ON THE INTEGRALITY OF THE TAYLOR COEFFICIENTS OF MIRROR MAPS

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Abstract. We show that the Taylor coefficients of the series $\mathbf{q}(z) = z\exp(\mathbf{G}(z)/\mathbf{F}(z))$ are integers, where $\mathbf{F}(z)$ and $\mathbf{G}(z) + \log(z)\mathbf{F}(z)$ are specific solutions of certain hypergeometric differential equations with maximal unipotent monodromy at $z = 0$. We also address the question of finding the largest integer u such that the Taylor coefficients of $(z^{-1}\mathbf{q}(z))^{1/u}$ are still integers. As consequences, we are able to prove numerous integrality results for the Taylor coefficients of mirror maps of Calabi–Yau complete intersections in weighted projective spaces, which improve and refine previous results by Lian and Yau, and by Zudilin. In particular, we prove the general "integrality" conjecture of Zudilin about these mirror maps. A further outcome of the present study is the determination of the Dwork–Kontsevich sequence $(u_N)_{N\ge1}$, where u_N is the largest integer such that $q(z)^{1/u_N}$ is a series with integer coefficients, where $q(z) = \exp(F(z)/G(z))$, $F(z) = \sum_{m=0}^{\infty}(Nm)!\,z^m/m!^N$ and $G(z) = \sum_{m=1}^{\infty}(H_{Nm} - H_m)(Nm)!\,z^m/m!^N$, with H_n denoting the n-th harmonic number, conditional on the conjecture that there are no prime number p and integer N such that the p-adic valuation of $H_N - 1$ is strictly greater than 3.

1. Introduction and statement of results

1.1. **Mirror maps.**

Mirror maps have appeared quite recently in mathematics and physics. Indeed, the term "mirror map" was coined in the late 1980s by physicists whose research in string theory led them to discover deep facts in algebraic geometry (e.g., pair of Calabi–Yau manifolds in "mirror," enumeration of rational curves on one of these Calabi–Yau manifolds in dimension 3). In a sense, mirror maps can be viewed as higher dimensional generalisations of certain classical modular forms, which naturally appear in low dimensions (see some examples below).

The purpose of the present article is to prove rather sharp integrality assertions for the Taylor coefficients of certain mirror maps coming from hypergeometric differential equations, which are Picard–Fuchs equations of suitable one parameter families of Calabi–Yau complete intersections in weighted projective spaces. The corresponding results (see Theorems 1–5) encompass integrality results on these mirror maps which exist in the literature, improving and refining them in numerous cases.

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Before getting deeper into the subject, it is beneficial to define at this point a special case of a mirror map (see just after (1.1) below), which will be studied in great detail in the present paper. For a real number $x > 0$ and an integer $m \geq 0$, set

$$H(x, m) := \sum_{n=0}^{m-1} \frac{1}{x+n},$$

where, by convention, the empty sum is 0. When $x = 1$, $H(1, m)$ is simply the m-th harmonic number H_m. For integers $k \geq 1$ and $N \geq 1$, let us define the power series

$$F_N(z) := \sum_{m=0}^{\infty} \frac{(Nm)!^k}{m!^{kN}} z^m$$

and

$$G_N(z) := k \sum_{m=1}^{\infty} \frac{(Nm)!^k}{m!^{kN}} \left(\sum_{j=1}^{N-1} H(j/N, m) - (N-1)H(1, m) \right) z^m,$$

which converge for $|z| < 1/N^{kN}$. ([1]) The functions $F_N(z)$ and $G_N(z) + \log(z)F_N(z)$ are solutions of the same hypergeometric differential equation, which is a special case of (1.18) below. (It is of maximal unipotent monodromy, i.e., $F_N(N^{-kN}z)$ is a hypergeometric function with only 1's as lower parameters. Its other solutions around $z = 0$ can then be obtained by Frobenius' method, see [29].) The function

$$q_N(z) := z \exp(G_N(z)/F_N(z)) \in z\mathbb{Q}[[z]] \tag{1.1}$$

is usually called the *canonical coordinate*, and its compositional inverse $z_N(q)$ is the prototype of a mirror map. In this paper, by abuse of terminology, we will also use the term "mirror map" for any canonical coordinate. ([2])

The case $N = 1$ is trivial because $G_1(z) = 0$ for any k. When $N = 2$ and $k = 1$, we have $F_2(z) = 1/\sqrt{1-4z}$ and $q_2(z) = 4z/(1+\sqrt{1-4z})^2$. For small values of N and k, modular forms rapidly enter the picture. Indeed, for $N = 2$ and $k = 2$, the compositional inverse of $q_2(z)$ is equal to $\lambda(q)/16 := q \prod_{n=1}^{\infty} \big((1+q^{2n})/(1+q^{2n-1})\big)^8$, which is a modular function of the variable τ, with $\Im(\tau) > 0$, defined by $q = \exp(2i\pi\tau)$. Furthermore, when $N = 2$ and $k = 3$, the compositional inverse of $q_2(z)$ is equal to $\lambda(q)(1-\lambda(q))/16$. See for example the discussion in [2, pp. 111–113] (and also for the importance of such facts in Diophantine approximation), and see [19, Sec. 3] for a discussion of the modularity of the case $N = 4, k = 1$. In fact, Doran [9] proved a result that enables one to describe all mirror maps of modular origin.

[1] Most of the time, the dependence on k will not be explicit in our notation for the sake of better readability. Furthermore, for the results of the present paper, it is sufficient to consider the series as *formal* power series.

[2] Canonical coordinates and mirror maps have distinct geometric meanings. However, in the number-theoretic study undertaken in the present paper, they play strictly the same role, because $(z^{-1}q(z))^{1/\tau} \in 1+z\mathbb{Z}[[z]]$ for some integer τ implies that $(q^{-1}z(q))^{1/\tau} \in 1+q\mathbb{Z}[[q]]$, and conversely. (See [18, Introduction].)

The most famous non-modular example of a mirror map is $q_5(z)$ (when $N = 5, k = 1$), which was used in the epoch-making paper by the physicists Candelas et al. [6]. Without going into details (see also [22, 24, 27] and the references therein), let us give a short explanation of the importance of $q_5(z)$. Starting from the family $\mathbf{M}$ of quintic hypersurfaces in $\mathbb{P}^4(\mathbb{C})$ defined by $\sum_{k=1}^5 x_k^5 - 5z\prod_{k=1}^5 x_k = 0$ (z being a complex parameter), Candelas et al. naturally associate another family $\mathbf{W}$ of manifolds (the "mirror of $\mathbf{M}$") which turn out to be Calabi–Yau. To $\mathbf{W}$, one can naturally associate a vector of periods (depending on z) which are solutions of the same differential equation (namely, the Picard–Fuchs equation of $\mathbf{W}$). This equation is simply the hypergeometric differential equation satisfied by $F_5(z)$ and $G_5(z) + \log(z)F_5(z)$ (case $N = 5$, $k = 1$ above). Then they observed the non-trivial property that the Taylor coefficients of $q_5(z)$ are integers. Furthermore, let us define the *Yukawa coupling* ([3])

$$K(q) := \frac{5}{1-5^5 z_5(q)} \cdot \frac{1}{F_5(z_5(q))^2} \cdot \left(\frac{q z_5'(q)}{z_5(q)}\right)^3 \in \mathbb{Q}[[q]], \tag{1.2}$$

where $z_5(q)$ is the compositional inverse of $q_5(z)$, and write it as $K(q) = 5 + \sum_{d=1}^\infty k_d \frac{q^d}{1-q^d}$, which is formally possible. Candelas et al. observed that the *instanton number* $n_d := k_d/d^3$ is an integer for all $d \geq 1$, which is already a non-trivial fact, but that furthermore n_d seems to be the number of rational curves of degree d lying on the initial quintic $\mathbf{M}$, thereby providing an effective algorithm to compute these numbers. These striking observations generated much interest amongst algebraic geometers, and this culminated in the work of Givental [12] and Lian et al. [20] where it is proved that, if for a given d the curves of degree d are all rigid, then there are n_d of them; see also the discussion in [27, p. 49]. In fact, the coincidence was proved to be true for $d \leq 9$, and the first difference occurs at $d = 10$ (see [8]). (However, we do not address questions related to Yukawa couplings in the present paper.)

1.2. **Integrality of mirror maps.** Deep results can also be proved without any appeal to algebraic geometry. Indeed, using p-adic methods, Lian and Yau were the first to prove that $q_5(z)$ has integral Taylor coefficients. In fact, in [19, Sec. 5, Theorem 5.5], they proved that $q_N(z) \in z\mathbb{Z}[[z]]$ for $k = 1$ and any N which is a prime number. Zudilin [30, Theorem 3] extended their result to any $k \geq 1$ and any N which is a prime power and made the following conjecture, which is also implicit in the cited articles of Lian and Yau, for example at the end of the introduction of [21]. In fact, the conjecture probably belongs to the folklore of mirror symmetry theory, and it seems that it had been left open for N not a prime power.

Conjecture 1. *For any integers $k \geq 1$ and $N \geq 1$, we have $q_N(z) \in z\mathbb{Z}[[z]]$.*

Such integrality questions for mirror maps and Yukawa couplings undoubtedly remain an important question for algebraic geometers, as is witnessed by the very recent preprint [28] (which is an elaborate version of [16]) on this subject. The mirror maps that are considered

[3] The Yukawa coupling is a geometric object whose definition in a specific situation can be found in [4, Definition 4.5.2]. In the present case, it can be computed as in (1.2).

in that paper comprise ours. When both approaches apply simultaneously, our results in Theorems 1–5 are stronger than Theorem 2 in [28, Sec. 1.3]. Indeed, we prove that certain mirror maps have integral Taylor coefficients, while in [28] the weaker statement is proved that mirror maps have Taylor coefficients in $\mathbb{Z}[1/n]$, where n is an integer parameter of geometric origin which is at least 2 (by assumption iii) just before Theorem 2 in [28]). On the other hand, the range of applicability of [28, Theorem 2] is much wider than ours. It is interesting to note that our approach as well as the one in [28] are heavily based on p-adic analysis, so that both are clearly close in spirit, although the exact methods that are applied are different.

In [21], Lian and Yau strengthened their result from [19] by proving an observation made by physicists:

$$\big(z^{-1}q_N(z)\big)^{1/N} \in \mathbb{Z}[[z]] \tag{1.3}$$

for $k=1$ and any prime N.

Our original goal was to settle Conjecture 1 and to prove (1.3) for arbitrary k and N. In the present paper, we shall accomplish much more: we establish refinements of the above integrality assertions which enable us to prove results that are even stronger and more general than Conjecture 1 or (1.3). In fact, our refinements will go into two different, only barely overlapping directions. One direction is inspired by a conjecture in Zudilin's paper [30], while the other seems to be entirely new. In the remainder of this introductory section, we describe these two directions, and we present our results. Their proofs will then be given in the subsequent sections.

1.3. **Refinements of** (1.3)**, part I.** We describe the second direction of refinement of (1.3) first. Starting point is the observation that

$$\sum_{j=1}^{N-1} H(j/N,m) - (N-1)H(1,m) = NH_{Nm} - NH_m.$$

Hence, with

$$G_{L,N}(z) := \sum_{m=1}^{\infty} H_{Lm}\,\frac{(Nm)!^k}{m!^{kN}}\,z^m$$

and $q_{L,N}(z) := \exp(G_{L,N}(z)/F_N(z))$, we have

$$q_N(z) = zq_{N,N}(z)^{kN}q_{1,N}(z)^{-kN}. \tag{1.4}$$

We are ready to state our first refinement of Conjecture 1 and (1.3).

Theorem 1. *For any integers $k,N\ge 1$ and $L\in\{1,2,\dots,N\}$, we have $q_{L,N}(z)\in\mathbb{Z}[[z]]$.*

Clearly, due to (1.4), Theorem 1 implies Conjecture 1 and (1.3).

However, much more is true. For any sequence $\mathbf{N} = (N_1, \dots, N_k)$ of positive integers (the N_j's are not necessarily distinct) and any integer $L \geq 1$, let us define the power series

$$F_{\mathbf{N}}(z) := \sum_{m=0}^{\infty} \left(\prod_{j=1}^{k} \frac{(N_j m)!}{m!^{N_j}} \right) z^m,$$

$$G_{L,\mathbf{N}}(z) := \sum_{m=1}^{\infty} H_{Lm} \left(\prod_{j=1}^{k} \frac{(N_j m)!}{m!^{N_j}} \right) z^m,$$

and the function $q_{L,\mathbf{N}}(z) := \exp\big(G_{L,\mathbf{N}}(z)/F_{\mathbf{N}}(z)\big)$. The series $F_{\mathbf{N}}(z)$ and $G_{\mathbf{N}}(z) + \log(z)F_{\mathbf{N}}(z)$, where $G_{\mathbf{N}}(z)$ is a suitable linear combination of the series $G_{L,\mathbf{N}}(z)$ (for different L's), are solutions of a hypergeometric differential equation with maximal unipotent monodromy (of type (1.18) below). That differential equation is the Picard–Fuchs equation of a one parameter family of mirror manifolds V' of a complete intersection V of k hypersurfaces $V_1, \dots, V_k$ of degrees $N_1, \dots, N_k$ in $\mathbb{P}^{d+k}(\mathbb{C})$: V is a family of Calabi–Yau manifolds if one chooses d equal to $\sum_{j=1}^{k} N_j - k - 1$. The mirrors V' are explicitly constructed in [4, Sec. 5.2].

Let us define $M_{\mathbf{N}} = \prod_{i=1}^{k} N_i!$ and, for $L \geq 1$, $V_{L,\mathbf{N}}$ as the largest integer such that $q_{L,\mathbf{N}}(z)^{1/V_{L,\mathbf{N}}} \in \mathbb{Z}[[z]]$. ([4]) While we are not able to determine $V_{L,\mathbf{N}}$ precisely, we shall prove the following result which, as should become clear from the discussion in Section 10, comes relatively close.

Theorem 2. *Let $\Theta_L := L!/\gcd(L!, L!\,H_L)$ be the denominator of H_L when written as a reduced fraction. Then, for any integers $N_1, \dots, N_k \geq 1$ and $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, we have $q_{L,\mathbf{N}}(z)^{\frac{\Theta_L}{M_{\mathbf{N}}}} \in \mathbb{Z}[[z]]$.*

Remarks 1. (a) For any integer $s \geq 1$, we have $q_{L,\mathbf{N}}(z)^{1/s} = 1 + s^{-1} M_{\mathbf{N}} H_L z + \mathcal{O}(z^2)$, and hence Theorem 2 is optimal when $L = 1$. This is not necessarily the case for other values of L and, in particular, Theorem 2 can sometimes be improved when $L = N_1 = \dots = N_k$ (see Theorem 3 below).

(b) Considering the case $k = 1$ and $L = N$ in Theorem 2, the first few values of $M_{(N)}/\Theta_N = \gcd(N!, N!\,H_N)$ (for $N \geq 1$) are

$$1, 1, 1, 2, 2, 36, 36, 144, 144, 1440, 1440, 17280, 17280, 241920, 3628800, 29030400, \dots \quad (1.5)$$

In the On-Line Encyclopedia of Integer Sequences [23], this sequence is entry `A056612`.

By forming a suitable product of the functions $q_{L,\mathbf{N}}(z)$, Theorem 2 implies the integrality of Taylor coefficients of the corresponding mirror maps $q_{\mathbf{N}}(z)$ of the mirror pair (V, V').

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[4]Let $q(z)$ be a given power series in $\mathbb{Z}[[z]]$, and let V be the largest integer with the property that $q(z)^{1/V} \in \mathbb{Z}[[z]]$. Then V carries complete information about *all* integers with that property: namely, the set of integers U with $q(z)^{1/U} \in \mathbb{Z}[[z]]$ consists of all divisors of V. Indeed, it is clear that all divisors of V belong to this set. Moreover, if U_1 and U_2 belong to this set, then also $\mathrm{lcm}(U_1, U_2)$ does (cf. [14, Lemma 5] for a simple proof based on Bézout's lemma).

According to [4, Prop. 5.1.2], the Yukawa coupling in this case is equal to

$$K(q) := \frac{N_1 N_2 \cdots N_k}{1-\lambda z_{\mathbf{N}}(q)} \cdot \frac{1}{F_{\mathbf{N}}(z_{\mathbf{N}}(q))^2} \cdot \left(\frac{q z'_{\mathbf{N}}(q)}{z_{\mathbf{N}}(q)}\right)^d,$$

where $\lambda = \prod_{j=1}^k N_j^{N_j}$ and $z_{\mathbf{N}}(q)$ is the compositional inverse of $q_{\mathbf{N}}(z)$. When $d = 3 = \sum_{j=1}^k N_j - k - 1$, the formal expansion $K(q) = K(0) + \sum_{m=1}^\infty k_m \frac{q^m}{1-q^m}$ enables one to count rational curves on the Calabi–Yau threefold V, at least for small values of the degree of the curves.

An outline of the proof of Theorem 2 is given in Section 3, with details being filled in in later sections. Since we shall refer to it below, we remark that an alternative way to define the integer Θ_L is via

$$\Theta_L = \prod_{p \le L} p^{-\min\{0, v_p(H_L)\}}, \tag{1.6}$$

where $v_p(\alpha)$ denotes the p-adic valuation of α. Here and in the sequel of the article, the letter p will always represent a prime number.

Due to (1.4), Theorem 2 has the following consequence for the original mirror map $q_N(z)$, thus improving significantly upon (1.3).

Corollary 1. *For all integers $k \ge 1$ and $N \ge 1$, we have*

$$\left(z^{-1} q_N(z)\right)^{\frac{\Theta_N}{N!^k kN}} \in \mathbb{Z}[[z]],$$

where $q_N(z)$ is the mirror map in (1.1).

In particular, in the emblematic case of the mirror map $q_5(z)$ of the quintic (case $N = 5$, $k = 1$), we obtain that $\left(z^{-1} q_5(z)\right)^{1/10} \in \mathbb{Z}[[z]]$, which improves on (1.3) by a factor of 2.

The next theorem presents the improvement of Theorem 2 for the case $L = N_1 = \cdots = N_k$ which was announced in Remark 1(a) above.

Theorem 3. *Let N be a positive integer, $\mathbf{N} = (N, N, \dots, N)$, with k occurrences of N, and let $\Xi_1 = 1$, $\Xi_7 = 1/140$, and, for $N \notin \{1, 7\}$,*

$$\Xi_N := \prod_{p \le N} p^{\min\{2+\xi(p,N), v_p(H_N)\}}, \tag{1.7}$$

where $\xi(p, N) = 1$ if p is a Wolstenholme prime (i.e., a prime p for which $v_p(H_{p-1}) \ge 3$ ([5])) or N is divisible by p, and $\xi(p, N) = 0$ otherwise. Then $q_{N,\mathbf{N}}(z)^{\frac{1}{\Xi_N N!^k}} \in \mathbb{Z}[[z]]$.

Remarks 2. For better comprehension, we discuss the meaning of the statement of Theorem 3 and its implications; in particular, we address some fine points of the definition of Ξ_N.

[5] Presently, only two such primes are known, namely 16843 and 2124679, and it is unknown whether there are infinitely many Wolstenholme primes or not.

(a) The case of $N = 1$ is trivial since $q_{1,(1,\dots,1)}(z) = 1/(1 - z)$. Furthermore, we have

$$\Xi_7 = \frac{1}{140} = 2^{v_2(H_7)} 5^{v_5(H_7)} 7^{v_7(H_7)},$$

which differs by a factor of 3 from the right-hand side of (1.7) with $N = 7$ (since $v_3(H_7) = v_3(\frac{363}{140}) = 1$).

(b) Since $q_{N,\mathbf{N}}(z) = 1 + H_N N!^k z + \mathcal{O}(z^2)$, it is clear that $q_{N,\mathbf{N}}(z)^{1/(p^{v_p(H_N)+1} N!^k)} \notin \mathbb{Z}[[z]]$, so that the exponent of p in the prime factorisation of $V_{N,\mathbf{N}}$ can be at most $v_p(H_N N!^k)$. In Theorem 3, this theoretically maximal exponent is further cut down. First of all, the number Ξ_N appearing there contains no prime factor $p > N$. Moreover, for primes p with $p \le N$ and $v_p(H_N) \ge 3$, the definition of Ξ_N cuts the theoretically maximal exponent $v_p(H_N N!^k)$ of p down to $2 + v_p(N!^k)$ respectively $3 + v_p(N!^k)$, depending on whether $\xi(p, N) = 0$ or $\xi(p, N) = 1$. In items (c)–(e) below, we address the question of how serious this cut is expected to be.

(c) Clearly, the minimum appearing in the exponent of p in the definition (1.7) of Ξ_N is $v_p(H_N)$ as long as $v_p(H_N) \le 2$. In other words, the exponent of p in the prime factorisation of Ξ_N depends largely on the p-adic behaviour of H_N. An extensive discussion of this topic, with many interesting results, can be found in [5]. We have as well computed a table of harmonic numbers H_N up to $N = 1000000$. ([6]) Indeed, the data suggest that pairs (p, N) with p prime, $p \le N$, and $v_p(H_N) \ge 3$ are not very frequent. More precisely, so far only five examples are known with $v_p(H_N) = 3$: four for $p = 11$, with $N = 848, 9338, 10583,$ and 35464717222268916272, and one for $p = 83$ with

$$\begin{aligned} N = {} & 79781079199360090066989143814676572961528399477699516786377994370\backslash \\ & 788396816921576769152458572350552007794214098216436918 18 \end{aligned} \tag{1.8}$$

(see [5, p. 289]; the value of N in (1.8), not printed in [5], was kindly communicated to us by David Boyd). There is no example known with $v_p(H_N) \ge 4$. It is, in fact, conjectured that no p and N exist with $v_p(H_N) \ge 4$. Some evidence for this conjecture (beyond mere computation) can be found in [5].

(d) Since in all the five examples for which $v_p(H_N) = 3$ we neither have $p \mid N$ (the gigantic number in (1.8) is congruent to 42 mod 83) nor that the prime p is a Wolstenholme prime, the exponent of p in the prime factorisation of Ξ_N in these cases is 2 instead of $v_p(H_N) = 3$.

(e) On the other hand, should there be a prime p and an integer N with $p \le N$, $v_p(H_N) \ge 3$, p a Wolstenholme prime or $p \mid N$, then the exponent of p in the prime factorisation of Ξ_N would be 3. However, no such examples are known. We conjecture that there are no such pairs (p, N). If this conjecture should turn out to be true, then, given $N \notin \{1, 7\}$, the definition of Ξ_N in (1.7) could be simplified to

$$\Xi_N := \prod_{p \le N} p^{\min\{2, v_p(H_N)\}}. \tag{1.9}$$

[6]The summary of the table is available at `http://www.mat.univie.ac.at/~kratt/artikel/H.html`.

In view of (1.6) and the fact that $M_{\mathbf{N}} = N!^k$ for the vector $\mathbf{N}$ in Theorem 3, this theorem improves upon Theorem 2 in the case $L = N$. Namely, Theorem 3 is *always* at least as strong as Theorem 2, and it is *strictly* stronger if $N \neq 7$ and $v_p(H_N) \geq 1$ for some prime p less than or equal to N. Indeed, the smallest $N \neq 7$ with that property is $N = 20$, in which case $v_5(H_{20}) = 1$.

We remark that strengthenings of Theorem 2 in the spirit of Theorem 3 for more general choices of the parameters can also be obtained by our techniques but are omitted here.

We outline the proof of Theorem 3 in Section 8, with the details being filled in in the subsequent Section 9. As we explain in Section 10, we conjecture that Theorem 3 cannot be improved if $k = 1$, that is, that for $k = 1$ the largest integer t_N such that $q_{N,N}(z)^{1/t_N} \in \mathbb{Z}[[z]]$ is exactly $\Xi_N N!$. Propositions 2 and 3 in Section 10 show that this conjecture would immediately follow if one could prove the conjecture from Remarks 2(c) above that there are no primes p and integers N with $v_p(H_N) \geq 4$.

Even if the series $q_{N,N}(z)$ appears in the identity (1.4), relating our mirror map $q_N(z)$ to the series $q_{L,N}(z)$ (with $L = 1$ and $L = N$), Theorem 3 does not imply an improvement over Corollary 1, for the coefficient of z in $(z^{-1}q_N(z))^{\Theta_N/(pN!^k kN)}$ is equal to $\frac{\Theta_N}{p}(H_N - 1)$, and, thus, it will not be integral for primes p with $v_p(H_N) > 0$. Still, there is an improvement of Corollary 1 in the spirit of Theorem 3. It involves primes p with $v_p(H_N - 1) > 0$ instead.

Theorem 4. *Let N be a positive integer with $N \geq 2$, and let*

$$\Omega_N := \prod_{p \leq N} p^{\min\{2+\omega(p,N), v_p(H_N-1)\}}, \tag{1.10}$$

where $\omega(p,N) = 1$ if p is a Wolstenholme prime or $N \equiv \pm 1 \bmod p$, and $\omega(p,N) = 0$ otherwise. Then $\big(z^{-1}q_N(z)\big)^{\frac{1}{\Omega_N N!^k kN}} \in \mathbb{Z}[[z]]$.

Remarks 3. Also here, some remarks are in order to get a better understanding of the above theorem.

(a) If $v_p(H_N) < 0$, then $v_p(H_N) = v_p(H_N - 1)$. Hence, differences in the prime factorisations of Ξ_N and Ω_N can only arise for primes p with $v_p(H_N) \geq 0$.

(b) Since $z^{-1}q_N(z) = 1 + (H_N - 1)N!^k kNz + \mathcal{O}(z^2)$, it is clear that

$$\big(z^{-1}q_N(z)\big)^{1/(p^{v_p(H_N-1)+1}N!^k kN)} \notin \mathbb{Z}[[z]].$$

If $\tilde{V}_N$ denotes the largest integer such that $\big(z^{-1}q_N(z)\big)^{1/\tilde{V}_N kN}$ is a series with integer coefficients, the exponent of p in $\tilde{V}_N$ can be at most $v_p\big((H_N - 1)N!^k\big)$. In Theorem 4, this theoretically maximal exponent is further cut down. Namely, as is the case for Ξ_N, the number Ω_N in (1.10) contains no prime factor $p > N$. Moreover, for primes p with $p \leq N$ and $v_p(H_N - 1) \geq 3$, the definition of Ω_N cuts the theoretically maximal exponent $v_p\big((H_N - 1)N!^k\big)$ of p down to $2 + v_p(N!^k)$ respectively $3 + v_p(N!^k)$, depending on whether $\omega(p,N) = 0$ or $\omega(p,N) = 1$.

(c) Concerning the question whether there are at all primes p and integers N with high values of $v_p(H_N - 1)$, we are not aware of any corresponding literature. Our table of harmonic numbers H_N mentioned in Remarks 2(c) does not contain any pair (p,N) with $v_p(H_N-1)\ge 3$. ([7]) In "analogy" to the conjecture mentioned in Remarks 2(c), we conjecture that no p and N exist with $v_p(H_N-1)\ge 4$. It may even be true that there are no p and N with $v_p(H_N-1)\ge 3$, in which case the definition of Ω_N in (1.10) could be simplified to

$$\Omega_N := \prod_{p\le N} p^{\min\{2,v_p(H_N-1)\}}. \tag{1.11}$$

In view of (1.6), Theorem 4 improves upon Corollary 1. Namely, Theorem 4 is *always* at least as strong as Corollary 1, and it is *strictly* stronger if $v_p(H_N-1)\ge 1$ for some prime p less than or equal to N. The smallest N with that property is $N=21$, in which case $v_5(H_{21}-1)=1$.

We sketch the proof of Theorem 4 in Section 11. We also explain in that section that we conjecture that Theorem 4 with $k=1$ is optimal, that is, that for $k=1$ the largest integer u_N such that $\big(z^{-1}q_N(z)\big)^{\frac{1}{Nu_N}}\in\mathbb{Z}[[z]]$ is exactly $\Omega_N N!$. Propositions 4 and 5 in Section 11 show that this conjecture would immediately follow if one could prove the conjecture from Remarks 3(c) above that there are no primes p and integers N with $v_p(H_N-1)\ge 4$. As a matter of fact, the sequence $(u_{2N})_{N\ge 1}$ appears also in the On-Line Encyclopedia of Integer Sequences [23], as sequence `A007757`, contributed by R. E. Borcherds under the denomination "Dwork–Kontsevich sequence" around 1995, without any reference or explicit formula for it, however. ([8])

1.4. **Refinements of** (1.3)**, part II.** We now move on to describe the other direction of refinement of Conjecture 1 and (1.3), inspired by Zudilin's paper [30]. For any given integer $N\ge 1$, let $r_1,r_2,\dots,r_d$ denote the integers in $\{1,2,\dots,N\}$ which are coprime to N. It is well-known that $d=\varphi(N)$, Euler's totient function, which is given by $\varphi(N)=N\prod_{p\mid N}\big(1-\frac{1}{p}\big)$. Set $C_N := N^{\varphi(N)}\prod_{p\mid N}p^{\varphi(N)/(p-1)}$, which is an integer because $p-1$ divides $\varphi(N)$ for any prime p dividing N. Let us also define the Pochhammer symbol $(\alpha)_m$ for complex numbers α and non-negative integers m by $(\alpha)_m := \alpha(\alpha+1)\cdots(\alpha+m-1)$ if $m\ge 1$ and $(\alpha)_0 := 1$. It can be proved (see [30, Lemma 1], or (1.15) together with Lemma 28(iii)) that, for any integer $m\ge 0$,

$$\mathbf{B}_N(m) := C_N^m \prod_{j=1}^{\varphi(N)} \frac{(r_j/N)_m}{m!} \tag{1.12}$$

is an integer. We will as well use $\mathbf{B}_{\mathbf{N}}(m) := \prod_{j=1}^{k}\mathbf{B}_{N_j}(m)$ for vectors $\mathbf{N}=(N_1,\dots,N_k)$ of positive integers. Zudilin also established another representation for $\mathbf{B}_N(m)$ (see [30, Lemma 4]). Namely, let $p_1,p_2,\dots,p_\ell$ denote the distinct prime factors of N, and let us

[7] The summary of the corresponding table, containing pairs (p,N) with $p\le N$ and $v_p(H_N-1)>0$, is available at `http://www.mat.univie.ac.at/~kratt/artikel/H1.html`.

[8] In private communication, both, Borcherds and Kontsevich could not remember where exactly this sequence and its denomination came from.

define the vectors of integers

$$(\alpha_j)_{j=1,\dots,\mu} := \left(N, \frac{N}{p_{j_1}p_{j_2}}, \frac{N}{p_{j_1}p_{j_2}p_{j_3}p_{j_4}}, \dots\right)_{1\le j_1<j_2<\dots\le \ell} \tag{1.13}$$

and

$$(\beta_j)_{j=1,\dots,\eta} := \left(\frac{N}{p_{j_1}}, \frac{N}{p_{j_1}p_{j_2}p_{j_3}}, \dots, 1, 1, \dots, 1\right)_{1\le j_1<j_2<\dots\le \ell}, \tag{1.14}$$

where $\alpha_1+\alpha_2+\dots+\alpha_\mu = \beta_1+\beta_2+\dots+\beta_\eta$. Then we have

$$\mathbf{B}_N(m) = \frac{\prod_{j=1}^{\mu}(\alpha_j m)!}{\prod_{j=1}^{\eta}(\beta_j m)!}. \tag{1.15}$$

For example, we have

$$\mathbf{B}_4(m) = \frac{(4m)!}{(2m)!\,m!^2},\ \mathbf{B}_6(m) = \frac{(6m)!}{(3m)!\,(2m)!\,m!},\ \mathbf{B}_{30}(m) = \frac{(30m)!\,(5m)!\,(3m)!\,(2m)!}{(15m)!\,(10m)!\,(6m)!\,m!^9}.$$

We now define the functions

$$\mathbf{H}_N(m) := \sum_{j=1}^{\varphi(N)} H(r_j/N, m) - \varphi(N)H(1,m),$$

$$\mathbf{F}_{\mathbf{N}}(z) := \sum_{m=0}^{\infty}\left(\prod_{j=1}^{k}\mathbf{B}_{N_j}(m)\right)z^m, \tag{1.16}$$

and

$$\mathbf{G}_{\mathbf{N}}(z) := \sum_{m=1}^{\infty}\left(\sum_{j=1}^{k}\mathbf{H}_{N_j}(m)\right)\left(\prod_{j=1}^{k}\mathbf{B}_{N_j}(m)\right)z^m.$$

We can now state Zudilin's conjecture from [30, p. 605].

Conjecture 2 (Zudilin)**.** *For any positive integers $N_1, N_2, \dots, N_k$, we have $\mathbf{q}_{\mathbf{N}}(z) := z\exp(\mathbf{G}_{\mathbf{N}}(z)/\mathbf{F}_{\mathbf{N}}(z)) \in z\mathbb{Z}[[z]]$.*

As explained by Zudilin, this conjecture implies Conjecture 1 because any function $q_N(z)$ is equal to $\mathbf{q}_{\mathbf{N}}(z)$ for a suitable $\mathbf{N}$. More precisely, for a given $N \ge 1$, the vector $\mathbf{N}$ to choose consists of all the positive divisors of N but 1. For example, we have $q_9(z) = \mathbf{q}_{(9,3)}(z)$, $q_{12}(z) = \mathbf{q}_{(12,6,4,3,2)}(z)$ and $q_{35}(z) = \mathbf{q}_{(7,5)}(z)$. However, Conjecture 2 does not imply any of the previous theorems. Zudilin proved that his conjecture holds under the condition that if a prime number divides $N_1N_2\cdots N_k$ then it also divides each N_j. This applies to the function $\mathbf{q}_{(9,3)}(z)$, but neither to $\mathbf{q}_{(12,6,4,3,2)}(z)$ nor to $\mathbf{q}_{(7,5)}(z)$, for example.

We claim that Conjecture 2 follows from the theorem below. For the statement of the theorem, for an integer $L \ge 1$, we need to define

$$\mathbf{G}_{L,\mathbf{N}}(z) := \sum_{m=1}^{\infty} H_{Lm}\left(\prod_{j=1}^{k}\mathbf{B}_{N_j}(m)\right)z^m.$$

Theorem 5. *For any integers* $N_1, N_2, \dots, N_k \geq 1$ *and* $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, *we have* $\mathbf{q}_{L,\mathbf{N}}(z) := \exp(\mathbf{G}_{L,\mathbf{N}}(z)/\mathbf{F}_{\mathbf{N}}(z)) \in \mathbb{Z}[[z]]$.

An outline of the proof of this theorem is given in Section 13, with details being filled in in subsequent sections.

To see that Theorem 5 implies Conjecture 2, we prove in Lemma 27 in Section 14 that, for a given N, we have

$$\mathbf{H}_N(m) = \sum_{j=1}^{\mu} \alpha_j H_{\alpha_j m} - \sum_{j=1}^{\eta} \beta_j H_{\beta_j m}. \tag{1.17}$$

Clearly, the α_j's and β_j's (defined in (1.13) and (1.14)) are less than N. Therefore $\sum_{j=1}^{k} \mathbf{H}_{N_j}(m)$ is a finite sum of terms of the form λH_{Lm}, where λ and L are integers with $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, and the claimed implication follows.

It is proved in [26] that there are exactly fourteen mirror maps $\mathbf{q}_{\mathbf{N}}(z)$ associated to complete intersections Calabi–Yau threefolds in weighted projective spaces: $\mathbf{q}_{(12)}(z)$, $\mathbf{q}_{(5)}(z)$, $\mathbf{q}_{(8)}(z)$, $\mathbf{q}_{(10)}(z)$, $\mathbf{q}_{(3,3)}(z)$, $\mathbf{q}_{(4,2,2)}(z)$, $\mathbf{q}_{(2,2,2,2)}(z)$, $\mathbf{q}_{(4,4)}(z)$, $\mathbf{q}_{(6,6)}(z)$, $\mathbf{q}_{(4,3)}(z)$, $\mathbf{q}_{(6,2,2)}(z)$, $\mathbf{q}_{(3,2,2)}(z)$, $\mathbf{q}_{(3,6)}(z)$, $\mathbf{q}_{(6,4)}(z)$. All but the first one were already described in [4, Sec. 6.4]. Theorem 5 implies that they all have integral Taylor coefficients, which is a new result for the last five of them.

More generally, for any $\mathbf{N}$, the functions $\mathbf{F}_{\mathbf{N}}(z)$ and $\mathbf{G}_{\mathbf{N}}(z) + \log(z)\mathbf{F}_{\mathbf{N}}(z)$ satisfy a hypergeometric differential equation $\mathbf{L}y = 0$ with maximal unipotent monodromy at the origin, where the differential operator $\mathbf{L}$ is defined by

$$\mathbf{L} := \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\varphi(N_1)+\cdots+\varphi(N_k)} - zC_{\mathbf{N}} \prod_{j=1}^{k} \prod_{i=1}^{\varphi(N_j)} \left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{r_{i,j}}{N_j}\right). \tag{1.18}$$

Here, $C_{\mathbf{N}} = C_{N_1} C_{N_2} \cdots C_{N_k}$ and the $r_{i,j} \in \{1, 2, \dots, N_j\}$ form the residue classes mod N_j which are coprime to N_j. This equation is the Picard–Fuchs equation of the mirror Calabi–Yau family of a one parameter family of Calabi–Yau complete intersections in a weighted projective space (see [7] and [15, Sec. 3]).

It is natural to expect refinements of Theorem 5 in the spirit of Theorem 2 and Theorem 4. We did not make a systematic research in this direction, but it could be interesting to do so. For example, in the case $k = 1$ and $\mathbf{N} = (6)$, it seems that the following relations are best possible: $\mathbf{q}_{1,(6)}(z)^{1/60}$, $\mathbf{q}_{2,(6)}(z)^{1/6}$, $\mathbf{q}_{3,(6)}(z)^{1/2}$, $\mathbf{q}_{4,(6)}(z)$, $\mathbf{q}_{5,(6)}(z)$ and $\mathbf{q}_{6,(6)}(z)$ are in $\mathbb{Z}[[z]]$.

As a first step towards such refinements, we prove in Lemma 30 in Section 14 that $\mathbf{B}_{\mathbf{N}}(1)$ always divides $\mathbf{B}_{\mathbf{N}}(m)$ for any $m \geq 1$ and any $\mathbf{N}$. Our techniques enable us to deduce that $\mathbf{q}_{1,\mathbf{N}}(z)^{1/\mathbf{B}_{\mathbf{N}}(1)} \in \mathbb{Z}[[z]]$, which proves the above assertion that $\mathbf{q}_{1,(6)}(z)^{1/60} \in \mathbb{Z}[[z]]$. In fact, this is optimal because $\mathbf{q}_{1,\mathbf{N}}(z) = 1 + \mathbf{B}_{\mathbf{N}}(1)z + \mathcal{O}(z^2)$. It turns out that $\mathbf{B}_{\mathbf{N}}(1)$ is a natural generalisation of the quantity $M_{\mathbf{N}}$ which appears in Theorem 2. However, for larger values of the parameter L, we do not know what the analogue of the quantity $M_{\mathbf{N}}/\Theta_L$ appearing in Theorem 2 would be.

Let us consider the hypergeometric functions F with rational parameters for which there exists a constant $C = C(F) > 0$ such that the Taylor coefficients of $F(Cz)$ are of the form

$$\frac{\prod_{j=1}^{r}(a_j m)!}{\prod_{j=1}^{s}(b_j m)!}, \tag{1.19}$$

where m, the a_j's and b_j's are non-negative integers, and $\sum_{j=1}^{r} a_j = \sum_{j=1}^{s} b_j$. A well-known theorem of Landau (cf. [26, Proposition 1.1]) determines the quotients of the form (1.19) which are integer-valued for all integers $m \geq 0$. The quantities $\mathbf{B_N}(m)$ are such examples but do not describe all these quotients, as demonstrated by the example $(3m)!/(m!\,(2m)!)$.

In our context, it is therefore natural to ask the following question: *what are the hypergeometric functions F (with z changed to Cz for a suitable C) such that*

(i) *their Taylor coefficients are integers and of the form* (1.19)*,*

(ii) *the associated "mirror map" type functions* (*that is, functions defined in a manner analogous to $\mathbf{q_N}(z)$*) *have integral Taylor coefficients?*

Of course, Condition (ii) has a sense only if the underlying differential equation has two solutions $F(z)$ and $G(z)+\log(z)F(z)$ with $F(z)$ and $G(z)$ holomorphic at 0, which in turn implies that at least one of the lower parameters of F is equal to 1. Rodriguez–Villegas proved in [26] a result which, once translated in our setting, says that the functions F with maximal unipotent monodromy (i.e., with only 1's as lower parameters) and satisfying (i) are exactly the functions $\mathbf{F_N}$ defined by (1.16). Theorem 5 proves that such functions also satisfy (ii). Numerical experiments seem to indicate that there are many examples of such functions F which are not of the form (1.16).

1.5. **Structure of the paper.** We now briefly review the organisation of the rest of the paper. Following the steps of previous authors, our approach for proving Theorems 1–5 uses p-adic analysis. In particular, we make essential use of Dwork's theory of formal congruences, which we recall in Section 2. Since the details of our proofs are involved, we provide brief outlines of the proofs of Theorems 2–5 (with Theorem 2 implying Theorem 1) in separate sections. Namely, Section 3 provides an outline of the proof of Theorem 2, while Section 13 provides an outline of the proof of Theorem 5. In both cases, the proof is reduced to a certain number of lemmas. The lemmas which are necessary for the proof of Theorem 2 are established in Sections 4–7, while those necessary for the proof of Theorem 5 are established in Sections 14–16. Finally, Section 8 contains an outline of the proof of Theorem 3, and Section 11 contains a sketch of the proof of Theorem 4. Both are largely based on arguments already used in the proof of Theorem 2. The outline of the proof of Theorem 3 requires again several auxiliary lemmas, whose proofs are postponed to a separate section, Section 9. The subsequent section, Section 10, reports on the evidence to believe (or not to believe) that the value t_N (defined in the next-to-last paragraph before Theorem 4) is given by $t_N = \Xi_N N!$, $N = 1, 2, \ldots$, while Section 12 addresses the question of whether the Dwork–Kontsevich sequence $(u_N)_{N\geq 1}$ (defined in the last paragraph in Section 1.3) is (or is not) given by $u_N = \Omega_N N!$, $N = 1, 2, \ldots$.

We draw the reader's attention to the fact that, while the general line of our approach follows that of previous authors (particularly [21]), there does arise a crucial difference (other than just technical complications): the reduction and rearrangement of the sums $C(a+Kp)$ in Sections 3 and 8, respectively $\mathbf{C}(a+Kp)$ in Section 13, via the congruence (3.3) requires a new reduction step, namely Lemma 5, congruence (8.5), and Lemma 24, respectively. In fact, the proofs of these two lemmas and the congruence form the most difficult parts of our paper. (In previous work, the use of (3.3) sufficed because the corresponding authors restricted themselves to N being prime or a prime power.)

We expect that (variations of) the detailed techniques that we present here can also serve to prove integrality results for other types of mirror maps, and in particular some of Observations 1–7 in [1, Sec. 4]. We also hope that our methods will turn out to be useful in the context of the very general multivariable mirror maps coming from the Gelfand–Kapranov–Zelevinsky hypergeometric series: see [4, Sec. 7.1], [15] and [25, Sec. 8] for numerous examples related to Calabi–Yau manifolds which are complete intersections in products of weighted projective spaces.

2. Dwork's theory of formal congruences

Since the Taylor coefficients of $F_{\mathbf{N}}(z)$ and $G_{L,\mathbf{N}}(z)$ (respectively $\mathbf{F}_{\mathbf{N}}(z)$ and $\mathbf{G}_{L,\mathbf{N}}(z)$) are explicit, we could also obtain an explicit formula for the coefficients of $q_{L,\mathbf{N}}(z)$ (respectively $\mathbf{q}_{L,\mathbf{N}}(z)$). Unfortunately, such a formula does not seem to be very useful to prove the desired integrality properties. Instead, we will follow the authors of the articles cited in the Introduction, who all relied on Dwork's theory.

First, consider a formal power series $S(z) \in \mathbb{Q}[[z]]$ and suppose that we want to prove that $S(z) \in \mathbb{Z}[[z]]$.

Lemma 1. *Let $S(z)$ be a power series in $\mathbb{Q}[[z]]$. If $S(z) \in \mathbb{Z}_p[[z]]$ for any prime number p, then $S(z) \in \mathbb{Z}[[z]]$.*

This is a consequence of the fact that, given $x \in \mathbb{Q}$, we have $x \in \mathbb{Z}$ if and only if $x \in \mathbb{Z}_p$ for all prime numbers p. Hence we can work in $\mathbb{Q}_p$ for any fixed prime p.

Lemma 2 ("Dwork's Lemma"). *Let $S(z) \in 1 + z\mathbb{Q}_p[[z]]$. Then, we have $S(z) \in 1 + z\mathbb{Z}_p[[z]]$ if and only if*

$$\frac{S(z^p)}{S(z)^p} \in 1 + pz\mathbb{Z}_p[[z]].$$

Proof. The proof is neither difficult nor long and can for example be found in the book of Lang [17, Ch. 14, p. 76]. Lang attributes this lemma to Dieudonné and Dwork. □

We now suppose that $S(z) = \exp(T(z)/\tau)$ for some $T(z) \in z\mathbb{Q}[[z]]$ and some integer $\tau \geq 1$. Dwork's Lemma implies the following result: τ being any fixed positive integer, we have $\exp\big(T(z)/\tau\big) \in 1 + z\mathbb{Z}_p[[z]]$ if and only if $T(z^p) - pT(z) \in p\tau z\mathbb{Z}_p[[z]]$. (See [19, Corollary 6.7] for a proof.) Since we will be interested in the case when $T(z) = g(z)/f(z)$ with $f(z) \in 1 + z\mathbb{Z}[[z]]$ and $g(z) \in z\mathbb{Q}[[z]]$, we state this result as follows.

Lemma 3. *Given two formal power series $f(z) \in 1 + z\mathbb{Z}[[z]]$ and $g(z) \in z\mathbb{Q}[[z]]$ and an integer $\tau \geq 1$, we have $\exp\big(g(z)/(\tau f(z))\big) \in 1 + z\mathbb{Z}_p[[z]]$ if and only if*

$$f(z)g(z^p) - p\,f(z^p)g(z) \in p\tau z\mathbb{Z}_p[[z]]. \tag{2.1}$$

Because of the special form of the functions which will play the role of $f(z)$ and $g(z)$, we will be able to deduce (2.1) from the following crucial result, also due to Dwork (see [11, Theorem 1.1]). We state it in slightly bigger generality than necessary (i.e., our functions will be independent of r).

Proposition 1 ("Dwork's Formal Congruences Theorem")**.** *For all integers $r \geq 0$, let $A_r : \mathbb{Z}_{\geq 0} \to \mathbb{Q}_p^{\times}$, $g_r : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_p \setminus \{0\}$ be mappings such that*

(i) $|A_r(0)|_p = 1$;

(ii) $A_r(m) \in g_r(m)\mathbb{Z}_p$;

(iii) for all integers $u, v, n, r, s \geq 0$ such that $0 \leq u < p^s$ and $0 \leq v < p$, we have

$$\frac{A_r(v + up + np^{s+1})}{A_r(v + up)} - \frac{A_{r+1}(u + np^s)}{A_{r+1}(u)} \in p^{s+1}\frac{g_{r+s+1}(n)}{g_r(v + up)}\mathbb{Z}_p. \tag{2.2}$$

Furthermore, let $F(z) = \sum_{m=0}^{\infty} A_0(m)z^m$, $G(z) = \sum_{m=0}^{\infty} A_1(m)z^m$, and

$$F_{m,s}(z) = \sum_{j=mp^s}^{(m+1)p^s-1} A_0(j)z^j, \quad G_{m,s}(z) = \sum_{j=mp^s}^{(m+1)p^s-1} A_1(j)z^j.$$

Then, for any integers $m, s \geq 0$, we have

$$G(z^p)F_{m,s+1}(z) - F(z)G_{m,s}(z^p) \in p^{s+1}g_s(m)\mathbb{Z}_p[[z]], \tag{2.3}$$

or, equivalently,

$$\sum_{j=mp^s}^{(m+1)p^s-1} \big(A_0(a + jp)A_1(K - j) - A_1(j)A_0(a + (K - j)p)\big) \in p^{s+1}g_s(m)\mathbb{Z}_p \tag{2.4}$$

for all a and K with $0 \leq a < p$ and $K \geq 0$.

Remarks 4. (a) Dwork proved his theorem with $A_r : \mathbb{Z}_{\geq 0} \to \mathbb{C}_p^{\times}$, $g_r : \mathbb{Z}_{\geq 0} \to \mathcal{O}_p \setminus \{0\}$ and $A_r(m) \in g_r(m)\mathcal{O}_p$ (where $\mathcal{O}_p$ is the ring of integers in $\mathbb{C}_p$). He obtained a result similar to (2.3) and (2.4), with $\mathcal{O}_p$ instead of $\mathbb{Z}_p$. In our more restrictive setting, (2.3) and (2.4) hold because $\big(p^{s+1}g_s(m)\mathcal{O}_p\big) \cap \mathbb{Q}_p = p^{s+1}g_s(m)\mathbb{Z}_p$.

(b) For any integers a and K with $0 \leq a < p$ and $K \geq 0$, the sum

$$\sum_{j=mp^s}^{(m+1)p^s-1} \big(A_0(a + jp)A_1(K - j) - A_1(j)A_0(a + (K - j)p)\big) \tag{2.5}$$

is exactly the $(a + pK)$-th Taylor coefficient of $G(z^p)F_{m,s+1}(z) - F(z)G_{m,s}(z^p)$, which explains the equivalence between the formal congruence (2.3) and the congruence (2.4). Note that in (2.5) the value of A_0 and A_1 at negative integers must be taken as 0.

(c) Most authors chose $g_s(m) = 1$ or a constant in m and s. We will use instead $g_s(m) = A_s(m)$: this choice has already been made by Dwork in [10, Sec. 2, p. 37].

(d) Dwork also applied his methods to the problems considered in the present paper. Indeed, he proved a result, namely [11, p. 311, Theorem 4.1], which implies that for any prime p that does not divide $N_1N_2\cdots N_k$, the mirror maps $\mathbf{q}_{\mathbf{N}}(z)$ have Taylor coefficients in $\mathbb{Z}_p$ (see [30, Proposition 2] for details). This fact was used by the authors of [18, 19, 30] who focussed essentially on the remaining case when p divides $N_1N_2\cdots N_k$. Our approach is different, for we make no distinction of this kind between prime numbers.

3. Outline of the proof of Theorem 2

In this section, we provide a brief outline of the proof of Theorem 2, reducing it to Lemmas 5–7. These lemmas are subsequently proved in Sections 5–7, with two auxiliary lemmas being the subject of the subsequent section.

By Dwork's Lemma (or rather its consequence given in Lemma 3), we want to prove that

$$F_{\mathbf{N}}(z)G_{L,\mathbf{N}}(z^p)-pF_{\mathbf{N}}(z^p)G_{L,\mathbf{N}}(z)\in p\frac{M_{\mathbf{N}}}{\Theta_L}z\mathbb{Z}_p[[z]].$$

We follow the presentation in [21] and we let $0\le a<p$ and $K\ge 0$.

The $(a+Kp)$-th Taylor coefficient of $F_{\mathbf{N}}(z)G_{L,\mathbf{N}}(z^p)-pF_{\mathbf{N}}(z^p)G_{L,\mathbf{N}}(z)$ is

$$C(a+Kp):=\sum_{j=0}^{K}B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j)(H_{L(K-j)}-pH_{La+Ljp}), \tag{3.1}$$

where $B_{\mathbf{N}}(m)=\prod_{j=1}^k B_{N_j}(m)$ with $B_N(m):=\frac{(Nm)!}{m!^N}$ (not to be confused with $\mathbf{B}_{\mathbf{N}}(m)$ and $\mathbf{B}_N(m)$). In view of Lemma 3, proving Theorem 2 is equivalent to proving that

$$C(a+Kp)\in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p \tag{3.2}$$

for all primes p and non-negative integers a and K with $0\le a<p$.

The following simple lemma will be frequently used in the sequel.

Lemma 4. *For all integers $m\ge 1$ and $N\ge 1$, we have*

$$B_N(m)\in N!\,\mathbb{Z}.$$

Proof. Set $U_m(N)=\frac{(Nm)!}{m!^N N!}$. For any $m,N\ge 1$, we have the trivial relation

$$U_m(N+1)=\binom{Nm+m-1}{m-1}U_m(N).$$

Therefore, since $U_m(1)=1$, the result follows by induction on N. $\square$

We deduce in particular that $B_{\mathbf{N}}(m)\in M_{\mathbf{N}}\mathbb{Z}$ for any $m\ge 1$.

Since

$$H_J=\sum_{j=1}^{\lfloor J/p\rfloor}\frac{1}{pj}+\sum_{\substack{j=1\\ p\nmid j}}^{J}\frac{1}{j},$$

we have

$$pH_J \equiv H_{\lfloor J/p \rfloor} \mod p\mathbb{Z}_p. \tag{3.3}$$

Applying this to $J = La + Ljp$, we get

$$pH_{La+Ljp} \equiv H_{\lfloor La/p \rfloor + Lj} \mod p\mathbb{Z}_p.$$

By Lemma 4, we infer

$$C(a+Kp) \equiv \sum_{j=0}^{K} B_{\mathbf{N}}(a+jp) B_{\mathbf{N}}(K-j)(H_{L(K-j)} - H_{\lfloor La/p \rfloor + Lj}) \mod pM_{\mathbf{N}}\mathbb{Z}_p. \tag{3.4}$$

Indeed, if $K \geq 1$ or $a \geq 1$, this is because $a + jp$ and $K - j$ cannot be simultaneously zero and therefore at least one of $B_{\mathbf{N}}(a+jp)$ or $B_{\mathbf{N}}(K-j)$ is divisible by $M_{\mathbf{N}}$ by Lemma 4. In the remaining case $K = a = j = 0$, we note that the difference of harmonic numbers in (3.4) is equal to 0, and therefore the congruence (3.4) holds trivially because $C(0) = 0$.

We now want to transform the sum on the right-hand side of (3.4) to a more manageable expression. In particular, we want to get rid of the floor function $\lfloor La/p \rfloor$. In order to achieve this, we will prove the following lemma in Section 5.

Lemma 5. *For any prime p, non-negative integers a and j with $0 \leq a < p$, positive integers $N_1, N_2, \ldots, N_k$, and $L \in \{1, 2, \ldots, \max(N_1, \ldots, N_k)\}$, we have*

$$B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\right) \in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p. \tag{3.5}$$

It follows from Eq. (3.4) and Lemma 5 that

$$C(a+Kp) \equiv \sum_{j=0}^{K} B_{\mathbf{N}}(a+jp) B_{\mathbf{N}}(K-j)\Big(H_{L(K-j)} - H_{Lj}\Big) \mod p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p \ ,$$

which can be rewritten as

$$C(a+Kp) \equiv -\sum_{j=0}^{K} H_{Lj}\Big(B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j) - B_{\mathbf{N}}(j)B_{\mathbf{N}}(a+(K-j)p)\Big) \mod p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p \ . \tag{3.6}$$

We now use a combinatorial lemma due to Dwork (see [11, Lemma 4.2]) which provides an alternative way to write the sum on the right-hand side of (3.6): namely, we have

$$\sum_{j=0}^{K} H_{Lj}\Big(B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j) - B_{\mathbf{N}}(j)B_{\mathbf{N}}(a+(K-j)p)\Big) = \sum_{s=0}^{r} \sum_{m=0}^{p^{r+1-s}-1} Y_{m,s}, \tag{3.7}$$

where r is such that $K < p^r$, and

$$Y_{m,s} := \big(H_{Lmp^s} - H_{L\lfloor m/p \rfloor p^{s+1}}\big) S(a, K, s, p, m),$$

the expression $S(a,K,s,p,m)$ being defined by

$$S(a,K,s,p,m) := \sum_{j=mp^s}^{(m+1)p^s-1} \big(B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j) - B_{\mathbf{N}}(j)B_{\mathbf{N}}(a+(K-j)p)\big).$$

In this expression for $S(a,K,s,p,m)$, it is assumed that $B_{\mathbf{N}}(n)=0$ for negative integers n.

It would suffice to prove that

$$Y_{m,s} \in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p \tag{3.8}$$

because (3.6) and (3.7) would then imply that $C(a+Kp) \in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p$, as desired.

We will prove (3.8) in the following manner. The expression for $S(a,K,s,p,m)$ is of the form considered in Proposition 1. The proposition will enable us to prove the following fact in Section 6.

Lemma 6. *For all primes p and non-negative integers a,m,s,K with $0 \le a < p$, we have*

$$S(a,K,s,p,m) \in p^{s+1}B_{\mathbf{N}}(m)\mathbb{Z}_p. \tag{3.9}$$

Furthermore, in Section 7 we shall prove the following lemma.

Lemma 7. *For all primes p, non-negative integers m, positive integers $N_1,N_2,\dots,N_k$, and $L \in \{1,2,\dots,\max(N_1,\dots,N_k)\}$, we have*

$$B_{\mathbf{N}}(m)\big(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}\big) \in \frac{M_{\mathbf{N}}}{p^s\Theta_L}\mathbb{Z}_p\ . \tag{3.10}$$

It is clear that (3.9) and (3.10) imply (3.8). This completes the outline of the proof of Theorem 2.

4. Two auxiliary lemmas

The proof of Lemma 6 in Section 6 requires two further auxiliary results, given in Lemmas 8 and 9 below, the proofs of which form the contents of this intermediary section. In the proof of the first lemma, and also in later proofs, we shall frequently make use of Legendre's formula for the p-adic valuation of $n!$, where n is a non-negative integer, which we recall here for convenience:

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor. \tag{4.1}$$

Lemma 8. *For all integers $N \ge 1$, $n \ge 0$, $s \ge 0$, $0 \le u < p^s$, p prime, we have*

$$\frac{B_N(u+np^s)}{B_N(u)} \in B_N(n)\,\mathbb{Z}_p,$$

where $B_N(m)$ is defined after (3.1).

Proof. We may rewrite the quotient in the assertion of the lemma as

$$\begin{aligned}\frac{(Nu+Nnp^s)!}{(u+np^s)!^N}\cdot\frac{u!^N}{(Nu)!}&=\frac{(Nu+Nnp^s)!}{(Nu)!\,(Nnp^s)!}\cdot\frac{(Nnp^s)!}{(np^s)!^N}\cdot\frac{(np^s)!^N u!^N}{(u+np^s)!^N}\\&=\binom{Nu+Nnp^s}{Nu}\cdot\frac{(Nnp^s)!}{(np^s)!^N}\cdot\binom{u+np^s}{u}^{-N}.\end{aligned}\tag{4.2}$$

The binomial coefficient $\binom{Nu+Nnp^s}{Nu}$ is an integer. For the multinomial coefficient we observe that

$$\begin{aligned}v_p\big((Nnp^s)!/(np^s)!^N\big)&=\sum_{k=1}^{\infty}\left(\left\lfloor\frac{Nnp^s}{p^k}\right\rfloor-N\left\lfloor\frac{np^s}{p^k}\right\rfloor\right)\\&=\sum_{k=1}^{\infty}\left(\left\lfloor\frac{Nn}{p^k}\right\rfloor-N\left\lfloor\frac{n}{p^k}\right\rfloor\right)\\&=v_p\big((Nn)!/n!^N\big),\end{aligned}$$

which means that

$$\frac{(Nnp^s)!}{(np^s)!^N}\in\frac{(Nn)!}{n!^N}\,\mathbb{Z}_p.$$

Hence, to conclude the proof of the lemma, it suffices to show that

$$v_p\left(\binom{u+np^s}{u}\right)=0.\tag{4.3}$$

In order to do so, we start with the formula

$$v_p\left(\binom{u+np^s}{u}\right)=\sum_{k=1}^{\infty}\left(\left\lfloor\frac{u+np^s}{p^k}\right\rfloor-\left\lfloor\frac{u}{p^k}\right\rfloor-\left\lfloor\frac{np^s}{p^k}\right\rfloor\right).$$

We distinguish between two cases.

(a) $k\le s$. In this case, $(np^s)/p^k$ is an integer and therefore

$$\left\lfloor\frac{u+np^s}{p^k}\right\rfloor-\left\lfloor\frac{u}{p^k}\right\rfloor-\left\lfloor\frac{np^s}{p^k}\right\rfloor=\left\lfloor\frac{u}{p^k}\right\rfloor+\frac{np^s}{p^k}-\left\lfloor\frac{u}{p^k}\right\rfloor-\frac{np^s}{p^k}=0.$$

(b) $k>s$. In this case, we have $\lfloor u/p^k\rfloor=0$ because $0\le u<p^s<p^k$.

Hence,

$$v_p\left(\binom{u+np^s}{u}\right)=\sum_{k=s+1}^{\infty}\left(\left\lfloor\frac{u+np^s}{p^k}\right\rfloor-\left\lfloor\frac{n}{p^{k-s}}\right\rfloor\right).$$

Since $0\le u/p^k<1$, we have

$$\left\lfloor\frac{u+np^s}{p^k}\right\rfloor-\left\lfloor\frac{n}{p^{k-s}}\right\rfloor\in\{0,1\},$$

so that it suffices to show that the value 1 cannot be attained. Arguing by contradiction, let us assume that, for some n, u, p, s, we have

$$\left\lfloor \frac{u+np^s}{p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k-s}} \right\rfloor = 1.$$

Writing, for simplicity, $X = \left\lfloor \frac{u+np^s}{p^k} \right\rfloor$, our assumption implies that

$$\frac{n}{p^{k-s}} < X \le \frac{u+np^s}{p^k}. \tag{4.4}$$

X being an integer, we thus have

$$\frac{n}{p^{k-s}} + \frac{1}{p^{k-s}} \le X. \tag{4.5}$$

Combining the right inequality of (4.4) with (4.5), we deduce that

$$\frac{n}{p^{k-s}} + \frac{1}{p^{k-s}} \le \frac{u}{p^k} + \frac{np^s}{p^k},$$

which means that $u \ge p^s$. This contradicts the hypothesis that $u < p^s$ and finishes the proof. □

During the proof of Lemma 6 in Section 6, we will also use certain properties of the p-adic gamma function Γ_p. This function is defined on integers $n \ge 1$ by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{k=1 \\ (k,p)=1}}^{n-1} k.$$

We will not consider its extension to $\mathbb{Q}_p$. In the following lemma, we collect the results on Γ_p that we shall need later on.

Lemma 9. (*i*) *For all integers $n \ge 1$, we have*

$$\frac{(np)!}{n!} = (-1)^{np+1} p^n \Gamma_p(1+np).$$

(*ii*) *For all integers $k \ge 1, n \ge 1, s \ge 0$, we have*

$$\Gamma_p(k+np^s) \equiv \Gamma_p(k) \mod p^s.$$

Proof. See [30, Lemma 7] for (*i*) and [17, p. 71, Lemma 1.1] for (*ii*). □

5. Proof of Lemma 5

The assertion is trivially true if $\lfloor La/p \rfloor = 0$, that is, if $0 \le a < p/L$. We may hence assume that $p/L \le a < p$ from now on. A further assumption upon which we agree without loss of generality for the rest of the proof is that $N_k = \max(N_1, \dots, N_k)$.

5.1. **First part: a weak version of Lemma 5.** In a first step, we prove that

$$B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\right)\in p\mathbb{Z}_p\ . \tag{5.1}$$

(The reader should note the absence of the term $M_{\mathbf{N}}/\Theta_L$ in comparison with (3.5).)

For the proof of (5.1), we note that the p-adic valuation of $B_{\mathbf{N}}(a+pj)$ is equal to

$$v_p\big(B_{\mathbf{N}}(a+pj)\big)=\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_i(a+pj)}{p^\ell}\right\rfloor-N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right).$$

Obviously, all the summands in this sum are non-negative, whence, in particular,

$$v_p\big(B_{\mathbf{N}}(a+pj)\big)\geq\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_k(a+pj)}{p^\ell}\right\rfloor-N_k\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right). \tag{5.2}$$

On the other hand, by definition of the harmonic numbers, we have

$$H_{Lj+\lfloor La/p\rfloor}-H_{Lj}=\frac{1}{Lj+1}+\frac{1}{Lj+2}+\cdots+\frac{1}{Lj+\lfloor La/p\rfloor}.$$

It therefore suffices to show that

$$v_p\big(B_{\mathbf{N}}(a+pj)\big)\geq 1+\max_{1\leq\varepsilon\leq\lfloor La/p\rfloor}v_p(Lj+\varepsilon). \tag{5.3}$$

The lower bound on the right-hand side of (5.2) can, in fact, be simplified since $0\leq a<p$; namely, we have

$$\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor=\left\lfloor\frac{j}{p^{\ell-1}}\right\rfloor. \tag{5.4}$$

For a given integer ε with $1\leq\varepsilon\leq\lfloor La/p\rfloor$, let $Lj+\varepsilon=p^d\beta$, where $d=v_p(Lj+\varepsilon)$. If we use this notation in (5.2), together with (5.4), we obtain

$$v_p\big(B_{\mathbf{N}}(a+pj)\big)\geq\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_ka}{p^\ell}-\frac{N_k\varepsilon}{Lp^{\ell-1}}+\frac{N_k\beta}{L}p^{d+1-\ell}\right\rfloor-N_k\left\lfloor-\frac{\varepsilon}{Lp^{\ell-1}}+\frac{\beta}{L}p^{d+1-\ell}\right\rfloor\right). \tag{5.5}$$

Since $\varepsilon\leq\lfloor La/p\rfloor$, we have $\frac{N_ka}{p^\ell}-\frac{N_k\varepsilon}{Lp^{\ell-1}}\geq 0$, whence

$$\left\lfloor\frac{N_ka}{p^\ell}-\frac{N_k\varepsilon}{Lp^{\ell-1}}+\frac{N_k\beta}{L}p^{d+1-\ell}\right\rfloor\geq\left\lfloor\frac{N_k\beta}{L}p^{d+1-\ell}\right\rfloor. \tag{5.6}$$

Clearly, we also have

$$\left\lfloor-\frac{\varepsilon}{Lp^{\ell-1}}+\frac{\beta}{L}p^{d+1-\ell}\right\rfloor\leq\left\lfloor\frac{\beta}{L}p^{d+1-\ell}\right\rfloor. \tag{5.7}$$

If we use (5.6) and (5.7) in (5.5), then we obtain

$$v_p\big(B_{\mathbf{N}}(a+pj)\big)\geq\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_k\beta}{L}p^{d+1-\ell}\right\rfloor-N_k\left\lfloor\frac{\beta}{L}p^{d+1-\ell}\right\rfloor\right). \tag{5.8}$$

Now we claim that $\beta p^{d+1-\ell}/L$ cannot be an integer. Indeed, if it were, then $L\gamma p^{\ell-1} = \beta p^d = Lj + \varepsilon$ for a suitable integer γ. It would follow that L divides ε, contradicting $1 \le \varepsilon \le La/p < L$. However, the fact that $\beta p^{d+1-\ell}/L$ is not an integer entails that

$$\frac{\beta}{L}p^{d+1-\ell} - \left\lfloor \frac{\beta}{L}p^{d+1-\ell} \right\rfloor \ge \frac{1}{L},$$

as long as $\ell \le d+1$. Multiplication of both sides of this inequality by N_k leads to the chain of inequalities

$$\frac{N_k\beta}{L}p^{d+1-\ell} - N_k \left\lfloor \frac{\beta}{L}p^{d+1-\ell} \right\rfloor \ge \frac{N_k}{L} \ge 1$$

(it is here where we use the assumption $L \le N_k = \max(N_1, \dots, N_k)$), whence

$$\left\lfloor \frac{N_k\beta}{L}p^{d+1-\ell} \right\rfloor - N_k \left\lfloor \frac{\beta}{L}p^{d+1-\ell} \right\rfloor \ge 1,$$

provided $\ell \le d+1$. Use of this estimation in (5.8) gives

$$v_p\big(B_{\mathbf{N}}(a+pj)\big) \ge d+1 = 1 + v_p(Lj+\varepsilon).$$

This completes the proof of (5.3), and, hence, of (5.1).

For later use, we record that we have in particular shown that for any

$$D \le 1 + \max_{1 \le \varepsilon \le \lfloor La/p \rfloor} v_p(Lj+\varepsilon)$$

we have

$$\sum_{\ell=2}^{D} \left(\left\lfloor \frac{N_k(a+pj)}{p^\ell} \right\rfloor - N_k \left\lfloor \frac{a+pj}{p^\ell} \right\rfloor \right) \ge D-1. \tag{5.9}$$

We now embark on the proof of (3.5) itself.

5.2. **Second part: the case $j = 0$.** In this case, we want to prove that

$$B_{\mathbf{N}}(a) H_{\lfloor La/p \rfloor} \in p\frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p \ , \tag{5.10}$$

or, using (3.3) (in the other direction), equivalently

$$B_{\mathbf{N}}(a) H_{La} \in \frac{M_{\mathbf{N}}}{\Theta_L}\mathbb{Z}_p \ , \tag{5.11}$$

The reader should keep in mind that we still assume that $p/L \le a < p$, so that, in particular, $a > 0$.

If $p > N_k = \max(N_1, \dots, N_k)$, then our claim, in the form (5.10), reduces to $B_{\mathbf{N}}(a)H_{\lfloor La/p \rfloor} \in p\mathbb{Z}_p$, which is indeed true because of (5.1) with $j = 0$.

Now let $p \le N_k$. Evidently, our claim, this time in the form (5.11), holds for $a = 1$. So, let $a \ge 2$ from now on.

In a similar way as we did for the expression in (5.1), we bound the p-adic valuation of the expression in (5.11) from below:

$$\begin{aligned}
v_p\big(B_{\mathbf{N}}(a)H_{La}\big) &= \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_i a}{p^\ell}\right\rfloor - N_i\left\lfloor\frac{a}{p^\ell}\right\rfloor\right) + v_p(H_{La})\\
&\geq \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i a}{p^\ell}\right\rfloor - \left\lfloor\log_p La\right\rfloor\\
&\geq \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{2N_i}{p^\ell}\right\rfloor - \left\lfloor\log_p Lp\right\rfloor\\
&\geq \left\lfloor\frac{2N_k}{p}\right\rfloor + \sum_{\ell=2}^{\infty}\left\lfloor\frac{2N_k}{p^\ell}\right\rfloor + \sum_{i=1}^{k-1}\sum_{\ell=1}^{\infty}\left\lfloor\frac{2N_i}{p^\ell}\right\rfloor - \left\lfloor\log_p L\right\rfloor - 1\\
&\geq \left\lfloor\frac{N_k}{p}\right\rfloor + \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i}{p^\ell}\right\rfloor - \left\lfloor\log_p L\right\rfloor - 1 \qquad (5.12)\\
&\geq \max\left\{1, \lfloor L/p\rfloor\right\} + \sum_{i=1}^{k} v_p(N_i!) - \left\lfloor\log_p L\right\rfloor - 1. \qquad (5.13)
\end{aligned}$$

If $p = 2$, then we can continue the estimation (5.13) as

$$v_2\big(B_{\mathbf{N}}(a)H_{La}\big) \geq \sum_{i=1}^{k} v_2(N_i!) - \lfloor\log_2 L\rfloor = v_2\big(M_{\mathbf{N}}/\Theta_L\big), \tag{5.14}$$

where we used Lemma 12 to obtain the equality. (In fact, at this point it was not necessary to consider the case $p = 2$ because $a < p$ and because we assumed $a \geq 2$. However, we shall re-use the present estimations later in the third part of the current proof, in a context where $a = 1$ is allowed.)

From now on let $p \geq 3$. We use the fact that

$$k \geq \left\lfloor\log_p k\right\rfloor + 2 \tag{5.15}$$

for all integers $k \geq 2$ and primes $p \geq 3$. Thus, in the case that $L \geq 2p$, the estimation (5.13) can be continued as

$$v_p\big(B_{\mathbf{N}}(a)H_{La}\big) \geq 1 + \left\lfloor\log_p \lfloor L/p\rfloor\right\rfloor + \sum_{i=1}^{k} v_p(N_i!) - \left\lfloor\log_p L\right\rfloor \geq \sum_{i=1}^{k} v_p(N_i!) = v_p(M_{\mathbf{N}}),$$

implying (5.14) in this case. If $p \leq L < 2p$, then the estimation (5.13) can be continued as

$$v_p\big(B_{\mathbf{N}}(a)H_{La}\big) \geq 1 + \sum_{i=1}^{k} v_p(N_i!) - 2 = v_p(M_{\mathbf{N}}/\Theta_L),$$

implying (5.14) in this case also. Finally, if $L < p$, it follows from (5.13) that

$$v_p\big(B_{\mathbf{N}}(a)H_{La}\big) \ge 1 + \sum_{i=1}^{k} v_p(N_i!) - 1 = v_p(M_{\mathbf{N}}),$$

implying (5.14) also in this final case. Everything combined, (5.11) follows immediately.

5.3. **Third part: the case $j > 0$.** Now let $j > 0$. If $p > N_k = \max(N_1, \dots, N_k)$, then (3.5) reduces to

$$B_{\mathbf{N}}(a + pj)\left(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}\right) \in p\mathbb{Z}_p\ ,$$

which is again true because of (5.1).

Now let $p \le N_k$. The reader should keep in mind that we still assume that $p/L \le a < p$, so that, in particular, $a > 0$. In a similar way as we did for the expression in (5.1), we bound the p-adic valuation of the expression in (3.5) from below. For the sake of convenience, we write T_1 for $\max_{1\le\varepsilon\le\lfloor La/p\rfloor} v_p(Lj + \varepsilon)$ and T_2 for $\lfloor \log_p(a + pj)\rfloor$. Since it is somewhat hidden where our assumption $j > 0$ enters the subsequent considerations, we point out to the reader that $j > 0$ implies that $T_2 \ge 1$; without this property the split of the sum over ℓ into subsums in the chain of inequalities below would be impossible. So, using the above notation, we have (the detailed explanations for the various steps are given immediately after the following chain of estimations)

$$\begin{aligned}
&v_p\Big(B_{\mathbf{N}}(a + pj)\left(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}\right)\Big)\\
&\quad= \sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_i(a+pj)}{p^\ell}\right\rfloor - N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right) + v_p\big(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}\big)\\
&\quad= \left\lfloor\frac{N_k(a+pj)}{p}\right\rfloor - N_k\left\lfloor\frac{a+pj}{p}\right\rfloor + \sum_{\ell=2}^{\min\{1+T_1,T_2\}}\left(\left\lfloor\frac{N_k(a+pj)}{p^\ell}\right\rfloor - N_k\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right)\\
&\qquad+ \sum_{\ell=\min\{1+T_1,T_2\}+1}^{\infty}\left(\left\lfloor\frac{N_k(a+pj)}{p^\ell}\right\rfloor - N_k\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right)\\
&\qquad+ \sum_{i=1}^{k-1}\sum_{\ell=1}^{\infty}\left(\left\lfloor\frac{N_i(a+pj)}{p^\ell}\right\rfloor - N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right) + v_p\big(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}\big)\\
&\quad\ge \left\lfloor\frac{N_k a}{p}\right\rfloor + \min\{1+T_1, T_2\} - 1 + \sum_{i=1}^{k}\sum_{\ell=T_2+1}^{\infty}\left(\left\lfloor\frac{N_i(a+pj)}{p^\ell}\right\rfloor - N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right)\\
&\qquad+ v_p\big(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}\big) \qquad\qquad (5.16)\\
&\quad\ge \left\lfloor\frac{N_k a}{p}\right\rfloor + T_1 + v_p\big(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}\big) + \min\{0, T_2 - T_1 - 1\}
\end{aligned}$$

$$+\sum_{i=1}^{k}\sum_{\ell=\lfloor\log_p(a+pj)\rfloor+1}^{\infty}\left(\left\lfloor\frac{N_i(a+pj)}{p^\ell}\right\rfloor-N_i\left\lfloor\frac{a+pj}{p^\ell}\right\rfloor\right) \tag{5.17}$$

$$\geq \max\{1,\lfloor L/p\rfloor\}+\min\{0,T_2-T_1-1\}+\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i}{p^\ell}\cdot\frac{a+pj}{p^{\lfloor\log_p(a+pj)\rfloor}}\right\rfloor \tag{5.18}$$

$$\geq \max\{1,\lfloor L/p\rfloor\}+\lfloor\log_p(a+pj)\rfloor-\lfloor\log_p\left(Lj+\lfloor La/p\rfloor\right)\rfloor-1$$
$$+\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i}{p^\ell}\cdot\frac{a+pj}{p^{\lfloor\log_p(a+pj)\rfloor}}\right\rfloor \tag{5.19}$$

$$\geq \max\{1,\lfloor L/p\rfloor\}+\lfloor\log_p j\rfloor-\lfloor\log_p\left(Lj+\lfloor La/p\rfloor\right)\rfloor+\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}\left\lfloor\frac{N_i}{p^\ell}\right\rfloor \tag{5.20}$$

$$\geq \max\{1,\lfloor L/p\rfloor\}+\lfloor\log_p j\rfloor-\lfloor\log_p L\rfloor-\left\lfloor\log_p\left(j+\frac{1}{L}\lfloor La/p\rfloor\right)\right\rfloor-1$$
$$+\sum_{i=1}^{k}v_p(N_i!) \tag{5.21}$$

$$\geq \max\{1,\lfloor L/p\rfloor\}-\lfloor\log_p L\rfloor-1+v_p(M_{\mathbf{N}}). \tag{5.22}$$

Here, we used (5.9) in order to get (5.16). To get (5.18), we used the inequalities

$$\left\lfloor\frac{N_k a}{p}\right\rfloor\geq\left\lfloor\frac{N_k}{p}\right\rfloor\geq\max\{1,\lfloor L/p\rfloor\} \tag{5.23}$$

and

$$T_1+v_p\big(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\big)\geq 0. \tag{5.24}$$

To get (5.19), we used that

$$T_2-T_1-1\geq\lfloor\log_p(a+pj)\rfloor-\lfloor\log_p\left(Lj+\lfloor La/p\rfloor\right)\rfloor-1$$

and

$$\lfloor\log_p(a+pj)\rfloor-\lfloor\log_p\left(Lj+\lfloor La/p\rfloor\right)\rfloor-1=\lfloor\log_p j\rfloor-\lfloor\log_p\left(Lj+\lfloor La/p\rfloor\right)\rfloor\leq 0,$$

so that

$$\min\{0,T_2-T_1-1\}\geq\lfloor\log_p(a+pj)\rfloor-\lfloor\log_p\left(Lj+\lfloor La/p\rfloor\right)\rfloor-1. \tag{5.25}$$

Next, to get (5.20), we used

$$\left\lfloor\frac{N_i}{p^\ell}\cdot\frac{a+pj}{p^{\lfloor\log_p(a+pj)\rfloor}}\right\rfloor\geq\left\lfloor\frac{N_i}{p^\ell}\right\rfloor. \tag{5.26}$$

To get (5.21), we used

$$\lfloor\log_p\left(Lj+\lfloor La/p\rfloor\right)\rfloor\leq\lfloor\log_p L\rfloor+\left\lfloor\log_p\left(j+\frac{1}{L}\lfloor La/p\rfloor\right)\right\rfloor+1. \tag{5.27}$$

Finally, we used $\frac{1}{L}\lfloor La/p\rfloor < 1$ in order to get (5.22).

If we now repeat the arguments after (5.13), then we see that the above estimation implies

$$v_p\Big(B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\right)\Big) \ge v_p\big(M_{\mathbf{N}}/\Theta_L\big). \tag{5.28}$$

This almost proves (3.5), our lower bound on the p-adic valuation of the number in (3.5) is just by 1 too low.

In order to establish that (3.5) is indeed correct, we assume by contradiction that all the inequalities in the estimations leading to (5.22) and finally to (5.28) are in fact equalities. In particular, the estimations in (5.23) hold with equality only if $a = 1$ and, if L should be at least p, also $\lfloor N_k/p\rfloor = \lfloor L/p\rfloor$. We shall henceforth assume both of these two conditions.

If we examine the arguments after (5.13) that led us from (5.22) to (5.28), then we see that they prove in fact

$$v_p\Big(B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\right)\Big) \ge 1+v_p\big(M_{\mathbf{N}}/\Theta_L\big) \tag{5.29}$$

except if:

Case 1: $p = 2$ and $\lfloor L/2\rfloor = 1$;
Case 2: $p \ge 3$ and $p \le L < 2p$;
Case 3: $p = 3$ and $\lfloor L/3\rfloor = 2$;
Case 4: $L < p$.

In all other cases, there holds either strict inequality in (5.15) with $k = \lfloor L/p\rfloor$, or one has $v_p(\Theta_L) \ge 1$ and is able to show

$$v_p\Big(B_{\mathbf{N}}(a+pj)\left(H_{Lj+\lfloor La/p\rfloor}-H_{Lj}\right)\Big) \ge v_p\big(M_{\mathbf{N}}\big),$$

so that (5.29) is satisfied, as desired. We now show that (5.29) holds as well in Cases 1–4, thus completing the proof of (3.5).

Case 1. Let first $p = 2$ and $L = 2$. We then have

$$\begin{aligned}\min\{0, T_2 - T_1 - 1\} &= \min\{0, \lfloor\log_2(2j+1)\rfloor - v_2(2j+1) - 1\}\\ &= \min\{0, \lfloor\log_2(2j+1)\rfloor - 1\} = 0 > -1,\end{aligned}$$

in contradiction to having equality in (5.25).

On the other hand, if $p = 2$ and $L = 3$ then, because of equality in the second estimation in (5.23), we must have $N_k = 3$. We have

$$H_{Lj+\lfloor La/p\rfloor} - H_{Lj} = H_{3j+1} - H_{3j} = \frac{1}{3j+1}.$$

If there holds equality in (5.25), then $Lj + \lfloor La/p\rfloor = 3j + 1$ must be a power of 2, say $3j + 1 = 2^e$ or, equivalently, $j = (2^e - 1)/3$. It follows that

$$\left\lfloor \frac{N_k}{p}\cdot\frac{a+pj}{p^{\lfloor\log_p(a+pj)\rfloor}}\right\rfloor = \left\lfloor \frac{3}{2}\cdot\frac{1+2j}{2^{\lfloor\log_2(1+2j)\rfloor}}\right\rfloor = \left\lfloor \frac{3}{2}\cdot\frac{2^{e+1}+1}{3\cdot 2^{e-1}}\right\rfloor = 2 > 1 = \left\lfloor\frac{3}{2}\right\rfloor = \left\lfloor\frac{N_k}{p}\right\rfloor,$$

in contradiction to having equality in (5.26) with $\ell = 1$.

CASE 2. Our assumptions $p \geq 3$ and $p \leq L < 2p$ imply

$$H_{Lj+\lfloor La/p \rfloor} - H_{Lj} = H_{Lj+1} - H_{Lj} = \frac{1}{Lj+1}.$$

Arguing as in the previous case, in order to have equality in (5.25), we must have $Lj+1 = f \cdot p^e$ for some positive integers e and f with $0 < f < p$. Thus, $j = (f \cdot p^e - 1)/L$ and $p < L$. (If $p = L$ then j would be non-integral.) It follows that

$$\left\lfloor \frac{N_k}{p} \cdot \frac{a+pj}{p^{\lfloor \log_p(a+pj) \rfloor}} \right\rfloor = \left\lfloor \frac{N_k}{p} \cdot \frac{f \cdot p^{e+1} + L - p}{L \cdot p^{\lfloor \log_p((f \cdot p^{e+1}+L-p)/L) \rfloor}} \right\rfloor. \tag{5.30}$$

If $f = 1$, then we obtain from (5.30) that

$$\left\lfloor \frac{N_k}{p} \cdot \frac{a+pj}{p^{\lfloor \log_p(a+pj) \rfloor}} \right\rfloor = \left\lfloor \frac{N_k}{p} \cdot \frac{p^{e+1}+L-p}{L \cdot p^{e-1}} \right\rfloor \geq \left\lfloor \frac{p^{e+1}+L-p}{p^e} \right\rfloor > 1 = \left\lfloor \frac{L}{p} \right\rfloor = \left\lfloor \frac{N_k}{p} \right\rfloor,$$

in contradiction with having equality in (5.26) with $\ell = 1$.

On the other hand, if $f \geq 2$, then we obtain from (5.30) that

$$\left\lfloor \frac{N_k}{p} \cdot \frac{a+pj}{p^{\lfloor \log_p(a+pj) \rfloor}} \right\rfloor \geq \left\lfloor \frac{f \cdot p^{e+1}+L-p}{p^{e+1}} \right\rfloor \geq f > 1 = \left\lfloor \frac{L}{p} \right\rfloor = \left\lfloor \frac{N_k}{p} \right\rfloor,$$

again in contradiction with having equality in (5.26) with $\ell = 1$.

CASE 3. Our assumptions $p = 3$ and $\lfloor L/3 \rfloor = 2$ imply

$$H_{Lj+\lfloor La/p \rfloor} - H_{Lj} = H_{Lj+2} - H_{Lj} = \frac{1}{Lj+1} + \frac{1}{Lj+2}.$$

Similar to the previous cases, in order to have equality in (5.25), we must have $Lj+\varepsilon = f \cdot 3^e$ for some positive integers ε, e, f with $0 < \varepsilon, f < 3$. The arguments from Case 2 can now be repeated almost verbatim. We leave the details to the reader.

CASE 4. If $L < p$, then $p/L > 1 = a$, a contradiction to the assumption that we made at the very beginning of this section.

This completes the proof of the lemma.

6. PROOF OF LEMMA 6

We want to use Proposition 1 with $A_r(m) = g_r(m) = B_{\mathbf{N}}(m)$ for all r. Clearly, the proposition would imply that $S(a, K, s, p, m) \in p^{s+1} B_{\mathbf{N}}(m)\mathbb{Z}_p$, and, thus, the claim. So, we need to verify the conditions (i)–(iii) in the statement of the proposition.

Condition (i) is true since $B_{\mathbf{N}}(0) = 1$. Condition (ii) holds by the definitions of $A_r(m)$ and $g_r(m)$. To check that Condition (iii) holds is more complicated. The proof will be decomposed in three steps.

6.1. **First step.** Straightforward computations imply that, for any fixed $j \in \{1, 2, \dots, k\}$, we have

$$\begin{aligned}\frac{B_{N_j}(v+up+np^{s+1})}{B_{N_j}(up+np^{s+1})} &= \frac{(N_jv+N_jup+N_jnp^{s+1})!}{(v+up+np^{s+1})!^{N_j}} \cdot \frac{(up+np^{s+1})!^{N_j}}{(N_jup+N_jnp^{s+1})!} \\ &= \frac{(N_jv+N_jup+N_jnp^{s+1})(N_jv-1+N_jup+N_jnp^{s+1})\cdots(N_jup+1+N_jnp^{s+1})}{(v+up+np^{s+1})^{N_j}(v-1+up+np^{s+1})^{N_j}\cdots(1+up+np^{s+1})^{N_j}} \\ &= \frac{(N_jv+N_jup)(N_jv-1+N_jup)\cdots(1+N_jup)+\mathcal{O}(p^{s+1})}{(v+up)^{N_j}(v-1+up)^{N_j}\cdots(1+up)^{N_j}+\mathcal{O}(p^{s+1})},\end{aligned}$$

where $\mathcal{O}(R)$ denotes an element of $R\mathbb{Z}_p$. Now we claim that this implies

$$\frac{B_{N_j}(v+up+np^{s+1})}{B_{N_j}(up+np^{s+1})} = \frac{B_{N_j}(v+up)}{B_{N_j}(up)} + \mathcal{O}(p^{s+1}). \tag{6.1}$$

Indeed, if $v = 0$, then this holds trivially. If $v > 0$, then, together with the hypothesis $v < p$, we infer that $(v+up)(v-1+up)\cdots(1+up)$ is not divisible by p, and thus we have

$$\begin{aligned}&\frac{1}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{N_j}+\mathcal{O}(p^{s+1})} \\ &\qquad\qquad = \frac{1}{\big((v+up)(v-1+up)^{N_j}\cdots(1+up)\big)^{N_j}}\big(1+\mathcal{O}(p^{s+1})\big).\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{(N_jv+N_jup)(N_jv-1+N_jup)\cdots(1+N_jup)+\mathcal{O}(p^{s+1})}{(v+up)^{N_j}(v-1+up)^{N_j}\cdots(1+up)^{N_j}+\mathcal{O}(p^{s+1})} \\ &\qquad = \frac{(N_jv+N_jup)(N_jv-1+N_jup)\cdots(1+N_jup)}{(v+up)^{N_j}(v-1+up)^{N_j}\cdots(1+up)^{N_j}} \\ &\qquad\qquad + \frac{\mathcal{O}(p^{s+1})}{(v+up)^{N_j}(v-1+up)^{N_j}\cdots(1+up)^{N_j}},\end{aligned}$$

which proves (6.1) because

$$\frac{1}{(v+up)(v-1+up)\cdots(1+up)} \in \mathbb{Z}_p \tag{6.2}$$

and

$$\frac{(N_jv+N_jup)(N_jv-1+N_jup)\cdots(1+N_jup)}{(v+up)^{N_j}(v-1+up)^{N_j}\cdots(1+up)^{N_j}} = \frac{B_{N_j}(v+up)}{B_{N_j}(up)}. \tag{6.3}$$

Note that (6.2) and (6.3) also imply that $B_{N_j}(v+up)/B_{N_j}(up) \in \mathbb{Z}_p$, a property that will be used below.

We deduce from (6.1) that

$$\prod_{j=1}^{k} \frac{B_{N_j}(v+up+np^{s+1})}{B_{N_j}(up+np^{s+1})} = \prod_{j=1}^{k}\left(\frac{B_{N_j}(v+up)}{B_{N_j}(up)} + \mathcal{O}(p^{s+1})\right). \tag{6.4}$$

By expanding the product on the right-hand side of (6.4) and using that

$$B_{N_j}(v+up)/B_{N_j}(up) \in \mathbb{Z}_p,$$

we obtain our first main equality:

$$\frac{B_{\mathbf{N}}(v+up+np^{s+1})}{B_{\mathbf{N}}(up+np^{s+1})} = \frac{B_{\mathbf{N}}(v+up)}{B_{\mathbf{N}}(up)} + \mathcal{O}(p^{s+1}). \tag{6.5}$$

6.2. **Second step.** The properties of Γ_p imply that

$$\begin{aligned}
\frac{B_{N_j}(up+np^{s+1})}{B_{N_j}(u+np^s)} &= (-1)^{N_j-1}\frac{\Gamma_p(1+N_jup+N_jnp^{s+1})}{\Gamma_p(1+up+np^{s+1})^{N_j}} &(6.6)\\
&= (-1)^{N_j-1}\frac{\Gamma_p(1+N_jup)+\mathcal{O}(p^{s+1})}{\Gamma_p(1+up)^{N_j}+\mathcal{O}(p^{s+1})} &(6.7)\\
&= (-1)^{N_j-1}\frac{\Gamma_p(1+N_jup)}{\Gamma_p(1+up)^{N_j}}\big(1+\mathcal{O}(p^{s+1})\big) &(6.8)\\
&= \frac{B_{N_j}(up)}{B_{N_j}(u)}\big(1+\mathcal{O}(p^{s+1})\big), &(6.9)
\end{aligned}$$

where (i) of Lemma 9 is used to see (6.6) and (6.9), and (ii) is used for (6.7). Equation (6.8) holds because $\Gamma_p(1+up)$ and $\Gamma_p(1+N_jup)$ are both not divisible by p. Taking the product over $j=1,2,\dots,k$, we obtain from (6.9) our second main equality:

$$\frac{B_{\mathbf{N}}(up+np^{s+1})}{B_{\mathbf{N}}(u+np^s)} = \frac{B_{\mathbf{N}}(up)}{B_{\mathbf{N}}(u)}\big(1+\mathcal{O}(p^{s+1})\big). \tag{6.10}$$

6.3. **Third step.** We now multiply the right-hand and left-hand sides of the main equalities (6.5) and (6.10): we get after simplification

$$\frac{B_{\mathbf{N}}(v+up+np^{s+1})}{B_{\mathbf{N}}(u+np^s)} = \frac{B_{\mathbf{N}}(v+up)}{B_{\mathbf{N}}(u)}\big(1+\mathcal{O}(p^{s+1})\big) + \frac{B_{\mathbf{N}}(up)}{B_{\mathbf{N}}(u)}\mathcal{O}(p^{s+1}).$$

We can rewrite this as

$$\begin{aligned}
\frac{B_{\mathbf{N}}(v+up+np^{s+1})}{B_{\mathbf{N}}(v+up)} &= \frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(u)}\big(1+\mathcal{O}(p^{s+1})\big) + \frac{B_{\mathbf{N}}(up)}{B_{\mathbf{N}}(u)}\cdot\frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(v+up)}\mathcal{O}(p^{s+1})\\
&= \frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(u)} + \frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(u)}\mathcal{O}(p^{s+1}) + \frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(v+up)}\mathcal{O}(p^{s+1}),
\end{aligned} \tag{6.11}$$

where the last line holds because $v_p\big(B_{\mathbf{N}}(up)/B_{\mathbf{N}}(u)\big)=0$.

If we compare (2.2) (with $A_r(m)=g_r(m)=B_{\mathbf{N}}(m)$) and (6.11), we see that it only remains to prove that we have

$$\frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(u)} \in \frac{B_{\mathbf{N}}(n)}{B_{\mathbf{N}}(v+up)}\mathbb{Z}_p \quad\text{and}\quad \frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(v+up)} \in \frac{B_{\mathbf{N}}(n)}{B_{\mathbf{N}}(v+up)}\mathbb{Z}_p. \tag{6.12}$$

The first membership relation follows from Lemma 8 because the latter implies the stronger property

$$\frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(u)} \in \left(\prod_{j=1}^{k} B_{N_j}(n)\right)\mathbb{Z}_p = B_{\mathbf{N}}(n)\,\mathbb{Z}_p. \tag{6.13}$$

For the second membership relation, we note that

$$\begin{aligned}\frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(v+up)} &= B_{\mathbf{N}}(u)\cdot\frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(u)}\cdot\frac{1}{B_{\mathbf{N}}(v+up)}\\ &\qquad\in B_{\mathbf{N}}(u)\,\frac{B_{\mathbf{N}}(n)}{B_{\mathbf{N}}(v+up)}\,\mathbb{Z}_p \subset \frac{B_{\mathbf{N}}(n)}{B_{\mathbf{N}}(v+up)}\,\mathbb{Z}_p,\end{aligned}$$

where we have used (6.13).

It now follows from (6.11) and (6.12) that we have

$$\frac{B_{\mathbf{N}}(v+up+np^{s+1})}{B_{\mathbf{N}}(v+up)} - \frac{B_{\mathbf{N}}(u+np^s)}{B_{\mathbf{N}}(u)} \in \frac{B_{\mathbf{N}}(n)}{B_{\mathbf{N}}(v+up)}\,\mathbb{Z}_p,$$

i.e., we have checked that hypothesis *(iii)* in Proposition 1 holds in our situation. We can therefore apply this proposition and obtain exactly the statement of the lemma.

7. Proof of Lemma 7

The claim is trivially true if p divides m. We may therefore assume that p does not divide m for the rest of the proof. Let us write $m = a + pj$, with $0 < a < p$. Then comparison with (3.5) shows that we are in a very similar situation here. Indeed, we may derive (3.10) from Lemma 5. In order to see this, we observe that

$$\begin{aligned}H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}} &= \sum_{\varepsilon=1}^{Lap^s} \frac{1}{Ljp^{s+1}+\varepsilon}\\ &= \sum_{\varepsilon=1}^{\lfloor La/p\rfloor} \frac{1}{Ljp^{s+1}+\varepsilon p^{s+1}} + \sum_{\substack{\varepsilon=1\\ p^{s+1}\nmid\varepsilon}}^{Lap^s} \frac{1}{Ljp^{s+1}+\varepsilon}\\ &= \frac{1}{p^{s+1}}\left(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}\right) + \sum_{\substack{\varepsilon=1\\ p^{s+1}\nmid\varepsilon}}^{Lap^s} \frac{1}{Ljp^{s+1}+\varepsilon}.\end{aligned}$$

Because of $v_p(x+y) \ge \min\{v_p(x), v_p(y)\}$, this implies

$$v_p(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}) \ge \min\{-1-s+v_p(H_{Lj+\lfloor La/p\rfloor} - H_{Lj}), -s\}.$$

It follows that

$$\begin{aligned}&v_p\Big(B_{\mathbf{N}}(m)\big(H_{Lmp^s} - H_{L\lfloor m/p\rfloor p^{s+1}}\big)\Big)\\ &\qquad\ge -1-s+\min\left\{v_p\Big(B_{\mathbf{N}}(a+pj)(H_{Lj+\lfloor La/p\rfloor} - H_{Lj})\Big), 1+v_p\big(B_{\mathbf{N}}(a+pj)\big)\right\}.\end{aligned}$$

Use of Lemmas 4 and 5 then completes the proof.

8. Outline of the proof of Theorem 3

In this section, we provide a brief outline of the proof of Theorem 3, reducing it to several lemmas and their corollaries. These are subsequently proved in the next section.

We must slightly "upgrade" the proof of Theorem 2 in the special case that $\mathbf{N} = (N, N, \dots, N)$ (with k occurrences of N) and $L = N$. To be precise, for all primes p, and for all non-negative integers K, a, and j with $0 \le a < p$, we must prove

$$C(a + Kp) = \sum_{j=0}^{K} B_{\mathbf{N}}(a + jp) B_{\mathbf{N}}(K - j)(H_{N(K-j)} - pH_{Na+Njp}) \in p\Xi_N N!^k \mathbb{Z}_p, \tag{8.1}$$

where $B_{\mathbf{N}}(m) = \frac{(Nm)!^k}{m!^{kN}}$, instead of "just" (3.2) for the present choice of parameters. We recall that, in view of a comparison of (1.6) with the definition of Ξ_N, there is only something to prove if $v_p(H_N) > 0$. Hence, Lemma 12 tells that there is nothing to prove if $p = 2$, and Lemma 13 together with Remarks 2(a) in the Introduction tells that, if $p = 3$, only the case $N = 22$ need to be considered. Therefore, in the remainder of this section, we always assume that p is a prime with $3 \le p \le N$ and $v_p(H_N) > 0$, and if $p = 3$ then $N = 22$.

There are two cases which can be treated directly: if $K = a = 0$, then $C(0) = 0$, and thus (8.1) holds trivially, whereas if $K = 0$ and $a = 1$, then $C(1) = -pB_{\mathbf{N}}(1)H_N = -pN!^k H_N$ and thus (8.1) holds by definition of Ξ_N. We therefore assume in addition $a + Kp \ge 2$ for the remainder of this section.

Going through the outline of the proof of Theorem 2 in Section 3, we see that, in order to establish (8.1), we need to prove corresponding stronger versions of (3.4) and Lemmas 5 and 7. To be precise, given non-negative integers m, K, a, and j with $0 \le a < p$ and $a + Kp \ge 2$, for $p \ge 5$ we should prove

$$C(a + Kp) \equiv \sum_{j=0}^{K} B_{\mathbf{N}}(a + jp) B_{\mathbf{N}}(K - j)(H_{N(K-j)} - H_{\lfloor Na/p \rfloor + Nj}) \mod p^3 N!^k \mathbb{Z}_p, \tag{8.2}$$

respectively, for $p = 3$ and $N = 22$,

$$C(a + 3K) \equiv \sum_{j=0}^{K} B_{\mathbf{N}}(a + 3j) B_{\mathbf{N}}(K - j)(H_{N(K-j)} - H_{\lfloor Na/3 \rfloor + Nj}) \mod 3^2 N!^k \mathbb{Z}_3, \tag{8.3}$$

respectively, if $v_p(H_N) \ge 3$ and p a Wolstenholme prime or $p \mid N$,

$$C(a + Kp) \equiv \sum_{j=0}^{K} B_{\mathbf{N}}(a + jp) B_{\mathbf{N}}(K - j)(H_{N(K-j)} - H_{\lfloor Na/p \rfloor + Nj}) \mod p^4 N!^k \mathbb{Z}_p, \tag{8.4}$$

and we need to prove

$$B_{\mathbf{N}}(a + pj) \left(H_{Nj + \lfloor Na/p \rfloor} - H_{Nj} \right) \in p\Xi_N N!^k \mathbb{Z}_p \tag{8.5}$$

and

$$B_{\mathbf{N}}(m)\big(H_{Nmp^s} - H_{N\lfloor m/p\rfloor p^{s+1}}\big) \in p^{-s}\Xi_N N!^k\mathbb{Z}_p \,. \tag{8.6}$$

If these five relations are used in Section 3 instead of their weaker counterparts (3.4), (3.5), and (3.10), respectively, then the proof in Section 3 yields (8.1), as required.

8.1. **Proof of** (8.2)–(8.4)**.** We recall that the congruence (3.4) followed from the congruence (3.3). Clearly, one cannot hope for improving (3.3) to $pH_J \equiv H_{\lfloor J/p\rfloor} \bmod p^3\mathbb{Z}_p$, there are counterexamples. However, if we combine Corollary 2 and Lemma 15(1) with $L = N$, then we see that

$$B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j)pH_{Na+Njp} \equiv B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j)H_{\lfloor Na/p\rfloor+Nj} \mod p^3N!^k\mathbb{Z}_p$$

as long as $a + Kp \geq 2$ and $a \neq 0$. Moreover, due to Lemma 4 and Corollary 3, the above congruence is even true if $a = 0$ and $p \geq 5$. This implies (8.2). The congruence (8.3) follows in the same way by the slightly weaker assertion for $p = 3$ in Corollary 3.

For the congruence (8.4), one needs to combine Corollary 2 and Lemma 15(4) with $L = N$, to see that

$$B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j)pH_{Na+Njp} \equiv B_{\mathbf{N}}(a+jp)B_{\mathbf{N}}(K-j)H_{\lfloor Na/p\rfloor+Nj} \mod p^4N!^k\mathbb{Z}_p \tag{8.7}$$

as long as $a + Kp \geq 2$ and $a \neq 0$. If $a = 0$, then the congruence (8.7) still holds as long as $j < K$ because of Lemma 4, Corollary 3, and the fact that the term $B_{\mathbf{N}}(K-j)$ contributes at least one factor p. The only remaining case to be discussed is $a = 0$ and $j = K$. If we apply the simple observation that $v_p(B_{\mathbf{N}}(p^eh)) = v_p(B_{\mathbf{N}}(h))$ for any positive integers e and h to $B_{\mathbf{N}}(a + jp) = B_{\mathbf{N}}(jp)$, then, making again appeal to Corollary 2 and Lemma 15(4) with $L = N$, we see that (8.7) holds as well as long as $j = K$ is no prime power. Finally, let $a = 0$ and $j = K$ be a prime power, $j = K = p^e$ say. If $e \geq 1$, then we may use Lemma 17 with $J = a + jp = a + Kp = p^{e+1}$ and $v_p(B_{\mathbf{N}}(p^{e+1})) = v_p(B_{\mathbf{N}}(1)) = v_p(N!^k)$ to conclude that (8.7) also holds in this case. On the other hand, if $j = K = 1$, then Lemma 18, the assumption that p is a Wolstenholme prime or that p divides N, the fact that $v_p(B_{\mathbf{N}}(p)) = v_p(B_{\mathbf{N}}(1)) = v_p(N!^k)$, altogether yield the congruence (8.7) in this case as well.

8.2. **Proof of** (8.5)**.** The proof of Lemma 5 in Section 5 can be used verbatim. The place where the necessary improvement is possible is (5.15) (which was used there with $k = \lfloor \log_p L\rfloor$). Clearly, we have $\lfloor 22/3\rfloor \geq \lfloor \log_3\lfloor 22/3\rfloor\rfloor + 5$. Moreover, we claim that in our more special context we have

$$\lfloor N/5\rfloor \geq \lfloor \log_5\lfloor N/5\rfloor\rfloor + 4 \tag{8.8}$$

if $p = 5$, we have

$$\lfloor N/p\rfloor \geq \lfloor \log_p\lfloor N/p\rfloor\rfloor + 5 \tag{8.9}$$

if $p > 5$, and we have

$$\lfloor N/p\rfloor \geq \lfloor \log_p\lfloor N/p\rfloor\rfloor + 6 \tag{8.10}$$

if $v_p(H_N) > 2$. Indeed, if $p = 5$ then Lemma 14 says that N must be at least 20 because otherwise $v_5(H_N) \leq 0$. The inequality (8.8) follows immediately. On the other hand, if $p = 11$ then it can be checked (by our table mentioned in Remarks 2(c) in the Introduction, for example) that N must be at least 77 because otherwise $v_{11}(H_N) \leq 0$. If p is different from 5 and 11 (in addition to being at least 5) then, according to Lemma 15(3), we must have $N/p \geq 5$. In both of the latter cases, the inequality (8.9) follows immediately. Finally, if $v_p(H_N) > 2$ then, according to Lemmas 14 and 15(4), we must have $p > 5$ and $N/p \geq 6$, thus implying (8.10).

The effect of the above improvement over (5.15) is that the proof in Section 5 shows that in our more special context we have

$$B_{\mathbf{N}}(a+pj)\left(H_{Nj+\lfloor Na/p \rfloor} - H_{Nj}\right) \in \begin{cases} p^2 N!^k \mathbb{Z}_p & \text{if } p = 5, \\ p^4 N!^k \mathbb{Z}_p & \text{if } v_p(H_N) > 2, \\ p^3 N!^k \mathbb{Z}_p & \text{otherwise.} \end{cases}$$

This implies (8.5) since $v_5(H_N) \leq 1$ according to Lemma 14.

8.3. **Proof of** (8.6)**.** Again, the proof of Lemma 7 in Section 7 can be used verbatim. The only differences arise at the very end. Namely, we have to use Corollary 2 combined with Lemma 15(1) instead of Lemma 4, and (8.2) instead of Lemma 5, to obtain our assertion for $m \geq 2$. In the remaining case $m = 1$, we compute directly:

$$B_{\mathbf{N}}(1) H_{Np^s} = N!^k \left(\frac{1}{p^s} H_N + \frac{1}{p^{s-1}} \sum_{\substack{\varepsilon=1 \\ p \nmid \varepsilon}}^{Np} \frac{1}{\varepsilon} + \frac{1}{p^{s-2}} \sum_{\substack{\varepsilon=1 \\ p \nmid \varepsilon}}^{Np^2} \frac{1}{\varepsilon} + \sum_{\substack{\varepsilon=1 \\ p^{s-2} \nmid \varepsilon}}^{Np^s} \frac{1}{\varepsilon} \right).$$

The last sum over ε is clearly an element of $p^{-s+3}\mathbb{Z}_p$. Moreover, if $p \geq 5$, Lemma 16 implies that the next-to-last expression between parentheses is an element of $p^{-s+4}\mathbb{Z}_p \subset p^{-s+3}\mathbb{Z}_p$, and that the second expression between parentheses is an element of $p^{-s+3}\mathbb{Z}_p$. If $p = 3$, it can be checked directly that the expression between parentheses is an element of $3^{-s+2}\mathbb{Z}_3$. Putting everything together, we conclude that

$$B_{\mathbf{N}}(1) H_{Np^s} \in p^{-s} N!^k \Xi_N \mathbb{Z}_p,$$

which finishes the proof of (8.6).

9. More auxiliary lemmas

In this section, we prove the auxiliary results necessary for the proof of Theorem 3, of which the outline was given in the previous section. These are, on the one hand, improvements of Lemma 4 (see Lemmas 10, 11 and Corollary 2), and, on the other hand, assertions addressing specific p-adic properties of harmonic numbers (see Lemmas 12–18 and Corollary 3). Some of the results of this section are also referred to in the next section.

We begin by two lemmas improving on Lemma 4.

Lemma 10. *For all positive integers N, a, and primes p with $2 \leq a < p$, we have*

$$v_p(B_{\mathbf{N}}(a)) \geq \left\lfloor \frac{N}{p} \right\rfloor + v_p(N!^k),$$

where $\mathbf{N} = (N, N, \ldots, N)$ (with k occurrences of N).

Proof. This was implicitly proved by the estimations leading to (5.12) in the case that $N_i = N$ for all i. □

Lemma 11. *For all positive integers N, a, j, and primes p with $1 \le a < p$, we have*

$$v_p(B_{\mathbf{N}}(a+jp)) \ge \left\lfloor \frac{N}{p} \right\rfloor + \min\{1+T_1, T_2\} - 1 + v_p(N!^k),$$

where $\mathbf{N} = (N, N, \dots, N)$ (with k occurrences of N), and where

$$T_1 = \max_{1 \le \varepsilon \le \lfloor Na/p \rfloor} v_p(Nj+\varepsilon) \quad \text{and} \quad T_2 = \left\lfloor \log_p(a+pj) \right\rfloor.$$

Proof. This is seen by going through the estimations leading to (5.22) in the case that $N_i = N$ for all i, without employing (5.23), (5.24), and (5.25). □

As a corollary to Lemmas 10 and 11, we obtain the following succinct p-adic estimation for $B_{\mathbf{N}}(m)$, which is needed in the proofs of (8.2), (8.4), and (8.6).

Corollary 2. *Let N and m be positive integers and p be a prime such that m is at least 2 and not divisible by p. Then we have*

$$v_p(B_{\mathbf{N}}(m)) \ge \left\lfloor \frac{N}{p} \right\rfloor + v_p(N!^k),$$

where $\mathbf{N} = (N, N, \dots, N)$ (with k occurrences of N).

The next three lemmas of this section provide elementary information on the p-adic valuation of harmonic numbers for $p = 2, 3, 5$ which is needed in the proof of Lemma 15 and is also referred to frequently at other places. (For example, Lemma 12 was used in the proof of (5.14).) The proofs are not difficult (cf. [5]) and are therefore omitted.

Lemma 12. *For all positive integers L, we have $v_2(H_L) = -\lfloor \log_2 L \rfloor$.*

Lemma 13. *We have $v_3(H_2) = v_3(H_7) = v_3(H_{22}) = 1$. For positive integers $L \notin \{2, 7, 22\}$, we have $v_3(H_L) \le 0$.*

Lemma 14. *We have $v_5(H_4) = 2$ and $v_5(H_{20}) = v_5(H_{24}) = 1$. For positive integers $L \notin \{4, 20, 24\}$, we have $v_5(H_L) \le 0$.*

Next, we record some properties of integers L and primes p for which $v_p(H_L) > 0$. These are needed throughout Section 8.

Lemma 15. *Let p be a prime, and let L be an integer with $p \le L$. Then the following assertions hold true:*

(1) *If $v_p(H_L) > 0$ then $L \ge 2p$.*
(2) *If $v_p(H_L) > 0$ and $p \ne 3$ then $L \ge 3p$.*
(3) *If $v_p(H_L) > 0$ and $p \notin \{3, 5, 11\}$ then $L \ge 5p$.*
(4) *If $v_p(H_L) > 2$ then $L \ge 6p$.*

Proof. We have

$$H_L = \frac{1}{p} H_{\lfloor L/p \rfloor} + \sum_{\substack{\varepsilon=1 \\ p \nmid \varepsilon}}^{L} \frac{1}{\varepsilon}.$$

Since the sum over ε is in $\mathbb{Z}_p$, in order to have $v_p(H_L) > 0$ we must have $v_p(H_{\lfloor L/p \rfloor}) > 0$. Clearly, $v_p(H_1) = 0$ so that (1) follows. If $\lfloor L/p \rfloor < 3$ and $p \neq 3$, then $v_p(H_{\lfloor L/p \rfloor})$ cannot be positive since $H_1 = 1$ and $H_2 = \frac{3}{2}$. This implies (2). If $\lfloor L/p \rfloor < 5$ and $p \notin \{3, 5, 11\}$, then, again, $v_p(H_{\lfloor L/p \rfloor})$ cannot be positive since, as we already noted, $H_1 = 1$ and $H_2 = \frac{3}{2}$, and since $H_3 = \frac{11}{6}$ and $H_4 = \frac{25}{12}$. This yields (3).

To see (4), we observe that, owing to Lemmas 12–14, we may assume that $p \notin \{2, 3, 5\}$. Furthermore, if $p = 11$ then, according to our table referred to in Remarks 2(c) in the Introduction (see also [5]), we have $L \geq 848$. Similarly, if $p = 137$ then, according to our table, we have $L > 500000$. The claim can now be established in the style of the proofs of (1)–(3) upon observing that $H_5 = \frac{137}{60}$. □

We turn to a slight generalisation of Wolstenholme's theorem on harmonic numbers. (We refer the reader to [13, Chapter VII] for information on this theorem, which corresponds to the case $r = 1$ in the lemma below.)

Lemma 16. *For all primes $p \geq 5$ and positive integers r, we have*

$$v_p(H_{rp-1} - H_{rp-p}) \geq 2.$$

Proof. By simple rearrangement, we have

$$\begin{aligned} H_{rp-1} - H_{rp-p} &= \sum_{\varepsilon=1}^{p-1} \frac{1}{rp - p + \varepsilon} = \sum_{\varepsilon=1}^{(p-1)/2} \left(\frac{1}{rp - p + \varepsilon} + \frac{1}{rp - \varepsilon} \right) \\ &= p(2r - 1) \sum_{\varepsilon=1}^{(p-1)/2} \frac{1}{(rp - p + \varepsilon)(rp - \varepsilon)}. \end{aligned}$$

It therefore suffices to consider the last sum over ε in $\mathbb{Z}/p\mathbb{Z}$ and show that it is $\equiv 0 \bmod p$. If we reduce this sum mod p, then we are left with

$$-\sum_{\varepsilon=1}^{(p-1)/2} \frac{1}{\varepsilon^2},$$

which is, up to the sign, the sum of all quadratic residues in $\mathbb{Z}/p\mathbb{Z}$, that is, equivalently,

$$-\sum_{\varepsilon=1}^{(p-1)/2} \varepsilon^2 = \frac{p(p-1)(p+1)}{24}.$$

Clearly, this is divisible by p for all primes $p \geq 5$. □

As a corollary, we obtain strengthenings of (3.3) that we need in the proof of (8.2) and (8.3).

Corollary 3. *For all primes $p \geq 5$ and positive integers J divisible by p, we have*

$$pH_J \equiv H_{J/p} \mod p^3\mathbb{Z}_p.$$

Moreover, for all positive integers J divisible by 3, we have

$$3H_J \equiv H_{J/3} \mod 3^2\mathbb{Z}_3.$$

Proof. By simple rearrangement, we have

$$pH_J - H_{J/p} = p\sum_{r=1}^{J/p}(H_{rp-1} - H_{rp-p}).$$

Due to Lemma 16, the p-adic valuation of this expression is at least 3 if $p \geq 5$. If $p = 3$, it is easily seen directly that the 3-adic valuation of this expression is at least 2. □

Further strengthenings of (3.3), needed in the proof of (8.4), are given in the final two lemmas of this section.

Lemma 17. *For all primes $p \geq 5$ and positive integers J divisible by p^2, we have*

$$pH_J \equiv H_{J/p} \mod p^5\mathbb{Z}_p.$$

Proof. Again, by simple rearrangement, we have

$$\begin{aligned} pH_J - H_{J/p} &= p\sum_{\substack{\varepsilon=1 \\ p\nmid\varepsilon}}^{J-1}\frac{1}{\varepsilon} = p\sum_{r=1}^{J/p^2}\sum_{\substack{\varepsilon=1 \\ p\nmid\varepsilon}}^{p^2-1}\frac{1}{rp^2 - p^2 + \varepsilon} \\ &= p^3\sum_{r=1}^{J/p^2}(2r-1)\sum_{\substack{\varepsilon=1 \\ p\nmid\varepsilon}}^{(p^2-1)/2}\frac{1}{(rp^2 - p^2 + \varepsilon)(rp^2 - \varepsilon)}. \end{aligned}$$

It therefore suffices to consider the last sum over ε in $\mathbb{Z}/p^2\mathbb{Z}$ and show that it is $\equiv 0 \mod p^2$. If we reduce this sum mod p^2, then we are left with

$$-\sum_{\substack{\varepsilon=1 \\ p\nmid\varepsilon}}^{(p^2-1)/2}\frac{1}{\varepsilon^2},$$

which is, up to the sign, the sum of all quadratic residues in $\mathbb{Z}/p^2\mathbb{Z}$, that is, equivalently,

$$-\sum_{\substack{\varepsilon=1 \\ p\nmid\varepsilon}}^{(p^2-1)/2}\varepsilon^2 = -\sum_{\varepsilon=1}^{(p^2-1)/2}\varepsilon^2 + \sum_{\varepsilon=1}^{(p-1)/2}(p\varepsilon)^2 = -\frac{p^2(p^2-1)(p^2+1)}{24} + p^2\frac{p(p-1)(p+1)}{24}.$$

Clearly, this is divisible by p^2 for all primes $p \geq 5$. □

Lemma 18. *For all primes $p \geq 5$ and positive integers N, we have*

$$pH_{pN} \equiv H_N \mod p^4\mathbb{Z}_p \tag{9.1}$$

if and only if p is a Wolstenholme prime or p divides N.

Proof. Using a rearrangement in the spirit of Lemma 16, we obtain

$$pH_{pN} - H_N = p\sum_{r=1}^{N}\sum_{\varepsilon=1}^{p-1}\frac{1}{rp-p+\varepsilon}.$$

We consider the sum over r in $\mathbb{Z}/p^3\mathbb{Z}$. This leads to

$$\begin{aligned}\sum_{r=1}^{N}\sum_{\varepsilon=1}^{p-1}(rp-p+\varepsilon)^{-1} &\equiv \sum_{r=1}^{N}\sum_{\varepsilon=1}^{p-1}\varepsilon^{-1}\big(1+p(r-1)\varepsilon^{-1}\big)^{-1}\\ &\equiv \sum_{r=1}^{N}\sum_{\varepsilon=1}^{p-1}\big(\varepsilon^{-1}-p(r-1)\varepsilon^{-2}+p^2(r-1)^2\varepsilon^{-3}\big)\\ &\equiv NH_{p-1} - p\binom{N}{2}H^{(2)}_{p-1} + p^2\frac{N(N-1)(2N-1)}{6}H^{(3)}_{p-1} \mod \mathbb{Z}/p^3\mathbb{Z},\end{aligned} \tag{9.2}$$

where $H^{(\alpha)}_m$ denotes the higher harmonic number defined by $H^{(\alpha)}_m = \sum_{n=1}^{m}\frac{1}{n^\alpha}$. By a rearrangement analogous to the one in the proof of Lemma 16, one sees that $v_p(H^{(3)}_{p-1}) \geq 1$, whence we may disregard the last term in the last line of (9.2). As it turns out, H_{p-1} and $H^{(2)}_{p-1}$ are directly related modulo $\mathbb{Z}/p^3\mathbb{Z}$. Namely, we have

$$\begin{aligned}H_{p-1} &\equiv \sum_{\varepsilon=1}^{p-1}\varepsilon^{-1} \equiv 2^{-1}\sum_{\varepsilon=1}^{p-1}\big(\varepsilon^{-1}+(p-\varepsilon)^{-1}\big) \equiv p2^{-1}\sum_{\varepsilon=1}^{p-1}\big(\varepsilon(p-\varepsilon)\big)^{-1}\\ &\equiv -p2^{-1}\sum_{\varepsilon=1}^{p-1}\varepsilon^{-2}\big(1+p\varepsilon^{-1}\big) \equiv -p2^{-1}\sum_{\varepsilon=1}^{p-1}\big(\varepsilon^{-2}+p\varepsilon^{-3}\big)\\ &\equiv -p2^{-1}H^{(2)}_{p-1} - p^2 2^{-1}H^{(3)}_{p-1} \mod \mathbb{Z}/p^3\mathbb{Z}.\end{aligned} \tag{9.3}$$

Again, we may disregard the last term. If we substitute this congruence in (9.2), then we obtain

$$\sum_{r=1}^{N}\sum_{\varepsilon=1}^{p-1}(rp-p+\varepsilon)^{-1} \equiv N^2H_{p-1} \mod \mathbb{Z}/p^3\mathbb{Z}.$$

Hence, the sum is congruent to zero mod $\mathbb{Z}/p^3\mathbb{Z}$ if and only if N is divisible by p (recall Lemma 16 with $r = 1$) or $v_p(H_{p-1}) \geq 3$, that is, if p is a Wolstenholme prime. This establishes the assertion of the lemma. □

10. The sequence $(t_N)_{N\ge 1}$

The purpose of this section is to report on the evidence for our conjecture that the largest integer t_N such that $q_{N,N}(z)^{1/t_N} \in \mathbb{Z}[[z]]$ is given by $t_N = \Xi_N N!$, that is, that Theorem 3 with $k = 1$ is optimal. We assume $k = 1$ throughout this section.

We first prove that there cannot be any prime number p larger than N which divides t_N.

Proposition 2. *Let p be a prime number and N a positive integer with $p > N$. Then there exists a positive integer $a < p$ such that*

$$v_p\big(B_N(a)H_{Na}\big) = 0, \tag{10.1}$$

where $B_N(m) = \frac{(Nm)!}{m!^N}$. In particular, p does not divide t_N.

Proof. If $N = 1$, we choose $a = 1$ to obtain $B_1(1)H_1 = 1$. On the other hand, if $N > 1$, we choose a to be the least integer such that $aN \ge p$. Since then $a < p$, we have

$$v_p\big(B_N(a)H_{Na}\big) = \sum_{\ell=1}^{\infty} \left(\left\lfloor \frac{Na}{p^\ell} \right\rfloor - N \left\lfloor \frac{a}{p^\ell} \right\rfloor \right) + v_p(H_{Na}) = 1 - 1 = 0.$$

To see that p cannot divide t_N, we observe that we have $C(a) = -pB_N(a)H_{Na}$ (for the sum $C(\cdot)$ defined in (3.1) in the case $k = 1$ and $L = N$) because $a < p$. Hence, the assertion (10.1) can be reformulated as $v_p\big(C(a)\big) = 1$. Since, because of $p > N$, we have $v_p(M_N) = v_p(N!) = 0$ and $v_p(\Xi_N) = 0$, it follows that

$$C(a) \notin p^2 \Xi_N N!\, \mathbb{Z}_p.$$

This means that one cannot increase the exponent of p in (8.1) (with $k = 1$) in our special case, and thus p cannot divide t_N. $\square$

So, if we hope to improve Theorem 3 with $k = 1$, then it must be by increasing exponents of prime numbers $p \le N$ in (1.7). It can be checked directly that the exponent of 3 cannot be increased if $N = 7$. (The reader should recall Remarks 2(a) in the Introduction.) According to Remarks 2(b), an improvement is therefore only possible if $v_p(H_N) > 2$ for some $p \le N$. Lemmas 12–14 in Section 9 tell that this does not happen with $p = 2, 3, 5$, so that the exponents of $2, 3, 5$ cannot be improved. (The same conclusion can also be drawn from [5] for many other prime numbers, but so far not for 83, for example.)

We already discussed in Remarks 2(c) whether at all there are primes p and integers N with $p \le N$ and $v_p(H_N) \ge 3$. We recall that, so far, only five examples are known, four of them involving $p = 11$.

The final result of this section shows that, if $v_p(H_N) = 3$, the exponent of p in the definition of Ξ_N in (1.7) cannot be increased so that Theorem 3 would still hold. (The reader should recall Remarks 2(b).)

Proposition 3. *Let p be a prime number and N a positive integer with $p \le N$ and $v_p(H_N) = 3$. If p is not a Wolstenholme prime and p does not divide N, then $q_{N,N}(z)^{1/p\Xi_N N!} \notin \mathbb{Z}[[z]]$.*

Proof. We assume that p is not a Wolstenholme prime and that p does not divide N. By Lemmas 12–14, we can furthermore assume that $p \geq 7$.

Going back to the outline of the proof of Theorem 3 in Section 8, we claim that

$$C(p) = \big(B_N(1)H_N - B_N(p)pH_{Np}\big) \notin p^4 N!\,\mathbb{Z}_p, \tag{10.2}$$

where $B_N(m) = \frac{(Nm)!}{m!^N}$. (The claim here is the non-membership relation; the equality holds by the definition of $C(\cdot)$ in (8.1).) This would imply that $C(p) \notin p^2 \Xi_N N!\,\mathbb{Z}_p$, and thus, by Lemma 3 (recall the outlines of the proofs of Theorems 2 and 3 in Sections 3 and 8, respectively), that $q_{N,N}(z)^{1/p\Xi_N N!} \notin \mathbb{Z}[[z]]$, as desired.

To establish (10.2), we consider

$$\begin{aligned} &H_N(B_N(1) - B_N(p)) \\ &\quad = H_N N!\bigg(1 - \frac{1}{(p-1)!^N}\big(1\cdot 2\cdots(p-1)\big)\big((p+1)\cdot(p+2)\cdots(2p-1)\big)\cdots \\ &\qquad\qquad \times\cdots\big((pN-p+1)\cdot(pN-p+2)\cdots(pN-1)\big)\bigg). \end{aligned}$$

Using $v_p(H_N) = 3$ and Wilson's theorem, we deduce

$$H_N(B_N(1) - B_N(p)) \in p^4 N!\,\mathbb{Z}_p. \tag{10.3}$$

However, by Lemma 18 and the fact that $v_p\big(B_N(p)\big) = v_p\big(B_N(1)\big) = v_p(N!)$, we obtain

$$B_N(1)H_N - B_N(p)pH_{pN} \not\equiv H_N(B_N(1) - B_N(p)) \mod p^4 N!\,\mathbb{Z}_p.$$

Together with (10.3), this yields (10.2). $\square$

To summarise the discussion of this section: if the conjecture in Remarks 2(c) that no prime p and integer N exists with $v_p(H_N) \geq 4$ should be true, then Theorem 3 with $k = 1$ is sharp, that is, the sequence $(t_N)_{N\geq 1}$ is given by $t_N = \Xi_N N!$.

11. Sketch of the proof of Theorem 4

In this section we discuss the proof of Theorem 4. Since it is completely analogous to the proof of Theorem 3 (see Section 8), we content ourselves to point out the differences. At the end of the section, we present analogues of Propositions 2 and 3, addressing the question of optimality of Theorem 4 with $k = 1$.

First of all, by (1.4), we have

$$\big(z^{-1}q_N(z)\big)^{1/kN} = \exp(\tilde{G}_N(z)/F_N(z)),$$

where $F_N(z)$ is the series from the Introduction and

$$\tilde{G}_N(z) := \sum_{m=1}^{\infty} \frac{(Nm)!^k}{m!^{kN}}\bigg(H_{Nm} - H_m\bigg)\, z^m.$$

As in Section 8, we must "upgrade" the proof of Theorem 2 in the special case that $\mathbf{N} = (N, N, \dots, N)$. Writing as before $B_{\mathbf{N}}(m) = \frac{(Nm)!^k}{m!^{kN}}$, we must consider the sum

$$\begin{aligned}\tilde{C}(a+Kp) &:= \sum_{j=0}^{K} B_{\mathbf{N}}(a+jp) B_{\mathbf{N}}(K-j)\big((H_{N(K-j)} - H_{K-j}) - p(H_{Na+Njp} - H_{a+jp})\big)\\ &= \sum_{j=0}^{K} B_{\mathbf{N}}(a+jp) B_{\mathbf{N}}(K-j)(H_{N(K-j)} - pH_{Na+Njp})\\ &\qquad\qquad - \sum_{j=0}^{K} B_{\mathbf{N}}(a+jp) B_{\mathbf{N}}(K-j)(H_{K-j} - pH_{a+jp}) \end{aligned} \tag{11.1}$$

and show that it is in $p\Omega_N N!^k\mathbb{Z}_p$ for all primes p, and for all non-negative integers K, a, and j with $0 \le a < p$. The special cases $K = a = 0$, respectively $K = 0$ and $a = 1$, are equally simple to be handled directly here. We leave their verification to the reader and assume $a + Kp \ge 2$ from now on.

Following the outline of the proof of Theorem 3 in Section 8, given non-negative integers m, K, a, and j with $0 \le a < p$ and $a + Kp \ge 2$, we should prove

$$\tilde{C}(a+Kp) \equiv \sum_{j=0}^{K} B_{\mathbf{N}}(a+jp) B_{\mathbf{N}}(K-j)\big((H_{N(K-j)} - H_{K-j}) - (H_{\lfloor Na/p\rfloor+Nj} - H_j)\big) \mod p^3 N!^k\mathbb{Z}_p, \tag{11.2}$$

respectively, if $v_p(H_N - 1) \ge 3$ and p a Wolstenholme prime or $N \equiv \pm 1 \mod p$,

$$\tilde{C}(a+Kp) \equiv \sum_{j=0}^{K} B_{\mathbf{N}}(a+jp) B_{\mathbf{N}}(K-j)\big((H_{N(K-j)} - H_{K-j}) - (H_{\lfloor Na/p\rfloor+Nj} - H_j)\big) \mod p^4 N!^k\mathbb{Z}_p, \tag{11.3}$$

and we need to prove

$$B_{\mathbf{N}}(a+pj)\left(H_{Nj+\lfloor Na/p\rfloor} - H_{Nj}\right) \in p\Omega_N N!^k\mathbb{Z}_p \tag{11.4}$$

and

$$B_{\mathbf{N}}(m)\big((H_{Nmp^s} - H_{mp^s}) - (H_{N\lfloor m/p\rfloor p^{s+1}} - H_{\lfloor m/p\rfloor p^{s+1}})\big) \in p^{-s}\Omega_N N!^k\mathbb{Z}_p\ . \tag{11.5}$$

As the second line of (11.1) shows, $\tilde{C}(a+Kp)$ is a difference of two sums $C(a+Kp)$ as in (3.1), one with $L = N$, the other with $L = 1$. Hence, all the arguments in Section 8 which are not based on the assumption that $v_p(H_N) > 0$ can be used with only little modification.

On the other hand, the assumption $v_p(H_N) > 0$ is not used at many places. First, we need substitutes for Lemmas 12–15.

Lemma 19. *For all positive integers $L \ge 2$, we have $v_2(H_L - 1) = -\lfloor \log_2 L\rfloor$.*

Lemma 20. *We have* $v_3(H_{66}-1) = v_3(H_{68}-1) = 1$. *For positive integers* $L \notin \{1, 66, 68\}$, *we have* $v_3(H_L - 1) \leq 0$.

Lemma 21. *We have* $v_5(H_3 - 1) = v_5(H_{21} - 1) = v_5(H_{23} - 1) = 1$. *For positive integers* $L \notin \{1, 3, 21, 23\}$, *we have* $v_5(H_L - 1) \leq 0$.

Lemma 22. *Let* p *be a prime, and let* L *be an integer with* $L \geq 2$ *and* $p \leq L$. *Then the following assertions hold true:*

(1) *If* $v_p(H_L - 1) > 0$ *then* $L \geq 4p$.
(2) *If* $v_p(H_L - 1) > 0$ *and* $p \neq 5$ *then* $L \geq 6p$.

These results can be proved in exactly the same way as Lemmas 12–15, respectively. In comparison to Lemma 15, the statement of Lemma 22 is in fact simpler, so that complications that arose in Section 8 (such as (8.3), for example) do not arise here.

Second, we need a substitute for Lemma 18.

Lemma 23. *For all primes* $p \geq 5$ *and positive integers* N, *we have*

$$p(H_{pN} - H_p) \equiv H_N - 1 \mod p^4\mathbb{Z}_p \tag{11.6}$$

if and only if p *is a Wolstenholme prime or* $N \equiv \pm 1 \bmod p$.

Proof. From the proof of Lemma 18, we know that

$$pH_{pN} - H_N \equiv pN^2 H_{p-1} \mod \mathbb{Z}/p^4\mathbb{Z}.$$

As a consequence, we obtain

$$p(H_{pN} - H_p) - (H_N - 1) \equiv p(N^2 - 1)H_{p-1} \mod \mathbb{Z}/p^4\mathbb{Z}.$$

The assertion of the lemma follows now immediately. $\square$

Finally, the computation for $m = 1$ in Subsection 8.3 must be replaced by

$$B_{\mathbf{N}}(1)(H_{Np^s} - H_{p^s}) = N!^k \left(\frac{1}{p^s}(H_N - 1) + \frac{1}{p^{s-1}} \sum_{\substack{\varepsilon=p+1 \\ p\nmid\varepsilon}}^{Np} \frac{1}{\varepsilon} + \frac{1}{p^{s-2}} \sum_{\substack{\varepsilon=p^2+1 \\ p\nmid\varepsilon}}^{Np^2} \frac{1}{\varepsilon} + \sum_{\substack{\varepsilon=p^s+1 \\ p^{s-2}\nmid\varepsilon}}^{Np^s} \frac{1}{\varepsilon} \right),$$

with the conclusion that

$$B_{\mathbf{N}}(1)(H_{Np^s} - H_{p^s}) \in p^{-s} N!^k \Omega_N \mathbb{Z}_p$$

being found in a completely analogous manner.

Altogether, this leads to a proof of Theorem 4.

12. The Dwork–Kontsevich sequence

In this section, we address the question of optimality of Theorem 4 when $k = 1$, that is, whether, given that $k = 1$, the largest integer u_N such that $\left(z^{-1}q_N(z)\right)^{\frac{1}{Nu_N}} \in \mathbb{Z}[[z]]$ is given by $\Omega_N N!$. Let us write $\tilde{q}_N(z)$ for $\left(z^{-1}q_N(z)\right)^{1/N}$ with k *being fixed to* 1. The first proposition shows that there cannot be any prime number p larger than N which divides

u_N. We omit the proof since it is completely analogous to the proof of Proposition 2 in Section 10.

Proposition 4. *Let p be a prime number and N a positive integer with $p > N \geq 2$. Then there exists a positive integer $a < p$ such that*

$$v_p\big(B_N(a)(H_{Na} - H_a)\big) = 0. \tag{12.1}$$

In particular, p does not divide u_N.

So, if we hope to improve Theorem 4 with $k = 1$, then it must be by increasing exponents of prime numbers $p \leq N$ in (1.10). According to Remarks 3(b) in the Introduction, an improvement is therefore only possible if $v_p(H_N - 1) > 2$ for some $p \leq N$.

The final result of this section shows that, if $v_p(H_N - 1) = 3$ (for which, however, so far no examples are known; see Remarks 3(b)), the exponent of p in the definition of Ω_N in (1.10) cannot be increased so that Theorem 4 with $k = 1$ would still hold.

Proposition 5. *Let p be a prime number and N a positive integer with $p \leq N$ and $v_p(H_N) = 3$. If p is not a Wolstenholme prime and $N \not\equiv \pm 1 \bmod p$, then $\tilde{q}_N(z)^{\frac{1}{p\Omega_N N!}} \notin \mathbb{Z}[[z]]$.*

Again, the proof is completely analogous to the proof of Proposition 3 in Section 10, which we therefore omit.

So, if the conjecture in Remarks 3(c) that no prime p and integer N exists with $v_p(H_N - 1) \geq 4$ should be true, then Theorem 4 with $k = 1$ is optimal, that is, the Dwork–Kontsevich sequence $(u_N)_{N\geq 1}$ is given by $u_N = \Omega_N N!$.

13. Outline of the proof of Theorem 5

In this section, we provide a brief outline of the proof of Theorem 5, reducing it to Lemmas 24–26. These lemmas are subsequently proved in Sections 15 and 16, with two auxiliary lemmas being the subject of the subsequent section.

We follow the strategy that we used in Section 3 to prove Theorem 2; that is, by the consequence of Dwork's Lemma given in Lemma 3, we want to prove that

$$\mathbf{F_N}(z)\mathbf{G}_{L,\mathbf{N}}(z^p) - p\mathbf{F_N}(z^p)\mathbf{G}_{L,\mathbf{N}}(z) \in pz\mathbb{Z}_p[[z]].$$

The $(a + Kp)$-th Taylor coefficient of $\mathbf{F_N}(z)\mathbf{G}_{L,\mathbf{N}}(z^p) - p\mathbf{F_N}(z^p)\mathbf{G}_{L,\mathbf{N}}(z)$ is

$$\mathbf{C}(a + Kp) := \sum_{j=0}^{K} \mathbf{B_N}(a + jp)\mathbf{B_N}(K - j)(H_{L(K-j)} - pH_{La+Ljp}),$$

where $\mathbf{B_N}(m) := \prod_{j=1}^{k} \mathbf{B}_{N_j}(m)$, the quantities $\mathbf{B}_{N_j}(m)$ being defined in (1.15). In view of Lemma 3, proving Theorem 5 is equivalent to proving that

$$\mathbf{C}(a + Kp) \in p\mathbb{Z}_p$$

for all primes p and non-negative integers a and K with $0 \le a < p$. Again using (3.3) with $J = La + Ljp$, we have

$$\mathbf{C}(a+Kp) \equiv \sum_{j=0}^{K} \mathbf{B_N}(a+jp)\mathbf{B_N}(K-j)(H_{L(K-j)} - H_{\lfloor La/p \rfloor + Lj}) \mod p\mathbb{Z}_p.$$

The analogue of Lemma 5 in the present context, which allows us to get rid of the floor function $\lfloor La/p \rfloor$ and rearrange the sum over j, is the following lemma. The proof can be found in Section 15.

Lemma 24. *For any prime p, non-negative integers a and j with $0 \le a < p$, positive integers $N_1, N_2, \dots, N_k$, and $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, we have*

$$\mathbf{B_N}(a+pj)\big(H_{Lj+\lfloor La/p \rfloor} - H_{Lj}\big) \in p\mathbb{Z}_p.$$

We now do the same rearrangements as those after Lemma 5 to conclude that

$$\mathbf{C}(a+Kp) \equiv -\sum_{s=0}^{r} \sum_{m=0}^{p^{r+1-s}-1} \mathbf{Y}_{m,s} \mod p\mathbb{Z}_p,$$

where r is such that $K < p^r$, and

$$\mathbf{Y}_{m,s} := \big(H_{Lmp^s} - H_{L\lfloor m/p \rfloor p^{s+1}}\big)\mathbf{S}(a, K, s, p, m),$$

the expression $\mathbf{S}(a, K, s, p, m)$ being defined by

$$\mathbf{S}(a, K, s, p, m) := \sum_{j=mp^s}^{(m+1)p^s-1} \big(\mathbf{B_N}(a+jp)\mathbf{B_N}(K-j) - \mathbf{B_N}(j)\mathbf{B_N}(a+(K-j)p)\big).$$

In this expression for $\mathbf{S}(a, K, s, p, m)$, it is assumed that $\mathbf{B_N}(n) = 0$ for negative integers n. If we prove that

$$\mathbf{Y}_{m,s} \in p\mathbb{Z}_p \tag{13.1}$$

for all m and s, then $\mathbf{C}(a+Kp) \in p\mathbb{Z}_p$, as desired.

Now, the last assertion follows from the following two lemmas, with the proof of the first given in Section 16, while the proof of the second is easily accomplished by a trivial adaptation of the proof of Lemma 7 given in Section 7, where we change all occurrences of $B_{\mathbf{N}}$ by $\mathbf{B_N}$ and apply Lemma 24 instead of Lemma 5. The use of Lemma 4 can be dropped without substitute. ([9]) Lemma 25 is the analogue of Lemma 6 in the present context, while Lemma 26 is the analogue of Lemma 7.

Lemma 25. *For all primes p and non-negative integers a, m, s, K with $0 \le a < p$, we have*

$$\mathbf{S}(a, K, s, p, m) \in p^{s+1}\mathbf{B_N}(m)\mathbb{Z}_p.$$

[9] To prove the refinement announced at the end of the Introduction that $\mathbf{q}_{1,\mathbf{N}}(z)^{1/\mathbf{B_N}(1)} \in \mathbb{Z}[[z]]$, the use of Lemma 4 must be everywhere replaced by the use of Lemma 30, the latter being proved in Section 14.

Lemma 26. *For all primes p, non-negative integers m, positive integers $N_1, N_2, \dots, N_k$, and $L \in \{1, 2, \dots, \max(N_1, \dots, N_k)\}$, we have*

$$\mathbf{B}_{\mathbf{N}}(m) \left(H_{Lmp^s} - H_{L\lfloor m/p \rfloor p^{s+1}} \right) \in \frac{1}{p^s} \mathbb{Z}_p.$$

It is clear that Lemmas 25 and 26 imply (13.1). This completes the outline of the proof of Theorem 5.

14. Further auxiliary lemmas

In this section, we establish four auxiliary results. The first one, Lemma 27, provides the observation (1.17) that reduces Zudilin's Conjecture 2 to Theorem 5. The second one, Lemma 28, is required for the proof of Lemma 24 in Section 15, while the third one, Lemma 29, is required for the proof of Lemma 25 in Section 16. The last result, Lemma 30, justifies an assertion made in the Introduction. Moreover, the proofs of Lemmas 29 and 30 make themselves use of Lemma 28.

Lemma 27. *Let m be a non-negative integer, and let N be a positive integer with associated parameters $\alpha_i, \beta_i, \mu, \eta$ (that is, given by (1.13) and (1.14), respectively). Then*

$$\mathbf{H}_N(m) = \sum_{j=1}^{\mu} \alpha_j H_{\alpha_j m} - \sum_{j=1}^{\eta} \beta_j H_{\beta_j m}.$$

Proof. For $N = 1$, we have $\mathbf{H}_1(m) = 0$, so that the assertion of the lemma holds trivially. Therefore, from now on, we assume $N \geq 2$.

We claim that, for any real number $m \geq 0$, we have

$$\frac{C_N^m}{\Gamma(m+1)^{\varphi(N)}} \prod_{j=1}^{\varphi(N)} \frac{\Gamma(m + r_j/N)}{\Gamma(r_j/N)} = \frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j m + 1)}{\prod_{j=1}^{\eta} \Gamma(\beta_j m + 1)}, \tag{14.1}$$

where $\Gamma(x)$ denotes the gamma function. This generalises Zudilin's identity (1.15) to real values of m. We essentially extend his proof to real m, using the well-known formula [3, p. 23, Theorem 1.5.2]

$$\Gamma(a)\, \Gamma\left(a + \frac{1}{n}\right) \Gamma\left(a + \frac{2}{n}\right) \cdots \Gamma\left(a + \frac{n-1}{n}\right) = n^{\frac{1}{2} - an} (2\pi)^{(n-1)/2} \Gamma(an), \tag{14.2}$$

valid for real numbers a and positive integers n such that aN is not an integer ≤ 0. Indeed, as in the Introduction, let $p_1, p_2, \dots, p_\ell$ denote the distinct prime factors of N. (It should be noted that there is at least one such prime factor due to our assumption $N \geq 2$.) Furthermore, for a subset J of $\{1, 2, \dots, \ell\}$, let p_J denote the product $\prod_{j \in J} p_j$ of corresponding prime factors of N. (In the case that $J = \emptyset$, the empty product must be interpreted as 1.) Then, by the principle of inclusion-exclusion, we can rewrite the left-hand side of (14.1) in the form

$$\frac{C_N^m}{\Gamma(m+1)^{\varphi(N)}}\cdot\frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even}}}\prod_{i=1}^{N/p_J}\Gamma\left(m+\frac{ip_J}{N}\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd}}}\prod_{i=1}^{N/p_J}\Gamma\left(m+\frac{ip_J}{N}\right)}\cdot\frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd}}}\prod_{i=1}^{N/p_J}\Gamma\left(\frac{ip_J}{N}\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even}}}\prod_{i=1}^{N/p_J}\Gamma\left(\frac{ip_J}{N}\right)}.$$

To each of the products over i, formula (14.2) can be applied. As a result, we obtain the expression

$$\frac{C_N^m}{\Gamma(m+1)^{\varphi(N)}}\cdot\frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even}}}\left(\frac{N}{p_J}\right)^{-\left(m+\frac{p_J}{N}\right)\frac{N}{p_J}}\Gamma\left(m\frac{N}{p_J}+1\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd}}}\left(\frac{N}{p_J}\right)^{-\left(m+\frac{p_J}{N}\right)\frac{N}{p_J}}\Gamma\left(m\frac{N}{p_J}+1\right)}\cdot\frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd}}}\Gamma(1)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even}}}\Gamma(1)}$$

$$=\frac{C_N^m}{\Gamma(m+1)^{\varphi(N)}}\cdot\frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even}}}\left(\frac{N}{p_J}\right)^{-mN/p_J}\Gamma\left(m\frac{N}{p_J}+1\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd}}}\left(\frac{N}{p_J}\right)^{-mN/p_J}\Gamma\left(m\frac{N}{p_J}+1\right)},\qquad(14.3)$$

where the simplification in the exponent of N/p_J is due to the fact that there are as many subsets of even cardinality of a given non-empty set as there are subsets of odd cardinality. Since, again by inclusion-exclusion,

$$\sum_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even}}}\frac{N}{p_J}-\sum_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd}}}\frac{N}{p_J}=N\prod_{p\mid N}\left(1-\frac{1}{p}\right)=\varphi(N),\qquad(14.4)$$

we have

$$\frac{1}{\Gamma(m+1)^{\varphi(N)}}\cdot\frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even}}}\Gamma\left(m\frac{N}{p_J}+1\right)}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd}}}\Gamma\left(m\frac{N}{p_J}+1\right)}=\frac{\prod_{j=1}^{\mu}\Gamma(\alpha_j m+1)}{\prod_{j=1}^{\eta}\Gamma(\beta_j m+1)}$$

and

$$\frac{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even}}}N^{-mN/p_J}}{\prod_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd}}}N^{-mN/p_J}}=N^{-m\varphi(N)}.$$

Finally, consider a fixed prime number dividing N, p_j say. Then, using again (14.4), we see that the exponent of p_j in the expression (14.3) is

$$-\frac{m}{p_j}\sum_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ odd},\,j\notin J}}\frac{N}{p_J}+\frac{m}{p_j}\sum_{\substack{J\subseteq\{1,2,\dots,\ell\}\\|J|\text{ even},\,j\notin J}}\frac{N}{p_J}=\frac{mN}{p_j}\prod_{\substack{p\mid N\\p\neq p_j}}\left(1-\frac{1}{p}\right)=\frac{m}{p_j}\frac{\varphi(N)}{1-\frac{1}{p_j}}=\frac{m\varphi(N)}{p_j-1}.$$

If all these observations are used in (14.3), we arrive at the right-hand side of (14.1).

Now, let us call $b(m)$ the function defined by both sides of (14.1), and let $\psi(x) = \Gamma'(x)/\Gamma(x)$ be the digamma function. We will use the well-known property (see [3, p. 13, Theorem 1.2.7]) that $\psi(x+n)-\psi(x) = H(x,n)$ for real numbers $x > 0$ and integers $n \geq 0$.

By taking the logarithmic derivative of the right hand side of (14.1), we have

$$\begin{aligned}
\frac{b'(m)}{b(m)} &= \sum_{j=1}^{\mu} \alpha_j \psi(\alpha_j m + 1) - \sum_{j=1}^{\eta} \beta_j \psi(\beta_j m + 1) \\
&= \sum_{j=1}^{\mu} \alpha_j \big(\psi(1) + H_{\alpha_j m}\big) - \sum_{j=1}^{\eta} \beta_j \big(\psi(1) + H_{\beta_j m}\big) \\
&= \sum_{j=1}^{\mu} \alpha_j H_{\alpha_j m} - \sum_{j=1}^{\eta} \beta_j H_{\beta_j m}, \qquad (14.5)
\end{aligned}$$

because $\sum_{j=1}^{\mu} \alpha_j = \sum_{j=1}^{\eta} \beta_j$. It also follows that $b'(0)/b(0) = 0$.

On the other hand, by taking the logarithmic derivative of the left hand side of (14.1), we also have

$$\frac{b'(m)}{b(m)} = \log(C_N) + \sum_{j=1}^{\varphi(N)} \psi(m + r_j/N) - \varphi(N)\psi(m+1).$$

Since $b'(0)/b(0) = 0$, we have $\log(C_N) = -\sum_{j=1}^{\varphi(N)} \psi(r_j/N) + \varphi(N)\psi(1)$ and therefore,

$$\begin{aligned}
\frac{b'(m)}{b(m)} &= \sum_{j=1}^{\varphi(N)} \big(\psi(m + r_j/N) - \psi(r_j/N)\big) - \varphi(N)\big(\psi(m+1) - \psi(1)\big) \\
&= \sum_{j=1}^{\varphi(N)} H(r_j/N, m) - \varphi(N) H(1, m) \\
&= \mathbf{H}_N(m). \qquad (14.6)
\end{aligned}$$

The lemma follows by equating the expressions (14.5) and (14.6) obtained for $b'(m)/b(m)$. □

Lemma 28. *For any integer $N \geq 1$ with associated parameters $\alpha_i, \beta_i, \mu, \eta$, the function*

$$\Delta(x) := \sum_{i=1}^{\mu} \lfloor \alpha_i x \rfloor - \sum_{i=1}^{\eta} \lfloor \beta_i x \rfloor$$

has the following properties:

(*i*) *Δ is 1-periodic. In particular, $\Delta(n) = 0$ for all integers n.*
(*ii*) *For all integers n, Δ is weakly increasing on intervals of the form $[n, n+1)$.*
(*iii*) *For all real numbers x, we have $\Delta(x) \geq 0$.*
(*iv*) *For all rational numbers $r \neq 0$ whose denominator is an element of $\{1, 2, \ldots, N\}$, we have $\Delta(r) \geq 1$.*

Remark 5. Clearly, the function Δ is a step function. The proof below shows that, in fact, all the jumps of Δ at non-integral places have the value $+1$ and occur exactly at rational numbers of the form r/N, with r coprime to N.

Proof. Property (i) follows from the equality $\sum_{i=1}^{\mu} \alpha_i = \sum_{i=1}^{\eta} \beta_i$ and the trivial fact that $\Delta(0) = 0$.

We turn our attention to property (ii). For convenience of notation, let

$$N = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$$

be the prime factorisation of N, where, as before, $p_1, p_2, \dots, p_\ell$ are the distinct prime factors of N, and where $e_1, e_2, \dots, e_\ell$ are positive integers.

As we already observed in the remark above, the function Δ is a step function. Moreover, jumps of Δ can only occur at values of x where some of the $\alpha_i x$, $1 \le i \le \mu$, or some of the $\beta_j x$, $1 \le j \le \eta$, (or both) are integers. At these values of x, the value of a (possible) jump is the difference between the number of i's for which $\alpha_i x$ is integral and the number of j's for which $\beta_j x$ is integral. In symbols, the value of the jump is

$$\#\{i : 1 \le i \le \mu \text{ and } \alpha_i x \in \mathbb{Z}\} - \#\{j : 1 \le j \le \eta \text{ and } \beta_j x \in \mathbb{Z}\}. \tag{14.7}$$

Let X be the place of a jump, X not being an integer. Then we can write X as

$$X = \frac{Z}{p_1^{f_1} p_2^{f_2} \cdots p_\ell^{f_\ell}},$$

where $f_1, f_2, \dots, f_\ell$ are non-negative integers, not all zero, and where Z is a non-zero integer relatively prime to $p_1^{f_1} p_2^{f_2} \cdots p_\ell^{f_\ell}$. Given

$$\alpha_i = p_1^{a_1} p_2^{a_2} \cdots p_\ell^{a_\ell}$$

with $e_1 + e_2 + \cdots + e_\ell - (a_1 + a_2 + \cdots + a_\ell)$ even and $0 \le e_k - a_k \le 1$ for each $k = 1, 2, \dots, \ell$, the number $\alpha_i X$ will be integral if and only if $a_k \ge f_k$ for all $k \in \{1, 2, \dots, \ell\}$. Similarly, given

$$\beta_j = p_1^{b_1} p_2^{b_2} \cdots p_\ell^{b_\ell}$$

with $e_1 + e_2 + \cdots + e_\ell - (b_1 + b_2 + \cdots + b_\ell)$ odd and $0 \le e_k - b_k \le 1$ for each $k = 1, 2, \dots, \ell$, the number $\beta_j X$ will be integral if and only if $b_k \ge f_k$ for all $k \in \{1, 2, \dots, \ell\}$. We do not have to take into account the β_j's which are 1, because $1 \cdot X = X$ is not an integer by assumption. For the generating function of vectors $(c_1, c_2, \dots, c_\ell)$ with $e_k \ge c_k \ge f_k$ and $e_k - c_k \le 1$, we have

$$\sum_{c_1=\max\{e_1-1,f_1\}}^{e_1} \cdots \sum_{c_\ell=\max\{e_\ell-1,f_\ell\}}^{e_\ell} z^{e_1+\cdots+e_\ell-(c_1+\cdots+c_\ell)} = \prod_{k=1}^{\ell} \left(1 + z \cdot \min\{1, e_k - f_k\}\right).$$

We obtain the difference in (14.7) (with X in place of x) by putting $z = -1$ on the left-hand side of this relation. The product on the right-hand side tells us that this difference is 0 in case that $e_k \ne f_k$ for some k, while it is 1 otherwise. Thus, all the jumps of the function Δ at non-integral places have the value $+1$.

Property (iii) follows now easily from (i) and (ii).

In order to prove (*iv*), we observe that the first jump of Δ in the interval $[0,1)$ occurs at $x = 1/N$. Thus, $\Delta(x) \geq 1$ for all x in $[1/N, 1)$. This implies in particular that $\Delta(r) \geq 1$ for all the above rational numbers r in the interval $[1/N, 1)$. That the same assertion holds in fact for *all* the above rational numbers r (not necessarily restricted to $[1/N, 1)$) follows now from the 1-periodicity of the function Δ. □

Lemma 29. *For any integers $m, r, w \geq 0$ such that $0 \leq w < p^r$, we have*

$$\frac{\mathbf{B_N}(w + mp^r)}{\mathbf{B_N}(m)} \in \mathbb{Z}_p. \tag{14.8}$$

Proof. We first show that we can assume that m is coprime to p. Indeed, let us write $m = hp^t$ with $\gcd(h, p) = 1$. We have to prove that

$$\frac{\mathbf{B_N}(w + hp^{r+t})}{\mathbf{B_N}(hp^t)} \in \mathbb{Z}_p.$$

Since $v_p\big(\mathbf{B_N}(hp^t)/\mathbf{B_N}(h)\big) = 0$ (as can be easily seen from (1.15) and Legendre's formula (4.1)), this amounts to prove that

$$\frac{\mathbf{B_N}(w + hp^{r+t})}{\mathbf{B_N}(h)} \in \mathbb{Z}_p,$$

which is the content of the lemma with $r + t$ instead of r and h instead of m, with $w < p^r < p^{r+t}$.

Therefore, from now on, we assume that $\gcd(m, p) = 1$ (however, this assumption will only be used after (14.11)). Since $v_p\big(\mathbf{B_N}(mp^r)/\mathbf{B_N}(m)\big) = 0$, we have to prove that

$$v_p\left(\frac{\mathbf{B_N}(w + mp^r)}{\mathbf{B_N}(mp^r)}\right) \geq 0$$

or, in an equivalent form, that

$$\sum_{j=1}^{k}\sum_{\ell=1}^{\infty}\Bigg(\bigg(\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}(w+mp^r)}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}(w+mp^r)}{p^\ell}\right\rfloor\bigg) \\ - \bigg(\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}mp^r}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}mp^r}{p^\ell}\right\rfloor\bigg)\Bigg) \geq 0, \tag{14.9}$$

where $\alpha_{i,j}, \beta_{i,j}, \mu_j, \eta_j$ are the parameters associated to N_j.

If $\ell \leq r$, then for any $j \in \{1, 2, \ldots, k\}$,

$$\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}mp^r}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}mp^r}{p^\ell}\right\rfloor = mp^{r-\ell}\left(\sum_{i=1}^{\mu_j}\alpha_{i,j} - \sum_{i=1}^{\eta_j}\beta_{i,j}\right) = 0.$$

Moreover,

$$\sum_{i=1}^{\mu_j}\left\lfloor\frac{\alpha_{i,j}(w+mp^r)}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_j}\left\lfloor\frac{\beta_{i,j}(w+mp^r)}{p^\ell}\right\rfloor \geq 0$$

because of Lemma 28(iii) with $N = N_j$. It therefore suffices to show

$$\sum_{j=1}^{k} \sum_{\ell=r+1}^{\infty} \left(\left(\sum_{i=1}^{\mu_j} \left\lfloor \frac{\alpha_{i,j}(w+mp^r)}{p^\ell} \right\rfloor - \sum_{i=1}^{\eta_j} \left\lfloor \frac{\beta_{i,j}(w+mp^r)}{p^\ell} \right\rfloor \right) \right.$$
$$\left. - \left(\sum_{i=1}^{\mu_j} \left\lfloor \frac{\alpha_{i,j}mp^r}{p^\ell} \right\rfloor - \sum_{i=1}^{\eta_j} \left\lfloor \frac{\beta_{i,j}mp^r}{p^\ell} \right\rfloor \right) \right) \geq 0, \quad (14.10)$$

(The reader should note the difference to (14.9) occurring in the summation bounds for ℓ.) For $\ell > r$, set $x_\ell = \{mp^r/p^\ell\}$ and $y_\ell = \{(w+mp^r)/p^\ell\}$. Using again $\sum_{i=1}^{\mu_j} \alpha_{i,j} - \sum_{i=1}^{\eta_j} \beta_{i,j} = 0$, we see that the left-hand side of (14.10) is equal to

$$\sum_{j=1}^{k} \sum_{\ell=r+1}^{\infty} \left(\left(\sum_{i=1}^{\mu_j} \lfloor \alpha_{i,j} y_\ell \rfloor - \sum_{i=1}^{\eta_j} \lfloor \beta_{i,j} y_\ell \rfloor \right) - \left(\sum_{i=1}^{\mu_j} \lfloor \alpha_{i,j} x_\ell \rfloor - \sum_{i=1}^{\eta_j} \lfloor \beta_{i,j} x_\ell \rfloor \right) \right). \quad (14.11)$$

We now claim that $x_\ell \leq y_\ell$ for $\ell > r$. To see this, we begin by the observation that, since m and p are coprime and $\ell > r$, the rational number $m/p^{\ell-r}$ is not an integer. It follows that

$$x_\ell + \frac{1}{p^{\ell-r}} = \left\{ \frac{m}{p^{\ell-r}} \right\} + \frac{1}{p^{\ell-r}} \leq 1.$$

Hence, since $w < p^r$, we infer that

$$x_\ell + \frac{w}{p^\ell} < 1.$$

On the other hand, we have

$$y_\ell = \left\{ \frac{w}{p^\ell} + \left\lfloor \frac{m}{p^{\ell-r}} \right\rfloor + x_\ell \right\} = \left\{ \frac{w}{p^\ell} + x_\ell \right\} = \frac{w}{p^\ell} + x_\ell.$$

Since $w \geq 0$, we obtain indeed $y_\ell \geq x_\ell$, as we claimed.

Using $x_\ell \leq y_\ell$ together with Lemma 28(ii), we see that, for $\ell > r$ and $j = 1, 2, \dots k$, we have

$$\sum_{i=1}^{\mu_j} \lfloor \alpha_{i,j} y_\ell \rfloor - \sum_{i=1}^{\eta_j} \lfloor \beta_{i,j} y_\ell \rfloor \geq \sum_{i=1}^{\mu_j} \lfloor \alpha_{i,j} x_\ell \rfloor - \sum_{i=1}^{\eta_j} \lfloor \beta_{i,j} x_\ell \rfloor,$$

which shows that the expression in (14.11) is non-negative, thus establishing (14.10) and also (14.9). This finishes the proof of the lemma. $\square$

We conclude this section with a result which was announced near the end of the Introduction. It is used nowhere else, but we give it for the sake of completeness. It is a generalisation of Lemma 4. By the same techniques used to prove Theorem 2, it enables one to prove that $\mathbf{q}_{1,\mathbf{N}}(z)^{1/\mathbf{B}_\mathbf{N}(1)} \in \mathbb{Z}[[z]]$ (see Footnote 9).

Lemma 30. *For any vector* $\mathbf{N}$ *and any integer* $m \geq 1$*, we have that* $\mathbf{B}_\mathbf{N}(1)$ *divides* $\mathbf{B}_\mathbf{N}(m)$*.*

Proof. Obviously, it is sufficient to prove the assertion for $k=1$ and $\mathbf{N}=(N)$. Let Δ be the function associated to N as defined in Lemma 28. We want to prove that, for any prime p, we have $v_p(\mathbf{B}_N(m)) \geq v_p(\mathbf{B}_N(1))$. We can assume that m and p are coprime because $v_p(\mathbf{B}_N(mp^t)) = v_p(\mathbf{B}_N(m))$ for any integers $m, t \geq 0$.

Now, when $\gcd(m,p)=1$, we have that

$$\begin{aligned} v_p(\mathbf{B}_N(m)) &= \sum_{\ell=1}^{\infty} \Delta(m/p^\ell) = \sum_{\ell=1}^{\infty} \Delta(\{m/p^\ell\}) \\ &\geq \sum_{\ell=1}^{\infty} \Delta(1/p^\ell) = v_p(\mathbf{B}_N(1)). \end{aligned}$$

Here, we used the 1-periodicity of Δ for the second equality. For the inequality, we used that $\{m/p^\ell\} \geq 1/p^\ell$ (because $\gcd(m,p)=1$ implies that m/p^ℓ is not an integer) and the the (partial) monotonicity of Δ described in Lemma 28(ii). □

15. Proof of Lemma 24

We follow the first part of the proof of Lemma 5. We write $\alpha_{i,m}$, $\beta_{i,m}$, μ_m, and η_m for the parameters associated to N_m, $m=1,2,\ldots,k$. We may assume that $\max(N_1,\ldots,N_k) = N_k$. Then, using again Lemma 28(iii),

$$\begin{aligned} v_p\big(\mathbf{B}_{\mathbf{N}}(a+pj)\big) &= \sum_{m=1}^{k}\sum_{\ell=1}^{\infty}\left(\sum_{i=1}^{\mu_m}\left\lfloor\frac{\alpha_{i,m}(a+pj)}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_m}\left\lfloor\frac{\beta_{i,m}(a+pj)}{p^\ell}\right\rfloor\right) \\ &\geq \sum_{\ell=1}^{\infty}\left(\sum_{i=1}^{\mu_k}\left\lfloor\frac{\alpha_{i,k}(a+pj)}{p^\ell}\right\rfloor - \sum_{i=1}^{\eta_k}\left\lfloor\frac{\beta_{i,k}(a+pj)}{p^\ell}\right\rfloor\right) = \sum_{\ell=1}^{\infty}\Delta_k\left(\frac{a+jp}{p^\ell}\right) \end{aligned}$$

with

$$\Delta_k(x) := \sum_{i=1}^{\mu_k}\lfloor\alpha_{i,k}x\rfloor - \sum_{i=1}^{\eta_k}\lfloor\beta_{i,k}x\rfloor.$$

We want to prove that

$$v_p\big(\mathbf{B}_{\mathbf{N}}(a+pj)\big) \geq 1 + v_p(Lj+\varepsilon) \tag{15.1}$$

for any integer ε such that $1 \leq \varepsilon \leq \lfloor La/p \rfloor$. We have

$$\frac{a+jp}{p^\ell} = \frac{a-p\varepsilon/L}{p^\ell} + \frac{pj+p\varepsilon/L}{p^\ell}.$$

15.1. **First step.** We claim that

$$\Delta_k\left(\frac{a+jp}{p^\ell}\right) \geq \Delta_k\left(\frac{pj+p\varepsilon/L}{p^\ell}\right). \tag{15.2}$$

To see this, we first observe that

$$\Delta_k\left(\frac{a+jp}{p^\ell}\right) = \Delta_k\left(\frac{a-p\varepsilon/L}{p^\ell} + \frac{pj+p\varepsilon/L}{p^\ell}\right) = \Delta_k\left(\frac{a-p\varepsilon/L}{p^\ell} + \left\{\frac{pj+p\varepsilon/L}{p^\ell}\right\}\right)$$

because Δ_k is 1-periodic.

We now claim that

$$0 \leq \frac{a - p\varepsilon/L}{p^\ell} + \left\{\frac{pj + p\varepsilon/L}{p^\ell}\right\} < 1. \tag{15.3}$$

Indeed, positivity is clear and we now concentrate on the upper bound. We write $j = up^{\ell-1} + v$ with $0 \leq v < p^{\ell-1}$. Hence,

$$\left\{\frac{pj + p\varepsilon/L}{p^\ell}\right\} = \left\{u + \frac{pv + p\varepsilon/L}{p^\ell}\right\} = \left\{\frac{v}{p^{\ell-1}} + \frac{p\varepsilon/L}{p^\ell}\right\}.$$

Since $0 \leq \varepsilon \leq \lfloor La/p \rfloor < L$, we have $0 \leq \frac{p\varepsilon/L}{p^\ell} < 1/p^{\ell-1}$ and therefore

$$0 \leq \frac{v}{p^{\ell-1}} + \frac{p\varepsilon/L}{p^\ell} < \frac{v}{p^{\ell-1}} + \frac{1}{p^{\ell-1}} \leq 1$$

(where the last inequality holds by definition of v), whence

$$\left\{\frac{pj + p\varepsilon/L}{p^\ell}\right\} = \frac{pv + p\varepsilon/L}{p^\ell}.$$

Therefore, we have

$$\frac{a - p\varepsilon/L}{p^\ell} + \left\{\frac{pj + p\varepsilon/L}{p^\ell}\right\} = \frac{a - p\varepsilon/L}{p^\ell} + \frac{pv + p\varepsilon/L}{p^\ell} = \frac{a}{p^\ell} + \frac{v}{p^{\ell-1}}.$$

Since $\frac{v}{p^{\ell-1}} < 1$ and $a < p$, we necessarily have

$$\frac{a}{p^\ell} + \frac{v}{p^{\ell-1}} < 1,$$

as desired.

Since $\frac{a-p\varepsilon/L}{p^\ell} \geq 0$, it follows from Lemma 28(i),(ii) (with $\Delta = \Delta_k$) and (15.3) that

$$\Delta_k\left(\frac{a - p\varepsilon/L}{p^\ell} + \left\{\frac{pj + p\varepsilon/L}{p^\ell}\right\}\right) \geq \Delta_k\left(\left\{\frac{pj + p\varepsilon/L}{p^\ell}\right\}\right) = \Delta_k\left(\frac{pj + p\varepsilon/L}{p^\ell}\right).$$

Thus, we have proved the claim (15.2) made at the beginning of this step.

15.2. **Second step.** Let us write $Lj + \varepsilon = \beta p^d$, where $d = v_p(Lj + \varepsilon)$, so that

$$\frac{pj + p\varepsilon/L}{p^\ell} = \frac{\beta p^{d+1-\ell}}{L}.$$

We have proved in the first step that

$$v_p\big(\mathbf{B}_{\mathbf{N}}(a + pj)\big) \geq \sum_{\ell=1}^{\infty} \Delta_k\left(\frac{\beta p^{d+1-\ell}}{L}\right). \tag{15.4}$$

By the same argument as the one that we used in the first part of the proof of Lemma 5 in Section 5, the rational number $\frac{\beta p^{d+1-\ell}}{L}$ is not an integer and for $\ell \leq d+1$, the denominator of $\frac{\beta p^{d+1-\ell}}{L}$ is at most L. Since $L \leq N_k$, it follows then from Lemma 28(iv), again with

$\Delta = \Delta_k$, that $\Delta_k(\beta p^{d+1-\ell}/L) \ge 1$ for any ℓ in $\{1, 2, \dots, d+1\}$. Use of this estimation in (15.4) gives

$$v_p\big(\mathbf{B_N}(a+pj)\big) \ge d+1 = 1 + v_p(Lj+\varepsilon).$$

This completes the proof of (15.1) and, hence, of Lemma 24.

16. Proof of Lemma 25

Again, we want to use Proposition 1, this time with $A_r(m) = g_r(m) = \mathbf{B_N}(m)$. Clearly, the proposition would imply that $\mathbf{S}(a, K, s, p, m) \in p^{s+1}\mathbf{B_N}(m)\mathbb{Z}_p$, and, thus, the claim. So, we need to verify the conditions (i)–(iii) in the statement of the proposition.

Conditions (i) and (ii) of Dwork's congruences theorem clearly hold, and we must check Condition (iii). To do the latter, we follow the method developed to prove Lemma 6 (see Section 6). The reader should recall that

$$\mathbf{B_N}(m) := \prod_{j=1}^{k} \mathbf{B}_{N_j}(m),$$

where $\mathbf{B}_{N_j}(m)$ is given by (1.12) or (1.15).

16.1. **First step.** Let us fix $j \in \{1, 2, \dots, k\}$ and set $D_N := N^{-\varphi(N)}C_N$, which is an integer. We claim that

$$\frac{\mathbf{B}_{N_j}(v+up+np^{s+1})}{\mathbf{B}_{N_j}(up+np^{s+1})} = \frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)} + \mathcal{O}(p^{s+1}). \tag{16.1}$$

To prove (16.1), we can use the same arguments as in the first step of the proof of Lemma 6, due to the identities ([10])

$$\frac{\mathbf{B}_{N_j}(v+up+np^{s+1})}{\mathbf{B}_{N_j}(up+np^{s+1})} = \frac{D_{N_j}^{v}\prod_{\ell=1}^{\varphi(N_j)}\prod_{i=1}^{v}\big(r_\ell+(i-1)N_j+uN_jp+nN_jp^{s+1}\big)}{\big((v+up+np^{s+1})(v-1+up+np^{s+1})\cdots(1+up+np^{s+1})\big)^{\varphi(N_j)}},$$

$$\frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)} = \frac{D_{N_j}^{v}\prod_{\ell=1}^{\varphi(N_j)}\prod_{i=1}^{v}\big(r_\ell+(i-1)N_j+uN_jp\big)}{\big((v+up)(v-1+up)\cdots(1+up)\big)^{\varphi(N_j)}}, \tag{16.2}$$

and to the fact that $(v+up)(v-1+up)\cdots(1+up)$ is not divisible by p.

A side result of (16.2) (which was actually used to prove (16.1)) is that

$$\frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)} \in \mathbb{Z}_p.$$

[10] They are immediate consequences of the definition (1.12) of $\mathbf{B}_N$. Zudilin used them in his proof of the following stronger version of (16.1):

$$\frac{\mathbf{B}_{N_j}(v+up+np^{s+1})}{\mathbf{B}_{N_j}(up+np^{s+1})} = \frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)}\big(1+\mathcal{O}(p^{s+1})\big).$$

However, for this, he assumes that p divides N_j (see [30, Eq. (35)]). Here, we do not assume that p divides N_j, and therefore we obtain the weaker equality (16.1), which is fortunately enough for our purposes.

We deduce from this fact and from (16.1) that

$$\prod_{j=1}^{k}\frac{\mathbf{B}_{N_j}(v+up+np^{s+1})}{\mathbf{B}_{N_j}(up+np^{s+1})}=\prod_{j=1}^{k}\left(\frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)}+\mathcal{O}(p^{s+1})\right)=\prod_{j=1}^{k}\frac{\mathbf{B}_{N_j}(v+up)}{\mathbf{B}_{N_j}(up)}+\mathcal{O}(p^{s+1}),$$

or, in other words,

$$\frac{\mathbf{B}_{\mathbf{N}}(v+up+np^{s+1})}{\mathbf{B}_{\mathbf{N}}(up+np^{s+1})}=\frac{\mathbf{B}_{\mathbf{N}}(v+up)}{\mathbf{B}_{\mathbf{N}}(up)}+\mathcal{O}(p^{s+1}). \tag{16.3}$$

16.2. **Second step.** Let us fix $j\in\{1,2,\dots,k\}$. Exactly as in the proof of the second step of Lemma 6, the properties of Γ_p imply that

$$\begin{aligned}\frac{\mathbf{B}_{N_j}(up+np^{s+1})}{\mathbf{B}_{N_j}(u+np^{s})}&=(-1)^{\mu_j-\eta_j}\frac{\prod_{i=1}^{\mu_j}\Gamma\big(1+\alpha_{i,j}(up+np^{s+1})\big)}{\prod_{i=1}^{\eta_j}\Gamma\big(1+\beta_{i,j}(up+np^{s+1})\big)}\\&=(-1)^{\mu_j-\eta_j}\frac{\prod_{i=1}^{\mu_j}\Gamma(1+\alpha_{i,j}up)+\mathcal{O}(p^{s+1})}{\prod_{i=1}^{\eta_j}\Gamma(1+\beta_{i,j}up)+\mathcal{O}(p^{s+1})}\\&=(-1)^{\mu_j-\eta_j}\frac{\prod_{i=1}^{\mu_j}\Gamma(1+\alpha_{i,j}up)}{\prod_{i=1}^{\eta_j}\Gamma(1+\beta_{i,j}up)}\big(1+\mathcal{O}(p^{s+1})\big)\\&=\frac{\mathbf{B}_{N_j}(up)}{\mathbf{B}_{N_j}(u)}\big(1+\mathcal{O}(p^{s+1})\big).\end{aligned}$$

Here we used again the fact that $\Gamma_p(m)$ is never divisible by p for any integer m.

Hence, taking the product over $j=1,2,\dots,k$, we obtain

$$\frac{\mathbf{B}_{\mathbf{N}}(up+np^{s+1})}{\mathbf{B}_{\mathbf{N}}(u+np^{s})}=\frac{\mathbf{B}_{\mathbf{N}}(up)}{\mathbf{B}_{\mathbf{N}}(u)}\big(1+\mathcal{O}(p^{s+1})\big). \tag{16.4}$$

16.3. **Third step.** We follow verbatim the beginning of the third step in the proof of Lemma 6 (i.e., we multiply the left-hand and right-hand sides of (16.3) and (16.4), etc.) to obtain

$$\frac{\mathbf{B}_{\mathbf{N}}(v+up+np^{s+1})}{\mathbf{B}_{\mathbf{N}}(v+up)}-\frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(u)}=\frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(u)}\mathcal{O}(p^{s+1})+\frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(v+up)}\mathcal{O}(p^{s+1}).$$

It remains to check that

$$\frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(u)}\in\frac{\mathbf{B}_{\mathbf{N}}(n)}{\mathbf{B}_{\mathbf{N}}(v+up)}\mathbb{Z}_p\quad\text{and}\quad\frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(v+up)}\in\frac{\mathbf{B}_{\mathbf{N}}(n)}{\mathbf{B}_{\mathbf{N}}(v+up)}\mathbb{Z}_p. \tag{16.5}$$

The first assertion in (16.5) can be rewritten as

$$\frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(n)}\cdot\frac{\mathbf{B}_{\mathbf{N}}(v+up)}{\mathbf{B}_{\mathbf{N}}(u)}\in\mathbb{Z}_p, \tag{16.6}$$

while the second assertion can be rewritten as

$$\frac{\mathbf{B}_{\mathbf{N}}(u+np^{s})}{\mathbf{B}_{\mathbf{N}}(n)}\in\mathbb{Z}_p. \tag{16.7}$$

Now, the assertion (16.7) is the special case $w = u$, $m = n$ and $r = s$ of Lemma 29, while (16.6) follows from (16.7) combined with the special case $w = v$, $m = u$ and $r = 1$ of Lemma 29.

This completes the proof of the lemma.

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