A NEW MATRIX INVERSE

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Abstract. We compute the inverse of a specific infinite-dimensional matrix, thus unifying a number of previous matrix inversions. Our inversion theorem is applied to derive a number of summation formulas of hypergeometric type.

1. Introduction. Let $F = (f_{nk})_{n,k\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers) be an infinite-dimensional lower-triangular matrix; i.e. $f_{nk} = 0$ unless $n \ge k$. The matrix $(f_{kl}^{-1})_{k,l\in\mathbb{Z}}$ is the inverse matrix of F if and only if

$$\sum_{n\ge k\ge l} f_{nk} f_{kl}^{-1} = \delta_{nl}$$

for all $n, l \in \mathbb{Z}$. Such matrix inversions are very important in many fields of combinatorics and special functions. For example, when dealing with combinatorial sums, application of the so-called "inverse relations" (see (4.1) and (4.2)), which base on matrix inversion, helps to simplify problems, or yields new identities. Riordan dedicated two chapters of his book [21] to inverse relations and its applications. Riordans inverse relations were classified and given a unified method of proof by Egorychev [7, ch.3]. Studying a specific class of inverse relations in a series of papers [12–15], Gould and Hsu [16] finally discovered a very general matrix inversion, which is equivalent to:

If

$$A_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + kb_j)}{(n-k)!} \tag{1.1)(1}$$

then

$$A_{kl}^{-1} = (-1)^{k-l}\frac{a_l + lb_l}{a_k + kb_k}\frac{\prod_{j=l+1}^{k}(a_j + kb_j)}{(k-l)!} . \tag{1.1)(2}$$

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(By convenience, products of the form $\prod_{j=u}^{u-1}$ are defined to be equal to 1, while for $u > v-1$ a product $\prod_{j=u}^{v-1}$ by definition is equal to 0.) The inverse pair (1.1) contains a lot of inverse relations of Gould–type (cf. [21, pp. 50–51]) and Abel–type (cf. [21, p. 99]). Carlitz [6] found a q-analogue of (1.1) which is equivalent to:

If

$$B_{nk} = q^{\binom{n-k}{2}} \frac{\prod_{j=k}^{n-1}(a_j + q^k b_j)}{(q;q)_{n-k}} \tag{1.2)(1}$$

then

$$B_{kl}^{-1} = (-1)^{k-l} \frac{a_l + q^l b_l}{a_k + q^k b_k} \frac{\prod_{j=l+1}^{k}(a_j + q^k b_j)}{(q;q)_{k-l}}\ , \tag{1.2)(2}$$

where $(a;q)_m = (1-a)(1-aq)\cdots(1-aq^{m-1})$. Special cases of (1.2) are of great significance in q-series theory. Andrews [1] showed that the Bailey transform [2, 3], used to prove identities of Rogers-Ramanujan type, is equivalent to the case $a_j = 1$ and $b_j = -bq^j$ of (1.2). Gessel and Stanton [11] used several specializations of the $a_j = 1$, $b_j = -bp^j$ case of (1.2) to derive a number of basic hypergeometric summations and transformations, and, once again, identities of Rogers–Ramanujan type.

In [4] Bressoud considers finite forms of Rogers–Ramanujan identities. The transform which he uses to prove them is equivalent to the matrix inversion [5]:

If

$$C_{nk} = \frac{(1-aq^{2k})(b;q)_{n+k}\,(ba^{-1};q)_{n-k}(ba^{-1})^k}{(1-a)(aq;q)_{n+k}\,(q;q)_{n-k}} \tag{1.3)(1}$$

then

$$C_{kl}^{-1} = \frac{(1-bq^{2l})(a;q)_{k+l}\,(ab^{-1};q)_{k-l}\,(ab^{-1})^l}{(1-b)(bq;q)_{k+l}\,(q;q)_{k-l}}\ . \tag{1.3)(2}$$

The purpose of this paper is to give the following generalization of (1.1)–(1.3), which will be proved in section 2:

Theorem. *Let $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$ and $(c_i)_{i\in\mathbb{Z}}$ be arbitrary sequences such that $c_i \neq c_j$ if $i \neq j$.*

If

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + c_k b_j)}{\prod_{j=k+1}^{n}(c_j - c_k)} \tag{1.4)(1}$$

then

$$f_{kl}^{-1} = \frac{a_l + c_l b_l}{a_k + c_k b_k} \frac{\prod_{j=l+1}^{k}(a_j + c_k b_j)}{\prod_{j=l}^{k-1}(c_j - c_k)}\ . \tag{1.4)(2}$$

In fact, (1.1) is the special case $c_k = k$, (1.2) is equivalent to the case $c_k = q^k$, and (1.3) is equivalent to the case $c_k = q^{-k} + aq^k$, $a_j = aq^{-j-1} + b^2q^{j-1}$, $b_j = -b/q$. Moreover, replacing c_k by $(1 + ac_k^2)/c_k$, a_j by $(a + b_j^2)/c_{j+1}$ and b_j by $-b_j/c_{j+1}$ in (1.4), we get after some simplification

Corollary. *With the assumption of the Theorem, $a \neq (c_j c_k)^{-1}$ for all $j, k \in \mathbb{Z}$, if*

$$g_{nk} = \frac{\prod_{j=k}^{n-1}(1-c_k b_j)\prod_{j=k}^{n-1}(b_j - ac_k)}{\prod_{j=k+1}^{n}(1-ac_k c_j)\prod_{j=k+1}^{n}(c_j - c_k)} \tag{1.5)(1}$$

then

$$g_{kl}^{-1} = \frac{(1-c_l b_l)(b_l - ac_l)}{(1-c_k b_k)(b_k - ac_k)} \frac{\prod_{j=l+1}^{k}(1-c_k b_j)\prod_{j=l+1}^{k}(b_j - ac_k)}{\prod_{j=l}^{k-1}(1-ac_k c_j)\prod_{j=l}^{k-1}(c_j - c_k)} \ . \tag{1.5)(2}$$

For $c_k = q^k$ the cases $b_j = bq^j$ and $b_j = bq^{-j}$ are equivalent to (1.3). In this sense the form (1.5) of (1.4) is the natural generalization of Bressoud's matrix inverse (1.3), such as (1.4) is the natural generalization of Gould's and Hsu's (1.1) and Carlitz's (1.2).

A special case of (1.5), namely (4.3), which involves rising q-factorials with two different bases, has been crucial in papers by Gasper and Rahman [8, 19]. They used inversion together with an indefinite bibasic sum to derive numerous beautiful bibasic, cubic, and quartic summation formulas for basic hypergeometric series (see section 3 for "hypergeometric" definitions). (They also extended this method to obtain bibasic, cubic, and quartic transformation formulas [9, 20; 10, sec. 3.6].) In section 4 we add two more applications of this inversion. We derive a summation formula (identity (4.8)) for series of hypergeometric type and as a by-product of the second application we obtain a basic hypergeometric transformation formula (identity (4.12)). The former contains an infinite family of summation formulas for very well-poised hypergeometric series. Besides, we use the opportunity to clearly demonstrate that what Gasper and Rahman do in [8, 19] is indeed inversion, though in disguise. This fact does not seem to be as accepted as it should be. (Their extension in [9, 20; 10, sec. 3.6] has an explanation in terms of "partial inversion".)

Finally, in section 5 we apply other special cases of (1.4) to obtain curious identities ((5.5), (5.9), (5.12)) which involve cubic, quartic, and quintic analogues of rising factorials, respectively.

2. Proof of Theorem. In [18] the author gave a method for solving Lagrange inversion problems. These are closely connected with the problem of inverting lower-triangular matrices. For convenience, by a formal Laurent series (fLs) we always mean a series of the form $\sum_{i\geq m} \alpha_i z^i$, for some $m \in \mathbb{Z}$. Given the fLs $a(z)$ and $b(z)$ we introduce the bilinear form $\langle\, ,\rangle$ by

$$\langle a(z), b(z)\rangle = \langle z^0\rangle a(z)\cdot b(z) \ ,$$

where $\langle z^0\rangle c(z)$ denotes the coefficient of z^0 in $c(z)$. Given any linear operator L acting on fLs, L^* denotes the adjoint of L with respect to $\langle\, ,\rangle$; i.e. $\langle La(z), b(z)\rangle = \langle a(z), L^* b(z)\rangle$ for all fLs $a(z)$ and $b(z)$. What we need is the following special case of [18, Theorem 1].

Lemma. *Let $F = (f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower-triangular matrix with $f_{kk} \neq 0$ for all $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, the formal Laurent series $f_k(z)$ and $\tilde{f}_k(z)$ are defined by $f_k(z) = \sum_{n\geq k} f_{nk} z^n$ and $\tilde{f}_k(z) = \sum_{l\leq k} f_{kl}^{-1} z^{-l}$, where $(f_{kl}^{-1})_{k,l\in\mathbb{Z}}$ is the uniquely determined inverse matrix of F. Suppose that for $k \in \mathbb{Z}$ holds*

$$U f_k(z) = c_k \cdot V f_k(z) \ , \tag{2.1}$$

where U, V are linear operators acting on formal Laurent series, U being bijective, and $(c_k)_{k\in\mathbb{Z}}$ is a sequence of constants. If $h_k(z)$ is a solution of

$$U^* h_k(z) = c_k \cdot V^* h_k(z) \ , \tag{2.2}$$

with $h_k(z) \not\equiv 0$ for all $k \in \mathbb{Z}$, then

$$\tilde{f}_k(z) = \frac{1}{\langle f_k(z), V^* h_k(z)\rangle} V^* h_k(z) \ . \tag{2.3}$$

In order to prove (1.4), we set $f_k(z) = \sum_{n\ge k} f_{nk} z^n$ with f_{nk} of (1.4)(1). Obviously, for $n \ge k$

$$(c_n - c_k) f_{nk} = (a_{n-1} + c_k b_{n-1}) f_{n-1,k} \ . \tag{2.4}$$

If we define the linear operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ by $\mathcal{A} z^k = a_k z^k, \mathcal{B} z^k = b_k z^k$ and $\mathcal{C} z^k = c_k z^k$, (2.4) may be rewritten in the form

$$(\mathcal{C} - z\mathcal{A}) f_k(z) = c_k (1 + z\mathcal{B}) f_k(z) \ , \tag{2.5}$$

valid for all $k \in \mathbb{Z}$. (2.5) is a system of equations of type (2.1) with $U = \mathcal{C} - z\mathcal{A}$ and $V = 1 + z\mathcal{B}$. The dual equations (2.2) for the auxiliary fLs $h_k(z)$ in this case read

$$(\mathcal{C}^* - \mathcal{A}^* z) h_k(z) = c_k (1 + \mathcal{B}^* z) h_k(z) \ . \tag{2.6}$$

Because of $\mathcal{A}^* z^{-k} = a_k z^{-k}$ etc., by comparing the coefficients of z^{-l} in (2.6) we obtain

$$(c_l - c_k) h_{kl} = (a_l + c_k b_l) h_{k,l+1} \ .$$

If we set $h_{kk} = 1$, we get

$$h_{kl} = \frac{\prod_{j=l}^{k-1} (a_j + c_k b_j)}{\prod_{j=l}^{k-1} (c_j - c_k)} \ .$$

Taking into account (2.3), here we have $\tilde{f}_k(z) = (1 + \mathcal{B}^* z) h_k(z)$. Hence again, comparing coefficients of z^{-l} leads to

$$f_{kl}^{-1} = h_{kl} + b_l h_{k,l+1} = \frac{a_l + c_l b_l}{a_k + c_k b_k} \frac{\prod_{j=l+1}^{k} (a_j + c_k b_j)}{\prod_{j=l}^{k-1} (c_j - c_k)} \ ,$$

which is exactly (1.4)(2).

3. Hypergeometric and basic hypergeometric notation. All the notation and terminology is adopted from [10, pp. 1–6]. The *(generalized) hypergeometric series* is defined by

$${}_rF_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{n!\,(b_1)_n \cdots (b_s)_n} z^n \ ,$$

where the rising factorial $(a)_n$ is given by $(a)_n := a(a+1)\cdots(a+n-1)$, $n \geq 1$, $(a)_0 := 1$. The gamma function can be used to extend rising factorials by defining $(a)_\beta = \lim_{\gamma\to a}\Gamma(\gamma+\beta)/\Gamma(\gamma)$, β arbitrary (cf. [17, p. 211f]). A hypergeometric series ${}_{r+1}F_r$ is called *very well-poised* if $a_i + b_i = 1 + a_0$ for $i = 1, 2, \dots, r$, and among the parameters a_i occurs $1 + a_0/2$. We use the standard abbreviation for very well-poised hypergeometric series,

$$ {}_{r+1}V_r(a_0; a_2, a_3, \dots, a_r; z) := {}_{r+1}F_r\left[\begin{matrix} a_0, 1+a_0/2, a_2, a_3, \dots, a_r \\ a_0/2, 1+a_0-a_2, 1+a_0-a_3, \dots, 1+a_0-a_r \end{matrix}; z\right]. $$

We shall also use the compact Gasper-Rahman notation

$$ (a_1, a_2, \dots, a_r)_n := (a_1)_n\,(a_2)_n\,\cdots\,(a_r)_n $$

and

$$ \Gamma\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}\right] := \frac{\Gamma(a_1)\,\Gamma(a_2)\,\cdots\,\Gamma(a_r)}{\Gamma(b_1)\,\Gamma(b_2)\,\cdots\,\Gamma(b_s)}\,. $$

Given a (fixed) complex number q with $|q| < 1$, the *basic hypergeometric series* is defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^n\,, $$

where, as before, the rising q-factorial $(a;q)_n$ is given by $(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$, $n \geq 1$, $(a)_0 := 1$. The infinite q-factorial $(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$ can be used to extend (finite) q-factorials by defining $(a;q)_\beta := (a;q)_\infty/(aq^\beta;q)_\infty$, β arbitrary. A basic hypergeometric series ${}_{r+1}\phi_r$ is called *very well-poised* if $a_i b_i = q a_0$ for $i = 1, 2, \dots, r$, and among the parameters a_i occur both $q\sqrt{a_0}$ and $-q\sqrt{a_0}$. We use the standard abbreviation for very well-poised basic hypergeometric series,

$$ {}_{r+1}W_r(a_0; a_3, a_4, \dots, a_r; z) := {}_{r+1}\phi_r\left[\begin{matrix} a_0, q\sqrt{a_0}, -q\sqrt{a_0}, a_3, a_4, \dots, a_r \\ \sqrt{a_0}, -\sqrt{a_0}, qa_0/a_2, qa_0/a_3, \dots, qa_0/a_r \end{matrix}; z\right]. $$

We use short notations in the basic hypergeometric context which are analogous to the hypergeometric ones.

All identities in our paper are subject to suitable conditions on the parameters such that the involved hypergeometric or basic hypergeometric series converge. We shall not state these conditions for each identity. The reader should consult [10, pp. 4–5].

4. Hypergeometric summation formulas from inversion. There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. [1, 11, 21]). If (f_{nk}) and (f_{kl}^{-1}) are lower triangular matrices being inverses of each other, then of course the following is true:

$$ \sum_{k=0}^{n} f_{nk} a_k = b_n \quad \textit{if and only if} \quad \sum_{l=0}^{k} f_{kl}^{-1} b_l = a_k. \tag{4.1} $$

If one side in (4.1) is known then the other produces another summation formula. There is also a less used dual version, which in Riordan's book [21] is called "rotated inversion". It reads

$$\sum_{n=k}^{\infty} f_{nk} B_n = A_k \quad \textit{if and only if} \quad \sum_{k=l}^{\infty} f_{kl}^{-1} A_k = B_l, \tag{4.2}$$

subject to suitable convergence conditions. Again, if one side in (4.2) is known then the other produces a possibly new identity.

What Gasper and Rahman do in [8, 19] is rotated inversion, though in disguise. Recall that the rotated inversion (4.2) bases on the orthogonal relation $\sum_{k=l}^{n} f_{nk} f_{kl}^{-1} = \delta_{nl}$. ($\delta_{nl}$ is the usual Kronecker delta.) Typically, Gasper and Rahman start with such an orthogonality relation, but with $l = 0$, i.e. $\sum_{k=0}^{n} f_{nk} f_{k0}^{-1} = \delta_{n0}$. (As inverse pair (f_{nk}), (f_{kl}^{-1}) they choose special cases of (4.3), with the roles of f_{nk} and f_{kl}^{-1} interchanged.) Then they multiply both sides by B_n, say, and sum over all n. This gives $\sum_{n=0}^{\infty} B_n \sum_{k=0}^{n} f_{nk} f_{k0}^{-1} = B_0$. Next the sums are interchanged to give $\sum_{k=0}^{\infty} f_{k0}^{-1} \sum_{n=k}^{\infty} f_{nk} B_n = B_0$. The inner sum in all cases is simplified by means of Bailey's nonterminating extension of Jackson's ${}_8\phi_7$-sum, to A_k say. Thus the identity $\sum_{k=0}^{\infty} f_{k0}^{-1} A_k = B_0$ is proved. Of course, the reader will have observed that this is exactly the proof of the left-to-right implication of (4.2), for $l = 0$. Hence, why not start with l generic? The answer is that there is no problem of doing the same with arbitrary l. But in the end it turns out that the resulting identity is not more general. In fact, a simple parameter substitution eliminates all occurences of l, and one is left with the identity for $l = 0$. This $l = 0$ phenomenon will be exemplified in the following application of the inversion (1.5).

Before we start, another digression appears to be of interest. Gasper and Rahman's indefinite bibasic sum [10, (3.6.13)], which is so important in their derivations, comes from a telescoping argument which is based on a mysterious factorization [10, (3.6.10)] of a difference of two four-term products into a four-term product. This factorization loses some of its mystery if one observes that it is just equivalent to the $n = 1$ case of Jackson's ${}_8\phi_7$-sum [10, (2.6.2)] (replace a by $adp^k q^{k-1}$, b by bp^k, c by cq^k, d by ap^k in Jackson's sum with $n = 1$).

In this section we use the inverse matrices of (1.5) with the choices $c_k = q^k$ and $b_j = bp^j$. This inversion can be written in the form,

if

$$f_{nk} = \frac{(bp^k q^k; p)_{n-k}\,(bp^k q^{-k}/a; p)_{n-k}}{(aq^{2k+1}; q)_{n-k}\,(q;q)_{n-k}}\;, \tag{4.3)(1}$$

then

$$f_{kl}^{-1} = p^{l-k}\frac{(1-bp^l q^l)(1-ap^{-l}q^l/b)}{(1-bp^k q^k)(1-ap^{-k}q^k/b)}\,\frac{(bp^k q^k; p^{-1})_{k-l}\,(bp^k q^{-k}/a; p^{-1})_{k-l}}{(aq^{2k-1}; q^{-1})_{k-l}\,(q^{-1};q^{-1})_{k-l}}\;. \tag{4.3)(2}$$

If in (4.3) we replace a by q^B, b by q^{B-C}, q by p^A, and let $p \to 1$, we obtain the inverse pair

If

$$(4.4)(1) \qquad f_{nk}=\frac{(AB-AC+(1+A)k)_{n-k}(-AC+(1-A)k)_{n-k}}{(B+2k+1)_{n-k}(n-k)!}\ ,$$

then

$$(4.4)(2) \quad f_{kl}^{-1}=\frac{(-AC+(1-A)l)\,(AB-AC+(1+A)l)}{(-AC+(1-A)k)\,(AB-AC+(1+A)k)} \frac{(-AB+AC-(1+A)k)_{k-l}(AC-(1-A)k)_{k-l}}{(1-B-2k)_{k-l}\,k!}\ .$$

For B_n we choose

$$(4.5) \qquad B_n=\frac{(D)_n}{(-B+AB-2AC+D)_n}.$$

Now we form the sum $A_k=\sum_{n=k}^{\infty} f_{nk}B_n$. In hypergeometric notation this is

$$A_k=\frac{(D)_k}{(-B+AB-2AC+D)_k}{}_3F_2\left[\begin{matrix}D+k,-AC+k-Ak,AB-AC+k+Ak\\ -B+AB-2AC+D+k,1+B+2k\end{matrix};1\right].$$

To the ${}_3F_2$ we apply a nonterminating extension of the Pfaff–Saalschütz summation [23, (2.4.4.4)]

$$(4.6) \quad {}_3F_2\left[\begin{matrix}a,b,c\\ d,1+a+b+c-d\end{matrix};1\right]=\Gamma\left[\begin{matrix}1+a-d,1+b-d,1+c-d,1+a+b+c-d\\ 1-d,1+b+c-d,1+a+c-d,1+a+b-d\end{matrix}\right] -\Gamma\left[\begin{matrix}d-1,1+a-d,1+b-d,1+c-d,1+a+b+c-d\\ 1-d,a,b,c,2+a+b+c-2d\end{matrix}\right] {}_3F_2\left[\begin{matrix}1+a-d,1+b-d,1+c-d\\ 2-d,2+a+b+c-2d\end{matrix};1\right],$$

thus obtaining

(4.7)

$$A_k=\Gamma\left[\begin{matrix}1+B-AB+2AC,1+B-AB+AC-D-Ak,1+B+AC-D+Ak\\ 1+B-AB+2AC-D-k,1+B-D+k,1+B+AC+k+Ak\end{matrix}\right] \Gamma\left[\begin{matrix}1+B+2k,-B+AB-2AC+D,D+k\\ 1+B-AB+AC+k-Ak,D,-B+AB-2AC+D+k\end{matrix}\right] -\Gamma\left[\begin{matrix}-1-B+AB-2AC+D+k,1+B-AB+2AC,1+B-AB+AC-D-Ak\\ 1+B-AB+2AC-D-k,-AC+k-Ak,AB-AC+k+Ak\end{matrix}\right] \Gamma\left[\begin{matrix}1+B+AC-D+Ak,1+B+2k,-B+AB-2AC+D\\ 2+2B-AB+2AC-D+k,D,-B+AB-2AC+D+k\end{matrix}\right] {}_3F_2\left[\begin{matrix}1+B-AB+2AC,1+B-AB+AC-D-Ak,1+B+AC-D+Ak\\ 2+B-AB+2AC-D-k,2+2B-AB+2AC-D+k\end{matrix};1\right]$$

By rotated inversion (4.2) the identity $\sum_{k=l}^{\infty} f_{kl}^{-1} A_k = B_l$, with f_{kl}^{-1}, A_k, B_l given by (4.4)(2), (4.7), (4.5), respectively, holds. The sum on the left-hand side splits into two sums, one of which is actually a double sum. In this double sum we interchange summations and simplify. In the resulting identity we replace B by $B-2l$, C by $C+(1-A)l/A$, and D by $D-l$. The effect of this last step is the same as if we would have set $l=0$. This is the $l=0$ phenomenon which was described before. Thus we arrive at the summation

$$(4.8)\quad \sum_{k=0}^{\infty} \frac{(B)_k (1+\frac{B}{2})_k (AB-AC)_{(1+A)k} (1+B+AC-D)_{Ak} (-AC)_{(1-A)k}}{k!\,(\frac{B}{2})_k (1+B+AC)_{(1+A)k} (1+AB-AC)_{Ak} (1+B-AB+AC)_{(1-A)k}} \frac{(1+B-AB+AC-D)_{-Ak} (D)_k}{(1-AC)_{-Ak} (1+B-D)_k}$$
$$-\Gamma\left[\begin{matrix} 1+B-AB+AC, 1+B+AC, 1+B-D, -1-B+AB-2AC+D \\ AB-AC, -AC, 2+2B-AB+2AC-D, D \end{matrix}\right]$$
$$\sum_{j=0}^{\infty} \frac{(1+B-AB+2AC)_j (1+B+AC-D)_j (1+B-AB+AC-D)_j}{j!\,(2+B-AB+2AC-D)_j (2+2B-AB+2AC-D)_j}$$
$$\sum_{k=0}^{\infty} \frac{(B)_k (1+\frac{B}{2})_k (-1-B+AB-2AC+D-j)_k}{k!\,(\frac{B}{2})_k (2+2B-AB+2AC-D+j)_k} \frac{(1+B+AC-D+j)_{Ak} (1+B-AB+AC-D+j)_{-Ak}}{(1+AB-AC)_{Ak} (1-AC)_{-Ak}}$$
$$=\Gamma\left[\begin{matrix} 1+B-AB+AC, 1+B+AC, 1+B-D, 1+B-AB+2AC-D \\ 1+B, 1+B-AB+2AC, 1+B-AB+AC-D, 1+B+AC-D \end{matrix}\right]$$

Unfortunately, the inner sum in the double sum cannot be evaluated in general. (It can for $A=1$. In this case even a q-analogue exists, see below.) If we terminate the first series by setting $D=-n$, n a nonnegative integer, then the term with the double sum vanishes because of the occurence of $\Gamma(D)$ in the denominator of this expression. The resulting summation can be rewritten as

$$(4.9)\quad \sum_{k=0}^{\infty} \frac{(B)_k (1+\frac{B}{2})_k (AB-AC)_{(A+1)k} (AC)_{Ak} (-B+AB-AC)_{(A-1)k}}{k!\,(\frac{B}{2})_k (1+B+AC)_{(A+1)k} (1+AB-AC)_{Ak} (1+AC)_{(A-1)k}}$$
$$\frac{(1+B+AC+n)_{Ak} (-n)_k}{(-B+AB-AC-n)_{Ak} (1+B+n)_k} = \frac{(1+B)_n (1+B-AB+2AC)_n}{(1+B+AC)_n (1+B-AB+AC)_n}.$$

If A is a positive integer (4.8) and (4.9) give summation formulas for (nonterminating, respectively terminating) very well-poised ${}_{4A+3}F_{4A+2}$-series. The case $A=1$ of (4.9) is almost equivalent to a known ${}_7F_6$-summation [23, (2.4.1.5)]. In fact, both formulas can be derived from each other by contiguous relations. Besides, identity (4.8) for a fixed A is equivalent to (4.8) with A replaced by $-A$. (Hence, the same is true for (4.9)). In fact, replacement of A by $-A$ is the same as replacement of C by $B-C$.

Letting n tend to ∞ in (4.9) leads to the summation

$$(4.10)\quad \sum_{k=0}^{\infty} \frac{(B)_k(1+\frac{B}{2})_k(AB-AC)_{(A+1)k}(AC)_{Ak}(-B+AB-AC)_{(A-1)k}}{k!\,(\frac{B}{2})_k(1+B+AC)_{(A+1)k}(1+AB-AC)_{Ak}(1+AC)_{(A-1)k}} = \Gamma\left[\begin{matrix} 1+B, 1+B-AB+2AC \\ 1+B+AC, 1+B-AB+AC \end{matrix}\right].$$

If A is a positive integer (4.10) gives a summation formula for nonterminating very well-poised ${}_{3A+2}F_{3A+1}$-series.

It is only the case $A=1$, where we are able to give a q-analogue of (4.8) (and hence of (4.9) and (4.10)). This is due to the absence of a suitable bibasic analogue of the Pfaff–Saalschütz formula. (Recently a bibasic analogue of the Pfaff–Saalschütz formula has been discovered by Singer [22], but it is not suited for our purposes.) To obtain the q-analogue of (4.8) in case $A=1$, start with the inverse pair (f_{nk}), (f_{kl}^{-1}) from (4.3), with $p=q$, $a=B$, $b=B/C$. For B_n choose $B_n = q^n(D;q)_n/(D/C^2;q)_n$. Then proceed as before. Instead of (4.6) use the nonterminating extension of the q-Pfaff–Saalschütz sum [10, (2.10.12)]. To evaluate the inner sum of the arising double sum, use the very well-poised ${}_6\phi_5$-summation [10, (2.7.1)]. What is finally obtained is the summation

$$(4.11)\quad {}_{10}W_9(B; \frac{\sqrt{B}}{\sqrt{C}}, -\frac{\sqrt{B}}{\sqrt{C}}, \frac{\sqrt{Bq}}{\sqrt{C}}, -\frac{\sqrt{Bq}}{\sqrt{C}}, \frac{BCq}{D}, C, D; q, q) - \frac{(Bq, 1/C, D, B/C, D/C^2, BCq^2/D; q)_\infty}{(Bq/D, BCq, D/C^2q, D/C, q, Bq/C; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} C^2q, C^2q/D, BCq/D \\ C^2q^2/D, BCq^2/D \end{matrix}; q, \frac{q}{C}\right] = \frac{(C^2q, Cq/D, BCq/D, Bq; q)_\infty}{(Cq, C^2q/D, Bq/D, BCq; q)_\infty}.$$

By comparing this identity with a special case of a transformation of Verma and Jain [24, (7.1); 10, Ex. 2.25, $a=B/C$, $b=qB/C^2$, $c=1$, $d=1$, $e=D$],

$${}_{10}W_9(B; \frac{\sqrt{B}}{\sqrt{C}}, -\frac{\sqrt{B}}{\sqrt{C}}, \frac{\sqrt{Bq}}{\sqrt{C}}, -\frac{\sqrt{Bq}}{\sqrt{C}}, C, D, \frac{BCq}{D}; q, q) + \frac{(Bq, Cq/D, B/C, D, C, Bq^2/D, BC^3q^3/D^2; q)_\infty}{(Cq, Bq/D, BCq, C^2q/D, D/Cq, Bq/C, BC^2q^3/D^2; q)_\infty}\, {}_{10}W_9(\frac{BC^2q^2}{D^2}; \frac{\sqrt{BC}q}{D}, -\frac{\sqrt{BC}q}{D}, \frac{\sqrt{BC}q^{\frac{3}{2}}}{D}, -\frac{\sqrt{BC}q^{\frac{3}{2}}}{D}, Cq, \frac{BCq}{D}, \frac{C^2q}{D}; q, q) = \frac{(Bq, Cq/D, C^2q, BCq/D; q)_\infty}{(Cq, Bq/D, BCq, C^2q/D; q)_\infty},$$

we obtain a transformation formula between a very well-poised ${}_{10}\phi_9$ and a ${}_3\phi_2$ series,

$${}_{10}W_9(\frac{BC^2q^2}{D^2}; \frac{\sqrt{BC}q}{D}, -\frac{\sqrt{BC}q}{D}, \frac{\sqrt{BC}q^{\frac{3}{2}}}{D}, -\frac{\sqrt{BC}q^{\frac{3}{2}}}{D}, Cq, \frac{C^2q}{D}, \frac{BCq}{D}; q, q) = \frac{(q/C, BCq^2/D, C^2q^2/D, BC^2q^3/D^2; q)_\infty}{(q, Cq^2/D, Bq^2/D, BC^3q^3/D^2; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} C^2q, C^2q/D, BCq/D \\ C^2q^2/D, BCq^2/D \end{matrix}; q, \frac{q}{C}\right],$$

or after having replaced B by BD^2/C^2q^2, C by B/C, D by qD/C, in that order,

$$(4.12)\quad {}_{10}W_9(B;\sqrt{C},-\sqrt{C},\sqrt{Cq},-\sqrt{Cq},D,\frac{Bq}{C},\frac{B^2}{CD};q,q) \\ = \frac{(Cq/B,Dq,B^2q/CD,Bq;q)_\infty}{(q,Bq/D,CDq/B,B^2q/C;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} B^2q/C^2, B^2/CD, D \\ B^2q/CD, Dq \end{matrix};q,\frac{Cq}{B}\right]$$

Mizan Rahman (private communication) showed me how to derive this transformation from standard basic hypergeometric transformation formulas by a sequence of series manipulations.

5. More identities. In this section we restrict ourselves to terminating series. Therefore here we use the "usual" inversion (4.1). We use special cases of (1.4) which are different from the one, (4.3), used in the previous section.

First, let us take $c_k = (k+c)^3$, $a_j = (j+a)^3$, $b_j = 1$. The inverse pair (1.4) in this special case reads

$$(5.1)(1)\qquad f_{nk} = \frac{(n+a)^3+(n+c)^3}{(k+a)^3+(k+c)^3}\frac{\prod_{j=k}^{n-1}((j+a)^3+(k+c)^3)}{\prod_{j=k+1}^{n}((j+c)^3-(k+c)^3)}$$

and

$$(5.1)(2)\qquad f_{kl}^{-1} = \frac{\prod_{j=l+1}^{k}((j+a)^3+(k+c)^3)}{\prod_{j=l}^{k-1}((j+c)^3-(k+c)^3)}\,.$$

Now choose

$$(5.2)\qquad b_l = \frac{(a+c,1+a/2+c/2,d,2a-c-d+1)_l}{(1,a/2+c/2,1+a+c-d,2c-a+d)_l}\,.$$

We form $a_k = \sum_{l=0}^{k} f_{kl}^{-1} b_l$, which can be written as

$$a_k = \frac{\prod_{j=1}^{k}((j+a)^3+(k+c)^3)}{\prod_{j=0}^{k-1}((j+c)^3-(k+c)^3)} \\ {}_7V_6(a+c;1+2a-c-d,d,c-c\omega-k\omega,c-c\omega^2-k\omega^2,-k;1).$$

Here, ω denotes a third root of unity. The ${}_7V_6$ can be evaluated by means of Dougall's sum [23, (2.3.4.4)]

$$(5.3)\quad {}_7V_6(a;b,c,d,1+2a-b-c-d+n,-n;1) \\ = \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}\,,$$

thus obtaining

$$a_k = \frac{(1+a+c)_{2k}\,(-a+2c)_k}{(1+a+c-d)_{2k}\,(-a+2c+d)_k}\frac{\prod_{j=1}^{k}((j+a-d)^3+(k+c)^3)}{\prod_{j=0}^{k-1}((j+c)^3-(k+c)^3)} \tag{5.4}$$

after simplification. Now, by the inversion (4.1), the identity $\sum_{k=0}^{n} f_{nk}a_k = b_n$ holds, with f_{nk}, a_k, b_n given in (5.1)(1), (5.4), (5.2), respectively. This identity reads

$$\sum_{k=0}^{n} \frac{(n+a)^3+(n+c)^3}{(k+a)^3+(k+c)^3}\frac{\prod_{j=k}^{n-1}((j+a)^3+(k+c)^3)}{\prod_{j=k+1}^{n}((j+c)^3-(k+c)^3)} \tag{5.5}$$
$$\frac{\prod_{j=1}^{k}((j+a-d)^3+(k+c)^3)}{\prod_{j=0}^{k-1}((j+c)^3-(k+c)^3)}\frac{(1+a+c)_{2k}\,(-a+2c)_k}{(1+a+c-d)_{2k}\,(-a+2c+d)_k}$$
$$= \frac{(a+c, 1+a/2+c/2, d, 2a-c-d+1)_n}{(1, a/2+c/2, 1+a+c-d, 2c-a+d)_n}.$$

The products which occur in the sum can be considered as cubic analogues of the rising factorials. For example, $\prod_{j=k}^{n-1}((j+a)^3+(k+c)^3)$ is a cubic analogue of $(2k+a+c)_{n-k} = \prod_{j=k}^{n-1}((j+a)+(k+c))$.

Next we choose $c_k = (k+c)^4$, $a_j = (j+a)^4$, $b_j = -1$. The inverse pair (1.4) now reads

$$f_{nk} = \frac{(n+a)^4-(n+c)^4}{(k+a)^4-(k+c)^4}\frac{\prod_{j=k}^{n-1}((j+a)^4-(k+c)^4)}{\prod_{j=k+1}^{n}((j+c)^4-(k+c)^4)} \tag{5.6)(1}$$

and

$$f_{kl}^{-1} = \frac{\prod_{j=l+1}^{k}((j+a)^4-(k+c)^4)}{\prod_{j=l}^{k-1}((j+c)^4-(k+c)^4)}\ . \tag{5.6)(2}$$

For b_l we take

$$b_l = \frac{(a+c, 1+a/2+c/2, 1-2c+2a)_l}{(1, a/2+c/2, 3c-a)_l}\ . \tag{5.7}$$

Then we form $a_k = \sum_{l=0}^{k} f_{kl}^{-1} b_l$. Again, the sum can be evaluated by means of Dougall's sum (5.3). Thus we get

$$a_k = \frac{(1+a+c)_{2k}}{(3c-a)_{2k}}\frac{\prod_{j=0}^{k-1}((j+2c-a)^4-(k+c)^4)}{\prod_{j=0}^{k-1}((j+c)^4-(k+c)^4)}\ . \tag{5.8}$$

The inversion (4.1) then yields

$$\sum_{k=0}^{n} \frac{(n+a)^4-(n+c)^4}{(k+a)^4-(k+c)^4}\frac{\prod_{j=k}^{n-1}((j+a)^4-(k+c)^4)}{\prod_{j=k+1}^{n}((j+c)^4-(k+c)^4)} \tag{5.9}$$
$$\frac{\prod_{j=0}^{k-1}((j+2c-a)^4-(k+c)^4)}{\prod_{j=0}^{k-1}((j+c)^4-(k+c)^4)}\frac{(1+a+c)_{2k}}{(3c-a)_{2k}}$$
$$= \frac{(a+c, 1+a/2+c/2, 1-2c-2a)_n}{(1, a/2+c/2, 1+a+c-d, 3c-a)_n}.$$

In this identity there occur quartic analogues of rising factorials.

Finally, we take $c_k = (k+c)^5$, $a_j = (j+a)^5$, $b_j = 1$. The inverse pair (1.4) then reads

(5.10)(1) $$f_{nk} = \frac{(n+a)^5 + (n+c)^5}{(k+a)^5 + (k+c)^5} \frac{\prod_{j=k}^{n-1}((j+a)^5 + (k+c)^5)}{\prod_{j=k+1}^{n}((j+c)^5 - (k+c)^5)}$$

and

(5.10)(2) $$f_{kl}^{-1} = \frac{\prod_{j=l+1}^{k}((j+a)^5 + (k+c)^5)}{\prod_{j=l}^{k-1}((j+c)^5 - (k+c)^5)} .$$

Now choose

(5.11) $$b_l = \frac{(a+c, 1+a/2+c/2)_l}{(1, a/2+c/2)_l} .$$

Then for c choose $(2a+1)/3$ and subsequently replace a by $3a+1$. If we do the same as in the previous two derivations, we finally obtain the following identity which contains quintic analogues of rising factorials,

(5.12) $$\sum_{k=0}^{n} \frac{(n+3a+1)^5 + (n+2a+1)^5}{(k+3a+1)^5 + (k+2a+1)^5} \frac{\prod_{j=k}^{n-1}((j+3a+1)^5 + (k+2a+1)^5)}{\prod_{j=k+1}^{n}((j+2a+1)^5 - (k+2a+1)^5)}$$
$$\frac{(5a+3)_{2k} \, (1+a+(1+2a+k)(\omega+\omega^2), 1+a+(1+2a+k)(\omega+\omega^3)}{\prod_{j=0}^{k-1}((j+2a+1)^5 - (k+2a+1)^5)}$$
$$\frac{1+a+(1+2a+k)(\omega+\omega^4))_k}{} = \frac{(5a+2, 5a/2+2)_n}{(1, 5a/2+1)_n}.$$

Here, ω denotes a fifth root of unity.

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