 53. This problem is essentially Problem 6 in [165].

Problem 56. *Let L and L' be two lattices (semilattices?) with $L' \subseteq L$. Furthermore, let R be a commutative ring, and let f be a function from L to R. Under which conditions is there a compact formula for the determinant*

$$\det_{x,y\in L'}\big(f(x\wedge_L y)\big), \tag{5.99}$$

where $\wedge_L$ is the meet operation in L?

By specialisation in Theorem 55, one can derive numerous corollaries. For example, a very attractive one is the evaluation of the "GCD determinant" due to Smith [174]. (In fact, Smith's result is a more general one for *factor closed subsets* of the positive integers.)

Theorem 57. *For any positive integer n, we have*

$$\det_{1\le i,j\le n}\big(\gcd(i,j)\big) = \prod_{i=1}^{n}\phi(i),$$

where ϕ denotes the Euler totient function. □

An interesting generalisation of Theorem 55 to posets was given by Altinişik, Sagan and Tuğlu [6]. Again, all undefined terminology can be found in [177, Ch. 3].

Theorem 58. *Let P be a finite poset, R be a commutative ring, and $f,g : P\times P \to R$ be two incidence functions, that is, $f(x,y)=0$ unless $x \le y$ in P, the same being true for g. Then*

$$\det_{x,y\in P}\left(\sum_{z\in P} f(z,x)g(z,y)\right) = \prod_{x\in P} f(x,x)g(x,x).$$

□

The reader is referred to Section 3 of [6] for the explanation why this theorem implies Lindström's.

5.9. Determinants for compositions. Our next family of determinants consists of determinants of matrices the rows and columns of which are indexed by *compositions*. Recall that a composition of a non-negative integer n is a vector $(\alpha_1, \alpha_2, \dots, \alpha_k)$ of non-negative integers such that $\alpha_1 + \alpha_2 + \dots + \alpha_k = n$, for some k. For a fixed k, let $\mathcal{C}(n,k)$ denote the corresponding set of compositions of n. While working on a problem in global optimisation, Brunat and Montes [30] discovered the following surprising determinant evaluation. It allowed them to show how to explicitly express a multivariable polynomial as a *difference of convex functions*. In the statement, we use standard multi-index notation: if $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_k)$ are two compositions, we let

$$\boldsymbol{\alpha}^{\boldsymbol{\beta}} := \alpha_1^{\beta_1} \alpha_2^{\beta_2} \cdots \alpha_k^{\beta_k}.$$

Theorem 59. *For any positive integers n and k, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathcal{C}(n,k)} \left(\boldsymbol{\alpha}^{\boldsymbol{\beta}} \right) = n^{\binom{n+k-1}{k}+k-1} \prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+k-i-1}{k-2}}, \tag{5.100}$$

where 0^0 is interpreted as 1. □

In recent joint work [29, 31], Brunat, Montes and the author showed that there is in fact a polynomial generalisation of this determinant evaluation.

Theorem 60. *Let $\mathbf{x} = (x_1, x_2, \dots, x_k)$ be a vector of indeterminates, and let λ be an indeterminate. Then, for any non-negative integers n and k, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathcal{C}(n,k)} \left((\mathbf{x} + \lambda \boldsymbol{\alpha})^{\boldsymbol{\beta}} \right) = \lambda^{(k-1)\binom{n+k-1}{k}} \left(|\mathbf{x}| + \lambda n \right)^{\binom{n+k-1}{k}} \prod_{i=1}^{n} i^{(k-1)\binom{n+k-i-1}{k-1}}, \tag{5.101}$$

where $\mathbf{x} + \lambda \boldsymbol{\alpha}$ is short for $(x_1 + \lambda\alpha_1, x_2 + \lambda\alpha_2, \dots, x_k + \lambda\alpha_k)$, and where $|\mathbf{x}| = x_1 + x_2 + \dots + x_k$.

As a matter of fact, there is actually a binomial variant which implies the above theorem. Extending our multi-index notation, let

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_k}{\beta_k}.$$

Theorem 61. *Let $\mathbf{x} = (x_1, x_2, \dots, x_k)$ be a vector of indeterminates, and let λ be an indeterminate. Then, using the notation from Theorem 60, for any non-negative integers n and k, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathcal{C}(n,k)} \left(\binom{\mathbf{x} + \lambda \boldsymbol{\alpha}}{\boldsymbol{\beta}} \right) = \lambda^{(k-1)\binom{n+k-1}{k}} \prod_{i=1}^{n} \left(\frac{|\mathbf{x}| + (\lambda - 1)n + i}{i} \right)^{\binom{n+k-i-1}{k-1}}. \tag{5.102}$$

The above theorem is proved in [29] by identification of factors (see "Method 3" in Section 4). Theorem 61 follows by extracting the highest homogeneous component in (5.101). I report that, if one naively replaces "compositions" by "integer partitions" in the above considerations, then the arising determinants do not have nice product formulae.

Another interesting determinant of a matrix with rows indexed by compositions appears in the work of Bergeron, Reutenauer, Rosas and Zabrocki [21, Theorem 4.8] on

Hopf algebras of non-commutative symmetric functions. It was used there to show that a certain set of generators of non-commutative symmetric functions were algebraically independent. To state their determinant evaluation, we need to introduce some notation. Given a composition $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n with all summands α_i positive, we let $D(\boldsymbol{\alpha}) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$. Furthermore, for two compositions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of n, we write $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ for the (unique) composition $\boldsymbol{\gamma}$ of n with $D(\boldsymbol{\gamma}) = D(\boldsymbol{\alpha}) \cup D(\boldsymbol{\beta})$. Finally, $\boldsymbol{\alpha}!$ is short for $\alpha_1!\,\alpha_2! \cdots \alpha_k!$.

Theorem 62. *Let* Comp(n) *denote the set of all compositions of n all summands of which are positive. Then*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathrm{Comp}(n)} \big((\boldsymbol{\alpha} \cup \boldsymbol{\beta})!\big) = \prod_{\boldsymbol{\gamma}\in\mathrm{Comp}(n)} \prod_{i=1}^{\ell(\boldsymbol{\gamma})} a_{\gamma_i}, \tag{5.103}$$

where $\ell(\boldsymbol{\gamma})$ is the number of summands (*components*) *of the composition $\boldsymbol{\gamma}$, and where a_m denotes the number of* indecomposable permutations *of m* (*cf.* [178, Ex. 5.13(b)]). *These numbers can be computed recursively by $a_1 = 1$ and*

$$a_n = n! - \sum_{i=1}^{n-1} a_i (n-1)!, \qquad n > 1.$$

□

As explained in [21], one proves the theorem by factoring the matrix in (5.103) in the form CDC^t, where C is the "incidence matrix" of "refinement of compositions," and where D is a diagonal matrix. Thus, in particular, the LU-factorisation of the matrix is determined.

5.10. Two partition determinants. On the surface, *integer partitions* (see below for their definition) seem to be very closely related to compositions, as they can be considered as "compositions where the order of the summands is without importance." However, experience shows that integer partitions are much more complex combinatorial objects than compositions. This may be the reason that the "composition determinants" from the preceding subsection do not seem to have analogues for integer partitions. Leaving aside this disappointment, here *is* a determinant of a matrix in which rows and columns are indexed by integer partitions. This determinant arose in work on *linear forms of values of the Riemann zeta function evaluated at positive integers*, although the traces of it have now been completely erased in the final version of the article [114]. (The symmetric function calculus in Section 12 of the earlier version [113] gives a vague idea where it may have come from.)

Recall that the *power symmetric function* of degree d in $x_1, x_2, \dots, x_k$ is given by $x_1^d + x_2^d + \cdots + x_k^d$, and is denoted by $p_d(x_1, x_2, \dots, x_k)$. (See [121, Ch. 1 and 2], [130, Ch. I] and [178, Ch. 7] for in-depth expositions of the theory of *symmetric functions*.) Then, while working on [114], Rivoal and the author needed to evaluate the determinant

$$\det_{\lambda,\mu\in\mathrm{Part}(n,k)} \big(p_\lambda(\mu_1, \mu_2, \dots, \mu_k)\big), \tag{5.104}$$

where Part(n, k) is the set of integer partitions of n with at most k parts, that is, the set of all possibilities to write n as a sum of non-negative integers, $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, with $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_k \ge 0$ (the non-zero λ_i's being called the *parts of* λ), and where

$$p_\lambda(x_1, x_2, \dots, x_k) = p_{\lambda_1}(x_1, \dots, x_k) p_{\lambda_2}(x_1, \dots, x_k) \cdots p_{\lambda_k}(x_1, \dots, x_k).$$

Following the advice given in Section 2, we went to the computer and let it calculate the prime factorisations of the values of this determinant for small values of n and k. Indeed, the prime factors turned out to be always small so that we were sure that a "nice" formula exists for the determinant. However, even more seemed to be true. Recall that, in order to facilitate a proof of a (still unknown) formula, it is (almost) always a good idea to try to introduce more parameters (see [109, Sec. 2]). This is what we did. It led us consider the following determinant,

$$\det_{\lambda,\mu \in \operatorname{Part}(n,k)} \big(p_\lambda(\mu_1 + X_1, \mu_2 + X_2, \dots, \mu_k + X_k) \big), \tag{5.105}$$

where $X_1, X_2, \dots, X_k$ are indeterminates. Here are some values of the determinant (5.105) for special values of n and k. For $n = k = 3$ we obtain

$$6\,(X_1 - X_2 + 2)(X_1 - X_3 + 1)(X_2 - X_3 + 1)(X_1 + X_2 + X_3 + 3)^4,$$

for $n = 4$ and $k = 3$ we obtain

$$8\,(X_1 - X_2 + 1)(X_1 - X_2 + 2)(X_1 - X_2 + 3)(X_1 - X_3 + 2)\\ \times (X_2 - X_3 + 1)(X_1 + X_2 + X_3 + 4)^7,$$

for $n = 5$ and $k = 3$ we get

$$8\,(X_1 - X_2 + 1)(X_1 - X_2 + 2)(X_1 - X_2 + 3)(X_1 - X_2 + 4)(X_1 - X_3 + 2)\\ \times (X_1 - X_3 + 3)(X_2 - X_3 + 1)(X_2 - X_3 + 2)(X_1 + X_2 + X_3 + 5)^{11},$$

for $n = 6$ and $k = 3$ we get

$$576\,(X_1 - X_2 + 1)(X_1 - X_2 + 2)^2(X_1 - X_2 + 3)^2(X_1 - X_2 + 4)(X_1 - X_2 + 5)\\ \times (X_1 - X_3 + 1)(X_1 - X_3 + 2)(X_1 - X_3 + 3)(X_1 - X_3 + 4)(X_2 - X_3 + 1)^2\\ \times (X_2 - X_3 + 2)^2(X_1 + X_2 + X_3 + 6)^{16},$$

for $n = 4$ and $k = 4$ we get

$$192\,(X_1 - X_2 + 1)(X_1 - X_2 + 2)(X_1 - X_2 + 3)(X_1 - X_3 + 2)(X_2 - X_3 + 1)\\ \times (X_1 - X_4 + 1)(X_2 - X_4 + 1)(X_3 - X_4 + 1)(X_1 + X_2 + X_3 + X_4 + 4)^7$$

while for $n = 5$ and $k = 4$ we get

$$48\,(X_1 - X_2 + 1)(X_1 - X_2 + 2)(X_1 - X_2 + 3)(X_1 - X_2 + 4)(X_1 - X_3 + 2)\\ \times (X_1 - X_3 + 3)(X_2 - X_3 + 1)(X_2 - X_3 + 2)(X_1 - X_4 + 2)\\ \times (X_2 - X_4 + 1)(X_3 - X_4 + 1)(X_1 + X_2 + X_3 + X_4 + 5)^{12}.$$

It is "therefore" evident that there will be one factor which is a power of $n + \sum_{i=1}^k X_i$, whereas the other factors will be of the form $X_i - X_j + c_{i,j}$, again raised to some power. In fact, the determinant is easy to compute for $k = 1$ because, in that case, it reduces to a special case of the Vandermonde determinant evaluation. However, we were not able to come even close to computing (5.105) in general. (As I already indicated, finally we managed to avoid the determinant evaluation in our work in [114].)

To proceed further, we remark that evaluating (5.105) is equivalent to evaluating the same determinant, but with the power symmetric functions p_λ replaced by the *Schur functions* s_λ, because the transition matrix between these two bases of symmetric functions is the *character table of the symmetric group* of the corresponding order (cf. [130, Ch. I, Sec. 7]), the determinant of which is known (see Theorem 64 below; since in our determinants (5.104) and (5.105) the indices range over all partitions of n *with at most* k *parts*, it is in fact the refinement given in Theorem 65 which we have to apply). The determinant with Schur functions has some advantages over the one with power symmetric functions since the former decomposes into a finer block structure. Alain Lascoux observed that, in fact, there is a generalisation of the Schur function determinant to *Graßmannian Schubert polynomials*, which contains another set of variables, $Y_1, Y_2, \dots, Y_{n+k-1}$. More precisely, given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, let $\mathbb{Y}_\lambda(X_1, X_2, \dots, X_k; Y_1, Y_2, \dots)$ denote the polynomial in the variables $X_1, X_2, \dots, X_k$ and $Y_1, Y_2, \dots$, defined by (see [121, Sections 1.4 and 9.7; the order of the B_{k_i} should be reversed in (9.7.2) and analogous places])

$$\mathbb{Y}_\lambda(X_1, X_2, \dots, X_k; Y_1, Y_2, \dots) := \det_{1\le i,j\le k}\left(S_{\lambda_i-i+j}(X_1, \dots, X_k; Y_1, \dots, Y_{\lambda_i+k-i})\right),$$

where the entries of the determinant are given by

$$\sum_{m=0}^{\infty} S_m(X_1, \dots, X_k; Y_1, \dots, Y_l)x^m = \frac{\prod_{i=1}^{l}(1-Y_ix)}{\prod_{i=1}^{k}(1-X_ix)}. \tag{5.106}$$

The Graßmannian Schubert polynomial $\mathbb{Y}_\lambda$ reduces to the Schur function s_λ when all the variables Y_i, $i = 1, 2, \dots$, are set equal to 0. Given these definitions, Alain Lascoux (private communication) established the following result.

Theorem 63. *Let* $X_1, X_2, \dots, X_k, Y_1, Y_2, \dots, Y_{n+k-1}$ *be indeterminates. Then,*

$$\det_{\lambda,\mu\in \operatorname{Part}(n,k)}\big(\mathbb{Y}_\lambda(\mu_1+X_1, \mu_2+X_2, \dots, \mu_k+X_k; Y_1, Y_2, \dots)\big)\\ = \Pi(n,k)\prod_{\sigma\in G(n,k)}\left(n+\sum_{i=1}^{k}X_i-\sum_{i=1}^{k}Y_{\sigma(i)}\right), \tag{5.107}$$

where $G(n,k)$ *denotes the set of all Graßmannian permutations,*[31] *the descent of which (if existent) is at* k*, and which contain at most* $n-1$ *inversions, and where* $\Pi(n,k)$ *is*

[31] A permutation σ in $\mathfrak{S}_\infty$ (the set of all permutations of the natural numbers $\mathbb{N}$ which fix all but a finite number of elements of $\mathbb{N}$) is called *Graßmannian* if $\sigma(i) < \sigma(i+1)$ for all i except possibly for one, the latter being called the *descent* of σ (see [129, p. 13]). An inversion of σ is a pair (i,j) such that $i < j$ but $\sigma(i) > \sigma(j)$. We remark that the number of Graßmannian permutations with descent (if existent) at k and at most $n-1$ inversions is equal to the number of partitions of at most $n-1$ with at most k parts (including the empty partition). More concretely, if we denote this number by $g_{n,k}$, the generating function of the numbers $g_{n,k}$ is given by

$$\sum_{n=1}^{\infty} g_{n,k}x^{n-1} = \frac{1}{1-q}\prod_{i=1}^{k}\frac{1}{1-q^i}.$$

given recursively by

$$\Pi(n,k)=\Pi(n,k-1)\Pi(n-k,k)\prod_{\mu\in\operatorname{Part}^+(n,k)}\prod_{j=1}^{k}(X_j+\mu_j-X_k), \tag{5.108}$$

$\operatorname{Part}^+(n,k)$ *denoting the set of partitions of n into* exactly k *(positive) parts, with initial conditions* $\Pi(n,k)=1$ *if* $k=1$ *or* $n\le 1$. *Explicitly,*

$$\Pi(n,k)=\prod_{\substack{w\\|w|_B<k-1}}\ \prod_{\mu\in\operatorname{Part}(n-|w|_Ak+\operatorname{inv}w,k-|w|_B)}\ \prod_{j=1}^{k-|w|_B}(X_j+\mu_j-X_{k-|w|_B}), \tag{5.109}$$

where the product over w runs over all finite-length words w with letters from $\{A,B\}$, including the empty word. The notation $|w|_B$ means the number of occurrences of B in w, with the analogous meaning for $|w|_A$. The quantity $\operatorname{inv}w$ *denotes the number of inversions of $w=w_1w_2\dots$, which is the number of pairs of letters (w_i,w_j), $i<j$, such that $w_i=B$ and $w_j=A$.*

We obtain the evaluation of the determinant (5.104) if we set $X_i=Y_i=0$ for all i in the above theorem and multiply by (5.114). Likewise, we obtain the evaluation of the determinant (5.105) if we set $Y_i=0$ for all i in the above theorem and multiply the result by (5.114).

For example, here is the determinant (5.107) for $n=3$ and $k=2$,

$$(X_1-X_2+2)(X_1+X_2-Y_1-Y_2+3)(X_1+X_2-Y_1-Y_3+3)\\ \times(X_1+X_2-Y_2-Y_3+3)(X_1+X_2-Y_1-Y_4+3),$$

and the following is the one for $n=k=3$,

$$(X_1-X_2+2)(X_1-X_3+1)(X_2-X_3+1)\\ \times(X_1+X_2+X_3-Y_1-Y_2-Y_3+3)(X_1+X_2+X_3-Y_1-Y_2-Y_4+3)\\ \times(X_1+X_2+X_3-Y_1-Y_3-Y_4+3)(X_1+X_2+X_3-Y_1-Y_2-Y_5+3).$$

In the sequel, I sketch Lascoux's proof of Theorem 63. The first step consists in applying the *Monk formula for double Schubert polynomials* in the case of Graßmannian Schubert polynomials (see [103]),

$$\left(\sum_{i=1}^k X_i-\sum_{i=1}^k Y_{\lambda_{k-i+1}+i}\right)\mathbb{Y}_\lambda=\mathbb{Y}_{\lambda+(1,0,0,\dots)}+\sum_\mu \mathbb{Y}_\mu,$$

where $\mathbb{Y}_\lambda$ is short for $\mathbb{Y}_\lambda(X_1,X_2,\dots,X_k;Y_1,Y_2,\dots)$, and where the sum on the right-hand side is over all partitions μ of the same size as $\lambda+(1,0,0,\dots)$ but lexicographically smaller. Clearly, by using this identity, appropriate row operations in the determinant (5.107) show that

$$n+\sum_{i=1}^k X_i-\sum_{i=1}^k Y_{\sigma(i)}, \tag{5.110}$$

is one of its factors, the relation between σ and λ being $\sigma(i)=\lambda_{k-i+1}+i$, $i=1,2,\dots,k$. In particular, σ can be extended (in a unique way) to a Graßmannian permutation. Moreover, doing these row operations, and taking out the factors of the form (5.110), we collect on the one hand the product in (5.107), and we may on the other hand reduce the determinant (5.107) to a determinant of the same form, in which, however, the partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ run over all partitions of size *at most* n with the *additional property that* $\lambda_1 = \lambda_2$ (instead of over all partitions from $\operatorname{Part}(n,k)$).

As it turns out, the determinant thus obtained is independent of the variables $Y_1, Y_2, \dots, Y_{n+k-1}$. Indeed, if we expand each Schubert polynomial $\mathbb{Y}_\lambda(\mu_1+X_1, \mu_2+X_2, \dots, \mu_k+X_k; Y_1, Y_2, \dots)$ in the determinant as a linear combination of Schur functions in $X_1, X_2, \dots, X_k$ with coefficients being polynomials in the $Y_1, Y_2, \dots, Y_{n+k-1}$, then, by also using that

$$S_1(\mu_1 + X_1, \mu_2 + X_2, \dots, \mu_k + X_k) = n + \sum_{i=1}^{k} X_i$$

is independent of μ, it is not difficult to see that one can eliminate all the Y_i's by appropriate row operations.

To summarise the current state of the discussion: we have already explained the occurrence of the product on the right-hand side of (5.107) as a factor of the determinant. Moreover, the remaining factor is given by a determinant of the same form as in (5.107),

$$\det_{\lambda,\mu} \left(\mathbb{Y}_\lambda(\mu_1 + X_1, \mu_2 + X_2, \dots, \mu_k + X_k; Y_1, Y_2, \dots)\right), \tag{5.111}$$

where, as before, μ runs over all partitions in $\operatorname{Part}(n,k)$, but where λ runs over partitions $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of size at most n, with the additional restriction that $\lambda_1 = \lambda_2$. The final observation was that this latter determinant is in fact independent of the Y_i's. This allows us to specify Y_1 arbitrarily, say $Y_1 = X_k$. Now another property of double Schubert polynomials, namely that (this follows from the definition of double Schubert polynomials by means of divided differences, see [121, (10.2.3)], and standard properties of divided differences)

$$\mathbb{Y}_{\rho+(1,1,\dots,1)}(X_1, X_2, \dots, X_k; Y_1, Y_2, \dots) = \mathbb{Y}_\rho(X_1, X_2, \dots, X_k; Y_2, \dots) \prod_{j=1}^{k} (X_j - Y_1) \tag{5.112}$$

comes in handy. (In the vector $(1,1,\dots,1)$ on the left-hand side of (5.112) there are k occurrences of 1.) Namely, if $Y_1 = X_k$, the matrix in (5.111) (of which the determinant is taken), $M(n,k; Y_1, Y_2, \dots)$ say, decomposes in block form. If λ is a partition with $\lambda_k > 0$ and μ is a partition with $\mu_k = 0$, then, because of (5.112), the corresponding entry in (5.111) vanishes. Furthermore, in the block where λ and μ are partitions with $\lambda_k = \mu_k = 0$, because of the definition (5.106) of the quantities $S_m(.)$ the corresponding entry reduces to

$$\begin{aligned}\mathbb{Y}_\lambda(\mu_1 + X_1, \mu_2 + X_2, \dots, \mu_{k-1} + X_{k-1}, X_k; X_k, Y_2, \dots)&\\ = \mathbb{Y}_\lambda(\mu_1 + X_1, \mu_2 + X_2, \dots, \mu_{k-1} + X_{k-1}; Y_2, \dots)&.\end{aligned}$$

In other words, this block is identical with $M(n, k-1; Y_2, \dots)$. Finally, in the block indexed by partitions λ and μ with $\lambda_k > 0$ and $\mu_k > 0$, we may use (5.112) to factor $\prod_{j=1}^{k}(X_j + \mu_j - X_k)$ out of the column indexed by μ, for all such μ. What remains is identical with $M(n-k, k; Y_2, \dots)$. Taking determinants, we obtain the recurrence (5.108). (Here we use again that the determinants in (5.111), that is, in particular, the determinants of $M(n,k;Y_1,\dots)$, $M(n,k-1;Y_2,\dots)$, and of $M(n-k,k;Y_2,\dots)$, are all independent of the Y_i's.) The explicit form (5.109) for $\Pi(n,k)$ can be easily derived by induction on n and k.

A few paragraphs above, we mentioned in passing another interesting determinant of a matrix the rows and columns of which are indexed by (integer) partitions: the *determinant of the character table of the symmetric group* $\mathfrak{S}_n$ (cf. [89, Cor. 6.5]). Since this is a classical and beautiful determinant evaluation which I missed to state in [109], I present it now in the theorem below. There, the notation $\lambda \vdash n$ stands for "λ is a partition of n." For all undefined notation, I refer the reader to standard texts on the representation theory of symmetric groups, as for example [89, 90, 159].

Theorem 64. *For partitions λ and ρ of n, let $\chi^\lambda(\rho)$ denote the value of the irreducible character χ^λ evaluated at a permutation of cycle type ρ. Then*

$$\det_{\lambda,\rho\vdash n}\left(\chi^\lambda(\rho)\right) = \prod_{\mu\vdash n}\prod_{i\ge1}\mu_i. \tag{5.113}$$

In words: the determinant of the character table of the symmetric group $\mathfrak{S}_n$ is equal to the products of all the parts of all the partitions of n. □

A refinement of this statement, where we restrict to partitions of n with at most k parts, is the following.

Theorem 65. *With the notation of Theorem 64, for all positive integers n and k, $n \ge k$, we have*

$$\det_{\lambda,\rho\in\mathrm{Part}(n,k)}\left(\chi^\lambda(\rho)\right) = \prod_{\mu\in\mathrm{Part}(n,k)}\prod_{i\ge1} m_i(\mu)!, \tag{5.114}$$

where $m_i(\mu)$ is the number of times i occurs as a part in the partition μ. □

This determinant evaluation follows from the decomposition of the *full* character table of $\mathfrak{S}_n$ in the form

$$\left(\chi^\lambda(\rho)\right)_{\lambda,\rho\vdash n} = L\cdot K^{-1}, \tag{5.115}$$

where $L=(L_{\lambda,\mu})_{\lambda,\mu\vdash n}$ is the transition matrix from power symmetric functions to monomial symmetric functions, and where $K=(K_{\lambda,\mu})_{\lambda,\mu\vdash n}$ is the *Kostka matrix*, the transition matrix from Schur functions to monomial symmetric functions (see [130, Ch. I, (6.12)]). For, if we order the partitions of n so that the partitions in $\mathrm{Part}(n,k)$ come before the partitions in $\mathrm{Part}(n,k+1)$, $k=1,2,\dots,n-1$, and within $\mathrm{Part}(n,k)$ lexicographically, then, with respect to this order, the matrix L is lower triangular and the matrix K, and hence also K^{-1}, is upper triangular. Furthermore, the matrix K, and hence also K^{-1}, is even block diagonal, the blocks along the diagonal being the ones which are formed by the rows and columns indexed by the partitions in $\mathrm{Part}(n,k)$, $k=1,2,\dots,n$. These facts together imply that the decomposition (5.115) restricts to the submatrices indexed by partitions in $\mathrm{Part}(n,k)$,

$$\left(\chi^\lambda(\rho)\right)_{\lambda,\rho\in\mathrm{Part}(n,k)} = (L_{\lambda,\mu})_{\lambda,\mu\in\mathrm{Part}(n,k)}\cdot(K_{\lambda,\mu})^{-1}_{\lambda,\mu\in\mathrm{Part}(n,k)}.$$

If we now take determinants on both sides, then, in view of $K_{\mu,\mu}=1$ and of

$$L_{\mu,\mu} = \prod_{i\ge1} m_i(\mu)!$$

for all μ, the theorem follows.

Further examples of nice determinant evaluations of tables of *characters of representations of symmetric groups and their double covers* can be found in [25, 142]. Determinants of tables of *characters of the alternating group* can be found in [24].

5.11. Elliptic determinant evaluations. In special functions theory there is currently a disease rapidly spreading, generalising the earlier mentioned q-disease (see Footnote 20). It could be called the *"elliptic disease."* Recall that, during the q-disease, we replaced every positive integer n by $1+q+q^2+\cdots+q^{n-1} = (1-q^n)/(1-q)$, and, more generally, shifted factorials $a(a+1)\cdots(a+k-1)$ by q-shifted factorials $(1-\alpha)(1-q\alpha)\cdots(1-\alpha q^{k-1})$. (Here, α takes the role of q^a, and one drops the powers of $1-q$ in order to ease notation.) Doing this with some "ordinary" identity, we arrived (hopefully) at its *q-analogue.* Now, once infected by the elliptic disease, we would replace every occurrence of a term $1-x$ (and, looking at the definition of q-shifted factorials, we can see that there will be many) by its *elliptic analogue* $\theta(x;p)$:

$$\theta(x) = \theta(x;p) = \prod_{j=0}^{\infty}(1-p^jx)(1-p^{j+1}/x).$$

Here, p is a complex number with $|p|<1$, which will be fixed throughout. Up to a trivial factor, $\theta(e^{2\pi ix};e^{2\pi i\tau})$ equals the *Jacobi theta function* $\theta_1(x|\tau)$ (cf. [189]). Clearly, $\theta(x)$ reduces to $1-x$ if $p=0$.

At first sight, one will be sceptical if this is a fruitful thing to do. After all, for working with the functions $\theta(x)$, the only identities which are available are the (trivial) *inversion formula*

$$\theta(1/x) = -\frac{1}{x}\,\theta(x), \tag{5.116}$$

the (trivial) *quasi-periodicity*

$$\theta(px) = -\frac{1}{x}\,\theta(x), \tag{5.117}$$

and *Riemann's* (highly non-trivial) *addition formula* (cf. [189, p. 451, Example 5])

$$\theta(xy)\,\theta(x/y)\,\theta(uv)\,\theta(u/v) - \theta(xv)\,\theta(x/v)\,\theta(uy)\,\theta(u/y) = \frac{u}{y}\,\theta(yv)\,\theta(y/v)\,\theta(xu)\,\theta(x/u). \tag{5.118}$$

Nevertheless, it has turned out recently that a surprising number of identities from the "ordinary" and from the "q-world" can be lifted to the elliptic level. This is particularly true for series of hypergeometric nature. We refer the reader to Chapter 11 of [72] for an account of the current state of the art in the theory of, as they are called now, *elliptic hypergeometric series.*

On the following pages, I give elliptic determinant evaluations a rather extensive coverage because, first of all, they were non-existent in [109] (with the exception of the mention of the papers [133, 134] by Milne), and, second, because I believe that the "elliptic research" is a research direction that will further prosper in the next future and will have numerous applications in many fields, also outside of just special functions theory and number theory. I further believe that the determinant evaluations presented in this subsection will turn out to be as fundamental as the determinant evaluations in Sections 2.1 and 2.2 in [109]. For some of them this belief is already a fact. For example, determinant evaluations involving elliptic functions have come into the picture in the theory of *multiple elliptic hypergeometric series*, see [95, 147, 152, 153, 155, 176, 187]. They have also an important role in the study of *Ruijsenaars operators* and related *integrable systems* [81, 157]. Furthermore, they have recently found applications in number theory to the problem of counting the number of *representations of an integer as a sum of triangular numbers* [154].

Probably the first elliptic determinant evaluation is due to Frobenius [69, (12)]. This identity has found applications to *Ruijsenaars operators* [157], to *multidimensional elliptic hypergeometric series* and *integrals* [95], [147] and to *number theory* [154]. For a generalisation to *higher genus Riemann surfaces*, see [60, Cor. 2.19]. Amdeberhan [7] observed that it can be easily proved using the condensation method (see "Method 2" in Section 4).

Theorem 66. *Let $x_1, x_2, \dots, x_n$, $a_1, a_2, \dots, a_n$ and t be indeterminates. Then there holds*

$$\det_{1\le i,j\le n}\left(\frac{\theta(ta_jx_i)}{\theta(t)\,\theta(a_jx_i)}\right)=\frac{\theta(ta_1\cdots a_nx_1\cdots x_n)}{\theta(t)}\frac{\prod\limits_{1\le i<j\le n} a_jx_j\,\theta(a_i/a_j)\,\theta(x_i/x_j)}{\prod\limits_{i,j=1}^{n}\theta(a_jx_i)}. \tag{5.119}$$

□

For $p=0$ and $t\to\infty$, this determinant identity reduces to Cauchy's evaluation (5.5) of the double alternant, and, thus, may be regarded as an "elliptic analogue" of the latter.

Okada [140, Theorem 1.1] has recently found an elliptic extension of Schur's evaluation (5.7) of a Cauchy-type Pfaffian. His proof works by the Pfaffian version of the condensation method.

Theorem 67. *Let $x_1, x_2, \dots, x_n$, t and w be indeterminates. Then there holds*

$$\operatorname*{Pf}_{1\le i,j\le 2n}\left(\frac{\theta(x_j/x_i)}{\theta(x_ix_j)}\frac{\theta(tx_ix_j)}{\theta(t)}\frac{\theta(wx_ix_j)}{\theta(w)}\right)\\
=\frac{\theta(tx_1\cdots x_{2n})}{\theta(t)}\frac{\theta(wx_1\cdots x_{2n})}{\theta(w)}\prod_{1\le i<j\le 2n}\frac{\theta(x_j/x_i)}{x_j\,\theta(x_ix_j)}. \tag{5.120}$$

□

The next group of determinant evaluations is from [156, Sec. 3]. As the Vandermonde determinant evaluation, or the other Weyl denominator formulae (cf. [109, Lemma 2]), are fundamental *polynomial* determinant evaluations, the evaluations in Lemma 68 below are equally fundamental in the elliptic domain as they can be considered as the elliptic analogues of the former. Indeed, Rosengren and Schlosser show that they *imply* the *Macdonald identities associated to affine root systems* [128], which are the affine analogues of the Weyl denominator formulae. In particular, in this way they obtain new proofs of the Macdonald identities.

In order to conveniently formulate Rosengren and Schlosser's determinant evaluations, we shall adopt the following terminology from [156]. For $0<|p|<1$ and $t\ne 0$, an A_{n-1} *theta function* f *of norm* t is a holomorphic function for $x \neq 0$ such that

$$f(px) = \frac{(-1)^n}{tx^n} f(x). \tag{5.121}$$

Moreover, if R denotes either of the root systems B_n, $B_n^\vee$, C_n, $C_n^\vee$, BC_n or D_n (see Footnote 26 and [84] for information on root systems), we call f an R *theta function* if

$$\begin{aligned}
f(px) &= -\frac{1}{p^{n-1}x^{2n-1}} f(x), && f(1/x) = -\frac{1}{x} f(x), && R = B_n,\\
f(px) &= -\frac{1}{p^{n}x^{2n}} f(x), && f(1/x) = -f(x), && R = B_n^\vee,\\
f(px) &= \frac{1}{p^{n+1}x^{2n+2}} f(x), && f(1/x) = -f(x), && R = C_n,\\
f(px) &= \frac{1}{p^{n-\frac{1}{2}}x^{2n}} f(x), && f(1/x) = -\frac{1}{x} f(x), && R = C_n^\vee,\\
f(px) &= \frac{1}{p^{n}x^{2n+1}} f(x), && f(1/x) = -\frac{1}{x} f(x), && R = BC_n,\\
f(px) &= \frac{1}{p^{n-1}x^{2n-2}} f(x), && f(1/x) = f(x), && R = D_n.
\end{aligned}$$

Given this definition, Rosengren and Schlosser [156, Lemma 3.2] show that a function f is an A_{n-1} theta function of norm t if and only if there exist constants C, $b_1, \dots, b_n$ such that $b_1 \cdots b_n = t$ and

$$f(x) = C\, \theta(b_1 x) \cdots \theta(b_n x),$$

and for the other six cases, they show that f is an R theta function if and only if there exist constants C, $b_1, \dots, b_{n-1}$ such that

$$\begin{aligned}
f(x) &= C\, \theta(x)\, \theta(b_1x)\, \theta(b_1/x) \cdots \theta(b_{n-1}x)\, \theta(b_{n-1}/x), && R = B_n,\\
f(x) &= C\, x^{-1}\theta(x^2;p^2)\, \theta(b_1x)\, \theta(b_1/x) \cdots \theta(b_{n-1}x)\, \theta(b_{n-1}/x), && R = B_n^\vee,\\
f(x) &= C\, x^{-1}\theta(x^2)\, \theta(b_1x)\, \theta(b_1/x) \cdots \theta(b_{n-1}x)\, \theta(b_{n-1}/x), && R = C_n,\\
f(x) &= C\, \theta(x;p^{\frac{1}{2}})\, \theta(b_1x)\, \theta(b_1/x) \cdots \theta(b_{n-1}x)\, \theta(b_{n-1}/x), && R = C_n^\vee,\\
f(x) &= C\, \theta(x)\, \theta(px^2;p^2)\, \theta(b_1x)\, \theta(b_1/x) \cdots \theta(b_{n-1}x)\, \theta(b_{n-1}/x), && R = BC_n,\\
f(x) &= C\, \theta(b_1x)\, \theta(b_1/x) \cdots \theta(b_{n-1}x)\, \theta(b_{n-1}/x), && R = D_n,
\end{aligned}$$

where $\theta(x) = \theta(x;p)$.

If one puts $p = 0$, then an A_{n-1} theta function of norm t becomes a polynomial of degree n such that the reciprocal of the product of its roots is equal to t. Similarly, if one puts $p = 0$, then a D_n theta function becomes a polynomial in $(x+1/x)$ of degree n. This is the specialisation of some of the following results which is relevant for obtaining the earlier Lemmas 16–20.

The elliptic extension of the Weyl denominator formulae is the following formula. (See [156, Prop. 3.4].)

Lemma 68. *Let $f_1, \dots, f_n$ be A_{n-1} theta functions of norm t. Then,*

$$\det_{1 \le i,j \le n} \left(f_j(x_i)\right) = C\, W_{A_{n-1}}(x), \tag{5.122}$$

for some constant C, where

$$W_{A_{n-1}}(x) = \theta(tx_1 \cdots x_n) \prod_{1 \le i < j \le n} x_j \theta(x_i/x_j).$$

Moreover, if R denotes either B_n, $B_n^\vee$, C_n, $C_n^\vee$, BC_n or D_n and $f_1, \dots, f_n$ are R theta functions, we have

$$\det_{1 \le i,j \le n} (f_j(x_i)) = C\, W_R(x), \tag{5.123}$$

for some constant C, where

$$\begin{aligned}
W_{B_n}(x) &= \prod_{i=1}^n \theta(x_i) \prod_{1 \le i < j \le n} x_i^{-1} \theta(x_i x_j)\, \theta(x_i/x_j),\\
W_{B_n^\vee}(x) &= \prod_{i=1}^n x_i^{-1} \theta(x_i^2; p^2) \prod_{1 \le i < j \le n} x_i^{-1} \theta(x_i x_j)\, \theta(x_i/x_j),\\
W_{C_n}(x) &= \prod_{i=1}^n x_i^{-1} \theta(x_i^2) \prod_{1 \le i < j \le n} x_i^{-1} \theta(x_i x_j)\, \theta(x_i/x_j),\\
W_{C_n^\vee}(x) &= \prod_{i=1}^n \theta(x_i; p^{\frac{1}{2}}) \prod_{1 \le i < j \le n} x_i^{-1} \theta(x_i x_j)\, \theta(x_i/x_j),\\
W_{BC_n}(x) &= \prod_{i=1}^n \theta(x_i)\, \theta(p x_i^2; p^2) \prod_{1 \le i < j \le n} x_i^{-1} \theta(x_i x_j)\, \theta(x_i/x_j),\\
W_{D_n}(x) &= \prod_{1 \le i < j \le n} x_i^{-1} \theta(x_i x_j)\, \theta(x_i/x_j).
\end{aligned}$$

□

Rosengren and Schlosser show in [156, Prop. 6.1] that the famous Macdonald identities for affine root systems [128] are equivalent to special cases of this lemma. We state the corresponding results below.

Theorem 69. *The following determinant evaluations hold:*

$$\det_{1 \le i,j \le n} \left(x_i^{j-1} \theta((-1)^{n-1} p^{j-1} t x_i^n; p^n) \right) = \frac{(p;p)_\infty^n}{(p^n;p^n)_\infty^n} W_{A_{n-1}}(x),$$

$$\det_{1 \le i,j \le n} \left(x_i^{j-n} \theta(p^{j-1} x_i^{2n-1}; p^{2n-1}) - x_i^{n+1-j} \theta(p^{j-1} x_i^{1-2n}; p^{2n-1}) \right) \\ = \frac{2(p;p)_\infty^n}{(p^{2n-1};p^{2n-1})_\infty^n} W_{B_n}(x),$$

$$\det_{1 \le i,j \le n} \left(x_i^{j-n-1} \theta(p^{j-1} x_i^{2n}; p^{2n}) - x_i^{n+1-j} \theta(p^{j-1} x_i^{-2n}; p^{2n}) \right) \\ = \frac{2(p^2;p^2)_\infty (p;p)_\infty^{n-1}}{(p^{2n};p^{2n})_\infty^n} W_{B_n^\vee}(x),$$

$$\det_{1\le i,j\le n}\left(x_i^{j-n-1}\theta(-p^jx_i^{2n+2};p^{2n+2})-x_i^{n+1-j}\theta(-p^jx_i^{-2n-2};p^{2n+2})\right)\\ =\frac{(p;p)_\infty^n}{(p^{2n+2};p^{2n+2})_\infty^n}\,W_{C_n}(x),$$

$$\det_{1\le i,j\le n}\left(x_i^{j-n}\theta(-p^{j-\frac12}x_i^{2n};p^{2n})-x_i^{n+1-j}\theta(-p^{j-\frac12}x_i^{-2n};p^{2n})\right)\\ =\frac{(p^{\frac12};p^{\frac12})_\infty(p;p)_\infty^{n-1}}{(p^{2n};p^{2n})_\infty^n}\,W_{C_n^\vee}(x),$$

$$\det_{1\le i,j\le n}\left(x_i^{j-n}\theta(-p^jx_i^{2n+1};p^{2n+1})-x_i^{n+1-j}\theta(-p^jx_i^{-2n-1};p^{2n+1})\right)\\ =\frac{(p;p)_\infty^n}{(p^{2n+1};p^{2n+1})_\infty^n}\,W_{BC_n}(x),$$

$$\det_{1\le i,j\le n}\left(x_i^{j-n}\theta(-p^{j-1}x_i^{2n-2};p^{2n-2})+x_i^{n-j}\theta(-p^{j-1}x_i^{2-2n};p^{2n-2})\right)\\ =\frac{4(p;p)_\infty^n}{(p^{2n-2};p^{2n-2})_\infty^n}\,W_{D_n}(x),\qquad n\ge 2.$$

□

Historically, aside from Frobenius' elliptic Cauchy identity (5.119), the subject of elliptic determinant evaluations begins with Warnaar's remarkable paper [187]. While the main subject of this paper is *elliptic hypergeometric series*, some elliptic determinant evaluations turn out to be crucial for the proofs of the results. Lemma 5.3 from [187] extends one of the basic determinant lemmas listed in [109], namely [109, Lemma 5], to the elliptic world, to which it reduces in the case $p=0$. We present this important elliptic determinant evaluation in the theorem below.

Theorem 70. *Let $x_1,x_2,\dots,x_n$, $a_1,a_2,\dots,a_n$ be indeterminates. For each $j=1,\dots,n$, let $P_j(x)$ be a D_j theta function. Then there holds*

$$\det_{1\le i,j\le n}\left(P_j(x_i)\prod_{k=j+1}^n\theta(a_kx_i)\,\theta(a_k/x_i)\right)\\ =\prod_{i=1}^nP_i(a_i)\prod_{1\le i<j\le n}a_jx_j^{-1}\,\theta(x_jx_i)\,\theta(x_j/x_i). \tag{5.124}$$

□

Warnaar used this identity to obtain a summation formula for a *multidimensional elliptic hypergeometric series*. Further related applications may be found in [152, 153, 155, 176]. The relevant special case of the above theorem is the following (see [187, Cor. 5.4]). It is the elliptic generalisation of [109, Theorem 28]. In the statement, we use the notation

$$(a;q,p)_m=\theta(a;p)\,\theta(aq;p)\cdots\theta(aq^{m-1};p), \tag{5.125}$$

which extends the notation for q-shifted factorials to the elliptic world.

Theorem 71. *Let $X_1, X_2, \dots, X_n$, A, B and C be indeterminates. Then, for any non-negative integer n, there holds*

$$\det_{1\le i,j\le n}\left(\frac{(AX_i;q,p)_{n-j}\,(AC/X_i;q,p)_{n-j}}{(BX_i;q,p)_{n-j}\,(BC/X_i;q,p)_{n-j}}\right)\\
=(Aq)^{\binom n2}\prod_{1\le i<j\le n}X_j\,\theta(X_i/X_j)\,\theta(C/X_iX_j)\prod_{i=1}^n\frac{(B/A;q,p)_{i-1}\,(ABCq^{2n-2i};q,p)_{i-1}}{(BX_i;q,p)_{n-1}\,(BC/X_i;q,p)_{n-1}}. \tag{5.126}$$

□

Theorem 29 from [109], which is slightly more general than [109, Theorem 28], can also be extended to an elliptic theorem by suitably specialising the variables in Theorem 70.

Theorem 72. *Let $X_1, X_2, \dots, X_n$, $Y_1, Y_2, \dots, Y_n$, A and B be indeterminates. Then, for any non-negative integer n, there holds*

$$\det_{1\le i,j\le n}\left(\frac{(X_iY_j;q,p)_j\,(AY_j/X_i;q,p)_j}{(BX_i;q,p)_j\,(AB/X_i;q,p)_j}\right)\\
=q^{2\binom n3}(AB)^{\binom n2}\prod_{1\le i<j\le n}\theta(X_jX_i/A)\,\theta(X_j/X_i)\\
\times\prod_{i=1}^n\frac{(ABY_iq^{i-2};q,p)_{i-1}\,(Y_i/Bq^{i-1};q,p)_{i-1}}{X_i^{i-1}\,(BX_i;q,p)_{n-1}\,(AB/X_i;q,p)_{n-1}}. \tag{5.127}$$

□

Another, very elegant, special case of Theorem 70 is the following elliptic Cauchy-type determinant evaluation. It was used by Rains [147, Sec. 3] in the course of deriving a $BC_n \leftrightarrow BC_m$ *integral transformation.*

Lemma 73. *Let $x_1, x_2, \dots, x_n$ and $a_1, a_2, \dots, a_n$ be indeterminates. Then there holds*

$$\det_{1\le i,j\le n}\left(\frac{1}{\theta(a_jx_i)\,\theta(a_j/x_i)}\right)=\frac{\displaystyle\prod_{1\le i<j\le n}a_jx_j^{-1}\,\theta(x_jx_i)\,\theta(x_j/x_i)\,\theta(a_ia_j)\,\theta(a_i/a_j)}{\displaystyle\prod_{i,j=1}^n\theta(a_jx_i)\,\theta(a_j/x_i)}. \tag{5.128}$$

□

The remaining determinant evaluations in the current subsection, with the exception of the very last one, are all due to Rosengren and Schlosser [156]. The first one is a further (however non-obvious) consequence of Theorem 70 (see [156, Cor. 4.3]). Two related determinant evaluations, restricted to the polynomial case, were applied in [162] and [164] to obtain *multidimensional matrix inversions* that played a major role in the derivation of new *summation formulae for multidimensional basic hypergeometric series.*

Theorem 74. *Let $x_1,x_2,\dots,x_n$, $a_1,a_2,\dots,a_{n+1}$, and b be indeterminates. For each $j=1,\dots,n+1$, let $P_j(x)$ be a D_j theta function. Then there holds*

$$P_{n+1}(b)\det_{1\le i,j\le n}\Bigg(P_j(x_i)\prod_{k=j+1}^{n+1}\big(\theta(a_kx_i)\,\theta(a_k/x_i)\big)\\ -\frac{P_{n+1}(x_i)}{P_{n+1}(b)}P_j(b)\prod_{k=j+1}^{n+1}\big(\theta(a_kb)\,\theta(a_k/b)\big)\Bigg)\\ =\prod_{i=1}^{n+1}P_i(a_i)\prod_{1\le i<j\le n+1}a_jx_j^{-1}\,\theta(x_jx_i)\,\theta(x_j/x_i),\quad (5.129)$$

where $x_{n+1}=b$. □

The next determinant evaluation is Theorem 4.4 from [156]. It generalises another basic determinant lemma listed in [109], namely Lemma 6 from [109], to the elliptic case. It looks as if it is a limit case of Warnaar's in Theorem 70. However, limits are very problematic in the elliptic world, and therefore it does not seem that Theorem 70 implies the theorem below. For a generalisation in a different direction than Theorem 70 see [182, Appendix B] (cf. also [156, Remark 4.6]).

Theorem 75. *Let $x_1,x_2,\dots,x_n$, $a_1,a_2,\dots,a_n$, and t be indeterminates. For each $j=1,\dots,n$, let $P_j(x)$ be an A_{j-1} theta function of norm $ta_1\cdots a_j$. Then there holds*

$$\det_{1\le i,j\le n}\left(P_j(x_i)\prod_{k=j+1}^{n}\theta(a_kx_i)\right)\\ =\frac{\theta(ta_1\cdots a_nx_1\cdots x_n)}{\theta(t)}\prod_{i=1}^{n}P_i(1/a_i)\prod_{1\le i<j\le n}a_jx_j\,\theta(x_i/x_j).\quad (5.130)$$

□

As is shown in [156, Cor. 4.8], this identity implies the following determinant evaluation.

Theorem 76. *Let $x_1,x_2,\dots,x_n$, $a_1,a_2,\dots,a_{n+1}$ and b be indeterminates. For each $j=1,\dots,n+1$, let $P_j(x)$ be an A_{j-1} theta function of norm $ta_1\cdots a_j$. Then there holds*

$$P_{n+1}(b)\det_{1\le i,j\le n}\left(P_j(x_i)\prod_{k=j+1}^{n+1}\theta(a_kx_i)-\frac{P_{n+1}(x_i)}{P_{n+1}(b)}P_j(b)\prod_{k=j+1}^{n+1}\theta(a_kb)\right)\\ =\frac{\theta(tba_1\cdots a_{n+1}x_1\cdots x_n)}{\theta(t)}\prod_{i=1}^{n+1}P_i(1/a_i)\prod_{1\le i<j\le n+1}a_jx_j\,\theta(x_i/x_j),\quad (5.131)$$

where $x_{n+1}=b$. □

By combining Lemma 73 and Theorem 75, a determinant evaluation similar to the one in Theorem 74, but different, is obtained in [156, Theorem 4.9].

Theorem 77. *Let $x_1, x_2, \dots, x_n$, $a_1, a_2, \dots, a_n$, and $c_1, \dots, c_{n+2}$ be indeterminates. For each $j = 1, \dots, n$, let $P_j(x)$ be an A_{j-1} theta function of norm $(c_1 \cdots c_{n+2} a_{j+1} \cdots a_n)^{-1}$. Then there holds*

$$\det_{1\le i,j\le n}\Bigg(x_i^{-n-1}P_j(x_i)\prod_{k=1}^{n+2}\theta(c_kx_i)\prod_{k=j+1}^{n}\theta(a_kx_i)\\ -x_i^{n+1}P_j(x_i^{-1})\prod_{k=1}^{n+2}\theta(c_kx_i^{-1})\prod_{k=j+1}^{n}\theta(a_kx_i^{-1})\Bigg)\\ =\frac{a_1\cdots a_n}{x_1\cdots x_n\,\theta(c_1\cdots c_{n+2}a_1\cdots a_n)}\prod_{i=1}^{n}P_i(1/a_i)\\ \times\prod_{1\le i<j\le n+2}\theta(c_ic_j)\prod_{i=1}^{n}\theta(x_i^2)\prod_{1\le i<j\le n}a_jx_i^{-1}\,\theta(x_ix_j)\,\theta(x_i/x_j). \tag{5.132}$$

□

The last elliptic determinant evaluation which I present here is a surprising elliptic extension of a determinant evaluation due to Andrews and Stanton [13, Theorem 8] (see [109, Theorem 42]) due to Warnaar [187, Theorem 4.17]. It is surprising because in the former there appear q-shifted factorials *and* q^2-shifted factorials at the same time, but nevertheless there exists an elliptic analogue, and to obtain it one only has to add the p everywhere in the shifted factorials to convert them to elliptic ones.

Theorem 78. *Let x and y be indeterminates. Then, for any non-negative integer n, there holds*

$$\det_{0\le i,j\le n-1}\left(\frac{(y/xq^i;q^2,p)_{i-j}\,(q/yxq^i;q^2,p)_{i-j}\,(1/x^2q^{2+4i};q^2,p)_{i-j}}{(q;q,p)_{2i+1-j}\,(1/yxq^{2i};q,p)_{i-j}\,(y/xq^{1+2i};q,p)_{i-j}}\right)\\ =\prod_{i=0}^{n-1}\frac{(x^2q^{2i+1};q,p)_i\,(xq^{3+i}/y;q^2,p)_i\,(yxq^{2+i};q^2,p)_i}{(x^2q^{2i+2};q^2,p)_i\,(q;q^2,p)_{i+1}\,(yxq^{1+i};q,p)_i\,(xq^{2+i}/y;q,p)_i}. \tag{5.133}$$

□

In closing this final subsection, I remind the reader that, as was already said before, many Hankel determinant evaluations involving elliptic functions can be found in [133] and [134].

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