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Determinant evaluations and $U(n)$ extensions of Heine's ${}_2\phi_1$-transformations

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Abstract. We give new proofs for $U(n)$ extensions of Heine's three ${}_2\phi_1$-transformation formulas that were recently discovered in another paper by the authors. These new proofs proceed by combining Heine's original transformation formulas with certain determinant evaluations. As a by-product we are able to generalize two of the $U(n)$ transformation formulas.

1. Introduction In the recent paper Gustafson and Krattenthaler [1994] we discovered $U(n)$ extensions of Heine's ${}_2\phi_1$-transformation formulas (see (2.1)–(2.3) for Heine's formulas) which involve $U(n)$ basic hypergeometric series of a new type. With the q-notation $(a)_n := \prod_{i=1}^{n}(1-aq^{i-1})$ (note that this differs from the standard notation $(a;q)_n$ in Gasper and Rahman's book (Gasper and Rahman [1990]) in that we drop the base q for the sake of being able to write our formulas more compactly), these transformation formulas are the following. Our $U(n)$ extension (Gustafson and Krattenthaler [1994], (1.7)) of the first Heine

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transformation reads

$$\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\prod_{i=1}^{n}\frac{(A)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(CX_i)_{k_i}}q^{k_i(1-i)}Z^{k_i}\right)$$
$$=\prod_{i=1}^{n}\frac{(Aq^{1-i})_\infty\,(BZX_i)_\infty}{(Zq^{1-i})_\infty\,(CX_i)_\infty}\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\right.$$
$$\left.\times\prod_{i=1}^{n}\frac{(Z)_{k_i}\,(CX_i/A)_{k_i}}{(q)_{k_i}\,(BZX_i)_{k_i}}q^{k_i(1-i)}A^{k_i}\right),\qquad(1.1)$$

provided the series terminate or $|q|<1$, $|Zq^{1-n}|<1$, $|Aq^{1-n}|<1$. Our $U(n)$ extension (Gustafson and Krattenthaler [1994], (1.8)) of the second Heine transformation reads

$$\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\prod_{i=1}^{n}\frac{(A)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(CX_i)_{k_i}}q^{k_i(1-i)}Z^{k_i}\right)$$
$$=\prod_{i=1}^{n}\frac{(Cq^{1-i}/B)_\infty\,(BZX_i)_\infty}{(Zq^{1-i})_\infty\,(CX_i)_\infty}\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\right.$$
$$\left.\times\prod_{i=1}^{n}\frac{(ABZ/C)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(BZX_i)_{k_i}}q^{k_i(1-i)}\left(\frac{C}{B}\right)^{k_i}\right),\qquad(1.2)$$

provided the series terminate or $|q|<1$, $|Zq^{1-n}|<1$, $|Cq^{1-n}/B|<1$. Finally, our $U(n)$ extension (Gustafson and Krattenthaler [1994], (1.9)) of the third Heine transformation reads

$$\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\prod_{i=1}^{n}\frac{(A)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(CX_i)_{k_i}}q^{k_i(1-i)}Z^{k_i}\right)$$
$$=\prod_{i=1}^{n}\frac{(ABZq^{1-i}/C)_\infty}{(Zq^{1-i})_\infty}\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\right.$$
$$\left.\times\prod_{i=1}^{n}\frac{(C/B)_{k_i}\,(CX_i/A)_{k_i}}{(q)_{k_i}\,(CX_i)_{k_i}}q^{k_i(1-i)}\left(\frac{ABZ}{C}\right)^{k_i}\right),\qquad(1.3)$$

provided the series terminate or $|q|<1$, $|Zq^{1-n}|<1$, $|ABZq^{1-n}/C|<1$. In particular, from these identities there also result $U(n)$ extensions of the q-Gauss summation formula and the q-binomial theorem, cf. Gustafson and Krattenthaler [1994].

We proved (1.1)–(1.3) in Gustafson and Krattenthaler [1994] by induction with respect to n. The purpose of this paper is to give new, direct proofs of these identities, avoiding induction. These new proofs, which are described in section 2 and are shorter than the original ones, proceed by a combination of Heine's original ${}_2\phi_1$-transformations and certain determinant evaluations. One determinant evaluation is simply the Vandermonde determinant. The other determinant evaluation, Lemma 1 in section 2, comes from a determinant lemma (Krattenthaler [1990a], Lemma 2.2) (see Lemma 2 in section 2) that has been already used successfully in a number of cases by the second author (Gessel and Krattenthaler [1995], Krattenthaler [1990a], Krattenthaler [1990b], Krattenthaler [1995]) in the computation of generating functions for tableaux and plane partitions. In Gessel and Krattenthaler [1995] this also lead to find $SU(n)$ basic hypergeometric identities. However, the particular technique that we use in the present paper seems to be new in the subject of multidimensional basic hypergeometric series. Hopefully, it will prove to be useful in other cases, too.

As a by-product of our new proofs, we are able to generalize the identities (1.2) and (1.3). We summarize the new (and old) results in the Theorem in section 2 (identities (2.4)–(2.6)), and subsequently prove them. Interestingly, in view of our new technique, the easiest to prove is the third identity, (2.6). It just requires the Vandermonde determinant and Heine's third ${}_2\phi_1$-transformation (which is displayed in (2.3)). Next in the hierarchy comes the second identity, (2.5). It requires a fancier determinant evaluation, Lemma 1, and Heine's second ${}_2\phi_1$-transformation (which is displayed in (2.2)). The most difficult one, again in view of our technique, is the first identity, (2.4). In fact, we do not know how to prove (2.4) in a way similar to the proofs of the other identities. However, once (2.5) and (2.6) are established, (2.4) follows by simply applying (2.5) and (2.6) in succession.

2. The $U(n)$ extensions of Heine's ${}_2\phi_1$-transformations and their proofs

Before we state the $U(n)$ extensions of Heine's ${}_2\phi_1$-transformation formulas that we are going to prove, it might be convenient to recall Heine's original ${}_2\phi_1$-transformations. These are (cf. Gasper and Rahman [1990], (1.4.1), (1.4.5), (1.4.6))

$$\sum_{k\ge0}\frac{(a)_k\,(b)_k}{(q)_k\,(c)_k}z^k=\frac{(a)_\infty\,(bz)_\infty}{(c)_\infty\,(z)_\infty}\sum_{k\ge0}\frac{(c/a)_k\,(z)_k}{(q)_k\,(bz)_k}a^k, \tag{2.1}$$

$$\sum_{k\ge0}\frac{(a)_k\,(b)_k}{(q)_k\,(c)_k}z^k=\frac{(c/b)_\infty\,(bz)_\infty}{(c)_\infty\,(z)_\infty}\sum_{k\ge0}\frac{(abz/c)_k\,(b)_k}{(q)_k\,(bz)_k}\left(\frac{c}{b}\right)^k, \tag{2.2}$$

$$\sum_{k\ge0}\frac{(a)_k\,(b)_k}{(q)_k\,(c)_k}z^k=\frac{(abz/c)_\infty}{(z)_\infty}\sum_{k\ge0}\frac{(c/a)_k\,(c/b)_k}{(q)_k\,(c)_k}\left(\frac{abz}{c}\right)^k. \tag{2.3}$$

Our $U(n)$ extensions of these identities are the following.

Theorem *There hold*

$$\begin{aligned}\sum_{k_1,\dots,k_n\ge0}&\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\prod_{i=1}^n\frac{(A)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(CX_i)_{k_i}}q^{k_i(1-i)}Z^{k_i}\right)\\&=\prod_{i=1}^n\frac{(Aq^{1-i})_\infty\,(BZX_i)_\infty}{(Zq^{1-i})_\infty\,(CX_i)_\infty}\sum_{k_1,\dots,k_n\ge0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\right.\\&\qquad\qquad\left.\times\prod_{i=1}^n\frac{(Z)_{k_i}\,(CX_i/A)_{k_i}}{(q)_{k_i}\,(BZX_i)_{k_i}}q^{k_i(1-i)}A^{k_i}\right),\end{aligned}\tag{2.4}$$

provided the series terminate or $|q|<1$, $|Zq^{1-n}|<1$, $|Aq^{1-n}|<1$,

$$\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\prod_{i=1}^n\frac{(A_i)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(CX_i)_{k_i}}q^{k_i(1-i)}Z^{k_i}\right)$$
$$=\prod_{i=1}^n\frac{(Cq^{1-i}/B)_\infty\,(BZX_i)_\infty}{(Zq^{1-i})_\infty\,(CX_i)_\infty}\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\right.$$
$$\left.\times\prod_{i=1}^n\frac{(A_iBZ/C)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(BZX_i)_{k_i}}q^{k_i(1-i)}\left(\frac{C}{B}\right)^{k_i}\right),\qquad(2.5)$$

provided the series terminate or $|q|<1$, $|Zq^{1-n}|<1$, $|Cq^{1-n}/B|<1$,

$$\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\prod_{i=1}^n\frac{(A_i)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(CA_iX_i)_{k_i}}q^{k_i(1-i)}Z^{k_i}\right)$$
$$=\prod_{i=1}^n\frac{(BZq^{1-i}/C)_\infty}{(Zq^{1-i})_\infty}\sum_{k_1,\dots,k_n\ge 0}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}\right.$$
$$\left.\times\prod_{i=1}^n\frac{(CA_i/B)_{k_i}\,(CX_i)_{k_i}}{(q)_{k_i}\,(CA_iX_i)_{k_i}}q^{k_i(1-i)}\left(\frac{BZ}{C}\right)^{k_i}\right),\qquad(2.6)$$

provided the series terminate or $|q|<1$, $|Zq^{1-n}|<1$, $|BZq^{1-n}/C|<1$.

Remark It should be noted that (2.4) is identical with (1.1), while (2.5) is more general than (1.2), and (2.6) is more general than (1.3). In fact, (1.2) comes out of (2.5) by setting $A_i\equiv A$, and (1.3) comes out of (2.6) by setting $A_i\equiv A$ and replacing C by C/A.

We start with the proof of the third identity.

Proof of (2.6) We have

$$\prod_{i=1}^n\left(X_iq^{k_i}\right)^{1-i}\cdot\prod_{1\le i<j\le n}\left(1-\frac{X_j}{X_i}q^{k_j-k_i}\right)=\prod_{1\le i<j\le n}\left(\frac{1}{X_jq^{k_j}}-\frac{1}{X_iq^{k_i}}\right)\qquad(2.7)$$
$$=\det_{1\le s,t\le n}\left(\left(\frac{1}{X_sq^{k_s}}\right)^{t-1}\right),\qquad(2.8)$$

the second line being due to Vandermonde's determinant. Hence we may write the left-hand side of (2.6), when multiplied by $\prod_{1\le i<j\le n}(1/X_j-1/X_i)$, as

$$\det_{1\le s,t\le n}\left(X_s^{1-t}\sum_{k_s\ge 0}\frac{(A_s)_{k_s}\,(BX_s)_{k_s}}{(q)_{k_s}\,(CA_sX_s)_{k_s}}\left(Zq^{1-t}\right)^{k_s}\right).$$

Now, to the sum inside the determinant we apply the third Heine transformation, (2.3), with $a=A_s$, $b=BX_s$, $c=CA_sX_s$, $z=Zq^{1-t}$. Thus we obtain

$$\det_{1\le s,t\le n}\left(X_s^{1-t}\frac{(BZq^{1-t}/C)_\infty}{(Zq^{1-t})_\infty}\sum_{k_s\ge 0}\frac{(CX_s)_{k_s}\,(CA_s/B)_{k_s}}{(q)_{k_s}\,(CA_sX_s)_{k_s}}\left(\frac{BZq^{1-t}}{C}\right)^{k_s}\right),$$

which, by taking factors out of the determinant and by using linearity of the determinant with respect to rows, equals

$$\prod_{i=1}^{n}\frac{(BZq^{1-i}/C)_\infty}{(Zq^{1-i})_\infty}$$
$$\times\sum_{k_1,\dots,k_n\ge0}\left(\prod_{i=1}^{n}\frac{(CX_i)_{k_i}\,(CA_i/B)_{k_i}}{(q)_{k_i}\,(CA_iX_i)_{k_i}}\left(\frac{BZ}{C}\right)^{k_i}\det_{1\le s,t\le n}\left((X_sq^{k_s})^{1-t}\right)\right).$$

By using (2.8), this last expression is turned into the right-hand side of (2.6), when multiplied by $\prod_{1\le i<j\le n}(1/X_j-1/X_i)$. $\square$

For the proof of (2.5) we need the following determinant evaluation.

Lemma 1 *Let $B,C,X_1,\dots,X_n$ be indeterminates. Then there holds*

$$\det_{1\le s,t\le n}\left(\frac{(BX_s)_{k_s+t-1}}{(CX_s)_{k_s+t}}\right)=B^{\binom n2}q^{\sum_{i=1}^n\binom{i-1}{2}}$$
$$\times\prod_{i=1}^{n}\left(\left(X_iq^{k_i}\right)^{n-1}\frac{(BX_i)_{k_i}\,(Cq/B)_{i-1}}{(CX_i)_{k_i+n}}\right)\prod_{1\le i<j\le n}\left(\frac{1}{X_jq^{k_j}}-\frac{1}{X_iq^{k_i}}\right).\qquad(2.9)$$

This determinant evaluation follows easily from a determinant lemma (Krattenthaler [1990a], Lemma 2.2), which we state here without proof.

Lemma 2 *Let $X_1,\dots,X_n$, $A_2,\dots,A_n$, and $B_2,\dots B_n$ be indeterminates. Then there holds*

$$\det_{1\le s,t\le n}\Big((X_s+A_n)\cdots(X_s+A_{t+1})(X_s+B_t)\cdots(X_s+B_2)\Big)$$
$$=\prod_{1\le i<j\le n}(X_i-X_j)\prod_{2\le i\le j\le n}(B_i-A_j).\qquad\square\qquad(2.10)$$

Proof of Lemma 1 By taking some factors out of the determinant, we write the left-hand side of (2.9) as

$$\prod_{i=1}^{n}\frac{(BX_i)_{k_i}}{(CX_i)_{k_i+n}}\det_{1\le s,t\le n}\Big((1-BX_sq^{k_s})\cdots(1-BX_sq^{k_s}q^{t-2})$$
$$\cdot(1-CX_sq^{k_s}q^t)\cdots(1-CX_sq^{k_s}q^{n-1})\Big)$$
$$=\prod_{i=1}^{n}\frac{(BX_i)_{k_i}}{(CX_i)_{k_i+n}}\left(X_iq^{k_i}\right)^{n-1}\det_{1\le s,t\le n}\left(\left(\frac{1}{X_sq^{k_s}}-B\right)\cdots\left(\frac{1}{X_sq^{k_s}}-Bq^{t-2}\right)\right.$$
$$\left.\cdot\left(\frac{1}{X_sq^{k_s}}-Cq^t\right)\cdots\left(\frac{1}{X_sq^{k_s}}-Cq^{n-1}\right)\right).$$

Now we apply Lemma 2 with the replacements $X_s\to1/X_sq^{k_s}$, $A_t\to-Cq^{t-1}$, $B_t\to-Bq^{t-2}$. A few simplifications then give the right-hand side of (2.9). $\square$

Now we are in the position to prove our second identity.

Proof of (2.5) By (2.7) and Lemma 1 with C replaced by Cq^{-n}, the left-hand side of (2.5), when multiplied by $\prod_{1\le i<j\le n}(1/X_j-1/X_i)$, equals

$$B^{-\binom{n}{2}}q^{-\sum_{i=1}^n\binom{i-1}{2}}\prod_{i=1}^n\frac{(CX_iq^{-n})_n}{(Cq^{1-n}/B)_{i-1}}\\ \times\det_{1\le s,t\le n}\left(\sum_{k_s\ge0}\frac{(A_s)_{k_s}\,(BX_s)_{k_s+t-1}}{(q)_{k_s}\,(CX_sq^{-n})_{k_s+t}}\left(X_sq^{k_s}\right)^{1-n}Z^{k_s}\right).$$

Now, to the sum inside the determinant we apply the second Heine transformation, (2.2), with $a=A_s$, $b=BX_sq^{t-1}$, $c=CX_sq^{t-n}$, $z=Zq^{1-n}$. Thus we obtain

$$B^{-\binom{n}{2}}q^{-\sum_{i=1}^n\binom{i-1}{2}}\prod_{i=1}^n\frac{(CX_iq^{-n})_n}{(Cq^{1-n}/B)_{i-1}}\det_{1\le s,t\le n}\left(\frac{(BX_s)_{t-1}}{(CX_sq^{-n})_t}X_s^{1-n}\right.\\ \left.\cdot\frac{(Cq^{1-n}/B)_\infty\,(BZX_sq^{t-n})_\infty}{(CX_sq^{t-n})_\infty\,(Zq^{1-n})_\infty}\sum_{k_s\ge0}\frac{(A_sBZ/C)_{k_s}\,(BX_sq^{t-1})_{k_s}}{(q)_{k_s}\,(BZX_sq^{t-n})_{k_s}}\left(\frac{Cq^{1-n}}{B}\right)^{k_s}\right)\\ =B^{-\binom{n}{2}}q^{-\sum_{i=1}^n\binom{i-1}{2}}\prod_{i=1}^n\frac{(Cq^{i-n}/B)_\infty\,(BZX_iq^{-n})_\infty}{(CX_i)_\infty\,(Zq^{1-n})_\infty}\\ \times\sum_{k_1,\dots,k_n\ge0}\left(\prod_{i=1}^n\frac{(A_iBZ/C)_{k_i}}{(q)_{k_i}}\left(\frac{C}{B}\right)^{k_i}\left(X_iq^{k_i}\right)^{1-n}\det_{1\le s,t\le n}\left(\frac{(BX_s)_{k_s+t-1}}{(BZX_sq^{-n})_{k_s+t}}\right)\right).$$

The determinant in the last line can be evaluated by means of Lemma 1, with C replaced by BZq^{-n}. Subsequently, a few simplifications lead to the right-hand side of (2.5), when multiplied by $\prod_{1\le i<j\le n}(1/X_j-1/X_i)$. $\square$

Finally, we are able to prove our first identity.

Proof of 2.4 To the left-hand side of (2.4) we apply (1.2) (which is (2.5) with $A_i\equiv A$). To the resulting series we apply (1.3) (which is (2.6) with $A_i\equiv A$ and C replaced by C/A), with the replacements $A\to ABZ/C$, $C\to BZ$, $Z\to C/B$. Thus we arrive at the right-hand side of (2.4). $\square$

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