CYCLIC SIEVING FOR GENERALISED NON-CROSSING PARTITIONS ASSOCIATED WITH COMPLEX REFLECTION GROUPS OF EXCEPTIONAL TYPE — THE DETAILS

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Dedicated to the memory of Herb Wilf

Abstract. We prove that the generalised non-crossing partitions associated with well-generated complex reflection groups of exceptional type obey two different cyclic sieving phenomena, as conjectured by Armstrong, and by Bessis and Reiner. This manuscript accompanies the paper *"Cyclic sieving for generalised non-crossing partitions associated with complex reflection groups of exceptional type"* [arχiv:1001.0028], for which it provides the computational details.

1. Introduction

In his memoir [3], Armstrong introduced *generalised non-crossing partitions* associated with finite (real) reflection groups, thereby embedding Kreweras' non-crossing partitions [22], Edelman's m-divisible non-crossing partitions [12], the non-crossing partitions associated with reflection groups due to Bessis [7] and Brady and Watt [11] into one uniform framework. Bessis and Reiner [10] observed that Armstrong's definition can be straightforwardly extended to *well-generated complex reflection groups* (see Section 2 for the precise definition). These generalised non-crossing partitions possess a wealth of beautiful properties, and they display deep and surprising relations to other combinatorial objects defined for reflection groups (such as the generalised cluster complex

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of Fomin and Reading [13], or the extended Shi arrangement and the geometric multichains of filters of Athanasiadis [5, 6]); see Armstrong's memoir [3] and the references given therein.

On the other hand, *cyclic sieving* is a phenomenon brought to light by Reiner, Stanton and White [29]. It extends the so-called "(-1)-phenomenon" of Stembridge [33, 34]. Cyclic sieving can be defined in three equivalent ways (cf. [29, Prop. 2.1]). The one which gives the name can be described as follows: given a set S of combinatorial objects, an action on S of a cyclic group $G = \langle g \rangle$ with generator g of order n, and a polynomial $P(q)$ in q with non-negative integer coefficients, we say that the triple (S, P, G) *exhibits the cyclic sieving phenomenon*, if the number of elements of S fixed by g^k equals $P(e^{2\pi ik/n})$. In [29] it is shown that this phenomenon occurs in surprisingly many contexts, and several further instances have been discovered since then.

In [3, Conj. 5.4.7] (also appearing in [10, Conj. 6.4]) and [10, Conj. 6.5], Armstrong, respectively Bessis and Reiner, conjecture that generalised non-crossing partitions for irreducible well-generated complex reflection groups exhibit two different cyclic sieving phenomena (see Sections 3 and 7 for the precise statements).

According to the classification of these groups due to Shephard and Todd [31], there are two infinite families of irreducible well-generated complex reflection groups, namely the groups $G(d, 1, n)$ and $G(e, e, n)$, where n, d, e are positive integers, and there are 26 exceptional groups. For the infinite families of types $G(d, 1, n)$ and $G(e, e, n)$, the two cyclic sieving conjectures follow from the results in [19].

The purpose of the present article is to prove the cyclic sieving conjectures of Armstrong, and of Bessis and Reiner, for the 26 exceptional types, thus completing the proof of these conjectures. Since the generalised non-crossing partitions feature a parameter m, from the outset this is *not* a finite problem. Consequently, we first need several auxiliary results to reduce the conjectures for each of the 26 exceptional types to a *finite* problem. Subsequently, we use Stembridge's *Maple* package `coxeter` [35] and the *GAP* package `CHEVIE` [14, 27] to carry out the remaining *finite* computations. It is interesting to observe that, for the verification of the type E_8 case, it is essential to use the decomposition numbers in the sense of [17, 18, 20] because, otherwise, the necessary computations would not be feasible in reasonable time with the currently available computer facilities. We point out that, for the special case where the aforementioned parameter m is equal to 1, the first cyclic sieving conjecture has been proved in a uniform fashion by Bessis and Reiner in [10]. (See [4] for a — non-uniform — proof of cyclic sieving for non-crossing partitions associated with *real* reflection groups under the action of the so-called Kreweras map, a special case of the second cyclic sieving phenomenon discussed in the present paper.) The crucial result on which the proof of Bessis and Reiner is based is (5.5) below, and it plays an important role in our reduction of the conjectures for the 26 exceptional groups to a finite problem.

Our paper is organised as follows. In the next section, we recall the definition of generalised non-crossing partitions for well-generated complex reflection groups and of decomposition numbers in the sense of [17, 18, 20], and we review some basic facts. The first cyclic sieving conjecture is subsequently stated in Section 3. In Section 4, we outline an elementary proof that the q-Fuß–Catalan number, which is the polynomial P in the cyclic sieving phenomena concerning the generalised non-crossing partitions for well-generated complex reflection groups, is always a polynomial with non-negative integer coefficients, as required by the definition of cyclic sieving. (The reader is referred to the first paragraph of Section 4 for comments on other approaches for establishing polynomiality with non-negative coefficients.) Section 5 contains the announced auxiliary results which, for the 26 exceptional types, allow a reduction of the conjecture to a finite problem. The remaining case-by-case verification of the conjecture is then carried out in Section 6. The second cyclic sieving conjecture is stated in Section 7. Section 8 contains the auxiliary results which, for the 26 exceptional types, allow a reduction of the conjecture to a finite problem, while Section 9 contains the remaining case-by-case verification of the conjecture.

2. Preliminaries

A *complex reflection group* is a group generated by (complex) reflections in $\mathbb{C}^n$. (Here, a reflection is a non-trivial element of $GL_n(\mathbb{C})$ which fixes a hyperplane pointwise and which has finite order.) We refer to [24] for an in-depth exposition of the theory complex reflection groups.

Shephard and Todd provided a complete classification of all *finite* complex reflection groups in [31] (see also [24, Ch. 8]). According to this classification, an arbitrary complex reflection group W decomposes into a direct product of *irreducible* complex reflection groups, acting on mutually orthogonal subspaces of the complex vector space on which W is acting. Moreover, the list of irreducible complex reflection groups consists of the infinite family of groups $G(m, p, n)$, where m, p, n are positive integers, and 34 exceptional groups, denoted $G_4, G_5, \dots, G_{37}$ by Shephard and Todd.

In this paper, we are only interested in finite complex reflection groups which are *well-generated*. A complex reflection group of rank n is called *well-generated* if it is generated by n reflections.[1] Well-generation can be equivalently characterised by a duality property due to Orlik and Solomon [28]. Namely, a complex reflection group of rank n has two sets of distinguished integers $d_1 \le d_2 \le \cdots \le d_n$ and $d_1^* \ge d_2^* \ge \cdots \ge d_n^*$, called its *degrees* and *codegrees*, respectively (see [24, p. 51 and Def. 10.27]). Orlik and Solomon observed, using case-by-case checking, that an irreducible complex reflection group W of rank n is well-generated if and only if its degrees and codegrees satisfy

$$d_i + d_i^* = d_n$$

for all $i = 1, 2, \dots, n$. The reader is referred to [24, App. D.2] for a table of the degrees and codegrees of all irreducible complex reflection groups. Together with the classification of Shephard and Todd [31], this constitutes a classification of well-generated complex reflection groups: the irreducible well-generated complex reflection groups are

— the two infinite families $G(d, 1, n)$ and $G(e, e, n)$, where d, e, n are positive integers,
— the exceptional groups $G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}$ of rank 2,
— the exceptional groups $G_{23} = H_3, G_{24}, G_{25}, G_{26}, G_{27}$ of rank 3,
— the exceptional groups $G_{28} = F_4, G_{29}, G_{30} = H_4, G_{32}$ of rank 4,
— the exceptional group G_{33} of rank 5,
— the exceptional groups $G_{34}, G_{35} = E_6$ of rank 6,
— the exceptional group $G_{36} = E_7$ of rank 7,

[1] We refer to [24, Def. 1.29] for the precise definition of "rank." Roughly speaking, the rank of a complex reflection group W is the minimal n such that W can be realized as reflection group on $\mathbb{C}^n$.

— and the exceptional group $G_{37} = E_8$ of rank 8.

In this list, we have made visible the groups $H_3, F_4, H_4, E_6, E_7, E_8$ which appear as exceptional groups in the classification of all irreducible *real* reflection groups (cf. [16]).

Let W be a well-generated complex reflection group of rank n, and let $T \subseteq W$ denote the set of *all* (complex) reflections in the group. Let $\ell_T : W \to \mathbb{Z}$ denote the word length in terms of the generators T. This word length is called *absolute length* or *reflection length*. Furthermore, we define a partial order $\le_T$ on W by

$$u \le_T w \quad \text{if and only if} \quad \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w). \tag{2.1}$$

This partial order is called *absolute order* or *reflection order*. As is well-known and easy to see, the equation in (2.1) is equivalent to the statement that every shortest representation of u by reflections occurs as an initial segment in some shortest product representation of w by reflections.

Now fix a (generalised) Coxeter element[2] $c \in W$ and a positive integer m. The *m-divisible non-crossing partitions $NC^m(W)$* are defined as the set

$$NC^m(W) = \big\{(w_0; w_1, \dots, w_m) : w_0 w_1 \cdots w_m = c \text{ and} \\ \ell_T(w_0) + \ell_T(w_1) + \dots + \ell_T(w_m) = \ell_T(c)\big\}.$$

A partial order is defined on this set by

$$(w_0; w_1, \dots, w_m) \le (u_0; u_1, \dots, u_m) \quad \text{if and only if} \quad u_i \le_T w_i \text{ for } 1 \le i \le m.$$

We have suppressed the dependence on c, since we understand this definition up to isomorphism of posets. To be more precise, it can be shown that any two Coxeter elements are related to each other by conjugation and (possibly) an automorphism on the field of complex numbers (see [32, Theorem 4.2] or [24, Cor. 11.25]), and hence the resulting posets $NC^m(W)$ are isomorphic to each other. If $m = 1$, then $NC^1(W)$ can be identified with the set $NC(W)$ of non-crossing partitions for the (complex) reflection group W as defined by Bessis and Corran (cf. [9] and [8, Sec. 13]; their definition extends the earlier definition by Bessis [7] and Brady and Watt [11] for real reflection groups).

The following result has been proved by a collaborative effort of several authors (see [8, Prop. 13.1]).

Theorem 1. *Let W be an irreducible well-generated complex reflection group, and let $d_1 \le d_2 \le \dots \le d_n$ be its degrees and $h := d_n$ its Coxeter number. Then*

$$|NC^m(W)| = \prod_{i=1}^{n} \frac{mh + d_i}{d_i}. \tag{2.2}$$

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[2]An element of an irreducible well-generated complex reflection group W of rank n is called a *Coxeter element* if it is *regular* in the sense of Springer [32] (see also [24, Def. 11.21]) and of order d_n. An element of W is called regular if it has an eigenvector which lies in no reflecting hyperplane of a reflection of W. It follows from an observation of Lehrer and Springer, proved uniformly by Lehrer and Michel [23] (see [24, Theorem 11.28]), that there is always a regular element of order d_n in an irreducible well-generated complex reflection group W of rank n. More generally, if a well-generated complex reflection group W decomposes as $W \cong W_1 \times W_2 \times \dots \times W_k$, where the W_i's are irreducible, then a Coxeter element of W is an element of the form $c = c_1 c_2 \cdots c_k$, where c_i is a Coxeter element of W_i, $i = 1, 2, \dots, k$. If W is a *real* reflection group, that is, if all generators in T have order 2, then the notion of generalised Coxeter element given above reduces to that of a Coxeter element in the classical sense (cf. [16, Sec. 3.16]).

Remark 1. (1) The number in (2.2) is called the *Fuß–Catalan number* for the reflection group W.

(2) If c is a Coxeter element of a well-generated complex reflection group W of rank n, then $\ell_T(c) = n$. (This follows from [8, Sec. 7].)

We conclude this section by recalling the definition of decomposition numbers from [17, 18, 20]. Although we need them here only for (very small) real reflection groups, and although, strictly speaking, they have been only defined for real reflection groups in [17, 18, 20], this definition can be extended to well-generated complex reflection groups without any extra effort, which we do now.

Given a well-generated complex reflection group W of rank n, types $T_1, T_2, \dots, T_d$ (in the sense of the classification of well-generated complex reflection groups) such that the sum of the ranks of the T_i's equals n, and a Coxeter element c, the *decomposition number* $N_W(T_1, T_2, \dots, T_d)$ is defined as the number of "minimal" factorisations $c = c_1c_2 \cdots c_d$, "minimal" meaning that $\ell_T(c_1) + \ell_T(c_2) + \cdots + \ell_T(c_d) = \ell_T(c) = n$, such that, for $i = 1, 2, \dots, d$, the type of c_i as a parabolic Coxeter element is T_i. (Here, the term "parabolic Coxeter element" means a Coxeter element in some parabolic subgroup. It follows from [30, Prop.6.3] that any element c_i is indeed a Coxeter element in a unique parabolic subgroup of W.[3] By definition, the type of c_i is the type of this parabolic subgroup.) Since any two Coxeter elements are related to each other by conjugation plus field automorphism, the decomposition numbers are independent of the choice of the Coxeter element c.

The decomposition numbers for real reflection groups have been computed in [17, 18, 20]. To compute the decomposition numbers for well-generated complex reflection groups is a task that remains to be done.

3. Cyclic sieving I

In this section we present the first cyclic sieving conjecture due to Armstrong [3, Conj. 5.4.7], and to Bessis and Reiner [10, Conj. 6.4].

Let $\phi : NC^m(W) \to NC^m(W)$ be the map defined by

$$(w_0; w_1, \dots, w_m) \mapsto \big((cw_mc^{-1})w_0(cw_mc^{-1})^{-1}; cw_mc^{-1}, w_1, w_2, \dots, w_{m-1}\big). \tag{3.1}$$

It is indeed not difficult to see that, if the $(m+1)$-tuple on the left-hand side is an element of $NC^m(W)$, then so is the $(m+1)$-tuple on the right-hand side. For $m = 1$, this action reduces to conjugation by the Coxeter element c (applied to w_1). Cyclic sieving arising from conjugation by c has been the subject of [10].

It is easy to see that ϕ^{mh} acts as the identity, where h is the Coxeter number of W (see (5.1) and Lemma 29 below). By slight abuse of notation, let C_1 be the cyclic group of order mh generated by ϕ. (The slight abuse consists in the fact that we insist on C_1 to be a cyclic group of order mh, while it may happen that the order of the action of ϕ given in (3.1) is actually a proper divisor of mh.)

[3] The uniqueness can be argued as follows: suppose that c_i were a Coxeter element in two parabolic subgroups of W, say U_1 and U_2. Then it must also be a Coxeter element in the intersection $U_1 \cap U_2$. On the other hand, the absolute length of a Coxeter element of a complex reflection group U is always equal to $\mathrm{rk}(U)$, the rank of U. (This follows from the fact that, for each element u of U, we have $\ell_T(u) = \mathrm{codim}\big(\ker(u - \mathrm{id})\big)$, with id denoting the identity element in U; see e.g. [30, Prop. 1.3]). We conclude that $\ell_T(c_i) = \mathrm{rk}(U_1) = \mathrm{rk}(U_2) = \mathrm{rk}(U_1 \cap U_2)$, This implies that $U_1 = U_2$.

Given these definitions, we are now in the position to state the first cyclic sieving conjecture of Armstrong, respectively of Bessis and Reiner. By the results of [19] and of this paper, it becomes the following theorem.

Theorem 2. *For an irreducible well-generated complex reflection group W and any $m \ge 1$, the triple $(NC^m(W), \operatorname{Cat}^m(W;q), C_1)$, where $\operatorname{Cat}^m(W;q)$ is the q-analogue of the Fuß–Catalan number defined by*

$$\operatorname{Cat}^m(W;q) := \prod_{i=1}^{n} \frac{[mh+d_i]_q}{[d_i]_q}, \tag{3.2}$$

exhibits the cyclic sieving phenomenon in the sense of Reiner, Stanton and White [29]. *Here, n is the rank of W, $d_1, d_2, \dots, d_n$ are the degrees of W, h is the Coxeter number of W, and $[\alpha]_q := (1-q^\alpha)/(1-q)$.*

Remark 2. We write $\operatorname{Cat}^m(W)$ for $\operatorname{Cat}^m(W;1)$.

By definition of the cyclic sieving phenomenon, we have to prove that $\operatorname{Cat}^m(W;q)$ is a polynomial in q with non-negative integer coefficients, and that

$$|\operatorname{Fix}_{NC^m(W)}(\phi^p)| = \operatorname{Cat}^m(W;q)\big|_{q=e^{2\pi i p/mh}}, \tag{3.3}$$

for all p in the range $0 \le p < mh$. The first fact is established in the next section, while the proof of the second is achieved by making use of several auxiliary results, given in Section 5, to reduce the proof to a finite problem, and a subsequent case-by-case analysis, which occupies Section 6.

4. The q-Fusz–Catalan numbers $\operatorname{Cat}^m(W;q)$

The purpose of this section is to provide an elementary, self-contained proof of the fact that, for all irreducible complex reflection groups W, the q-Fuß–Catalan number $\operatorname{Cat}^m(W;q)$ is a polynomial in q with non-negative integer coefficients. For most of the groups, this is a known property. However, aside from the fact that, for many of the known cases, the proof is very indirect and uses deep algebraic results on rational Cherednik algebras, there still remained some cases where this property had not been formally established. The reader is referred to the Theorem in Section 1.6 of [15], which says that, under the assumption of a certain rank condition ([15, Hypothesis 2.4]), the q-Fuß–Catalan number $\operatorname{Cat}^m(W;q)$ is a Hilbert series of a finite-dimensional quotient of the ring of invariants of W and also the graded character of a finite-dimensional irreducible representation of a spherical rational Cherednik algebra associated with W. At present, this rank condition has been proven for all irreducible well-generated complex reflection groups apart from $G_{17}, G_{18}, G_{29}, G_{33}, G_{34}$; see [25, Tables 8 and 9, column "rank"], and the recent paper [26], which establishes the result in the case of G_{32}.

In the sequel, aside from the standard notation $[\alpha]_q = (1-q^\alpha)/(1-q)$ for q-integers, we shall also use the q-binomial coefficient, which is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} 1, & \text{if } k=0, \\ \frac{[n]_q\,[n-1]_q\cdots[n-k+1]_q}{[k]_q\,[k-1]_q\cdots[1]_q}, & \text{if } k>0. \end{cases}$$

We begin with several auxiliary results.

Proposition 3. *For all non-negative integers n and k, the q-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is a polynomial in q with non-negative integer coefficients.*

Proof. This is a well-known fact, which can be derived either from the recurrence relation(s) satisfied by the q-binomial coefficients (generalising Pascal's recurrence relation for binomial coefficients; cf. [1, eqs. (3.3.3) and (3.3.4)]), or from the fact that the q-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is the generating function for (integer) partitions with at most k parts all of which are at most $n-k$ (cf. [1, Theorem 3.1]). □

Proposition 4. *For all non-negative integers m and n, the q-Fuß–Catalan number of type A_n,*

$$\frac{1}{[(m+1)n+1]_q} \begin{bmatrix} (m+1)n+1 \\ n \end{bmatrix}_q ,$$

is a polynomial in q with non-negative integer coefficients.

Proof. This follows by replacing n by $mn+1$ and j by n in Theorem 2 of [2]. □

Lemma 5. *If a and b are coprime positive integers, then*

$$\frac{[ab]_q}{[a]_q\,[b]_q} \tag{4.1}$$

is a polynomial in q of degree $(a-1)(b-1)$, all of whose coefficients are in $\{0,1,-1\}$. Moreover, if one disregards the coefficients which are 0, then $+1$'s and (-1)'s alternate, and the constant coefficient as well as the leading coefficient of the polynomial equal $+1$.

Proof. Let $\Phi_n(q)$ denote the n-th cyclotomic polynomial in q. Using the classical formula

$$1-q^n = \prod_{d\mid n} \Phi_d(q),$$

we see that

$$\frac{(1-q)(1-q^{ab})}{(1-q^a)(1-q^b)} = \prod_{\substack{d_1\mid a,\, d_1\neq 1\\ d_2\mid a,\, d_2\neq 1}} \Phi_{d_1 d_2}(q),$$

so that, manifestly, the expression in (4.1) is a polynomial in q. The claim concerning the degree of this polynomial is obvious.

In order to establish the claim on the coefficients, we start with a sub-expression of (4.1),

$$\frac{(1-q^{ab})}{(1-q^a)(1-q^b)} = \left(\sum_{i=0}^{b-1} q^{ia}\right)\left(\sum_{j=0}^{\infty} q^{jb}\right) = \sum_{k=0}^{\infty} C_k q^k, \tag{4.2}$$

say. The assumption that a and b are coprime implies that $0 \le C_k \le 1$ for $k \le (a-1)(b-1)$. Multiplying both sides of (4.2) by $1-q$, we obtain the equation

$$\frac{[ab]_q}{[a]_q\,[b]_q} = (1-q) \sum_{k=0}^{(a-1)(b-1)} C_k q^k + (1-q) \sum_{k=(a-1)(b-1)+1}^{\infty} C_k q^k. \tag{4.3}$$

By our previous observation on the coefficients C_k with $k \le (a-1)(b-1)$, it is obvious that the coefficients of the first expression on the right-hand side of (4.3) are alternately

$+1$ and -1, when 0's are disregarded. Since we already know that the left-hand side is a polynomial in q of degree $(a-1)(b-1)$, we may ignore the second expression.

The proof is concluded by observing that the claims on the constant and leading coefficients are obvious. $\square$

Corollary 6. *Let a and b be coprime positive integers, and let γ be an integer with $\gamma \ge (a-1)(b-1)$. Then the expression*

$$\frac{[\gamma]_q\,[ab]_q}{[a]_q\,[b]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. Let

$$\frac{[ab]_q}{[a]_q\,[b]_q} = \sum_{k=0}^{(a-1)(b-1)} D_k q^k.$$

We then have

$$\frac{[\gamma]_q\,[ab]_q}{[a]_q\,[b]_q} = \sum_{N=0}^{(a-1)(b-1)+\gamma-1} q^N \sum_{k=\max\{0,N-\gamma+1\}}^{N} D_k. \tag{4.4}$$

If $N \le \gamma-1$, then, by Lemma 5, the sum over k on the right-hand side of (4.4) equals $1-1+1-1+\cdots$, which is manifestly non-negative. On the other hand, if $N > \gamma-1$, then we may rewrite the sum over k on the right-hand side of (4.4) as

$$\sum_{k=\max\{0,N-\gamma+1\}}^{N} D_k = \sum_{k=N-\gamma+1}^{(a-1)(b-1)} D_k = \sum_{k=0}^{(a-1)(b-1)+\gamma-1-N} D_{(a-1)(b-1)-k}.$$

Again, by Lemma 5, this sum equals $1-1+1-1+\cdots$, which is manifestly non-negative. $\square$

Lemma 7. *Let α and β be positive integers with $\alpha \ge 6$ and $\beta \ge 8$. Then the expression*

$$[\alpha]_{q^3}\,[\beta]_{q^4}\,\frac{[72]_q\,[3]_q\,[4]_q}{[8]_q\,[9]_q\,[12]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[72]_q\,[3]_q\,[4]_q}{[8]_q\,[9]_q\,[12]_q} = (1-q^3+q^9-q^{15}+q^{18})(1-q^4+q^8-q^{12}+q^{16}-q^{20}+q^{24}-q^{28}+q^{32}).$$

It should be observed that both factors on the right-hand side have the property that coefficients are in $\{0,1,-1\}$ and that $(+1)$'s and (-1)'s alternate, if one disregards the coefficients which are 0. If we now apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_{q^3}$ times the first factor is a polynomial in q with non-negative integer coefficients, as is $[\beta]_{q^4}$ times the second factor. Taken together, this establishes the claim. $\square$

Lemma 8. *Let α and β be positive integers with $\alpha \geq 26$ and $\beta \geq 8$. Then the expression*

$$[\alpha]_q \, [\beta]_{q^4} \, \frac{[15]_q}{[3]_q \, [5]_q} \frac{[72]_q \, [3]_q \, [4]_q}{[8]_q \, [9]_q \, [12]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\begin{aligned} &\frac{[15]_q \, [72]_q \, [4]_q}{[5]_q \, [8]_q \, [9]_q \, [12]_q} \\ &\quad = (1 - q + q^5 - q^6 + q^9 - q^{11} + q^{12} - q^{13} + q^{14} - q^{15} + q^{17} - q^{20} + q^{21} - q^{25} + q^{26}) \\ &\qquad\qquad \times (1 - q^4 + q^8 - q^{12} + q^{16} - q^{20} + q^{24} - q^{28} + q^{32}). \end{aligned}$$

Again, if we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_q$ times the first factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_{q^4}$ times the second factor. Taken together, this establishes the claim. $\square$

Lemma 9. *Let α and β be positive integers with $\alpha \geq 18$ and $\beta \geq 3$. Then the expression*

$$[\alpha]_{q^3} \, [\beta]_{q^4} \, \frac{[90]_q \, [3]_q \, [4]_q}{[5]_q \, [6]_q \, [9]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\begin{aligned} &\frac{[90]_q \, [3]_q \, [4]_q}{[5]_q \, [6]_q \, [9]_q} \\ &\quad = (1 - q^3 + q^9 - q^{12} + q^{18} - q^{21} + q^{27} - q^{33} + q^{36} - q^{42} + q^{45} - q^{51} + q^{54}) \\ &\qquad\qquad \times (1 - q^4 + q^5 + q^6 - q^9 + q^{11} + q^{12} - q^{14} + q^{17} + q^{18} - q^{19} + q^{23}) \\ &\quad = (1 - q^3 + q^9 - q^{12} + q^{18} - q^{21} + q^{27} - q^{33} + q^{36} - q^{42} + q^{45} - q^{51} + q^{54}) \\ &\qquad\qquad \times \big(1 - q^4 + q^{12} + q^5(1 - q^4 + q^{12}) + q^6(1 - q^8 + q^{12}) + q^{11}(1 - q^8 + q^{12})\big). \end{aligned}$$

Again, if we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_{q^3}$ times the first factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_{q^4}$ times the second factor. Taken together, this establishes the claim. $\square$

Lemma 10. *Let α and β be positive integers with $\alpha \geq 20$ and $\beta \geq 18$. Then the expression*

$$[\alpha]_q \, [\beta]_{q^3} \, \frac{[90]_q \, [3]_q}{[5]_q \, [6]_q \, [9]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[90]_q\,[3]_q}{[5]_q\,[6]_q\,[9]_q} = (1-q^3+q^9-q^{12}+q^{18}-q^{21}+q^{27}-q^{33}+q^{36}-q^{42}+q^{45}-q^{51}+q^{54}) \times (1-q+q^5-q^7+q^{10}-q^{13}+q^{15}-q^{19}+q^{20}).$$

Again, if we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_{q^3}$ times the second factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_{q^3}$ times the first factor. Taken together, this establishes the claim. □

Lemma 11. *Let α be a positive integer with $\alpha \ge 26$. Then the expression*

$$[\alpha]_q \frac{[15]_q}{[3]_q\,[5]_q} \frac{[12]_{q^3}}{[3]_{q^3}\,[4]_{q^3}}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[15]_q}{[3]_q\,[5]_q} \frac{[12]_{q^3}}{[3]_{q^3}\,[4]_{q^3}} = 1-q+q^5-q^6+q^9-q^{11}+q^{12}-q^{13}+q^{14}-q^{15}+q^{17}-q^{20}+q^{21}-q^{25}+q^{26}.$$

Once again, the coefficients of the polynomial on the right-hand side are in $\{0,1,-1\}$ and $(+1)$'s and (-1)'s alternate, if one disregards the coefficients which are 0. The argument from the proof of Corollary 6 then implies that this is a polynomial in q with non-negative integer coefficients. □

Lemma 12. *Let α be a positive integer with $\alpha \ge 14$. Then the expression*

$$[\alpha]_q \frac{[15]_q}{[3]_q\,[5]_q} \frac{[6]_{q^3}}{[2]_{q^3}\,[3]_{q^3}}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[15]_q}{[3]_q\,[5]_q} \frac{[6]_{q^3}}{[2]_{q^3}\,[3]_{q^3}} = 1-q+q^5-q^7+q^9-q^{13}+q^{14}.$$

Repeating the arguments of the previous proof, this establishes the claim. □

Lemma 13. *Let α and β be positive integers with $\alpha \ge 30$ and $\beta \ge 20$. Then the expression*

$$[\alpha]_q\,[\beta]_{q^2} \frac{[84]_q\,[2]_q}{[4]_q\,[6]_q\,[7]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\begin{aligned}&\frac{[84]_q\,[2]_q}{[4]_q\,[6]_q\,[7]_q}\\&\quad=(1-q+q^6-q^8+q^{12}-q^{15}+q^{18}-q^{22}+q^{24}-q^{29}+q^{30})\\&\qquad\times(1-q^2+q^4-q^6+q^8-q^{10}+q^{12}-q^{14}+q^{16}-q^{18}+q^{20}-q^{22}\\&\qquad\qquad+q^{24}-q^{26}+q^{28}-q^{30}+q^{32}-q^{34}+q^{36}-q^{38}+q^{40}).\end{aligned}$$

Once again, if we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_q$ times the first factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_{q^2}$ times the second factor. Taken together, this establishes the claim. □

Lemma 14. *Let α and β be positive integers with $\alpha \ge 24$ and $\beta \ge 68$. Then the expression*

$$[\alpha]_q\,[\beta]_q\,\frac{[105]_q}{[3]_q\,[5]_q\,[7]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\begin{aligned}&\frac{[105]_q}{[3]_q\,[5]_q\,[7]_q}\\&\quad=(1-q+q^5-q^6+q^7-q^8+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}-q^{16}\\&\qquad\qquad+q^{17}-q^{18}+q^{19}-q^{23}+q^{24})\\&\qquad\times(1-q+q^3-q^4+q^6-q^7+q^9-q^{10}+q^{12}-q^{13}+q^{15}-q^{16}+q^{18}-q^{19}\\&\qquad\qquad+q^{21}-q^{22}+q^{24}-q^{25}+q^{27}-q^{28}+q^{30}-q^{31}+q^{33}-q^{34}+q^{35}-q^{37}\\&\qquad\qquad+q^{38}-q^{40}+q^{41}-q^{43}+q^{44}-q^{46}+q^{47}-q^{49}+q^{50}-q^{52}+q^{53}-q^{55}\\&\qquad\qquad+q^{56}-q^{58}+q^{59}-q^{61}+q^{62}-q^{64}+q^{65}-q^{67}+q^{68}).\end{aligned}$$

Once again, if we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_q$ times the first factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_q$ times the second factor. Taken together, this establishes the claim. □

Lemma 15. *Let α and β be positive integers with $\alpha \ge 24$ and $\beta \ge 34$. Then the expression*

$$[\alpha]_q\,[\beta]_q\,\frac{[70]_q}{[2]_q\,[5]_q\,[7]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\begin{aligned}\frac{[70]_q}{[2]_q\,[5]_q\,[7]_q}&\\ &=(1-q+q^5-q^6+q^7-q^8+q^{10}-q^{11}+q^{12}-q^{13}\\ &\qquad+q^{14}-q^{16}+q^{17}-q^{18}+q^{19}-q^{23}+q^{24})\\ &\quad\times(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}\\ &\qquad+q^{14}-q^{15}+q^{16}-q^{17}+q^{18}-q^{19}+q^{20}-q^{21}+q^{22}-q^{23}+q^{24}-q^{25}\\ &\qquad+q^{26}-q^{27}+q^{28}-q^{29}+q^{30}-q^{31}+q^{32}-q^{33}+q^{34}).\end{aligned}$$

Also here, if we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_q$ times the first factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_q$ times the second factor. Taken together, this establishes the claim. $\square$

Lemma 16. *Let α and β be positive integers with $\alpha\ge 4$ and $\beta\ge 2$. Then the expression*

$$[\alpha]_{q^2}\,[\beta]_{q^5}\,\frac{[30]_q\,[2]_q\,[3]_q\,[5]_q}{[6]_q\,[10]_q\,[15]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[30]_q\,[2]_q\,[3]_q\,[5]_q}{[6]_q\,[10]_q\,[15]_q}=1+q-q^3-q^4-q^5+q^7+q^8.$$

If we multiply this expression by $[\alpha]_{q^2}$, then, for $\alpha=4$ we obtain

$$1+q+q^2-q^5-q^9+q^{12}+q^{13}+q^{14},$$

for $\alpha=5$ we obtain

$$1+q+q^2-q^5+q^8-q^{11}+q^{14}+q^{15}+q^{16},$$

and, for $\alpha\ge 6$, we obtain

$$1+q+q^2-q^5+q^8+q^{10}+p_1(q)+q^{2\alpha-4}+q^{2\alpha-2}-q^{2\alpha+1}+q^{2\alpha+4}+q^{2\alpha+5}+q^{2\alpha+6},$$

where $p_1(q)$ is a polynomial in q with non-negative coefficients of order at least 11 and degree at most $2\alpha-5$. In all cases it is obvious that the product of the result and $[\beta]_{q^5}$, with $\beta\ge 2$, is a polynomial in q with non-negative coefficients. $\square$

Lemma 17. *Let α and β be positive integers with $\alpha\ge 14$ and $\beta\ge 2$. Then the expression*

$$[\alpha]_q\,[\beta]_{q^5}\,\frac{[14]_q}{[2]_q\,[7]_q}\,\frac{[30]_q\,[2]_q\,[3]_q\,[5]_q}{[6]_q\,[10]_q\,[15]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[14]_q}{[2]_q\,[7]_q}\,\frac{[30]_q\,[2]_q\,[3]_q\,[5]_q}{[6]_q\,[10]_q\,[15]_q}=1-q^3-q^5+q^6+q^7+q^8-q^9-q^{11}+q^{14}.$$

If we multiply this expression by $[\alpha]_q$, then, for $\alpha \geq 14$, we obtain

$$1+q+q^2-q^5+q^7+2q^8+q^9+q^{10}+p_2(q) \\ +q^{\alpha+3}+q^{\alpha+4}+2q^{\alpha+5}+q^{\alpha+6}-q^{\alpha+8}+q^{\alpha+11}+q^{\alpha+12}+q^{\alpha+13},$$

where $p_2(q)$ is a polynomial in q with non-negative coefficients of order at least 11 and degree at most $\alpha+2$. It is obvious that the product of the result and $[\beta]_{q^5}$, with $\beta \geq 2$, is a polynomial in q with non-negative coefficients. $\square$

Lemma 18. *Let α and β be positive integers with $\alpha \geq 32$ and $\beta \geq 12$. Then the expression*

$$[\alpha]_q\,[\beta]_{q^2}\,\frac{[35]_q}{[5]_q\,[7]_q}\frac{[30]_q\,[2]_q\,[3]_q\,[5]_q}{[6]_q\,[10]_q\,[15]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[35]_q}{[5]_q\,[7]_q}\frac{[30]_q\,[2]_q\,[3]_q\,[5]_q}{[6]_q\,[10]_q\,[15]_q} = 1-q^2-q^3+q^5+q^6+q^7-q^8-2q^9+q^{11}+q^{12}-q^{16}+q^{20}+q^{21} \\ -2q^{23}-q^{24}+q^{25}+q^{26}+q^{27}-q^{29}-q^{30}+q^{32}.$$

If we multiply this expression by $[\alpha]_q$, then, for $\alpha \geq 32$, we obtain

$$1+q-q^3-q^4+q^6+2q^7+q^8-q^9-q^{10}+q^{12}+q^{13}+q^{14}+q^{15}+q^{20}+2q^{21}+2q^{22} \\ -q^{24}+q^{26}+2q^{27}+2q^{28}+q^{29}+p_3(q)+q^{\alpha+2}+2q^{\alpha+4}+2q^{\alpha+5}+q^{\alpha+6}-q^{\alpha+8} \\ +2q^{\alpha+9}+2q^{\alpha+10}+q^{\alpha+11}+q^{\alpha+16}+q^{\alpha+17}+q^{\alpha+18}+q^{\alpha+19}-q^{\alpha+21}-q^{\alpha+22}+q^{\alpha+23} \\ +2q^{\alpha+24}+q^{\alpha+25}-q^{\alpha+27}-q^{\alpha+28}+q^{\alpha+30}+q^{\alpha+31},$$

where $p_3(q)$ is a polynomial in q with non-negative coefficients of order at least 30 and degree at most $\alpha+1$. It is obvious that the product of the result and $[\beta]_{q^2}$, with $\beta \geq 12$, is a polynomial in q with non-negative coefficients. $\square$

Lemma 19. *Let α and β be positive integers with $\alpha \geq 16$ and $\beta \geq 2$. Then the expression*

$$[\alpha]_{q^2}\,[\beta]_{q^5}\,\frac{[60]_q\,[2]_q\,[3]_q\,[5]_q}{[10]_q\,[12]_q\,[15]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[60]_q\,[2]_q\,[3]_q\,[5]_q}{[10]_q\,[12]_q\,[15]_q} = 1+q-q^3-q^4-q^5-q^6+q^8+q^9+q^{10}+q^{11}+q^{12}-q^{14}-q^{15}-q^{16} \\ -q^{17}-q^{18}+q^{20}+q^{21}+q^{22}+q^{23}+q^{24}-q^{26}-q^{27}-q^{28}-q^{29}+q^{31}+q^{32}.$$

If we multiply this expression by $[\alpha]_{q^2}$, then, for $\alpha \geq 16$, we obtain

$$1+q+q^2-q^5-q^6-q^7+q^{10}+q^{11}+2q^{12}+q^{13}+q^{14} \\ -q^{17}-q^{18}-q^{19}+q^{22}+q^{23}+2q^{24}+p_4(q)+s_4(q),$$

where $p_4(q)$ is a polynomial in q with non-negative coefficients of order at least 25 and degree at most $2\alpha+5$ and $s_4(q)$ completes the above expression to a symmetric polynomial. It is obvious that the product of the result and $[\beta]_{q^5}$, with $\beta \ge 2$, is a polynomial in q with non-negative coefficients. $\square$

Lemma 20. *Let α and β be positive integers with $\alpha \ge 56$ and $\beta \ge 4$. Then the expression*

$$[\alpha]_q\,[\beta]_{q^2}\,\frac{[35]_q}{[5]_q\,[7]_q}\,\frac{[60]_q\,[2]_q\,[3]_q\,[5]_q}{[10]_q\,[12]_q\,[15]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\begin{aligned}\frac{[35]_q}{[5]_q\,[7]_q}\,\frac{[60]_q\,[2]_q\,[3]_q\,[5]_q}{[10]_q\,[12]_q\,[15]_q} &= 1-q^2-q^3+q^5+q^7-q^9+q^{12}-q^{13}+q^{15}-q^{16}-q^{18}\\ &\quad +q^{19}+q^{20}+q^{22}-2q^{23}+q^{27}-q^{28}+q^{29}-2q^{33}+q^{34}+q^{36}+q^{37}-q^{38}\\ &\quad -q^{40}+q^{41}-q^{43}+q^{44}-q^{47}+q^{49}+q^{51}-q^{53}-q^{54}+q^{56}.\end{aligned}$$

If we multiply this expression by $[\alpha]_q$, then, for $\alpha \ge 56$, we obtain

$$\begin{aligned}1+q-q^3-q^4+q^7+q^8+q^{12}+q^{15}-q^{18}+q^{20}+q^{21}+2q^{22}+q^{27}&\\ +q^{29}+q^{30}+q^{31}+q^{32}-q^{33}+q^{36}+2q^{37}+q^{38}+p_5(q)+s_5(q)&,\end{aligned}$$

where $p_5(q)$ is a polynomial in q with non-negative coefficients of order at least 39 and degree at most $\alpha+16$ and $s_5(q)$ completes the above expression to a symmetric polynomial. It is obvious that the product of the result and $[\beta]_{q^2}$, with $\beta \ge 4$, is a polynomial in q with non-negative coefficients. $\square$

Lemma 21. *Let α and β be positive integers with $\alpha \ge 38$ and $\beta \ge 2$. Then the expression*

$$[\alpha]_q\,[\beta]_{q^5}\,\frac{[14]_q}{[2]_q\,[7]_q}\,\frac{[60]_q\,[2]_q\,[3]_q\,[5]_q}{[10]_q\,[12]_q\,[15]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[14]_q}{[2]_q\,[7]_q}\,\frac{[60]_q\,[2]_q\,[3]_q\,[5]_q}{[10]_q\,[12]_q\,[15]_q} = 1-q^3-q^5+q^7+q^8-q^{13}+q^{19}-q^{25}+q^{30}+q^{31}-q^{33}-q^{35}+q^{38}.$$

If we multiply this expression by $[\alpha]_q$, then, for $\alpha \ge 38$, we obtain

$$1+q+q^2-q^5-q^6+q^8+q^9+q^{10}+q^{11}+p_6(q)+s_6(q),$$

where $p_6(q)$ is a polynomial in q with non-negative coefficients of order at least 12 and degree at most $\alpha+25$ and $s_6(q)$ completes the above expression to a symmetric polynomial. It is obvious that the product of the result and $[\beta]_{q^2}$, with $\beta \ge 2$, is a polynomial in q with non-negative coefficients. $\square$

Lemma 22. *Let α and β be positive integers with $\alpha \ge 30$ and $\beta \ge 26$. Then the expression*

$$[\alpha]_q\,[\beta]_{q^3}\,\frac{[126]_q\,[3]_q}{[6]_q\,[7]_q\,[9]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[126]_q\,[3]_q}{[6]_q\,[7]_q\,[9]_q} = (1-q+q^6-q^8+q^{12}-q^{15}+q^{18}-q^{22}+q^{24}-q^{29}+q^{30})$$
$$\times(1-q^3+q^9-q^{12}+q^{18}-q^{21}+q^{27}-q^{30}+q^{36}-q^{39}+q^{42}-q^{48}+q^{51}-q^{57}+q^{60}-q^{66}+q^{69}-q^{75}+q^{78}).$$

If we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_q$ times the first factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_{q^3}$ times the second factor. Taken together, this establishes the claim. ☐

Lemma 23. *Let α and β be positive integers with $\alpha \geq 66$ and $\beta \geq 54$. Then the expression*

$$[\alpha]_q\,[\beta]_{q^3}\,\frac{[252]_q\,[3]_q}{[7]_q\,[9]_q\,[12]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\frac{[252]_q\,[3]_q}{[7]_q\,[9]_q\,[12]_q}$$
$$=(1-q+q^7-q^8+q^{12}-q^{13}+q^{14}-q^{15}+q^{19}-q^{20}+q^{21}-q^{22}+q^{24}-q^{25}+q^{26}-q^{27}+q^{28}-q^{29}+q^{31}-q^{32}+q^{33}-q^{34}+q^{35}-q^{37}+q^{38}-q^{39}+q^{40}-q^{41}+q^{42}-q^{44}+q^{45}-q^{46}+q^{47}-q^{51}+q^{52}-q^{53}+q^{54}-q^{58}+q^{59}-q^{65}+q^{66})$$
$$\times(1-q^3+q^9-q^{12}+q^{18}-q^{21}+q^{27}-q^{30}+q^{36}-q^{39}+q^{45}-q^{48}+q^{54}-q^{57}+q^{63}-q^{66}+q^{72}-q^{75}+q^{81}-q^{87}+q^{90}-q^{96}+q^{99}-q^{105}+q^{108}-q^{114}+q^{117}-q^{123}+q^{126}-q^{132}+q^{135}-q^{141}+q^{144}-q^{150}+q^{153}-q^{159}+q^{162}).$$

If we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_q$ times the first factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_{q^3}$ times the second factor. Taken together, this establishes the claim. ☐

Lemma 24. *Let α and β be positive integers with $\alpha \geq 54$ and $\beta \geq 34$. Then the expression*

$$[\alpha]_q\,[\beta]_{q^2}\,\frac{[140]_q\,[2]_q}{[4]_q\,[7]_q\,[10]_q}$$

is a polynomial in q with non-negative integer coefficients.

Proof. We have

$$\begin{aligned}\frac{[140]_q\,[2]_q}{[4]_q\,[7]_q\,[10]_q}\\
&=(1-q+q^7-q^8+q^{10}-q^{11}+q^{14}-q^{15}+q^{17}-q^{18}+q^{20}-q^{22}+q^{24}-q^{25}\\
&\qquad+q^{27}-q^{29}+q^{30}-q^{32}+q^{34}-q^{36}+q^{37}-q^{39}+q^{40}-q^{43}+q^{44}-q^{46}\\
&\qquad+q^{47}-q^{53}+q^{54})\\
&\quad\times(1-q^2+q^4-q^6+q^8-q^{10}+q^{12}-q^{14}+q^{16}-q^{18}+q^{20}-q^{22}+q^{24}\\
&\qquad-q^{26}+q^{28}-q^{30}+q^{32}-q^{34}+q^{36}-q^{38}+q^{40}-q^{42}+q^{44}-q^{46}\\
&\qquad+q^{48}-q^{50}+q^{52}-q^{54}+q^{56}-q^{58}+q^{60}-q^{62}+q^{64}-q^{66}+q^{68}),\end{aligned}$$

If we apply the same idea as in the proof of Corollary 6, then we see that $[\alpha]_q$ times the first factor on the the right-hand side of the above equation is a polynomial in q with non-negative integer coefficients, as is $[\beta]_{q^2}$ times the second factor. Taken together, this establishes the claim. □

We are now ready for the proof of the main result of this section.

Theorem 25. *For all irreducible well-generated complex reflection groups and positive integers m, the q-Fuß–Catalan number $\operatorname{Cat}^m(W;q)$ is a polynomial in q with non-negative integer coefficients.*

Proof. First, let $W=A_n$. In this case, the degrees are $2,3,\dots,n+1$, and hence

$$\operatorname{Cat}^m(A_n;q)=\frac{1}{[(m+1)n+1]_q}\begin{bmatrix}(m+1)n+1\\ n\end{bmatrix}_q,$$

which, by Proposition 4, is a polynomial in q with non-negative integer coefficients.

Next, let $W=G(d,1,n)$. In this case, the degrees are $d,2d,\dots,nd$, and hence

$$\operatorname{Cat}^m(G(d,1,n);q)=\begin{bmatrix}(m+1)n\\ n\end{bmatrix}_{q^d},$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

Now, let $W=G(e,e,n)$. In this case, the degrees are $e,2e,\dots,(n-1)e,n$, and hence

$$\begin{aligned}\operatorname{Cat}^m(G(e,e,n);q)&=\frac{[m(n-1)e+n]_q}{[n]_q}\prod_{i=1}^{n-1}\frac{[m(n-1)e+ie]_q}{[ie]_q}\\
&=\begin{bmatrix}(m+1)(n-1)\\ n-1\end{bmatrix}_{q^e}+q^n[e]_{q^n}\begin{bmatrix}(m+1)(n-1)\\ n\end{bmatrix}_{q^e},\end{aligned}$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

It remains to verify the claim for the exceptional groups.

For $W=G_4$, the degrees are $4,6$, and hence

$$\operatorname{Cat}^m(G_4;q)=\frac{[6m+4]_q\,[6m+6]_q}{[4]_q\,[6]_q}=\frac{1}{[3m+4]_{q^2}}\begin{bmatrix}3m+4\\ 3\end{bmatrix}_{q^2},$$

which, by Proposition 4, is a polynomial in q with non-negative integer coefficients.

For $W = G_5$, the degrees are $6, 12$, and hence

$$\mathrm{Cat}^m(G_5; q) = \frac{[12m+6]_q\,[12m+12]_q}{[6]_q\,[12]_q} = \begin{bmatrix} 2m+2 \\ 2 \end{bmatrix}_{q^6},$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

For $W = G_6$, the degrees are $4, 12$, and hence

$$\mathrm{Cat}^m(G_6; q) = \frac{[12m+4]_q\,[12m+12]_q}{[4]_q\,[12]_q} = [3m+1]_{q^4}\,[m+1]_{q^{12}},$$

which is manifestly a polynomial in q with non-negative integer coefficients.

For $W = G_8$, the degrees are $8, 12$, and hence

$$\mathrm{Cat}^m(G_8; q) = \frac{[12m+8]_q\,[12m+12]_q}{[8]_q\,[12]_q} = \frac{1}{[3m+4]_{q^4}} \begin{bmatrix} 3m+4 \\ 3 \end{bmatrix}_{q^4},$$

which, by Proposition 4, is a polynomial in q with non-negative integer coefficients.

For $W = G_9$, the degrees are $8, 24$, and hence

$$\mathrm{Cat}^m(G_9; q) = \frac{[24m+8]_q\,[24m+24]_q}{[8]_q\,[24]_q} = [3m+1]_{q^8}\,[m+1]_{q^{24}},$$

which is manifestly a polynomial in q with non-negative integer coefficients.

For $W = G_{10}$, the degrees are $12, 24$, and hence

$$\mathrm{Cat}^m(G_{10}; q) = \frac{[24m+12]_q\,[24m+24]_q}{[12]_q\,[24]_q} = \begin{bmatrix} 2m+2 \\ 2 \end{bmatrix}_{q^{12}},$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

For $W = G_{14}$, the degrees are $6, 24$, and hence

$$\mathrm{Cat}^m(G_{14}; q) = \frac{[24m+6]_q\,[24m+24]_q}{[6]_q\,[24]_q} = [4m+1]_{q^6}\,[m+1]_{q^{24}},$$

which is manifestly a polynomial in q with non-negative integer coefficients.

For $W = G_{16}$, the degrees are $20, 30$, and hence

$$\mathrm{Cat}^m(G_{16}; q) = \frac{[30m+20]_q\,[30m+30]_q}{[20]_q\,[30]_q} = \frac{1}{[3m+4]_{q^{10}}} \begin{bmatrix} 3m+4 \\ 3 \end{bmatrix}_{q^{10}},$$

which, by Proposition 4, is a polynomial in q with non-negative integer coefficients.

For $W = G_{17}$, the degrees are $20, 60$, and hence

$$\mathrm{Cat}^m(G_{17}; q) = \frac{[60m+20]_q\,[60m+60]_q}{[20]_q\,[60]_q} = [3m+1]_{q^{20}}\,[m+1]_{q^{60}},$$

which is manifestly a polynomial in q with non-negative integer coefficients.

For $W = G_{18}$, the degrees are $30, 60$, and hence

$$\mathrm{Cat}^m(G_{18}; q) = \frac{[60m+30]_q\,[60m+60]_q}{[30]_q\,[60]_q} = \begin{bmatrix} 2m+2 \\ 2 \end{bmatrix}_{q^{30}},$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

For $W=G_{20}$, the degrees are $12,30$, and hence

$$\begin{aligned}\operatorname{Cat}^m(G_{20};q)&=\frac{[30m+12]_q\,[30m+30]_q}{[12]_q\,[30]_q}\\&=\begin{cases}\left[\frac{5m+2}{2}\right]_{q^{12}}\,[m+1]_{q^{30}}\,, & \text{if } m \text{ is even,}\\ [5m+2]_{q^6}\,\left[\frac{m+1}{2}\right]_{q^{60}}\,\frac{[10]_{q^6}}{[2]_{q^6}\,[5]_{q^6}}, & \text{if } m \text{ is odd,}\end{cases}\end{aligned}$$

which, by Corollary 6, are polynomials in q with non-negative integer coefficients in both cases.

For $W=G_{21}$, the degrees are $12,60$, and hence

$$\operatorname{Cat}^m(G_{21};q)=\frac{[60m+12]_q\,[60m+60]_q}{[12]_q\,[60]_q}=[5m+1]_{q^{12}}\,[m+1]_{q^{60}},$$

which is manifestly a polynomial in q with non-negative integer coefficients.

For $W=G_{23}=H_3$, the degrees are $2,6,10$, and hence

$$\begin{aligned}\operatorname{Cat}^m(H_3;q)&=\frac{[10m+2]_q\,[10m+6]_q\,[10m+10]_q}{[2]_q\,[6]_q\,[10]_q}\\&=\begin{cases}[5m+2]_{q^2}\,\left[\frac{5m+3}{3}\right]_{q^6}\,[m+1]_{q^{10}}\,, & \text{if } m\equiv 0 \pmod 3,\\ \left[\frac{5m+1}{3}\right]_{q^6}\,[5m+3]_{q^2}\,[m+1]_{q^{10}}\,, & \text{if } m\equiv 1 \pmod 3,\\ [5m+2]_{q^2}\,[5m+3]_{q^2}\,\left[\frac{m+1}{3}\right]_{q^{30}}\,\frac{[15]_{q^2}}{[3]_{q^2}\,[5]_{q^2}}, & \text{if } m\equiv 2 \pmod 3,\end{cases}\end{aligned}$$

which, by Corollary 6, are polynomials in q with non-negative integer coefficients in all cases.

For $W=G_{24}$, the degrees are $4,6,14$, and hence

$$\operatorname{Cat}^m(G_{24};q)=\frac{[14m+4]_q\,[14m+6]_q\,[14m+14]_q}{[4]_q\,[6]_q\,[14]_q}.$$

We have

$$\operatorname{Cat}^m(G_{24};q)=\begin{cases}\left[\frac{7m}{2}+1\right]_{q^4}\,\left[\frac{14m}{6}+1\right]_{q^6}\,[m+1]_{q^{14}}\,, & \text{if } m\equiv 0 \pmod 6,\\ \left[\frac{7m+2}{3}\right]_{q^6}\,\left[\frac{7m+3}{2}\right]_{q^4}\,[m+1]_{q^{14}}\,, & \text{if } m\equiv 1 \pmod 6,\\ \left[\frac{7m}{2}+1\right]_{q^4}\,[7m+3]_{q^2}\,\left[\frac{m+1}{3}\right]_{q^{42}}\,\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}, & \text{if } m\equiv 2 \pmod 6,\\ [7m+2]_{q^2}\,\left[\frac{7m}{3}+1\right]_{q^6}\,\left[\frac{m+1}{2}\right]_{q^{28}}\,\frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}}, & \text{if } m\equiv 3 \pmod 6,\\ \left[\frac{7m+2}{6}\right]_{q^{12}}\,\frac{[6]_{q^2}}{[2]_{q^2}\,[3]_{q^2}}\,[7m+3]_{q^2}\,[m+1]_{q^{14}}\,, & \text{if } m\equiv 4 \pmod 6,\\ [7m+2]_{q^2}\,\left[\frac{7m+3}{2}\right]_{q^4}\,\left[\frac{m+1}{3}\right]_{q^{42}}\,\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}, & \text{if } m\equiv 5 \pmod 6,\end{cases}$$

which, by Corollary 6, are polynomials in q with non-negative integer coefficients in all cases.

For $W=G_{25}$, the degrees are $6,9,12$, and hence

$$\operatorname{Cat}^m(G_{25};q)=\frac{[12m+6]_q\,[12m+9]_q\,[12m+12]_q}{[6]_q\,[9]_q\,[12]_q}=\frac{1}{[4m+5]_{q^3}}\begin{bmatrix}4m+5\\4\end{bmatrix}_{q^3},$$

which, by Proposition 4, is a polynomial in q with non-negative integer coefficients.

For $W = G_{26}$, the degrees are $6, 12, 18$, and hence

$$\operatorname{Cat}^m(G_{26}; q) = \frac{[18m+6]_q\,[18m+12]_q\,[18m+18]_q}{[6]_q\,[12]_q\,[18]_q} = \begin{bmatrix} 3m+3 \\ 3 \end{bmatrix}_{q^6},$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

For $W = G_{27}$, the degrees are $6, 12, 30$, and hence

$$\operatorname{Cat}^m(G_{27}; q) = \frac{[30m+6]_q\,[30m+12]_q\,[30m+30]_q}{[6]_q\,[12]_q\,[30]_q} = [m+1]_{q^{30}} \begin{bmatrix} 5m+2 \\ 2 \end{bmatrix}_{q^6},$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

For $W = G_{28} = F_4$, the degrees are $2, 6, 8, 12$, and hence

$$\begin{aligned} \operatorname{Cat}^m(F_4; q) &= \frac{[12m+2]_q\,[12m+6]_q\,[12m+8]_q\,[12m+12]_q}{[2]_q\,[6]_q\,[8]_q\,[12]_q} \\ &= \begin{cases} [6m+1]_{q^2}\ [2m+1]_{q^6}\ \left[\frac{3}{2}m+1\right]_{q^8}\ [m+1]_{q^{12}}\,, & \text{if } m \text{ is even,} \\ [6m+1]_{q^2}\ [2m+1]_{q^6}\ \left[\frac{m+1}{2}\right]_{q^{24}}\ [3m+2]_{q^4}\ \dfrac{[6]_{q^4}}{[2]_{q^4}\,[3]_{q^4}}, & \text{if } m \text{ is odd,} \end{cases} \end{aligned}$$

which in both cases is a polynomial in q with non-negative integer coefficients; in the second case this is due to Corollary 6.

For $W = G_{29}$, the degrees are $4, 8, 12, 20$, and hence

$$\begin{aligned} \operatorname{Cat}^m(G_{29}; q) &= \frac{[20m+4]_q\,[20m+8]_q\,[20m+12]_q\,[20m+20]_q}{[4]_q\,[8]_q\,[12]_q\,[20]_q} \\ &= [m+1]_{q^{20}} \begin{bmatrix} 5m+3 \\ 3 \end{bmatrix}_{q^4}, \end{aligned}$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

For $W = G_{30} = H_4$, the degrees are $2, 12, 20, 30$, and hence

$$\operatorname{Cat}^m(H_4; q) = \frac{[30m+2]_q\,[30m+12]_q\,[30m+20]_q\,[30m+30]_q}{[2]_q\,[12]_q\,[20]_q\,[30]_q}.$$

If m is even, then we have

$$\operatorname{Cat}^m(H_4; q) = [15m+1]_{q^2}\ \left[\tfrac{5}{2}m+1\right]_{q^{12}}\ \left[\tfrac{3}{2}m+1\right]_{q^{20}}\ [m+1]_{q^{30}}\,,$$

which is manifestly a polynomial in q with non-negative integer coefficients. On the other hand, if m is odd, then we may write

$$\operatorname{Cat}^m(H_4; q) = \left[\tfrac{15m+1}{2}\right]_{q^4}\ [5m+2]_{q^6}\ [3m+2]_{q^{10}}\ \left[\tfrac{m+1}{2}\right]_{q^{60}}\ \frac{[30]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[6]_{q^6}\,[10]_{q^2}\,[15]_{q^2}},$$

which, by Lemma 16, is a polynomial in q with non-negative integer coefficients.

For $W = G_{32}$, the degrees are $12, 18, 24, 30$, and hence

$$\begin{aligned} \operatorname{Cat}^m(G_{32}; q) &= \frac{[30m+12]_q\,[30m+18]_q\,[30m+24]_q\,[30m+30]_q}{[12]_q\,[18]_q\,[24]_q\,[30]_q} \\ &= \frac{1}{[5m+6]_{q^6}} \begin{bmatrix} 5m+6 \\ 5 \end{bmatrix}_{q^6}, \end{aligned}$$

which, by Proposition 4, is a polynomial in q with non-negative integer coefficients.

For $W = G_{33}$, the degrees are $4, 6, 10, 12, 18$, and hence

$$\mathrm{Cat}^m(G_{33}; q) = \frac{[18m+4]_q\,[18m+6]_q\,[18m+10]_q\,[18m+12]_q\,[18m+18]_q}{[4]_q\,[6]_q\,[10]_q\,[12]_q\,[18]_q}.$$

If $m \equiv 0 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{9}{2}m+1\right]_{q^4}\,[3m+1]_{q^6}\,\left[\tfrac{9}{5}m+1\right]_{q^{10}}\,\left[\tfrac{3}{2}m+1\right]_{q^{12}}\,[m+1]_{q^{18}}\,,$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 1 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{3m+1}{2}\right]_{q^{12}}\,\left[\tfrac{9m+5}{2}\right]_{q^4}\,\left[\tfrac{3m+2}{5}\right]_{q^{30}}\,[m+1]_{q^{18}}\,[9m+2]_{q^2}\,\frac{[15]_{q^2}}{[3]_{q^2}\,[5]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 2 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{9m+2}{10}\right]_{q^{20}}\,[3m+1]_{q^6}\,\left[\tfrac{3}{2}m+1\right]_{q^{12}}\,[m+1]_{q^{18}}\,[9m+5]_{q^2}\,\frac{[10]_{q^2}}{[2]_{q^2}\,[5]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 3 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{3m+1}{10}\right]_{q^{60}}\,\left[\tfrac{9m+5}{2}\right]_{q^4}\,[3m+2]_{q^6}\,[m+1]_{q^{18}}\,[9m+2]_{q^2}\,\frac{[30]_{q^2}}{[5]_{q^2}\,[6]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 4 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{9}{2}m+1\right]_{q^4}\,[3m+1]_{q^6}\,\left[\tfrac{3}{2}m+1\right]_{q^{12}}\,\left[\tfrac{m+1}{5}\right]_{q^{90}}\,[9m+5]_{q^2}\,\frac{[45]_{q^2}}{[5]_{q^2}\,[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 5 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{3m+1}{2}\right]_{q^{12}}\,\left[\tfrac{9m+5}{10}\right]_{q^{20}}\,[3m+2]_{q^6}\,[m+1]_{q^{18}}\,[9m+2]_{q^2}\,\frac{[10]_{q^2}}{[2]_{q^2}\,[5]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 6 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{9}{2}m+1\right]_{q^4}\,[3m+1]_{q^6}\,\left[\tfrac{3m+2}{10}\right]_{q^{60}}\,[m+1]_{q^{18}}\,[9m+5]_{q^2}\,\frac{[30]_{q^2}}{[5]_{q^2}\,[6]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 7 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{9m+2}{5}\right]_{q^{10}}\,\left[\tfrac{3m+1}{2}\right]_{q^{12}}\,\left[\tfrac{9m+5}{2}\right]_{q^4}\,[3m+2]_{q^6}\,[m+1]_{q^{18}}\,,$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 8 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33}; q) = \left[\tfrac{9}{2}m+1\right]_{q^4}\,\left[\tfrac{3m+1}{5}\right]_{q^{30}}\,\left[\tfrac{3}{2}m+1\right]_{q^{12}}\,[m+1]_{q^{18}}\,[9m+5]_{q^2}\,\frac{[15]_{q^2}}{[3]_{q^2}\,[5]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. Finally, if $m \equiv 9 \pmod{10}$, then we have

$$\mathrm{Cat}^m(G_{33};q) = \left[\tfrac{3m+1}{2}\right]_{q^{12}} \left[\tfrac{9m+5}{2}\right]_{q^4} [3m+2]_{q^6} \left[\tfrac{m+1}{5}\right]_{q^{90}} [9m+2]_{q^2} \frac{[45]_{q^2}}{[5]_{q^2}\,[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients.

For $W = G_{34}$, the degrees are $6, 12, 18, 24, 30, 42$, and hence

$$\begin{aligned}\mathrm{Cat}^m(G_{34};q) &= \frac{[42m+6]_q\,[42m+12]_q\,[42m+18]_q\,[42m+24]_q\,[42m+30]_q\,[42m+42]_q}{[6]_q\,[12]_q\,[18]_q\,[24]_q\,[30]_q\,[42]_q}\\ &= [m+1]_{q^{42}} \begin{bmatrix} 7m+5 \\ 5 \end{bmatrix}_{q^6},\end{aligned}$$

which, by Proposition 3, is a polynomial in q with non-negative integer coefficients.

For $W = G_{35} = E_6$, the degrees are $2, 5, 6, 8, 9, 12$, and hence

$$\mathrm{Cat}^m(E_6;q) = \frac{[12m+2]_q\,[12m+5]_q\,[12m+6]_q\,[12m+8]_q\,[12m+9]_q\,[12m+12]_q}{[2]_q\,[5]_q\,[6]_q\,[8]_q\,[9]_q\,[12]_q}.$$

If $m \equiv 0 \pmod{30}$, then we have

$$\mathrm{Cat}^m(E_6;q) = [6m+1]_{q^2} \left[\tfrac{12m+5}{5}\right]_{q^5} [2m+1]_{q^6} \left[\tfrac{3m+2}{2}\right]_{q^8} \left[\tfrac{4m+3}{3}\right]_{q^9} [m+1]_{q^{12}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 1 \pmod{30}$, then we have

$$\begin{aligned}\mathrm{Cat}^m(E_6;q) = [6m+1]_{q^2} \left[\tfrac{2m+1}{3}\right]_{q^{18}} [4m+3]_{q^3} \frac{[6]_{q^3}}{[2]_{q^3}[3]_{q^3}}&\\ \times \left[\tfrac{3m+2}{5}\right]_{q^{20}} [12m+5]_q \frac{[20]_q}{[4]_q\,[5]_q} \left[\tfrac{m+1}{2}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}&.\end{aligned}$$

If one decomposes $[6m+1]_{q^2}$ as $[3m+1]_{q^4} + q^2[3m]_{q^4}$, then one sees that, by Corollary 6, the above expression is a polynomial in q with non-negative integer coefficients. If $m \equiv 2 \pmod{30}$, then we have

$$\begin{aligned}\mathrm{Cat}^m(E_6;q) = [6m+1]_{q^2} [12m+5]_q \left[\tfrac{2m+1}{5}\right]_{q^{30}} \frac{[30]_q}{[5]_q[6]_q}&\\ \times \left[\tfrac{3m+2}{2}\right]_{q^8} [4m+3]_{q^3} \left[\tfrac{m+1}{3}\right]_{q^{36}} \frac{[12]_{q^3}}{[3]_{q^3}[4]_{q^3}}&,\end{aligned}$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 3 \pmod{30}$, then we have

$$\begin{aligned}\mathrm{Cat}^m(E_6;q) = [6m+1]_{q^2} [12m+5]_q [2m+1]_{q^6} [3m+2]_{q^4}&\\ \times \left[\tfrac{4m+3}{15}\right]_{q^{45}} \frac{[45]_q}{[5]_q[9]_q} \left[\tfrac{m+1}{2}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}&,\end{aligned}$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 4 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = \left[\tfrac{12m+2}{5}\right]_{q^5}\left[\tfrac{12m+5}{2}\right]_{q^2}\left[\tfrac{2m+1}{3}\right]_{q^{18}}\frac{[6]_{q^3}}{[2]_{q^3}[3]_{q^3}} \times \left[\tfrac{3m+2}{2}\right]_{q^8}[4m+3]_{q^3}[m+1]_{q^{12}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 5 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}\left[\tfrac{12m+5}{5}\right]_{q^5}[2m+1]_{q^6} \times [3m+2]_{q^4}[4m+3]_{q^3}\left[\tfrac{m+1}{6}\right]_{q^{72}}\frac{[72]_q\,[3]_q\,[4]_q}{[8]_q\,[9]_q\,[12]_q},$$

which, by Lemma 7, is a polynomial in q with non-negative integer coefficients. If $m \equiv 6 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q[2m+1]_{q^6}\left[\tfrac{3m+2}{10}\right]_{q^{40}}\frac{[40]_q}{[5]_q[8]_q}\left[\tfrac{4m+3}{3}\right]_{q^9}[m+1]_{q^{12}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 7 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = \left[\tfrac{6m+1}{2}\right]_{q^4}[12m+5]_q\left[\tfrac{2m+1}{15}\right]_{q^{90}} \times \frac{[90]_q\,[3]_q\,[4]_q}{[5]_q\,[6]_q\,[9]_q}[3m+2]_{q^4}[4m+3]_{q^3}\left[\tfrac{m+1}{2}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}\,[3]_{q^4}},$$

which, by Corollary 6 and Lemma 9, is a polynomial in q with non-negative integer coefficients. If $m \equiv 8 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q[2m+1]_{q^6}\left[\tfrac{3m+2}{2}\right]_{q^8} \times \left[\tfrac{4m+3}{5}\right]_{q^{15}}\frac{[15]_q}{[3]_q\,[5]_q}\left[\tfrac{m+1}{3}\right]_{q^{36}}\frac{[12]_{q^3}}{[3]_{q^3}[4]_{q^3}},$$

which, by Lemma 11, is a polynomial in q with non-negative integer coefficients. If $m \equiv 9 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = \left[\tfrac{12m+2}{5}\right]_{q^5}\left[\tfrac{12m+5}{2}\right]_{q^2}[2m+1]_{q^6}[3m+2]_{q^4}\left[\tfrac{4m+3}{3}\right]_{q^9}\left[\tfrac{m+1}{2}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 10 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}\left[\tfrac{12m+5}{5}\right]_{q^5}\left[\tfrac{2m+1}{3}\right]_{q^{18}}\frac{[6]_{q^3}}{[2]_{q^3}[3]_{q^3}}\left[\tfrac{3m+2}{2}\right]_{q^8}[4m+3]_{q^3}[m+1]_{q^{12}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 11 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q[2m+1]_{q^6} \\ \times \left[\tfrac{3m+2}{5}\right]_{q^{20}} \frac{[20]_q}{[4]_q\,[5]_q}[4m+3]_{q^3}\left[\tfrac{m+1}{6}\right]_{q^{72}} \frac{[72]_q\,[3]_q\,[4]_q}{[8]_q\,[9]_q\,[12]_q}.$$

If one decomposes $[6m+1]_{q^2}$ as $[3m+1]_{q^4}+q^2[3m]_{q^4}$, then one sees that, by Corollary 6 and Lemma 7, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 12 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q\left[\tfrac{2m+1}{5}\right]_{q^{30}} \frac{[30]_q}{[5]_q[6]_q}\left[\tfrac{3m+2}{2}\right]_{q^8}\left[\tfrac{4m+3}{3}\right]_{q^9}[m+1]_{q^{12}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 13 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q\left[\tfrac{2m+1}{3}\right]_{q^{18}} \frac{[6]_{q^3}}{[2]_{q^3}[3]_{q^3}} \\ \times [3m+2]_{q^4}\left[\tfrac{4m+3}{5}\right]_{q^{15}} \frac{[15]_q}{[3]_q\,[5]_q}\left[\tfrac{m+1}{2}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}},$$

which, by Lemma 12, is a polynomial in q with non-negative integer coefficients. If $m \equiv 14 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = \left[\tfrac{12m+2}{5}\right]_{q^5}\left[\tfrac{12m+5}{2}\right]_{q^2}[2m+1]_{q^6}\left[\tfrac{3m+2}{2}\right]_{q^8}[4m+3]_{q^3}\left[\tfrac{m+1}{3}\right]_{q^{36}} \frac{[12]_{q^3}}{[3]_{q^3}[4]_{q^3}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 15 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}\left[\tfrac{12m+5}{5}\right]_{q^5}[2m+1]_{q^6}[3m+2]_{q^4}\left[\tfrac{4m+3}{3}\right]_{q^9}\left[\tfrac{m+1}{2}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 16 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q\left[\tfrac{2m+1}{3}\right]_{q^{18}} \frac{[6]_{q^3}}{[2]_{q^3}[3]_{q^3}} \\ \times \left[\tfrac{3m+2}{10}\right]_{q^{40}} \frac{[40]_q}{[5]_q[8]_q}[4m+3]_{q^3}[m+1]_{q^{12}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 17 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q\left[\tfrac{2m+1}{5}\right]_{q^{30}} \frac{[30]_q}{[5]_q\,[6]_q} \\ \times [3m+2]_{q^4}[4m+3]_{q^3}\left[\tfrac{m+1}{6}\right]_{q^{72}} \frac{[72]_q\,[3]_q\,[4]_q}{[8]_q\,[9]_q\,[12]_q},$$

which, by Corollary 6 and Lemma 7, is a polynomial in q with non-negative integer coefficients. If $m \equiv 18 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q[2m+1]_{q^6}\left[\tfrac{3m+2}{2}\right]_{q^8}\left[\tfrac{4m+3}{15}\right]_{q^{45}}\frac{[45]_q}{[5]_q[9]_q}[m+1]_{q^{12}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 19 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = \left[\tfrac{12m+2}{5}\right]_{q^5}\left[\tfrac{12m+5}{2}\right]_{q^2}\left[\tfrac{2m+1}{3}\right]_{q^{18}}\frac{[6]_{q^3}}{[2]_{q^3}[3]_{q^3}}\\ \times [3m+2]_{q^4}[4m+3]_{q^3}\left[\tfrac{m+1}{2}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 20 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}\left[\tfrac{12m+5}{5}\right]_{q^5}[2m+1]_{q^6}\left[\tfrac{3m+2}{2}\right]_{q^8}[4m+3]_{q^3}\left[\tfrac{m+1}{3}\right]_{q^{36}}\frac{[12]_{q^3}}{[3]_{q^3}[4]_{q^3}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 21 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q[2m+1]_{q^6}\\ \times \left[\tfrac{3m+2}{5}\right]_{q^{20}}\frac{[20]_q}{[4]_q[5]_q}\left[\tfrac{4m+3}{3}\right]_{q^9}\left[\tfrac{m+1}{2}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 22 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q\left[\tfrac{2m+1}{15}\right]_{q^{90}}\frac{[90]_q[3]_q}{[5]_q[6]_q[9]_q}\\ \times \left[\tfrac{3m+2}{2}\right]_{q^8}[4m+3]_{q^3}[m+1]_{q^{12}},$$

which, by Lemma 10, is a polynomial in q with non-negative integer coefficients. If $m \equiv 23 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q[2m+1]_{q^6}\\ \times [3m+2]_{q^4}\left[\tfrac{4m+3}{5}\right]_{q^{15}}\frac{[15]_q}{[3]_q[5]_q}\left[\tfrac{m+1}{6}\right]_{q^{72}}\frac{[72]_q[3]_q[4]_q}{[8]_q[9]_q[12]_q},$$

which, by Lemma 8, is a polynomial in q with non-negative integer coefficients. If $m \equiv 24 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = \left[\tfrac{12m+2}{5}\right]_{q^5}\left[\tfrac{12m+5}{2}\right]_{q^2}[2m+1]_{q^6}\left[\tfrac{3m+2}{2}\right]_{q^8}\left[\tfrac{4m+3}{3}\right]_{q^9}[m+1]_{q^{12}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 25 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2} \left[\tfrac{12m+5}{5}\right]_{q^5} \left[\tfrac{2m+1}{3}\right]_{q^{18}} \frac{[6]_{q^3}}{[2]_{q^3}[3]_{q^3}} \\ \times [3m+2]_{q^4}[4m+3]_{q^3} \left[\tfrac{m+1}{2}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 26 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q[2m+1]_{q^6} \\ \times \left[\tfrac{3m+2}{10}\right]_{q^{40}} \frac{[40]_q}{[5]_q[8]_q}[4m+3]_{q^3} \left[\tfrac{m+1}{3}\right]_{q^{36}} \frac{[12]_{q^3}}{[3]_{q^3}[4]_{q^3}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 27 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q \left[\tfrac{2m+1}{5}\right]_{q^{30}} \frac{[30]_q}{[5]_q[6]_q} \\ \times [3m+2]_{q^4} \left[\tfrac{4m+3}{3}\right]_{q^9} \left[\tfrac{m+1}{2}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 28 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = [6m+1]_{q^2}[12m+5]_q \left[\tfrac{2m+1}{3}\right]_{q^{18}} \frac{[6]_{q^3}}{[2]_{q^3}[3]_{q^3}} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^8} \left[\tfrac{4m+3}{5}\right]_{q^{15}} \frac{[15]_q}{[3]_q[5]_q}[m+1]_{q^{12}},$$

which, by Lemma 12, is a polynomial in q with non-negative integer coefficients. If $m \equiv 29 \pmod{30}$, then we have

$$\operatorname{Cat}^m(E_6;q) = \left[\tfrac{6m+1}{5}\right]_{q^{10}} \frac{[10]_q}{[2]_q[5]_q}[12m+5]_q[2m+1]_{q^6} \\ \times [3m+2]_{q^4}[4m+3]_{q^3} \left[\tfrac{m+1}{6}\right]_{q^{72}} \frac{[72]_q[3]_q[4]_q}{[8]_q[9]_q[12]_q},$$

which, by Corollary 6 and Lemma 7, is a polynomial in q with non-negative integer coefficients.

For $W = G_{36} = E_7$, the degrees are $2, 6, 8, 10, 12, 14, 18$, and hence

$$\operatorname{Cat}^m(E_7;q) = \frac{[18m+2]_q[18m+6]_q[18m+8]_q[18m+10]_q}{[2]_q[6]_q[8]_q[10]_q} \\ \times \frac{[18m+12]_q[18m+14]_q[18m+18]_q}{[12]_q[14]_q[18]_q}.$$

If $m \equiv 0 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8}\left[\tfrac{9m+5}{5}\right]_{q^{10}}\left[\tfrac{3m+2}{2}\right]_{q^{12}}\left[\tfrac{9m+7}{7}\right]_{q^{14}}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 1 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2}\left[\tfrac{9m+5}{7}\right]_{q^{14}}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 2 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}}\left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2}$$
$$\times\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 3 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}}\left[\tfrac{3m+1}{10}\right]_{q^{60}}\frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}}[9m+4]_{q^2}$$
$$\times\left[\tfrac{9m+5}{4}\right]_{q^8}[3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 4 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{20}\right]_{q^{40}}\frac{[20]_{q^2}}{[4]_{q^2}[5]_{q^2}}[9m+5]_{q^2}$$
$$\times\left[\tfrac{3m+2}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 5 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{7}\right]_{q^{14}}\left[\tfrac{9m+5}{5}\right]_{q^{10}}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 6 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}[3m+1]_{q^6}\left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2}$$
$$\times\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 7 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2}$$
$$\times[9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 8 \pmod{140}$, then we have

$$\mathrm{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{5}\right]_{q^{30}} \frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}} \left[\tfrac{9m+4}{4}\right]_{q^8} \\ \times \left[\tfrac{9m+5}{7}\right]_{q^{14}} \left[\tfrac{3m+2}{2}\right]_{q^{12}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 9 \pmod{140}$, then we have

$$\mathrm{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{14}\right]_{q^{84}} \frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}} \left[\tfrac{9m+4}{5}\right]_{q^{10}} \\ \times [9m+5]_{q^2} [3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 10 \pmod{140}$, then we have

$$\mathrm{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}} [3m+1]_{q^6} \left[\tfrac{9m+4}{2}\right]_{q^4} \left[\tfrac{9m+5}{5}\right]_{q^{10}} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 11 \pmod{140}$, then we have

$$\mathrm{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \left[\tfrac{9m+5}{4}\right]_{q^8} \\ \times \left[\tfrac{3m+2}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 12 \pmod{140}$, then we have

$$\mathrm{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{28}\right]_{q^{56}} \frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{12}} \left[\tfrac{9m+7}{5}\right]_{q^{10}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 13 \pmod{140}$, then we have

$$\mathrm{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{10}\right]_{q^{60}} \frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}} [9m+4]_{q^2} \\ \times [9m+5]_{q^2} [3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 14 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{10}\right]_{q^{20}}\frac{[10]_{q^2}}{[2]_{q^2}\,[5]_{q^2}} \\ \times [9m+5]_{q^2}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}\left[\tfrac{9m+7}{7}\right]_{q^{14}}[m+1]_{q^{18}}.$$

If one decomposes $[9m+5]_{q^2}$ as $[\frac{9m}{2}+3]_{q^4}+q^2[\frac{9m}{2}+2]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 15 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2} \\ \times \left[\tfrac{9m+5}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}[3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 16 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}\left[\tfrac{3m+1}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}}\left[\tfrac{9m+4}{4}\right]_{q^8}[9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 17 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}}\left[\tfrac{3m+1}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+4]_{q^2}\left[\tfrac{9m+5}{2}\right]_{q^4} \\ \times [3m+2]_{q^6}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 18 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{5}\right]_{q^{30}}\frac{[15]_{q^2}}{[3]_{q^2}\,[5]_{q^2}} \\ \times \left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2}\left[\tfrac{3m+2}{28}\right]_{q^{168}}\frac{[84]_{q^2}\,[2]_{q^2}}{[4]_{q^2}\,[6]_{q^2}\,[7]_{q^2}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6 and Lemma 13, is a polynomial in q with non-negative integer coefficients. If $m \equiv 19 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}[9m+5]_{q^2} \\ \times [3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 20 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8}\left[\tfrac{9m+5}{5}\right]_{q^{10}}\left[\tfrac{3m+2}{2}\right]_{q^{12}} \times [9m+7]_{q^2}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 21 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}\left[\tfrac{3m+1}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+4]_{q^2}\left[\tfrac{9m+5}{2}\right]_{q^4} \times [3m+2]_{q^6}\left[\tfrac{9m+7}{7}\right]_{q^{14}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 22 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{2}\right]_{q^4}\left[\tfrac{9m+5}{7}\right]_{q^{14}}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} \times \left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 23 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{35}\right]_{q^{210}}\frac{[105]_{q^2}}{[3]_{q^2}[5]_{q^2}[7]_{q^2}}[9m+4]_{q^2}[9m+5]_{q^2} \times [3m+2]_{q^6}[9m+7]_{q^2}\left[\tfrac{m+1}{2}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}},$$

which, by Corollary 6 and Lemma 14, is a polynomial in q with non-negative integer coefficients. If $m \equiv 24 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}}[3m+1]_{q^6}\left[\tfrac{9m+4}{20}\right]_{q^{40}}\frac{[20]_{q^2}}{[4]_{q^2}[5]_{q^2}}[9m+5]_{q^2} \times \left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 25 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{2}\right]_{q^{12}} \times [9m+4]_{q^2}\left[\tfrac{9m+5}{5}\right]_{q^{10}}\left[\tfrac{3m+2}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 26 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} [3m+1]_{q^6} \left[\tfrac{9m+4}{14}\right]_{q^{28}} \frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}\,[3]_{q^4}} [9m+7]_{q^2} [m+1]_{q^{18}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4}+q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 27 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \\ \times [9m+5]_{q^2} [3m+2]_{q^6} \left[\tfrac{9m+7}{5}\right]_{q^{10}} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 28 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{5}\right]_{q^{30}} \frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}} \left[\tfrac{9m+4}{4}\right]_{q^8} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{12}} \left[\tfrac{9m+7}{7}\right]_{q^{14}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 29 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{2}\right]_{q^{12}} \left[\tfrac{9m+4}{5}\right]_{q^{10}} \left[\tfrac{9m+5}{7}\right]_{q^{14}} [3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 30 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}} \left[\tfrac{9m+4}{2}\right]_{q^4} \left[\tfrac{9m+5}{5}\right]_{q^{10}} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}\,[3]_{q^4}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 31 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{35}\right]_{q^{70}} \frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \\ \times \left[\tfrac{9m+5}{4}\right]_{q^8} [3m+2]_{q^6} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 32 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8} \\ \times [9m+5]_{q^2}\left[\tfrac{3m+2}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 33 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{10}\right]_{q^{60}}\frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}}\left[\tfrac{9m+4}{7}\right]_{q^{14}} \\ \times [9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 34 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{10}\right]_{q^{20}}\frac{[10]_{q^2}}{[2]_{q^2}[5]_{q^2}}[9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4}+q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 35 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2}\left[\tfrac{9m+5}{5}\right]_{q^{10}}[3m+2]_{q^6}\left[\tfrac{9m+7}{7}\right]_{q^{14}}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 36 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8}\left[\tfrac{9m+5}{7}\right]_{q^{14}}\left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 37 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[9m+4]_{q^2} \\ \times [9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{20}\right]_{q^{40}}\frac{[20]_{q^2}}{[4]_{q^2}[5]_{q^2}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 38 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}}\left[\tfrac{3m+1}{5}\right]_{q^{30}}\frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}}\left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 39 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{5}\right]_{q^{10}}[9m+5]_{q^2} \\ \times\left[\tfrac{3m+2}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 40 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}} \\ \times\left[\tfrac{9m+5}{5}\right]_{q^{10}}\left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 41 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2} \\ \times[9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 42 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6} \\ \times\left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}\left[\tfrac{9m+7}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 43 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{10}\right]_{q^{60}}\frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}}[9m+4]_{q^2} \\ \times\left[\tfrac{9m+5}{7}\right]_{q^{14}}[3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 44 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}}\left[\tfrac{9m+4}{20}\right]_{q^{40}}\frac{[20]_{q^2}}{[4]_{q^2}[5]_{q^2}}[9m+5]_{q^2} \\ \times\left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 45 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2}\left[\tfrac{9m+5}{5}\right]_{q^{10}}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 46 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}[3m+1]_{q^6} \times \left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2}\left[\tfrac{3m+2}{28}\right]_{q^{168}}\frac{[84]_{q^2}\,[2]_{q^2}}{[4]_{q^2}\,[6]_{q^2}\,[7]_{q^2}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Lemma 13, is a polynomial in q with non-negative integer coefficients. If $m \equiv 47 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{7}\right]_{q^{14}}[9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 48 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{5}\right]_{q^{30}}\frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}}\left[\tfrac{9m+4}{4}\right]_{q^8}[9m+5]_{q^2} \times \left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 49 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}\left[\tfrac{9m+4}{5}\right]_{q^{10}}\left[\tfrac{9m+5}{2}\right]_{q^4} \times [3m+2]_{q^6}\left[\tfrac{9m+7}{7}\right]_{q^{14}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 50 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{2}\right]_{q^4} \times \left[\tfrac{9m+5}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 51 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}\left[\tfrac{3m+1}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[9m+4]_{q^2} \times \left[\tfrac{9m+5}{4}\right]_{q^8}[3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 52 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8}[9m+5]_{q^2}\left[\tfrac{3m+2}{2}\right]_{q^{12}}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 53 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{10}\right]_{q^{60}} \frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}} [9m+4]_{q^2} \\ \times [9m+5]_{q^2} \left[\tfrac{3m+2}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}} \left[\tfrac{9m+7}{4}\right]_{q^8} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 54 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{70}\right]_{q^{140}} \frac{[70]_{q^2}}{[2]_{q^2}[5]_{q^2}[7]_{q^2}} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+7]_{q^2} [m+1]_{q^{18}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4} + q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6 and Lemma 15, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 55 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \left[\tfrac{9m+5}{5}\right]_{q^{10}} [3m+2]_{q^6} \\ \times [9m+7]_{q^2} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 56 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} [3m+1]_{q^6} \left[\tfrac{9m+4}{4}\right]_{q^8} [9m+5]_{q^2} \left[\tfrac{3m+2}{2}\right]_{q^{12}} \left[\tfrac{9m+7}{7}\right]_{q^{14}} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 57 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{2}\right]_{q^4} \left[\tfrac{3m+1}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} \\ \times [9m+4]_{q^2} \left[\tfrac{9m+5}{7}\right]_{q^{14}} [3m+2]_{q^6} \left[\tfrac{9m+7}{5}\right]_{q^{10}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 58 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{35}\right]_{q^{210}} \frac{[105]_{q^2}}{[3]_{q^2}[5]_{q^2}[7]_{q^2}} \left[\tfrac{9m+4}{2}\right]_{q^4} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6 and Lemma 14, is a polynomial in q with non-negative integer coefficients. If $m \equiv 59 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}} \left[\tfrac{3m+1}{2}\right]_{q^{12}} \left[\tfrac{9m+4}{5}\right]_{q^{10}} \left[\tfrac{9m+5}{4}\right]_{q^8} [3m+2]_{q^6} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 60 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8}\left[\tfrac{9m+5}{5}\right]_{q^{10}} \\ \times \left[\tfrac{3m+2}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 61 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{7}\right]_{q^{14}}[9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 62 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6} \\ \times \left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}\left[\tfrac{9m+7}{5}\right]_{q^{10}}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 63 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{10}\right]_{q^{60}}\frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}}[9m+4]_{q^2} \\ \times [9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{7}\right]_{q^{14}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 64 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{20}\right]_{q^{40}}\frac{[20]_{q^2}}{[4]_{q^2}[5]_{q^2}} \\ \times \left[\tfrac{9m+5}{7}\right]_{q^{14}}\left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 65 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[9m+4]_{q^2} \\ \times \left[\tfrac{9m+5}{5}\right]_{q^{10}}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 66 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}[3m+1]_{q^6}\left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 67 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \\ \times [9m+5]_{q^2} \left[\tfrac{3m+2}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}} \left[\tfrac{9m+7}{5}\right]_{q^{10}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 68 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{5}\right]_{q^{30}} \frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}} \\ \times \left[\tfrac{9m+4}{28}\right]_{q^{56}} \frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}} [9m+5]_{q^2} \left[\tfrac{3m+2}{2}\right]_{q^{12}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 69 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{2}\right]_{q^{12}} \left[\tfrac{9m+4}{5}\right]_{q^{10}} \\ \times [9m+5]_{q^2} [3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 70 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{2}\right]_{q^4} \left[\tfrac{9m+5}{5}\right]_{q^{10}} \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} \\ \times \left[\tfrac{9m+7}{7}\right]_{q^{14}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 71 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{20}\right]_{q^{40}} \frac{[20]_{q^2}}{[4]_{q^2}[5]_{q^2}} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \\ \times \left[\tfrac{9m+5}{7}\right]_{q^{14}} [3m+2]_{q^6} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 72 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}} \left[\tfrac{9m+4}{4}\right]_{q^8} \\ \times [9m+5]_{q^2} \left[\tfrac{3m+2}{2}\right]_{q^{12}} \left[\tfrac{9m+7}{5}\right]_{q^{10}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 73 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}} \left[\tfrac{3m+1}{10}\right]_{q^{60}} \frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}} [9m+4]_{q^2} \\ \times [9m+5]_{q^2} [3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 74 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{10}\right]_{q^{20}} \frac{[10]_{q^2}}{[2]_{q^2}\,[5]_{q^2}} \\ \times [9m+5]_{q^2} \left[\tfrac{3m+2}{28}\right]_{q^{168}} \frac{[84]_{q^2}\,[2]_{q^2}}{[4]_{q^2}\,[6]_{q^2}\,[7]_{q^2}} [9m+7]_{q^2} [m+1]_{q^{18}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4} + q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6 and Lemma 13, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 75 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{2}\right]_{q^{12}} \left[\tfrac{9m+4}{7}\right]_{q^{14}} \left[\tfrac{9m+5}{5}\right]_{q^{10}} [3m+2]_{q^6} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 76 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} [3m+1]_{q^6} \left[\tfrac{9m+4}{4}\right]_{q^8} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{12}} [9m+7]_{q^2} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 77 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+4]_{q^2} \\ \times \left[\tfrac{9m+5}{2}\right]_{q^4} [3m+2]_{q^6} \left[\tfrac{9m+7}{35}\right]_{q^{70}} \frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 78 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{5}\right]_{q^{30}} \frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}} \\ \times \left[\tfrac{9m+4}{2}\right]_{q^4} \left[\tfrac{9m+5}{7}\right]_{q^{14}} \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 79 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{14}\right]_{q^{84}} \frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}} \left[\tfrac{9m+4}{5}\right]_{q^{10}} [9m+5]_{q^2} \\ \times [3m+2]_{q^6} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 80 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}} [3m+1]_{q^6} \left[\tfrac{9m+4}{4}\right]_{q^8} \left[\tfrac{9m+5}{5}\right]_{q^{10}} \left[\tfrac{3m+2}{2}\right]_{q^{12}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 81 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \\ \times [9m+5]_{q^2} \left[\tfrac{3m+2}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}} \left[\tfrac{9m+7}{4}\right]_{q^8} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 82 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{14}\right]_{q^{28}} \frac{[14]_{q^2}}{[2]_{q^2}[7]_{q^2}} \\ \times [9m+5]_{q^2} \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} \left[\tfrac{9m+7}{5}\right]_{q^{10}} [m+1]_{q^{18}}.$$

If one decomposes $[9m+5]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4}+q^2[\frac{9m}{2}+2]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 83 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{10}\right]_{q^{60}} \frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}} [9m+4]_{q^2} [9m+5]_q \\ \times [3m+2]_{q^6} [9m+7]_{q^2} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 84 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{20}\right]_{q^{40}} \frac{[20]_{q^2}}{[4]_{q^2}[5]_{q^2}} \\ \times [9m+5]_{q^2} \left[\tfrac{3m+2}{2}\right]_{q^{12}} \left[\tfrac{9m+7}{7}\right]_{q^{14}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 85 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{2}\right]_{q^{12}} \\ \times [9m+4]_{q^2} \left[\tfrac{9m+5}{35}\right]_{q^{70}} \frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}} [3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 86 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} \left[\tfrac{3m+1}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}} \left[\tfrac{9m+4}{2}\right]_{q^4} [9m+5]_{q^2} \times \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 87 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \left[\tfrac{9m+5}{4}\right]_{q^8} [3m+2]_{q^6} \left[\tfrac{9m+7}{5}\right]_{q^{10}} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 88 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{5}\right]_{q^{30}} \frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}} \left[\tfrac{9m+4}{4}\right]_{q^8} [9m+5]_{q^2} \times \left[\tfrac{3m+2}{14}\right]_{q^{84}} \frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 89 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{2}\right]_{q^{12}} \left[\tfrac{9m+4}{35}\right]_{q^{70}} \frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}} \times [9m+5]_{q^2} [3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 90 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{2}\right]_{q^4} \left[\tfrac{9m+5}{5}\right]_{q^{10}} \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} \times [9m+7]_{q^2} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 91 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \left[\tfrac{9m+5}{4}\right]_{q^8} [3m+2]_{q^6} \left[\tfrac{9m+7}{7}\right]_{q^{14}} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 92 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{4}\right]_{q^8} \left[\tfrac{9m+5}{7}\right]_{q^{14}} \left[\tfrac{3m+2}{2}\right]_{q^{12}} \left[\tfrac{9m+7}{5}\right]_{q^{10}} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 93 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{35}\right]_{q^{210}} \frac{[105]_{q^2}}{[3]_{q^2}[5]_{q^2}[7]_{q^2}} [9m+4]_{q^2} \\ \times [9m+5]_{q^2}[3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} \left[\tfrac{m+1}{2}\right]_{q^{36}} \frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}},$$

which, by Corollary 6 and Lemma 14, is a polynomial in q with non-negative integer coefficients. If $m \equiv 94 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}} [3m+1]_{q^6} \left[\tfrac{9m+4}{10}\right]_{q^{20}} \frac{[10]_{q^2}}{[2]_{q^2}[5]_{q^2}} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+7]_{q^2}[m+1]_{q^{18}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4}+q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 95 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{2}\right]_{q^{12}} [9m+4]_{q^2} \left[\tfrac{9m+5}{5}\right]_{q^{10}} \\ \times \left[\tfrac{3m+2}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}} [9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 96 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} [3m+1]_{q^6} \left[\tfrac{9m+4}{28}\right]_{q^{56}} \frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{12}} [9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 97 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+4]_{q^2} \left[\tfrac{9m+5}{2}\right]_{q^4} \\ \times [3m+2]_{q^6} \left[\tfrac{9m+7}{5}\right]_{q^{10}} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 98 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{5}\right]_{q^{30}} \frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}} \\ \times \left[\tfrac{9m+4}{2}\right]_{q^4} [9m+5]_{q^2} \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} \left[\tfrac{9m+7}{7}\right]_{q^{14}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 99 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{5}\right]_{q^{10}}\left[\tfrac{9m+5}{7}\right]_{q^{14}}[3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 100 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}}\left[\tfrac{9m+4}{4}\right]_{q^8} \\ \times \left[\tfrac{9m+5}{5}\right]_{q^{10}}\left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 101 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2} \\ \times [9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 102 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6} \\ \times \left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2}\left[\tfrac{3m+2}{28}\right]_{q^{168}}\frac{[84]_{q^2}\,[2]_{q^2}}{[4]_{q^2}\,[6]_{q^2}\,[7]_{q^2}}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which, by Lemma 13, is a polynomial in q with non-negative integer coefficients. If $m \equiv 103 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{10}\right]_{q^{60}}\frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}}\left[\tfrac{9m+4}{7}\right]_{q^{14}}[9m+5]_{q^2} \\ \times [3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 104 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{20}\right]_{q^{40}}\frac{[20]_{q^2}}{[4]_{q^2}[5]_{q^2}}[9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 105 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} \\ \times [9m+4]_{q^2}\left[\tfrac{9m+5}{10}\right]_{q^{20}}\frac{[10]_{q^2}}{[2]_{q^2}\,[5]_{q^2}}[3m+2]_{q^6}\left[\tfrac{9m+7}{7}\right]_{q^{14}}[m+1]_{q^{18}}.$$

If one decomposes $[9m+1]_{q^2}$ as $[\frac{9m+1}{2}]_{q^4}+q^2[\frac{9m+1}{2}]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m\equiv 106 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q)=\left[\tfrac{9m+1}{5}\right]_{q^{10}}[3m+1]_{q^6}\left[\tfrac{9m+4}{2}\right]_{q^4}\left[\tfrac{9m+5}{7}\right]_{q^{14}}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}\\
\times[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m\equiv 107 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q)=\left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[9m+4]_{q^2}\\
\times[9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m\equiv 108 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q)=\left[\tfrac{9m+1}{7}\right]_{q^{14}}\left[\tfrac{3m+1}{5}\right]_{q^{30}}\frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}}\left[\tfrac{9m+4}{4}\right]_{q^8}[9m+5]_{q^2}\\
\times\left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m\equiv 109 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q)=[9m+1]_{q^2}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{5}\right]_{q^{10}}\\
\times[9m+5]_{q^2}\left[\tfrac{3m+2}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m\equiv 110 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q)=[9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}[7]_{q^2}}\\
\times\left[\tfrac{9m+5}{5}\right]_{q^{10}}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}[m+1]_{q^{18}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4}+q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m\equiv 111 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q)=\left[\tfrac{9m+1}{5}\right]_{q^{10}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2}\left[\tfrac{9m+5}{4}\right]_{q^8}[3m+2]_{q^6}\\
\times[9m+7]_{q^2}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 112 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8} \times [9m+5]_{q^2}\left[\tfrac{3m+2}{2}\right]_{q^{12}}\left[\tfrac{9m+7}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 113 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{10}\right]_{q^{60}}\frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}}[9m+4]_{q^2} \times \left[\tfrac{9m+5}{7}\right]_{q^{14}}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 114 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}}\left[\tfrac{9m+4}{10}\right]_{q^{20}}\frac{[10]_{q^2}}{[2]_{q^2}[5]_{q^2}}[9m+5]_{q^2} \times \left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}[m+1]_{q^{18}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4}+q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 115 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2} \times \left[\tfrac{9m+5}{5}\right]_{q^{10}}[3m+2]_{q^6}[9m+7]_{q}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 116 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8}[9m+5]_{q^2} \times \left[\tfrac{3m+2}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 117 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}\left[\tfrac{9m+4}{7}\right]_{q^{14}}\left[\tfrac{9m+5}{2}\right]_{q^4} \times [3m+2]_{q^6}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 118 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} \left[\tfrac{3m+1}{5}\right]_{q^{30}} \frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}} \\ \times \left[\tfrac{9m+4}{2}\right]_{q^4} [9m+5]_{q^2} \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} [9m+7]_{q^2} \left[\tfrac{m+1}{7}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 119 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{2}\right]_{q^{12}} \left[\tfrac{9m+4}{5}\right]_{q^{10}} [9m+5]_{q^2} [3m+2]_{q^6} \left[\tfrac{9m+7}{7}\right]_{q^{14}} [m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 120 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2} [3m+1]_{q^6} \left[\tfrac{9m+4}{4}\right]_{q^8} \\ \times \left[\tfrac{9m+5}{35}\right]_{q^{70}} \frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}} \left[\tfrac{3m+2}{2}\right]_{q^{12}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 121 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}} \left[\tfrac{3m+1}{14}\right]_{q^{84}} \frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}} [9m+4]_{q^2} \\ \times [9m+5]_{q^2} [3m+2]_{q^6} \left[\tfrac{9m+7}{4}\right]_{q^8} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 122 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}} [3m+1]_{q^6} \\ \times \left[\tfrac{9m+4}{2}\right]_{q^4} [9m+5]_{q^2} \left[\tfrac{3m+2}{4}\right]_{q^{24}} \frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}} \left[\tfrac{9m+7}{5}\right]_{q^{10}} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 123 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8} \left[\tfrac{3m+1}{10}\right]_{q^{60}} \frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}} [9m+4]_{q^2} [9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}} [9m+7]_{q^2} [m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 124 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}}[9m+5]_{q^2} \times \left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}\left[\tfrac{m+1}{5}\right]_{q^{90}}\frac{[45]_{q^2}}{[5]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 125 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{2}\right]_{q^{12}} \times [9m+4]_{q^2}\left[\tfrac{9m+5}{5}\right]_{q^{10}}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 126 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}[3m+1]_{q^6} \times \left[\tfrac{9m+4}{2}\right]_{q^4}[9m+5]_{q^2}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}\left[\tfrac{9m+7}{7}\right]_{q^{14}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 127 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}[9m+4]_{q^2}\left[\tfrac{9m+5}{7}\right]_{q^{14}}[3m+2]_{q^6}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 128 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{35}\right]_{q^{210}}\frac{[105]_{q^2}}{[3]_{q^2}[5]_{q^2}[7]_{q^2}}\left[\tfrac{9m+4}{4}\right]_{q^8}[9m+5]_{q^2} \times \left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Lemma 14, is a polynomial in q with non-negative integer coefficients. If $m \equiv 129 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{7}\right]_{q^{14}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{5}\right]_{q^{10}}[9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{4}\right]_{q^8}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 130 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6} \times \left[\tfrac{9m+4}{2}\right]_{q^4}\left[\tfrac{9m+5}{5}\right]_{q^{10}}\left[\tfrac{3m+2}{28}\right]_{q^{168}}\frac{[84]_{q^2}\,[2]_{q^2}}{[4]_{q^2}\,[6]_{q^2}\,[7]_{q^2}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Lemma 13, is a polynomial in q with non-negative integer coefficients. If $m \equiv 131 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{5}\right]_{q^{10}}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{7}\right]_{q^{14}}\left[\tfrac{9m+5}{4}\right]_{q^8}[3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 132 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8} \\ \times [9m+5]_{q^2}\left[\tfrac{3m+2}{2}\right]_{q^{12}}\left[\tfrac{9m+7}{5}\right]_{q^{10}}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 133 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{10}\right]_{q^{60}}\frac{[30]_{q^2}}{[5]_{q^2}[6]_{q^2}}[9m+4]_{q^2} \\ \times [9m+5]_{q^2}[3m+2]_{q^6}\left[\tfrac{9m+7}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 134 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}[3m+1]_{q^6}\left[\tfrac{9m+4}{10}\right]_{q^{20}}\frac{[10]_{q^2}}{[2]_{q^2}\,[5]_{q^2}} \\ \times \left[\tfrac{9m+5}{7}\right]_{q^{14}}\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}[m+1]_{q^{18}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4}+q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 135 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[9m+4]_{q^2} \\ \times \left[\tfrac{9m+5}{5}\right]_{q^{10}}[3m+2]_{q^6}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 136 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{35}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}[7]_{q^2}}[3m+1]_{q^6}\left[\tfrac{9m+4}{4}\right]_{q^8}[9m+5]_{q^2} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{12}}[9m+7]_{q^2}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 137 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+4]_{q^2}\left[\tfrac{9m+5}{2}\right]_{q^4} \\ \times \left[\tfrac{3m+2}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}}\left[\tfrac{9m+7}{5}\right]_{q^{10}}[m+1]_{q^{18}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 138 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = [9m+1]_{q^2}\left[\tfrac{3m+1}{5}\right]_{q^{30}}\frac{[15]_{q^2}}{[3]_{q^2}[5]_{q^2}}\left[\tfrac{9m+4}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}[7]_{q^2}}[9m+5]_{q^2}\\ \times\left[\tfrac{3m+2}{4}\right]_{q^{24}}\frac{[6]_{q^4}}{[2]_{q^4}[3]_{q^4}}[9m+7]_{q^2}[m+1]_{q^{18}}.$$

If one decomposes $[9m+7]_{q^2}$ as $[\frac{9m}{2}+4]_{q^4}+q^2[\frac{9m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 139 \pmod{140}$, then we have

$$\operatorname{Cat}^m(E_7;q) = \left[\tfrac{9m+1}{4}\right]_{q^8}\left[\tfrac{3m+1}{2}\right]_{q^{12}}\left[\tfrac{9m+4}{5}\right]_{q^{10}}[9m+5]_{q^2}\\ \times[3m+2]_{q^6}[9m+7]_{q^2}\left[\tfrac{m+1}{7}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients.

For $W = G_{37} = E_8$, the degrees are $2, 8, 12, 14, 18, 20, 24, 30$, and hence

$$\operatorname{Cat}^m(E_7;q) = \frac{[30m+2]_q\,[30m+8]_q\,[30m+12]_q\,[30m+14]_q}{[2]_q\,[8]_q\,[12]_q\,[14]_q}\\ \times\frac{[30m+18]_q\,[30m+20]_q\,[30m+24]_q\,[30m+30]_q}{[18]_q\,[20]_q\,[24]_q\,[30]_q}.$$

If $m \equiv 0 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{2}\right]_{q^{12}}\left[\tfrac{15m+7}{7}\right]_{q^{14}}\\ \times\left[\tfrac{5m+3}{3}\right]_{q^{18}}\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{4}\right]_{q^{24}}[m+1]_{q^{30}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 1 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}\left[\tfrac{5m+2}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}[7]_{q^2}}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{4}\right]_{q^{24}}\\ \times[3m+2]_{q^{10}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 2 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}}[15m+7]_{q^2}[5m+3]_{q^6}\\ \times\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 3 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4} \left[\tfrac{15m+4}{7}\right]_{q^{14}} [5m+2]_{q^6} \left[\tfrac{15m+7}{4}\right]_{q^8} \left[\tfrac{5m+3}{6}\right]_{q^{36}} \frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}} \\ \times [3m+2]_{q^{10}} [5m+4]_{q^6} \left[\tfrac{m+1}{4}\right]_{q^{120}} \frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 4 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{4}\right]_{q^8} \left[\tfrac{5m+2}{2}\right]_{q^{12}} \\ \times [15m+7]_{q^2} [5m+3]_{q^6} \left[\tfrac{3m+2}{14}\right]_{q^{140}} \frac{[70]_{q^2}}{[7]_{q^2}[10]_{q^2}} \left[\tfrac{5m+4}{12}\right]_{q^{72}} \frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 5 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8} [15m+4]_{q^2} \left[\tfrac{5m+2}{3}\right]_{q^{18}} \left[\tfrac{15m+7}{2}\right]_{q^4} \left[\tfrac{5m+3}{28}\right]_{q^{168}} \frac{[84]_{q^2}}{[7]_{q^2}\,[12]_{q^2}} \\ \times [3m+2]_{q^{10}} [5m+4]_{q^6} \left[\tfrac{m+1}{2}\right]_{q^{60}} \frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 6 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{7}\right]_{q^{14}} \left[\tfrac{15m+4}{2}\right]_{q^4} \left[\tfrac{5m+2}{4}\right]_{q^{24}} [15m+7]_{q^2} \\ \times \left[\tfrac{5m+3}{3}\right]_{q^{18}} \left[\tfrac{3m+2}{4}\right]_{q^{40}} \frac{[10]_{q^4}}{[2]_{q^4}\,[5]_{q^4}} \left[\tfrac{5m+4}{2}\right]_{q^{12}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 7 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4} [15m+4]_{q^2} [5m+2]_{q^6} \left[\tfrac{15m+7}{2}8\right]_{q^{56}} \frac{[28]_{q^2}}{[4]_{q^2}\,[7]_{q^2}} \left[\tfrac{5m+3}{2}\right]_{q^{12}} \\ \times [3m+2]_{q^{10}} \left[\tfrac{5m+4}{3}\right]_{q^{18}} \left[\tfrac{m+1}{4}\right]_{q^{120}} \frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 8 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{4}\right]_{q^8} \left[\tfrac{5m+2}{42}\right]_{q^{252}} \frac{[126]_{q^2}[3]_{q^2}}{[6]_{q^2}[7]_{q^2}[9]_{q^2}} [15m+7]_{q^2} [5m+3]_{q^6} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{20}} \left[\tfrac{5m+4}{4}\right]_{q^{24}} [m+1]_{q^{30}},$$

which, by Lemma 22, is a polynomial in q with non-negative integer coefficients. If $m \equiv 9 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = {} & \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}\,[4]_{q^6}}\\ & \times [3m+2]_{q^{10}}\left[\tfrac{5m+4}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},\end{aligned}$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 10 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = {} & [15m+1]_{q^2}\left[\tfrac{15m+4}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+2}{4}\right]_{q^{24}}\\ & \times [15m+7]_{q^2}[5m+3]_{q^6}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}\,[5]_{q^4}}\left[\tfrac{5m+4}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}}[m+1]_{q^{30}}.\end{aligned}$$

If one decomposes $[15m+7]_{q^2}$ as $[\frac{15m}{2}+4]_{q^4} + q^2[\frac{15m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 11 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = {} & \left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{2}\right]_{q^{12}}\\ & \times \left[\tfrac{3m+2}{7}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}\,[7]_{q^2}}[5m+4]_{q^6}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},\end{aligned}$$

which, by Corollary 6 and Lemma 20, is a polynomial in q with non-negative integer coefficients. If $m \equiv 12 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = {} & [15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{2}\right]_{q^{12}}\\ & \times [15m+7]_{q^2}\left[\tfrac{5m+3}{21}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}}\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{4}\right]_{q^{24}}[m+1]_{q^{30}},\end{aligned}$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 13 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = {} & \left[\tfrac{15m+1}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}\,[7]_{q^2}}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{4}\right]_{q^{24}}\\ & \times [3m+2]_{q^{10}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},\end{aligned}$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 14 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = {} & [15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}}\left[\tfrac{15m+7}{7}\right]_{q^{14}}\\ & \times [5m+3]_{q^6}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}[m+1]_{q^{30}},\end{aligned}$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 15 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4} [15m+4]_{q^2} \left[\tfrac{5m+2}{7}\right]_{q^{42}} \frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}} \left[\tfrac{15m+7}{4}\right]_{q^8} \left[\tfrac{5m+3}{6}\right]_{q^{36}} \frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}} \\ \times [3m+2]_{q^{10}} [5m+4]_{q^6} \left[\tfrac{m+1}{4}\right]_{q^{120}} \frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 16 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{4}\right]_{q^8} \left[\tfrac{5m+2}{2}\right]_{q^{12}} [15m+7]_{q^2} [5m+3]_{q^6} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{20}} \left[\tfrac{5m+4}{84}\right]_{q^{504}} \frac{[252]_{q^2}\,[3]_{q^2}}{[7]_{q^2}\,[9]_{q^2}\,[12]_{q^2}} [m+1]_{q^{30}},$$

which, by Lemma 23, is a polynomial in q with non-negative integer coefficients. If $m \equiv 17 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8} \left[\tfrac{15m+4}{7}\right]_{q^{14}} \left[\tfrac{5m+2}{3}\right]_{q^{18}} \left[\tfrac{15m+7}{2}\right]_{q^4} \left[\tfrac{5m+3}{4}\right]_{q^{24}} \\ \times [3m+2]_{q^{10}} [5m+4]_{q^6} \left[\tfrac{m+1}{2}\right]_{q^{60}} \frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 18 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{2}\right]_{q^4} \left[\tfrac{5m+2}{4}\right]_{q^{24}} [15m+7]_{q^2} \left[\tfrac{5m+3}{3}\right]_{q^{18}} \\ \left[\tfrac{3m+2}{28}\right]_{q^{280}} \frac{[140]_{q^2}[2]_{q^2}}{[4]_{q^2}[7]_{q^2}[10]_{q^2}} \left[\tfrac{5m+4}{2}\right]_{q^{12}} [m+1]_{q^{30}},$$

which, by Lemma 24, is a polynomial in q with non-negative integer coefficients. If $m \equiv 19 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4} [15m+4]_{q^2} [5m+2]_{q^6} \left[\tfrac{15m+7}{4}\right]_{q^8} \left[\tfrac{5m+3}{1}4\right]_{q^{84}} \frac{[42]_{q^2}}{[6]_{q^2}\,[7]_{q^2}} \\ \times [3m+2]_{q^{10}} \left[\tfrac{5m+4}{3}\right]_{q^{18}} \left[\tfrac{m+1}{4}\right]_{q^{120}} \frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 20 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{7}\right]_{q^{14}} \left[\tfrac{15m+4}{4}\right]_{q^8} \left[\tfrac{5m+2}{6}\right]_{q^{36}} \frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}} [15m+7]_{q^2} [5m+3]_{q^6} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{20}} \left[\tfrac{5m+4}{4}\right]_{q^{24}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 21 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+3}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}\,[4]_{q^6}} \times [3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 17, is a polynomial in q with non-negative integer coefficients. If $m \equiv 22 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{28}\right]_{q^{168}}\frac{[84]_{q^2}}{[7]_{q^2}[12]_{q^2}}[15m+7]_{q^2} \times [5m+3]_{q^6}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 23 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{2}\right]_{q^{12}} \times [3m+2]_{q^{10}}\left[\tfrac{5m+4}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 24 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}}\left[\tfrac{5m+2}{2}\right]_{q^{12}}[15m+7]_{q^2} \times \left[\tfrac{5m+3}{3}\right]_{q^{18}}\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{4}\right]_{q^{24}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 25 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{4}\right]_{q^{24}} \times \left[\tfrac{3m+2}{7}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Lemma 18, is a polynomial in q with non-negative integer coefficients. If $m \equiv 26 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}}[15m+7]_{q^2}\left[\tfrac{5m+3}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}} \times \left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}[m+1]_{q^{30}}.$$

If one decomposes $[15m+1]_{q^2}$ as $[5m+1]_{q^6} + q^2[5m]_{q^6} + q^4[5m]_{q^6}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If

$m \equiv 27 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{14}\right]_{q^{28}} \frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}} [15m+4]_{q^2} [5m+2]_{q^6} \left[\tfrac{15m+7}{4}\right]_{q^8} \left[\tfrac{5m+3}{6}\right]_{q^{36}} \frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}} \\ \times [3m+2]_{q^{10}} [5m+4]_{q^6} \left[\tfrac{m+1}{4}\right]_{q^{120}} \frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 21, is a polynomial in q with non-negative integer coefficients. If $m \equiv 28 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{4}\right]_{q^8} \left[\tfrac{5m+2}{2}\right]_{q^{12}} \left[\tfrac{15m+7}{7}\right]_{q^{14}} \\ \times [5m+3]_{q^6} \left[\tfrac{3m+2}{2}\right]_{q^{20}} \left[\tfrac{5m+4}{12}\right]_{q^{72}} \frac{[12]_{q^6}}{[3]_{q^6}\,[4]_{q^6}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 29 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8} [15m+4]_{q^2} \left[\tfrac{5m+2}{21}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}\,[9]_{q^2}} \left[\tfrac{15m+7}{2}\right]_{q^4} \left[\tfrac{5m+3}{4}\right]_{q^{24}} \\ \times [3m+2]_{q^{10}} [5m+4]_{q^6} \left[\tfrac{m+1}{2}\right]_{q^{60}} \frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 30 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{2}\right]_{q^4} \left[\tfrac{5m+2}{4}\right]_{q^{24}} [15m+7]_{q^2} \left[\tfrac{5m+3}{3}\right]_{q^{18}} \\ \left[\tfrac{3m+2}{4}\right]_{q^{40}} \frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}} \left[\tfrac{5m+4}{14}\right]_{q^{84}} \frac{[42]_{q^2}}{[6]_{q^2}\,[7]_{q^2}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 31 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4} \left[\tfrac{15m+4}{7}\right]_{q^{14}} [5m+2]_{q^6} \left[\tfrac{15m+7}{4}\right]_{q^8} \left[\tfrac{5m+3}{2}\right]_{q^{12}} \\ \times [3m+2]_{q^{10}} \left[\tfrac{5m+4}{3}\right]_{q^{18}} \left[\tfrac{m+1}{4}\right]_{q^{120}} \frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 32 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{4}\right]_{q^8} \left[\tfrac{5m+2}{6}\right]_{q^{36}} \frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}} [15m+7]_{q^2} [5m+3]_{q^6} \\ \times \left[\tfrac{3m+2}{14}\right]_{q^{140}} \frac{[70]_{q^2}}{[7]_{q^2}[10]_{q^2}} \left[\tfrac{5m+4}{4}\right]_{q^{24}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 33 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{84}\right]_{q^{504}}\frac{[252]_{q^2}\,[3]_{q^2}}{[7]_{q^2}\,[9]_{q^2}\,[12]_{q^2}}$$
$$\times [3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Lemmas 16 and 23, is a polynomial in q with non-negative integer coefficients. If $m \equiv 34 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{7}\right]_{q^{14}}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{4}\right]_{q^{24}}[15m+7]_{q^2}$$
$$[5m+3]_{q^6}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 35 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+3}{2}\right]_{q^{12}}$$
$$\times [3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 36 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[15m+7]_{q^2}\left[\tfrac{5m+3}{3}\right]_{q^{18}}$$
$$\times \left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{4}\right]_{q^{24}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 37 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{4}\right]_{q^{24}}$$
$$\times [3m+2]_{q^{10}}\left[\frac{5m+4}{21}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}\,[9]_{q^2}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 38 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+2}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}}$$
$$\times [15m+7]_{q^2}[5m+3]_{q^6}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}[m+1]_{q^{30}}.$$

If one decomposes $[15m+7]_{q^2}$ as $[\frac{15m}{2}+4]_{q^4}+q^2[\frac{15m}{2}+3]_{q^4}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv$

39 (mod 84), then we have

$$\operatorname{Cat}^m(E_8;q)=\left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}}\\ \times\left[\tfrac{3m+2}{7}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}\,[7]_{q^2}}[5m+4]_{q^6}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 20, is a polynomial in q with non-negative integer coefficients. If $m\equiv 40$ (mod 84), then we have

$$\operatorname{Cat}^m(E_8;q)=[15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{2}\right]_{q^{12}}[15m+7]_{q^2}\left[\tfrac{5m+3}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\\ \times\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}}[m+1]_{q^{30}}.$$

If one decomposes $[15m+7]_{q^2}$ as $[5m+1]_{q^6}+q^2[5m]_{q^6}+q^4[5m]_{q^6}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m\equiv 41$ (mod 84), then we have

$$\operatorname{Cat}^m(E_8;q)=\left[\tfrac{15m+1}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}\,[7]_{q^2}}[15m+4]_{q^2}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{4}\right]_{q^{24}}\\ \times[3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m\equiv 42$ (mod 84), then we have

$$\operatorname{Cat}^m(E_8;q)=[15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{4}\right]_{q^{24}}\left[\tfrac{15m+7}{7}\right]_{q^{14}}\\ \times\left[\tfrac{5m+3}{3}\right]_{q^{18}}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}\,[5]_{q^4}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m\equiv 43$ (mod 84), then we have

$$\operatorname{Cat}^m(E_8;q)=\left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}\left[\tfrac{5m+2}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{2}\right]_{q^{12}}\\ \times[3m+2]_{q^{10}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m\equiv 44$ (mod 84), then we have

$$\operatorname{Cat}^m(E_8;q)=[15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}}[15m+7]_{q^2}[5m+3]_{q^6}\\ \times\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{28}\right]_{q^{168}}\frac{[84]_{q^2}}{[7]_{q^2}[12]_{q^2}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 45 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}\left[\tfrac{15m+4}{7}\right]_{q^{14}}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}\,[4]_{q^6}}$$
$$\times [3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 46 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{4}\right]_{q^{24}}[15m+7]_{q^2}$$
$$[5m+3]_{q^6}\left[\tfrac{3m+2}{28}\right]_{q^{280}}\frac{[140]_{q^2}[2]_{q^2}}{[4]_{q^2}[7]_{q^2}[10]_{q^2}}\left[\tfrac{5m+4}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}}[m+1]_{q^{30}},$$

which, by Corollary 6 and Lemma 24, is a polynomial in q with non-negative integer coefficients. If $m \equiv 47 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}\,[7]_{q^2}}$$
$$\times [3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 48 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{7}\right]_{q^{14}}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{2}\right]_{q^{12}}[15m+7]_{q^2}$$
$$\left[\tfrac{5m+3}{3}\right]_{q^{18}}\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{4}\right]_{q^{24}}[m+1]_{q^{30}},$$

which is manifestly a polynomial in q with non-negative integer coefficients. If $m \equiv 49 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+3}{4}\right]_{q^{24}}$$
$$\times [3m+2]_{q^{10}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 50 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{28}\right]_{q^{168}}\frac{[84]_{q^2}}{[7]_{q^2}[12]_{q^2}}[15m+7]_{q^2}$$
$$\times [5m+3]_{q^6}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}\left[\tfrac{m+1}{3}\right]_{q^{90}}\frac{[15]_{q^6}}{[3]_{q^6}[5]_{q^6}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 51 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}}\\ \times [3m+2]_{q^{10}}\left[\tfrac{5m+4}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 52 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}}\left[\tfrac{5m+2}{2}\right]_{q^{12}}[15m+7]_{q^2}\\ \times [5m+3]_{q^6}\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 53 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{4}\right]_{q^{24}}\\ \times \left[\tfrac{3m+2}{7}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}\,[7]_{q^2}}[5m+4]_{q^6}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Lemma 18, is a polynomial in q with non-negative integer coefficients. If $m \equiv 54 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{4}\right]_{q^{24}}[15m+7]_{q^2}\\ \left[\tfrac{5m+3}{21}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}[9]_{q^2}}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}\,[5]_{q^4}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 55 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{2}\right]_{q^{12}}\\ \times [3m+2]_{q^{10}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Lemma 21, is a polynomial in q with non-negative integer coefficients. If $m \equiv 56 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}}\left[\tfrac{15m+7}{7}\right]_{q^{14}}[5m+3]_{q^6}\\ \times \left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{4}\right]_{q^{24}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 57 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}\left[\tfrac{5m+2}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}\,[4]_{q^6}}$$
$$\times [3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 58 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{4}\right]_{q^{24}}[15m+7]_{q^2}[5m+3]_{q^6}$$
$$\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{42}\right]_{q^{252}}\frac{[126]_{q^2}[3]_{q^2}}{[6]_{q^2}[7]_{q^2}[9]_{q^2}}[m+1]_{q^{30}},$$

which, by Lemma 22, is a polynomial in q with non-negative integer coefficients. If $m \equiv 59 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4}\left[\tfrac{15m+4}{7}\right]_{q^{14}}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{2}\right]_{q^{12}}$$
$$\times [3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 60 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{2}\right]_{q^{12}}[15m+7]_{q^2}\left[\tfrac{5m+3}{3}\right]_{q^{18}}$$
$$\times \left[\tfrac{3m+2}{14}\right]_{q^{140}}\frac{[70]_{q^2}}{[7]_{q^2}[10]_{q^2}}\left[\tfrac{5m+4}{4}\right]_{q^{24}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 61 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{28}\right]_{q^{168}}\frac{[84]_{q^2}}{[7]_{q^2}\,[12]_{q^2}}$$
$$\times [3m+2]_{q^{10}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 62 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{7}\right]_{q^{14}}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}}[15m+7]_{q^2}[5m+3]_{q^6}$$
$$\times \left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 63 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+3}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}}\\ \times[3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},\end{aligned}$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 64 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{14}\right]_{q^{84}}\frac{[42]_{q^2}}{[6]_{q^2}[7]_{q^2}}[15m+7]_{q^2}\\ \times[5m+3]_{q^6}\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}}[m+1]_{q^{30}},\end{aligned}$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 65 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{4}\right]_{q^{24}}\\ \times[3m+2]_{q^{10}}\left[\tfrac{5m+4}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},\end{aligned}$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 66 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+2}{4}\right]_{q^{24}}[15m+7]_{q^2}\\ \times\left[\tfrac{5m+3}{3}\right]_{q^{18}}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}\,[5]_{q^4}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}[m+1]_{q^{30}},\end{aligned}$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 67 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{2}\right]_{q^{12}}\\ \times\left[\tfrac{3m+2}{7}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}\,[7]_{q^2}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},\end{aligned}$$

which, by Lemma 20, is a polynomial in q with non-negative integer coefficients. If $m \equiv 68 \pmod{84}$, then we have

$$\begin{aligned}\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}}\\ \times[15m+7]_{q^2}\left[\tfrac{5m+3}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}}\left[\tfrac{3m+2}{2}\right]_{q^{20}}\left[\tfrac{5m+4}{4}\right]_{q^{24}}[m+1]_{q^{30}}.\end{aligned}$$

If one decomposes $[15m+1]_{q^2}$ as $[5m+1]_{q^6}+q^2[5m]_{q^6}+q^4[5m]_{q^6}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m\equiv 69 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q)=\left[\tfrac{15m+1}{28}\right]_{q^{56}}\frac{[28]_{q^2}}{[4]_{q^2}\,[7]_{q^2}}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}\,[4]_{q^6}}\\ \times[3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m\equiv 70 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q)=[15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{4}\right]_{q^{24}}\left[\tfrac{15m+7}{7}\right]_{q^{14}}\\ \times[5m+3]_{q^6}\left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m\equiv 71 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q)=\left[\tfrac{15m+1}{2}\right]_{q^4}[15m+4]_{q^2}\left[\tfrac{5m+2}{21}\right]_{q^{126}}\frac{[63]_{q^2}}{[7]_{q^2}\,[9]_{q^2}}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{2}\right]_{q^{12}}\\ \times[3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m\equiv 72 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q)=[15m+1]_{q^2}\left[\tfrac{15m+4}{4}\right]_{q^8}\left[\tfrac{5m+2}{2}\right]_{q^{12}}[15m+7]_{q^2}\left[\tfrac{5m+3}{3}\right]_{q^{18}}\left[\tfrac{3m+2}{2}\right]_{q^{20}}\\ \left[\tfrac{5m+4}{28}\right]_{q^{168}}\frac{[84]_{q^2}}{[7]_{q^2}[12]_{q^2}}[m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m\equiv 73 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q)=\left[\tfrac{15m+1}{4}\right]_{q^8}\left[\tfrac{15m+4}{7}\right]_{q^{14}}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{4}\right]_{q^{24}}\\ \times[3m+2]_{q^{10}}\left[\tfrac{5m+4}{3}\right]_{q^{18}}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m\equiv 74 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q)=[15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{4}\right]_{q^{24}}[15m+7]_{q^2}[5m+3]_{q^6}\\ \times\left[\tfrac{3m+2}{28}\right]_{q^{280}}\frac{[140]_{q^2}[2]_{q^2}}{[4]_{q^2}[7]_{q^2}[10]_{q^2}}\left[\tfrac{5m+4}{2}\right]_{q^{12}}\left[\tfrac{m+1}{3}\right]_{q^{90}}\frac{[15]_{q^6}}{[3]_{q^6}[5]_{q^6}},$$

which, by Corollary 6 and Lemma 24, is a polynomial in q with non-negative integer coefficients. If $m \equiv 75 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4} [15m+4]_{q^2} [5m+2]_{q^6} \left[\tfrac{15m+7}{4}\right]_{q^8} \left[\tfrac{5m+3}{42}\right]_{q^{252}} \frac{[126]_{q^2}\,[3]_{q^2}}{[6]_{q^2}\,[7]_{q^2}\,[9]_{q^2}} \\ \times [3m+2]_{q^{10}} [5m+4]_{q^6} \left[\tfrac{m+1}{4}\right]_{q^{120}} \frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Lemmas 19 and 22, is a polynomial in q with non-negative integer coefficients. If $m \equiv 76 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{7}\right]_{q^{14}} \left[\tfrac{15m+4}{4}\right]_{q^8} \left[\tfrac{5m+2}{2}\right]_{q^{12}} [15m+7]_{q^2} \\ \times [5m+3]_{q^6} \left[\tfrac{3m+2}{2}\right]_{q^{20}} \left[\tfrac{5m+4}{12}\right]_{q^{72}} \frac{[12]_{q^6}}{[3]_{q^6}[4]_{q^6}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 77 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8} [15m+4]_{q^2} \left[\tfrac{5m+2}{3}\right]_{q^{18}} \left[\tfrac{15m+7}{14}\right]_{q^{28}} \frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}} \left[\tfrac{5m+3}{4}\right]_{q^{24}} \\ \times [3m+2]_{q^{10}} [5m+4]_{q^6} \left[\tfrac{m+1}{2}\right]_{q^{60}} \frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 16, is a polynomial in q with non-negative integer coefficients. If $m \equiv 78 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{2}\right]_{q^4} \left[\tfrac{5m+2}{28}\right]_{q^{168}} \frac{[84]_{q^2}}{[7]_{q^2}[12]_{q^2}} [15m+7]_{q^2} \left[\tfrac{5m+3}{3}\right]_{q^{18}} \\ \times \left[\tfrac{3m+2}{4}\right]_{q^{40}} \frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}} \left[\tfrac{5m+4}{2}\right]_{q^{12}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 79 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{2}\right]_{q^4} [15m+4]_{q^2} [5m+2]_{q^6} \left[\tfrac{15m+7}{4}\right]_{q^8} \left[\tfrac{5m+3}{2}\right]_{q^{12}} \\ \times [3m+2]_{q^{10}} \left[\tfrac{5m+4}{21}\right]_{q^{126}} \frac{[63]_{q^2}}{[7]_{q^2}\,[9]_{q^2}} \left[\tfrac{m+1}{4}\right]_{q^{120}} \frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 19, is a polynomial in q with non-negative integer coefficients. If $m \equiv 80 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2} \left[\tfrac{15m+4}{28}\right]_{q^{56}} \frac{[28]_{q^2}}{[4]_{q^2}[7]_{q^2}} \left[\tfrac{5m+2}{6}\right]_{q^{36}} \frac{[6]_{q^6}}{[2]_{q^6}[3]_{q^6}} [15m+7]_{q^2} [5m+3]_{q^6} \\ \times \left[\tfrac{3m+2}{2}\right]_{q^{20}} \left[\tfrac{5m+4}{4}\right]_{q^{24}} [m+1]_{q^{30}},$$

which, by Corollary 6, is a polynomial in q with non-negative integer coefficients. If $m \equiv 81 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{4}\right]_{q^8}[15m+4]_{q^2}[5m+2]_{q^6}\left[\tfrac{15m+7}{2}\right]_{q^4}\left[\tfrac{5m+3}{12}\right]_{q^{72}}\frac{[12]_{q^6}}{[3]_{q^6}\,[4]_{q^6}} \\ \times \left[\tfrac{3m+2}{7}\right]_{q^{70}}\frac{[35]_{q^2}}{[5]_{q^2}\,[7]_{q^2}}[5m+4]_{q^6}\left[\tfrac{m+1}{2}\right]_{q^{60}}\frac{[30]_{q^2}[2]_{q^2}[3]_{q^2}[5]_{q^2}}{[6]_{q^2}[10]_{q^2}[15]_{q^2}},$$

which, by Corollary 6 and Lemma 18, is a polynomial in q with non-negative integer coefficients. If $m \equiv 82 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = [15m+1]_{q^2}\left[\tfrac{15m+4}{2}\right]_{q^4}\left[\tfrac{5m+2}{4}\right]_{q^{24}}[15m+7]_{q^2}\left[\tfrac{5m+3}{7}\right]_{q^{42}}\frac{[21]_{q^2}}{[3]_{q^2}\,[7]_{q^2}} \\ \left[\tfrac{3m+2}{4}\right]_{q^{40}}\frac{[10]_{q^4}}{[2]_{q^4}[5]_{q^4}}\left[\tfrac{5m+4}{6}\right]_{q^{36}}\frac{[6]_{q^6}}{[2]_{q^6}\,[3]_{q^6}}[m+1]_{q^{30}}.$$

If one decomposes $[15m+1]_{q^2}$ as $[5m+1]_{q^6} + q^2[5m]_{q^6} + q^4[5m]_{q^6}$, then one sees that, by Corollary 6, this is a polynomial in q with non-negative integer coefficients. If $m \equiv 83 \pmod{84}$, then we have

$$\operatorname{Cat}^m(E_8;q) = \left[\tfrac{15m+1}{14}\right]_{q^{28}}\frac{[14]_{q^2}}{[2]_{q^2}\,[7]_{q^2}}[15m+4]_{q^2}\left[\tfrac{5m+2}{3}\right]_{q^{18}}\left[\tfrac{15m+7}{4}\right]_{q^8}\left[\tfrac{5m+3}{2}\right]_{q^{12}} \\ \times [3m+2]_{q^{10}}[5m+4]_{q^6}\left[\tfrac{m+1}{4}\right]_{q^{120}}\frac{[60]_{q^2}\,[2]_{q^2}\,[3]_{q^2}\,[5]_{q^2}}{[10]_{q^2}\,[12]_{q^2}\,[15]_{q^2}},$$

which, by Corollary 6 and Lemma 21, is a polynomial in q with non-negative integer coefficients.

□

5. Auxiliary results I

This section collects several auxiliary results which allow us to reduce the problem of proving Theorem 2, or the equivalent statement (3.3), for the 26 exceptional groups listed in Section 2 to a finite problem. While Lemmas 27 and 28 cover special choices of the parameters, Lemmas 26 and 30 afford an inductive procedure. More precisely, if we assume that we have already verified Theorem 2 for all groups of smaller rank, then Lemmas 26 and 30, together with Lemmas 27 and 31, reduce the verification of Theorem 2 for the group that we are currently considering to a finite problem; see Remark 3. The final lemma of this section, Lemma 32, disposes of complex reflection groups with a special property satisfied by their degrees.

Let $p = am + b$, $0 \le b < m$. We have

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^{a+1}w_{m-b+1}c^{-a-1}, c^{a+1}w_{m-b+2}c^{-a-1}, \dots, c^{a+1}w_m c^{-a-1}, \\ c^a w_1 c^{-a}, \dots, c^a w_{m-b} c^{-a}\big), \quad (5.1)$$

where $*$ stands for the element of W which is needed to complete the product of the components to c.

Lemma 26. *It suffices to check* (3.3) *for p a divisor of mh. More precisely, let p be a divisor of mh, and let k be another positive integer with* $\gcd(k, mh/p) = 1$, *then we have*

$$\operatorname{Cat}^m(W;q)\big|_{q=e^{2\pi ip/mh}} = \operatorname{Cat}^m(W;q)\big|_{q=e^{2\pi ikp/mh}} \tag{5.2}$$

and

$$|\operatorname{Fix}_{NC^m(W)}(\phi^p)| = |\operatorname{Fix}_{NC^m(W)}(\phi^{kp})|. \tag{5.3}$$

Proof. For (5.2), this follows immediately from

$$\lim_{q\to\zeta}\frac{[\alpha]_q}{[\beta]_q} = \begin{cases} \frac{\alpha}{\beta} & \text{if } \alpha\equiv\beta\equiv 0 \pmod d,\\ 1 & \text{otherwise,}\end{cases} \tag{5.4}$$

where ζ is a primitive d-th root of unity and α,β are non-negative integers such that $\alpha\equiv\beta \pmod d$.

In order to establish (5.3), suppose that $x\in\operatorname{Fix}_{NC^m(W)}(\phi^p)$, that is, $x\in NC^m(W)$ and $\phi^p(x)=x$. It obviously follows that $\phi^{kp}(x)=x$, so that $x\in\operatorname{Fix}_{NC^m(W)}(\phi^{kp})$. To establish the converse, note that, if $\gcd(k,mh/p)=1$, then there exists k' with $k'k\equiv 1 \pmod{\frac{mh}{p}}$. It follows that, if $x\in\operatorname{Fix}_{NC^m(W)}(\phi^{kp})$, that is, if $x\in NC^m(W)$ and $\phi^{kp}(x)=x$, then $x=\phi^{k'kp}(x)=\phi^p(x)$, whence $x\in\operatorname{Fix}_{NC^m(W)}(\phi^p)$. □

Lemma 27. *Let p be a divisor of mh. If p is divisible by m, then* (3.3) *is true.*

Proof. According to (5.1), the action of ϕ^p on $NC^m(W)$ is described by

$$\phi^p\big((w_0;w_1,\dots,w_m)\big) = \big(*;c^{p/m}w_1c^{-p/m},\dots,c^{p/m}w_mc^{-p/m}\big).$$

Hence, if $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, then each individual w_i must be fixed under conjugation by $c^{p/m}$.

Using the notation $W'=\operatorname{Cent}_W(c^{p/m})$, the previous observation means that $w_i\in W'$, $i=1,2,\dots,m$. Springer [32, Theorem 4.2] (see also [24, Theorem 11.24(iii)]) proved that W' is a well-generated complex reflection group whose degrees coincide with those degrees of W that are divisible by mh/p. It was furthermore shown in [10, Lemma 3.3] that

$$NC(W)\cap W' = NC(W'). \tag{5.5}$$

Hence, the tuples $(w_0;w_1,\dots,w_m)$ fixed by ϕ^p are in fact identical with the elements of $NC^m(W')$, which implies that

$$|\operatorname{Fix}_{NC^m(W)}(\phi^p)| = |NC^m(W')|. \tag{5.6}$$

Application of Theorem 1 with W replaced by W' and of the "limit rule" (5.4) then yields that

$$|NC^m(W')| = \prod_{\substack{1\le i\le n\\ \frac{mh}{p}\mid d_i}}\frac{mh+d_i}{d_i} = \operatorname{Cat}^m(W;q)\big|_{q=e^{2\pi ip/mh}}. \tag{5.7}$$

Combining (5.6) and (5.7), we obtain (3.3). This finishes the proof of the lemma. □

Lemma 28. *Equation* (3.3) *holds for all divisors p of m.*

Proof. Using (5.4) and the fact that the degrees of irreducible well-generated complex reflection groups satisfy $d_i < h$ for all $i < n$, we see that

$$\operatorname{Cat}^m(W;q)\big|_{q=e^{2\pi i p/mh}} = \begin{cases} m+1 & \text{if } m=p, \\ 1 & \text{if } m \neq p. \end{cases}$$

On the other hand, if $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, then, because of the action (5.1), we must have $w_1 = w_{p+1} = \cdots = w_{m-p+1}$ and $w_1 = c w_{m-p+1} c^{-1}$. In particular, $w_1 \in \operatorname{Cent}_W(c)$. By the theorem of Springer cited in the proof of Lemma 27, the subgroup $\operatorname{Cent}_W(c)$ is itself a complex reflection group whose degrees are those degrees of W that are divisible by h. The only such degree is h itself, hence $\operatorname{Cent}_W(c)$ is the cyclic group generated by c. Moreover, by (5.5), we obtain that $w_1 = \varepsilon$, the identity element of W, or $w_1 = c$. Therefore, for $m = p$ the set $\operatorname{Fix}_{NC^m(W)}(\phi^p)$ consists of the $m+1$ elements $(w_0; w_1, \dots, w_m)$ obtained by choosing $w_i = c$ for a particular i between 0 and m, all other w_j's being equal to ε, while, for $m \neq p$, we have

$$\operatorname{Fix}_{NC^m(W)}(\phi^p) = \big\{(c; \varepsilon, \dots, \varepsilon)\big\},$$

whence the result. □

Lemma 29. *Let W be an irreducible well-generated complex reflection group all of whose degrees are divisible by d. Then each element of W is fixed under conjugation by $c^{h/d}$.*

Proof. By the theorem of Springer cited in the proof of Lemma 27, the subgroup $W' = \operatorname{Cent}_W(c^{h/d})$ is itself a complex reflection group whose degrees are those degrees of W that are divisible by d. Thus, by our assumption, the degrees of W' coincide with the degrees of W, and hence W' must be equal to W. Phrased differently, each element of W is fixed under conjugation by $c^{h/d}$, as claimed. □

Lemma 30. *Let W be an irreducible well-generated complex reflection group of rank n, and let $p = m_1 h_1$ be a divisor of mh, where $m = m_1 m_2$ and $h = h_1 h_2$. Without loss of generality, we assume that* $\gcd(h_1, m_2) = 1$. *Suppose that Theorem* 2 *has already been verified for all irreducible well-generated complex reflection groups with rank $< n$. If h_2 does not divide all degrees d_i, then Equation* (3.3) *is satisfied.*

Proof. Let us write $h_1 = a m_2 + b$, with $0 \le b < m_2$. The condition $\gcd(h_1, m_2) = 1$ translates into $\gcd(b, m_2) = 1$. From (5.1), we infer that

$$\begin{aligned} &\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ &\quad = \big(*; c^{a+1} w_{m-m_1 b+1} c^{-a-1}, c^{a+1} w_{m-m_1 b+2} c^{-a-1}, \dots, c^{a+1} w_m c^{-a-1}, \\ &\qquad\qquad c^a w_1 c^{-a}, \dots, c^a w_{m-m_1 b} c^{-a}\big). \end{aligned} \tag{5.8}$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$\begin{aligned} w_i &= c^{a+1} w_{i+m-m_1 b} c^{-a-1}, \quad i = 1, 2, \dots, m_1 b, \\ w_i &= c^a w_{i-m_1 b} c^{-a}, \quad i = m_1 b + 1, m_1 b + 2, \dots, m, \end{aligned}$$

which, after iteration, implies in particular that

$$w_i = c^{b(a+1)+(m_2-b)a} w_i c^{-b(a+1)-(m_2-b)a} = c^{h_1} w_i c^{-h_1}, \quad i = 1, 2, \dots, m.$$

It is at this point where we need $\gcd(b, m_2) = 1$. The last equation shows that each w_i, $i = 1, 2, \ldots, m$, and thus also w_0, lies in $\mathrm{Cent}_W(c^{h_1})$. By the theorem of Springer cited in the proof of Lemma 27, this centraliser subgroup is itself a complex reflection group, W' say, whose degrees are those degrees of W that are divisible by $h/h_1 = h_2$. Since, by assumption, h_2 does not divide *all* degrees, W' has rank strictly less than n. Again by assumption, we know that Theorem 2 is true for W', so that in particular,

$$|\operatorname{Fix}_{NC^m(W')}(\phi^p)| = \operatorname{Cat}^m(W'; q)\big|_{q=e^{2\pi i p/mh}}.$$

The arguments above together with (5.5) show that $\operatorname{Fix}_{NC^m(W)}(\phi^p) = \operatorname{Fix}_{NC^m(W')}(\phi^p)$. On the other hand, using (5.4) it is straightforward to see that

$$\operatorname{Cat}^m(W; q)\big|_{q=e^{2\pi i p/mh}} = \operatorname{Cat}^m(W'; q)\big|_{q=e^{2\pi i p/mh}}.$$

This proves (3.3) for our particular p, as required. $\square$

Lemma 31. *Let W be an irreducible well-generated complex reflection group of rank n, and let $p = m_1 h_1$ be a divisor of mh, where $m = m_1 m_2$ and $h = h_1 h_2$. We assume that* $\gcd(h_1, m_2) = 1$. *If $m_2 > n$ then*

$$\operatorname{Fix}_{NC^m(W)}(\phi^p) = \big\{(c; \varepsilon, \ldots, \varepsilon)\big\}.$$

Proof. Let us suppose that $(w_0; w_1, \ldots, w_m) \in \operatorname{Fix}_{NC^m(W)}(\phi^p)$ and that there exists a $j \geq 1$ such that $w_j \neq \varepsilon$. By (5.8), it then follows for such a j that also $w_k \neq \varepsilon$ for all $k \equiv j - l m_1 b \pmod{m}$, where, as before, b is defined as the unique integer with $h_1 = a m_2 + b$ and $0 \leq b < m_2$. Since, by assumption, $\gcd(b, m_2) = 1$, there are exactly m_2 such k's which are distinct mod m. However, this implies that the sum of the absolute lengths of the w_i's, $0 \leq i \leq m$, is at least $m_2 > n$, a contradiction to Remark 1.(2). $\square$

Remark 3. (1) If we put ourselves in the situation of the assumptions of Lemma 30, then we may conclude that equation (3.3) only needs to be checked for pairs (m_2, h_2) subject to the following restrictions:

$$m_2 \geq 2, \quad \gcd(h_1, m_2) = 1, \quad \text{and } h_2 \text{ divides all degrees of } W. \tag{5.9}$$

Indeed, Lemmas 27 and 30 together imply that equation (3.3) is always satisfied in all other cases.

(2) Still putting ourselves in the situation of Lemma 30, if $m_2 > n$ and $m_2 h_2$ does not divide any of the degrees of W, then equation (3.3) is satisfied. Indeed, Lemma 31 says that in this case the left-hand side of (3.3) equals 1, while a straightforward computation using (5.4) shows that in this case the right-hand side of (3.3) equals 1 as well.

(3) It should be observed that this leaves a finite number of choices for m_2 to consider, whence a finite number of choices for (m_1, m_2, h_1, h_2). Altogether, there remains a finite number of choices for $p = h_1 m_1$ to be checked.

Lemma 32. *Let W be an irreducible well-generated complex reflection group of rank n with the property that $d_i \mid h$ for $i = 1, 2, \ldots, n$. Then Theorem 2 is true for this group W.*

Proof. By Lemma 26, we may restrict ourselves to divisors p of mh.

Suppose that $e^{2\pi ip/mh}$ is a d_i-th root of unity for some i. In other words, mh/p divides d_i. Since d_i is a divisor of h by assumption, the integer mh/p also divides h. But this is equivalent to saying that m divides p, and equation (3.3) holds by Lemma 27.

Now assume that mh/p does not divide any of the d_i's. Then, by (5.4), the right-hand side of (3.3) equals 1. On the other hand, $(c;\varepsilon,\dots,\varepsilon)$ is always an element of $\mathrm{Fix}_{NC^m(W)}(\phi^p)$. To see that there are no others, we make appeal to the classification of all irreducible well-generated complex reflection groups, which we recalled in Section 2. Inspection reveals that all groups satisfying the hypotheses of the lemma have rank $n \le 2$. Except for the groups contained in the infinite series $G(d,1,n)$ and $G(e,e,n)$ for which Theorem 2 has been established in [19], these are the groups $G_5, G_6, G_9, G_{10}, G_{14}, G_{17}, G_{18}, G_{21}$. We now discuss these groups case by case, keeping the notation of Lemma 30. In order to simplify the argument, we note that Lemma 31 implies that equation (3.3) holds if $m_2 > 2$, so that in the following arguments we always may assume that $m_2 = 2$.

Case G_5. The degrees are $6, 12$, and therefore Remark 3.(1) implies that equation (3.3) is always satisfied.

Case G_6. The degrees are $4, 12$, and therefore, according to Remark 3.(1), we need only consider the case where $h_2 = 4$ and $m_2 = 2$, that is, $p = 3m/2$. Then (5.8) becomes

$$\phi^p\big((w_0;w_1,\dots,w_m)\big) = (*; c^2 w_{\frac{m}{2}+1}c^{-2}, c^2 w_{\frac{m}{2}+2}c^{-2}, \dots, c^2 w_m c^{-2}, cw_1c^{-1}, \dots, cw_{\frac{m}{2}}c^{-1}\big). \tag{5.10}$$

If $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p and not equal to $(c;\varepsilon,\dots,\varepsilon)$, there must exist an i with $1 \le i \le \frac{m}{2}$ such that $\ell_T(w_i) = \ell_T(w_{\frac{m}{2}+i}) = 1$, $w_{\frac{m}{2}+i} = cw_ic^{-1}$, $w_iw_{\frac{m}{2}+i} = w_icw_ic^{-1} = c$, and all w_j, with $j \ne i, \frac{m}{2}+i$, equal ε. However, with the help of the *GAP* package `CHEVIE` [14, 27], one verifies that there is no w_i in G_6 such that

$$\ell_T(w_i) = 1 \quad \text{and} \quad w_icw_ic^{-1} = c$$

are simultaneously satisfied. Hence, the left-hand side of (3.3) is equal to 1, as required.

Case G_9. The degrees are $8, 24$, and therefore, according to Remark 3.(1), we need only consider the case where $h_2 = 8$ and $m_2 = 2$, that is, $p = 3m/2$. This is the same p as for G_6. Again, `CHEVIE` finds no solution. Hence, the left-hand side of (3.3) is equal to 1, as required.

Case G_{10}. The degrees are $12, 24$, and therefore Remark 3.(1) implies that equation (3.3) is always satisfied.

Case G_{14}. The degrees are $6, 24$, and therefore Remark 3.(1) implies that equation (3.3) is always satisfied.

Case G_{17}. The degrees are $20, 60$, and therefore, according to Remark 3.(1), we need only consider the cases where $h_2 = 20$ or $h_2 = 4$. In the first case, $p = 3m/2$, which is the same p as for G_6. Again, `CHEVIE` finds no solution. In the second case, $p = 15m/2$. Then (5.8) becomes

$$\begin{aligned}\phi^p&\big((w_0;w_1,\dots,w_m)\big)\\ &= (*; c^8 w_{\frac{m}{2}+1}c^{-8}, c^8 w_{\frac{m}{2}+2}c^{-8}, \dots, c^8 w_m c^{-8}, c^7w_1c^{-7}, \dots, c^7w_{\frac{m}{2}}c^{-7}\big).\end{aligned} \tag{5.11}$$

By Lemma 29, every element of $NC(W)$ is fixed under conjugation by c^3, and, thus, on elements fixed by ϕ^p, the above action of ϕ^p reduces to the one in (5.10). This action was already discussed in the first case. Hence, in both cases, the left-hand side of (3.3) is equal to 1, as required.

Case G_{18}. The degrees are $30, 60$, and therefore Remark 3.(1) implies that equation (3.3) is always satisfied.

Case G_{21}. The degrees are $12, 60$, and therefore, according to Remark 3.(1), we need only consider the cases where $h_2 = 12$ or $h_2 = 4$. In the first case, $p = 5m/2$, so that (5.8) becomes

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^3 w_{\frac{m}{2}+1} c^{-3}, c^3 w_{\frac{m}{2}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{m}{2}} c^{-2}\big). \quad (5.12)$$

If $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p and not equal to $(c; \varepsilon, \dots, \varepsilon)$, there must exist an i with $1 \le i \le \frac{m}{2}$ such that $\ell_T(w_i) = 1$ and $w_i c^2 w_i c^{-2} = c$. However, with the help of the *GAP* package CHEVIE [14, 27], one verifies that there is no such solution to this equation. In the second case, $p = 15m/2$. Then (5.8) becomes the action in (5.11). By Lemma 29, every element of $NC(W)$ is fixed under conjugation by c^5, and, thus, on elements fixed by ϕ^p, the action of ϕ^p in (5.11) reduces to the one in the first case. Hence, in both cases, the left-hand side of (3.3) is equal to 1, as required.

This completes the proof of the lemma. □

6. Case-by-case verification of Theorem 2

In the sequel we write ζ_d for a primitive d-th root of unity.

Case G_4. The degrees are $4, 6$, and hence we have

$$\mathrm{Cat}^m(G_4; q) = \frac{[6m+6]_q\,[6m+4]_q}{[6]_q\,[4]_q}.$$

Let ζ be a $6m$-th root of unity. In what follows, we abbreviate the assertion that "ζ is a primitive d-th root of unity" as "$\zeta = \zeta_d$." The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_4; q) = m+1, \quad \text{if } \zeta = \zeta_6, \zeta_3, \tag{6.1a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_4; q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta = \zeta_4,\ 2 \mid m, \tag{6.1b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_4; q) = \mathrm{Cat}^m(G_4), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{6.1c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_4; q) = 1, \quad \text{otherwise.} \tag{6.1d}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.1). The only cases not covered by Lemmas 27 and 28 are the ones in (6.1b) and (6.1d). On the other hand, the only case left to consider according to Remark 3 is the case where $h_2 = m_2 = 2$, that is the case (6.1b) where $p = 3m/2$. In particular, m must be divisible by 2. The action of ϕ^p is the same as the one in (5.10). With the help of CHEVIE, one finds that each of the 3 (complex) reflections in G_4 which are less than the (chosen) Coxeter element is a valid choice for w_i, and each of these choices gives rise to $m/2$ elements in $NC^m(G_4)$ since the index i ranges from 1 to $m/2$.

Hence, in total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(G_4)}(\phi^p)$, which agrees with the limit in (6.1b).

Case G_8. The degrees are $8, 12$, and hence we have

$$\mathrm{Cat}^m(G_8;q) = \frac{[12m+12]_q\,[12m+8]_q}{[12]_q\,[8]_q}.$$

Let ζ be a $12m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_8;q) = m+1, \quad \text{if } \zeta = \zeta_{12}, \zeta_6, \zeta_3, \tag{6.2a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_8;q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta = \zeta_8,\ 2 \mid m, \tag{6.2b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_8;q) = \mathrm{Cat}^m(G_8), \quad \text{if } \zeta = \zeta_4, -1, 1, \tag{6.2c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_8;q) = 1, \quad \text{otherwise.} \tag{6.2d}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.2). The only cases not covered by Lemmas 27 and 28 are the ones in (6.2b) and (6.2d). On the other hand, the only case left to consider according to Remark 3 is the case where $h_2 = 4$ and $m_2 = 2$, that is the case (6.2b) where $p = 3m/2$. In particular, m must be divisible by 2. The action of ϕ^p is the same as the one in (5.10). With the help of `CHEVIE`, one finds that each of the 3 (complex) reflections in G_8 which are less than the (chosen) Coxeter element is a valid choice for w_i, and each of these choices gives rise to $m/2$ elements in $NC^m(G_8)$ since the index i ranges from 1 to $m/2$.

Hence, in total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(G_8)}(\phi^p)$, which agrees with the limit in (6.2b).

Case G_{16}. The degrees are $20, 30$, and hence we have

$$\mathrm{Cat}^m(G_{16};q) = \frac{[30m+30]_q\,[30m+20]_q}{[30]_q\,[20]_q}.$$

Let ζ be a $30m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{16};q) = m+1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \zeta_6, \zeta_3, \tag{6.3a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{16};q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta = \zeta_{20}, \zeta_4,\ 2 \mid m, \tag{6.3b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{16};q) = \mathrm{Cat}^m(G_{16}), \quad \text{if } \zeta = \zeta_{10}, \zeta_5, -1, 1, \tag{6.3c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{16};q) = 1, \quad \text{otherwise.} \tag{6.3d}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.3). The only cases not covered by Lemmas 27 and 28 are the ones in (6.3b) and (6.3d). On the other hand, the only cases left to consider according to Remark 3 are the cases where $h_2 = 10$ and $m_2 = 2$, respectively $h_2 = m_2 = 2$. Both cases belong to (6.3b). In the first case, we have $p = 3m/2$, while in the second case we have $p = 15m/2$. In particular, m must be divisible by 2. In the first case, the action of ϕ^p is the same as the one in (5.10). With the help of `CHEVIE`, one finds that each of the 3 (complex) reflections in G_{16} which are less than the (chosen) Coxeter element is a valid choice for w_i, and each of these choices gives rise to $m/2$ elements in $NC^m(G_{16})$

since the index i ranges from 1 to $m/2$. On the other hand, if $p = 15m/2$, then the action of ϕ^p is the same as the one in (5.11). By Lemma 29, every element of $NC(W)$ is fixed under conjugation by c^3, and, thus, on elements fixed by ϕ^p, the action of ϕ^p reduces to the one in the first case.

Hence, in total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\operatorname{Fix}_{NC^m(G_{16})}(\phi^p)$, which agrees with the limit in (6.3b).

Case G_{20}. The degrees are $12, 30$, and hence we have

$$\operatorname{Cat}^m(G_{20}; q) = \frac{[30m+30]_q\,[30m+12]_q}{[30]_q\,[12]_q}.$$

Let ζ be a $30m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{20}; q) = m+1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \zeta_{10}, \zeta_5, \tag{6.4a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{20}; q) = \tfrac{5m+2}{2}, \quad \text{if } \zeta = \zeta_{12}, \zeta_4,\ 2 \mid m, \tag{6.4b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{20}; q) = \operatorname{Cat}^m(G_{20}), \quad \text{if } \zeta = \zeta_6, \zeta_3, -1, 1, \tag{6.4c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{20}; q) = 1, \quad \text{otherwise.} \tag{6.4d}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.4). The only cases not covered by Lemmas 27 and 28 are the ones in (6.4b) and (6.4d). On the other hand, the only cases left to consider according to Remark 3 are the cases where $h_2 = 6$ and $m_2 = 2$, respectively $h_2 = m_2 = 2$. Both cases belong to (6.4b). In the first case, we have $p = 5m/2$, while in the second case we have $p = 15m/2$. In particular, m must be divisible by 2. In the first case, the action of ϕ^p is the same as the one in (5.12). With the help of `CHEVIE`, one finds that each of the 5 (complex) reflections in G_{20} which are less than the (chosen) Coxeter element is a valid choice for w_i, and each of these choices gives rise to $m/2$ elements in $NC^m(G_{20})$ since the index i ranges from 1 to $m/2$. On the other hand, if $p = 15m/2$, then the action of ϕ^p is the same as the one in (5.11). By Lemma 29, every element of $NC(W)$ is fixed under conjugation by c^5, and, thus, on elements fixed by ϕ^p, the action of ϕ^p reduces to the one in the first case.

Hence, in total, we obtain $1 + 5\frac{m}{2} = \frac{5m+2}{2}$ elements in $\operatorname{Fix}_{NC^m(G_{20})}(\phi^p)$, which agrees with the limit in (6.4b).

Case $G_{23} = H_3$. The degrees are $2, 6, 10$, and hence we have

$$\operatorname{Cat}^m(H_3; q) = \frac{[10m+10]_q\,[10m+6]_q\,[10m+2]_q}{[10]_q\,[6]_q\,[2]_q}.$$

Let ζ be a $10m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(H_3;q)=m+1,\quad \text{if } \zeta=\zeta_{10},\zeta_5, \tag{6.5a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(H_3;q)=\tfrac{10m+6}{6},\quad \text{if } \zeta=\zeta_6,\zeta_3,\ 3\mid m, \tag{6.5b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(H_3;q)=\operatorname{Cat}^m(H_3),\quad \text{if } \zeta=-1 \text{ or } \zeta=1, \tag{6.5c}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(H_3;q)=1,\quad \text{otherwise.} \tag{6.5d}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.5). The only cases not covered by Lemmas 27 and 28 are the ones in (6.5b) and (6.5d). By Lemma 26, we are free to choose $p=5m/3$ if $\zeta=\zeta_6$, respectively $p=10m/3$ if $\zeta=\zeta_3$. In both cases, m must be divisible by 3.

We start with the case that $p=5m/3$. From (5.1), we infer

$$\phi^p\big((w_0;w_1,\dots,w_m)\big)=(*;c^2w_{\frac{m}{3}+1}c^{-2},c^2w_{\frac{m}{3}+2}c^{-2},\dots,c^2w_mc^{-2},cw_1c^{-1},\dots,cw_{\frac{m}{3}}c^{-1}\big). \tag{6.6}$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^2w_{\frac{m}{3}+i}c^{-2},\quad i=1,2,\dots,\tfrac{2m}{3}, \tag{6.7a}$$

$$w_i=cw_{i-\frac{2m}{3}}c^{-1},\quad i=\tfrac{2m}{3}+1,\tfrac{2m}{3}+2,\dots,m. \tag{6.7b}$$

There are two distinct possibilities for choosing the w_i's, $1\le i\le m$: either all the w_i's are equal to ε, or there is an i with $1\le i\le\frac{m}{3}$ such that

$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{3}})=\ell_T(w_{i+\frac{2m}{3}})=1.$$

Writing t_1,t_2,t_3 for $w_i,w_{i+\frac{m}{3}},w_{i+\frac{2m}{3}}$, respectively, the equations (6.7) reduce to

$$t_1=c^2t_2c^{-2}, \tag{6.8a}$$

$$t_2=c^2t_3c^{-2}, \tag{6.8b}$$

$$t_3=ct_1c^{-1}. \tag{6.8c}$$

One of these equations is in fact superfluous: if we substitute (6.8b) and (6.8c) in (6.8a), then we obtain $t_1=c^5t_1c^{-5}$ which is automatically satisfied due to Lemma 29 with $d=2$.

Since $(w_0;w_1,\dots,w_m)\in NC^m(H_3)$, we must have $t_1t_2t_3=c$. Combining this with (6.8), we infer that

$$t_1(c^{-2}t_1c^2)(ct_1c^{-1})=c. \tag{6.9}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains five solutions for t_1 in this equation:

$$t_1\in\big\{[2],\ [3],\ [2,1,2],\ [1,2,3,2,1],\ [1,3,2,1,2,1,3]\big\}. \tag{6.10}$$

Here we have used the short notation of `coxeter`: if $\{s_1,s_2,s_3\}$ is a simple system of generators of H_3, corresponding to the Dynkin diagram displayed in Figure 1, then $[j_1,j_2,\dots,j_k]$ stands for the element $s_{j_1}s_{j_2}\dots s_{j_k}$.

We claim that each of the above five solutions gives rise to $m/3$ elements of $\operatorname{Fix}_{NC^m(H_3)}(\phi^p)$. Indeed, given t_1, the elements t_2 and t_3 can be computed by (6.8a) and (6.8c), and there are $m/3$ possibilities to choose the index i for w_i.

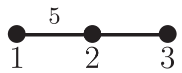

FIGURE 1. The Dynkin diagram for H_3

In total, we obtain $1+5\frac{m}{3}=\frac{10m+6}{6}$ elements in $\mathrm{Fix}_{NC^m(H_3)}(\phi^p)$, which agrees with the limit in (6.5b).

In the case that $p=10m/3$, we infer from (5.1) that

$$\begin{aligned}&\phi^p\big((w_0;w_1,\dots,w_m)\big)\\&\quad=\big(*;c^4w_{\frac{2m}{3}+1}c^{-4},c^4w_{\frac{2m}{3}+2}c^{-4},\dots,c^4w_mc^{-4},c^3w_1c^{-3},\dots,c^3w_{\frac{2m}{3}}c^{-3}\big).\end{aligned}\tag{6.11}$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^4w_{\frac{2m}{3}+i}c^{-4},\quad i=1,2,\dots,\tfrac{m}{3},\tag{6.12a}$$

$$w_i=c^3w_{i-\frac{m}{3}}c^{-3},\quad i=\tfrac{m}{3}+1,\tfrac{m}{3}+2,\dots,m.\tag{6.12b}$$

There are two distinct possibilities for choosing the w_i's, $1\le i\le m$: either all the w_i's are equal to ε, or there is an i with $1\le i\le \frac{m}{3}$ such that

$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{3}})=\ell_T(w_{i+\frac{2m}{3}})=1.$$

Writing t_1,t_2,t_3 for $w_i,w_{i+\frac{m}{3}},w_{i+\frac{2m}{3}}$, respectively, the equations (6.12) reduce to

$$t_1=c^4t_3c^{-4},\tag{6.13a}$$

$$t_2=c^3t_1c^{-3},\tag{6.13b}$$

$$t_3=c^3t_2c^{-3}.\tag{6.13c}$$

One of these equations is in fact superfluous: if we substitute (6.13b) and (6.13c) in (6.13a), then we obtain $t_1=c^{10}t_1c^{-10}$ which is automatically satisfied since $c^{10}=\varepsilon$.

Since $(w_0;w_1,\dots,w_m)\in NC^m(H_3)$, we must have $t_1t_2t_3=c$. Combining this with (6.13), we infer that

$$t_1(c^3t_1c^{-3})(c^{-4}t_1c^4)=c.\tag{6.14}$$

Using that $c^5t_1c^{-5}=t_1$, due to Lemma 29 with $d=2$, we see that this equation is equivalent with (6.9). Therefore, we are facing exactly the same enumeration problem here as for $p=5m/3$, and, consequently, the number of solutions to (6.14) is the same, namely $\frac{5m+3}{3}$, as required.

Finally, we turn to (6.5d). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2=1$ and $m_2=3$, $h_2=m_2=2$, respectively $h_2=2$ and $m_2=3$. These correspond to the choices $p=10m/3$, $p=5m/2$, respectively $p=5m/3$, out of which only $p=5m/2$ has not yet been discussed and belongs to the current case. The corresponding action of ϕ^p is given by (5.12). A computation with Stembridge's *Maple* package `coxeter` [35] finds no solution. Hence, the left-hand side of (3.3) is equal to 1, as required.

CASE G_{24}. The degrees are $4, 6, 14$, and hence we have

$$\operatorname{Cat}^m(G_{24};q)=\frac{[14m+14]_q\,[14m+6]_q\,[14m+4]_q}{[14]_q\,[6]_q\,[4]_q}.$$

Let ζ be a $14m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{24};q)=m+1,\quad \text{if } \zeta=\zeta_{14},\zeta_7, \tag{6.15a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{24};q)=\tfrac{7m+3}{3},\quad \text{if } \zeta=\zeta_6,\zeta_3,\ 3\mid m, \tag{6.15b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{24};q)=\tfrac{7m+2}{2},\quad \text{if } \zeta=\zeta_4,\ 2\mid m, \tag{6.15c}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{24};q)=\operatorname{Cat}^m(G_{24}),\quad \text{if } \zeta=-1 \text{ or } \zeta=1, \tag{6.15d}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{24};q)=1,\quad \text{otherwise}. \tag{6.15e}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.15). The only cases not covered by Lemmas 27 and 28 are the ones in (6.15b), (6.15c), and (6.15e).

We first consider (6.15b). By Lemma 26, we are free to choose $p=7m/3$ if $\zeta=\zeta_6$, respectively $p=14m/3$ if $\zeta=\zeta_3$. In both cases, m must be divisible by 3.

We start with the case that $p=7m/3$. From (5.1), we infer

$$\begin{aligned}&\phi^p\big((w_0;w_1,\dots,w_m)\big)\\&\qquad=\big(*;c^3w_{\frac{2m}{3}+1}c^{-3},c^3w_{\frac{2m}{3}+2}c^{-3},\dots,c^3w_mc^{-3},c^2w_1c^{-2},\dots,c^2w_{\frac{2m}{3}}c^{-2}\big).\end{aligned}$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^3w_{\frac{2m}{3}+i}c^{-3},\quad i=1,2,\dots,\tfrac{m}{3}, \tag{6.16a}$$

$$w_i=c^2w_{i-\frac{m}{3}}c^{-2},\quad i=\tfrac{m}{3}+1,\tfrac{m}{3}+2,\dots,m. \tag{6.16b}$$

There are two distinct possibilities for choosing the w_i's, $1\le i\le m$: either all the w_i's are equal to ε, or there is an i with $1\le i\le\frac{m}{3}$ such that

$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{3}})=\ell_T(w_{i+\frac{2m}{3}})=1.$$

Writing t_1,t_2,t_3 for $w_i,w_{i+\frac{m}{3}},w_{i+\frac{2m}{3}}$, respectively, the equations (6.16) reduce to

$$t_1=c^3t_3c^{-3}, \tag{6.17a}$$

$$t_2=c^2t_1c^{-2}, \tag{6.17b}$$

$$t_3=c^2t_2c^{-2}. \tag{6.17c}$$

One of these equations is in fact superfluous: if we substitute (6.17b) and (6.17c) in (6.17a), then we obtain $t_1=c^7t_1c^{-7}$ which is automatically satisfied due to Lemma 29 with $d=2$.

Since $(w_0;w_1,\dots,w_m)\in NC^m(G_{24})$, we must have $t_1t_2t_3=c$. Combining this with (6.17), we infer that

$$t_1(c^2t_1c^{-2})(c^4t_1c^{-4})=c. \tag{6.18}$$

With the help of `CHEVIE`, one obtains 7 solutions for t_1 in this equation:

$$t_1\in\big\{[1],\,[2],\,[3],\,[15],\,[16],\,[19],\,[21]\big\}, \tag{6.19}$$

each of them giving rise to $m/3$ elements of $\mathrm{Fix}_{NC^m(G_{24})}(\phi^p)$ since i ranges from 1 to $m/3$. Here we have used the short notation of `CHEVIE`: $[j_1, j_2, \dots, j_k]$ stands for the element $r_{j_1} r_{j_2} \dots r_{j_k}$, where r_i is the i-th (complex) reflection corresponding to the i-th root in the internal ordering of the roots of G_{24} in `CHEVIE`.

In total, we obtain $1 + 7\frac{m}{3} = \frac{7m+3}{3}$ elements in $\mathrm{Fix}_{NC^m(G_{24})}(\phi^p)$, which agrees with the limit in (6.15b).

In the case that $p = 14m/3$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^5 w_{\frac{m}{3}+1} c^{-5}, c^5 w_{\frac{m}{3}+2} c^{-5}, \dots, c^5 w_m c^{-5}, c^4 w_1 c^{-4}, \dots, c^4 w_{\frac{m}{3}} c^{-4}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^5 w_{\frac{m}{3}+i} c^{-5}, \quad i = 1, 2, \dots, \tfrac{2m}{3}, \tag{6.20a}$$

$$w_i = c^4 w_{i-\frac{2m}{3}} c^{-4}, \quad i = \tfrac{2m}{3}+1, \tfrac{2m}{3}+2, \dots, m. \tag{6.20b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{3}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 1.$$

Writing t_1, t_2, t_3 for $w_i, w_{i+\frac{m}{3}}, w_{i+\frac{2m}{3}}$, respectively, the equations (6.20) reduce to

$$t_1 = c^5 t_2 c^{-5}, \tag{6.21a}$$

$$t_2 = c^5 t_3 c^{-5}, \tag{6.21b}$$

$$t_3 = c^4 t_1 c^{-4}. \tag{6.21c}$$

One of these equations is in fact superfluous: if we substitute (6.21b) and (6.21c) in (6.21a), then we obtain $t_1 = c^{14} t_1 c^{-14}$ which is automatically satisfied since $c^{14} = \varepsilon$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{24})$, we must have $t_1 t_2 t_3 = c$. Combining this with (6.21), we infer that

$$t_1 (c^9 t_1 c^{-9})(c^{-4} t_1 c^4) = c. \tag{6.22}$$

Using that $c^7 t_1 c^{-7} = t_1$, due to Lemma 29 with $d = 2$, we see that this equation is equivalent with (6.18). Therefore, we are facing exactly the same enumeration problem here as for $p = 7m/3$, and, consequently, the number of solutions to (6.22) is the same, namely $\frac{7m+3}{3}$, as required.

Our next case is (6.15c). By Lemma 26, we are free to choose $p = 7m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^4 w_{\frac{m}{2}+1} c^{-4}, c^4 w_{\frac{m}{2}+2} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_1 c^{-3}, \dots, c^3 w_{\frac{m}{2}} c^{-3}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^4 w_{\frac{m}{2}+i} c^{-4}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.23a}$$

$$w_i = c^3 w_{i-\frac{m}{2}} c^{-3}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.23b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1.$$

Writing t_1, t_2 for $w_i, w_{i+\frac{m}{2}}$, respectively, the equations (6.23) reduce to

$$t_1 = c^4 t_2 c^{-4}, \tag{6.24a}$$

$$t_2 = c^3 t_1 c^{-3}. \tag{6.24b}$$

One of these equations is in fact superfluous: if we substitute (6.24b) in (6.24a), then we obtain $t_1 = c^7 t_1 c^{-7}$ which is automatically satisfied due to Lemma 29 with $d = 2$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{24})$, we must have $t_1 t_2 \le_T c$, where $\le_T$ is the partial order defined in (2.1). Combining this with (6.24), we infer that

$$t_1(c^3 t_1 c^{-3}) \le_T c. \tag{6.25}$$

With the help of `CHEVIE`, one obtains 7 solutions for t_1 in this relation:

$$t_1 \in \{[5], [6], [7], [9], [12], [29], [32]\}, \tag{6.26}$$

each of them giving rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(G_{24})}(\phi^p)$ since i ranges from 1 to $m/2$. Here we have used again the short notation of `CHEVIE` referring to the internal ordering of the roots of G_{24} in `CHEVIE`.

In total, we obtain $1 + 7\frac{m}{2} = \frac{7m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(G_{24})}(\phi^p)$, which agrees with the limit in (6.15c).

Finally, we turn to (6.15e). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 1$ and $m_2 = 3$, $h_2 = m_2 = 2$, and $h_2 = 2$ and $m_2 = 3$. These correspond to the choices $p = 14m/3$, $p = 7m/2$, respectively $p = 7m/3$, all of which have already been discussed as they do not belong to (6.15e). Hence, (3.3) must necessarily hold, as required.

Case G_{25}. The degrees are $6, 9, 12$, and hence we have

$$\mathrm{Cat}^m(G_{25}; q) = \frac{[12m+12]_q\,[12m+9]_q\,[12m+6]_q}{[12]_q\,[9]_q\,[6]_q}.$$

Let ζ be a $12m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{25}; q) = m+1, \quad \text{if } \zeta = \zeta_{12}, \zeta_4, \tag{6.27a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{25}; q) = \tfrac{4m+3}{3}, \quad \text{if } \zeta = \zeta_9,\ 3 \mid m, \tag{6.27b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{25}; q) = (m+1)(2m+1), \quad \text{if } \zeta = \zeta_6, -1 \tag{6.27c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{25}; q) = \mathrm{Cat}^m(G_{25}), \quad \text{if } \zeta = \zeta_3, 1, \tag{6.27d}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{25}; q) = 1, \quad \text{otherwise.} \tag{6.27e}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.27). The only cases not covered by Lemmas 27 and 28 are the ones in (6.27b) and (6.27e).

We first consider (6.27b). By Lemma 26, we are free to choose $p = 4m/3$. In particular, m must be divisible by 3. From (5.1), we infer

$$\begin{aligned}\phi^p\big((w_0; w_1, \dots, w_m)\big)\\ &= \big(*; c^2 w_{\frac{2m}{3}+1} c^{-2}, c^2 w_{\frac{2m}{3}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{2m}{3}} c^{-1}\big).\end{aligned}$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{2m}{3}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{3}, \tag{6.28a}$$

$$w_i = c w_{i-\frac{m}{3}} c^{-1}, \quad i = \tfrac{m}{3}+1, \tfrac{m}{3}+2, \dots, m. \tag{6.28b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{3}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 1.$$

Writing t_1, t_2, t_3 for $w_i, w_{i+\frac{m}{3}}, w_{i+\frac{2m}{3}}$, respectively, the equations (6.28) reduce to

$$t_1 = c^2 t_3 c^{-2}, \tag{6.29a}$$

$$t_2 = c t_1 c^{-1}, \tag{6.29b}$$

$$t_3 = c t_2 c^{-1}. \tag{6.29c}$$

One of these equations is in fact superfluous: if we substitute (6.29b) and (6.29c) in (6.29a), then we obtain $t_1 = c^4 t_1 c^{-4}$ which is automatically satisfied due to Lemma 29 with $d = 3$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{25})$, we must have $t_1 t_2 t_3 = c$. Combining this with (6.29), we infer that

$$t_1 (c t_1 c^{-1})(c^2 t_1 c^{-2}) = c. \tag{6.30}$$

With the help of CHEVIE, one obtains four solutions for t_1 in this equation:

$$t_1 \in \big\{[1],\ [2],\ [3],\ [14]\big\}, \tag{6.31}$$

each of them giving rise to $m/3$ elements of $\mathrm{Fix}_{NC^m(G_{25})}(\phi^p)$ since i ranges from 1 to $m/3$. Here we have used again the short notation of CHEVIE referring to the internal ordering of the roots of G_{25} in CHEVIE.

In total, we obtain $1 + 4\frac{m}{3} = \frac{4m+3}{3}$ elements in $\mathrm{Fix}_{NC^m(G_{25})}(\phi^p)$, which agrees with the limit in (6.27b).

Finally, we turn to (6.27e). By Remark 3, the only choice for h_2 and m_2 to be considered are $h_2 = m_2 = 3$. This corresponds to the choice $p = 4m/3$, which has already been discussed as they do not belong to (6.27e). Hence, (3.3) must necessarily hold, as required.

Case G_{26}. The degrees are $6, 12, 18$, and hence we have

$$\mathrm{Cat}^m(G_{26}; q) = \frac{[18m+18]_q\,[18m+12]_q\,[18m+6]_q}{[18]_q\,[12]_q\,[6]_q}.$$

Let ζ be a $14m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{26}; q) = m+1, \quad \text{if } \zeta = \zeta_{18}, \zeta_9, \tag{6.32a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{26}; q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta = \zeta_{12}, \zeta_4,\ 2 \mid m, \tag{6.32b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{26}; q) = \mathrm{Cat}^m(G_{26}), \quad \text{if } \zeta = \zeta_6, \zeta_3, -1, 1, \tag{6.32c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{26}; q) = 1, \quad \text{otherwise}. \tag{6.32d}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.32). The only cases not covered by Lemmas 27 and 28 are the ones in (6.32b) and (6.32d).

We first consider (6.32b). By Lemma 26, we are free to choose $p = 3m/2$ if $\zeta = \zeta_{12}$, respectively $p = 9m/2$ if $\zeta = \zeta_4$. In both cases, m must be divisible by 2.

We start with the case that $p = 3m/2$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{2}+1} c^{-2}, c^2 w_{\frac{m}{2}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{2}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{2}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.33a}$$

$$w_i = c w_{i-\frac{m}{2}} c^{-1}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.33b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1.$$

Writing t_1, t_2 for $w_i, w_{i+\frac{m}{2}}$, respectively, the equations (6.33) reduce to

$$t_1 = c^2 t_2 c^{-2}, \tag{6.34a}$$

$$t_2 = c^1 t_1 c^{-1}. \tag{6.34b}$$

One of these equations is in fact superfluous: if we substitute (6.34b) in (6.34a), then we obtain $t_1 = c^3 t_1 c^{-3}$ which is automatically satisfied due to Lemma 29 with $d = 6$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{26})$, we must have $t_1 t_2 \le_T c$. Combining this with (6.34), we infer that

$$t_1(c t_1 c^{-1}) \le_T c. \tag{6.35}$$

With the help of `CHEVIE`, one obtains three solutions for t_1 in this equation:

$$t_1 \in \big\{[2],\ [3],\ [12]\big\},$$

each of them giving rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(G_{26})}(\phi^p)$ since i ranges from 1 to $m/2$. Here we have again used the short notation of `CHEVIE` referring to the internal ordering of the roots of G_{26} in `CHEVIE`.

In total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(G_{26})}(\phi^p)$, which agrees with the limit in (6.32b).

In the case that $p = 9m/2$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big)$$
$$= (*; c^5 w_{\frac{m}{2}+1} c^{-5}, c^5 w_{\frac{m}{2}+2} c^{-5}, \dots, c^5 w_m c^{-5}, c^4 w_1 c^{-4}, \dots, c^4 w_{\frac{m}{2}} c^{-4}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^5 w_{\frac{m}{2}+i} c^{-5}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.36a}$$

$$w_i = c^4 w_{i-\frac{m}{2}} c^{-4}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.36b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{3}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1.$$

Writing t_1, t_2 for $w_i, w_{i+\frac{m}{2}}$, respectively, the equations (6.36) reduce to

$$t_1 = c^5 t_2 c^{-5}, \tag{6.37a}$$

$$t_2 = c^4 t_1 c^{-4}. \tag{6.37b}$$

One of these equations is in fact superfluous: if we substitute (6.37b) in (6.37a), then we obtain $t_1 = c^9 t_1 c^{-9}$ which is automatically satisfied due to Lemma 29 with $d = 2$. Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{26})$, we must have $t_1 t_2 \le_T c$. Combining this with (6.37), we infer that

$$t_1(c^4 t_1 c^{-4}) \le_T c. \tag{6.38}$$

Using that $c^3 t_1 c^{-3} = t_1$, due to Lemma 29 with $d = 6$, we see that this equation is equivalent with (6.35). Therefore, we are facing exactly the same enumeration problem here as for $p = 3m/2$, and, consequently, the number of solutions to (6.38) is the same, namely $\frac{3m+2}{2}$, as required.

Finally, we turn to (6.32d). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 6$ and $m_2 = 2$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 3m/2$, respectively $p = 9m/2$, all of which have already been discussed as they do not belong to (6.32d). Hence, (3.3) must necessarily hold, as required.

Case G_{27}. The degrees are $6, 12, 30$, and hence we have

$$\operatorname{Cat}^m(G_{27}; q) = \frac{[30m+30]_q\,[30m+12]_q\,[30m+6]_q}{[30]_q\,[12]_q\,[6]_q}.$$

Let ζ be a $14m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{27}; q) = m+1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \zeta_{10}, \zeta_5, \tag{6.39a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{27}; q) = \tfrac{5m+2}{2}, \quad \text{if } \zeta = \zeta_{12}, \zeta_4,\ 2 \mid m, \tag{6.39b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{27}; q) = \operatorname{Cat}^m(G_{27}), \quad \text{if } \zeta = \zeta_6, \zeta_3, -1, 1, \tag{6.39c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{27}; q) = 1, \quad \text{otherwise.} \tag{6.39d}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.39). The only cases not covered by Lemmas 27 and 28 are the ones in (6.39b) and (6.39d).

We first consider (6.39b). By Lemma 26, we are free to choose $p = 5m/2$ if $\zeta = \zeta_{12}$, respectively $p = 15m/2$ if $\zeta = \zeta_4$. In both cases, m must be divisible by 2.

We start with the case that $p = 5m/2$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = \big(*; c^3 w_{\frac{m}{2}+1} c^{-3}, c^3 w_{\frac{m}{2}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{m}{2}} c^{-2}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{m}{2}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.40a}$$

$$w_i = c^2 w_{i-\frac{m}{2}} c^{-2}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.40b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1.$$

Writing t_1, t_2 for $w_i, w_{i+\frac{m}{2}}$, respectively, the equations (6.40) reduce to

$$t_1 = c^3 t_2 c^{-3}, \tag{6.41a}$$

$$t_2 = c^2 t_1 c^{-2}. \tag{6.41b}$$

One of these equations is in fact superfluous: if we substitute (6.41b) in (6.41a), then we obtain $t_1 = c^5 t_1 c^{-5}$ which is automatically satisfied due to Lemma 29 with $d = 6$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{27})$, we must have $t_1 t_2 \le_T c$. Combining this with (6.41), we infer that

$$t_1(c^2 t_1 c^{-2}) \le_T c. \tag{6.42}$$

With the help of CHEVIE, one obtains five solutions for t_1 in this equation:

$$t_1 \in \{[1], [2], [15], [16], [28]\},$$

each of them giving rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(G_{27})}(\phi^p)$ since i ranges from 1 to $m/2$. Here we have used the short notation of CHEVIE referring to the internal ordering of the roots of G_{27} in CHEVIE.

In total, we obtain $1 + 5\frac{m}{2} = \frac{5m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(G_{27})}(\phi^p)$, which agrees with the limit in (6.39b).

In the case that $p = 15m/2$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = (*; c^8 w_{\frac{m}{2}+1} c^{-8}, c^8 w_{\frac{m}{2}+2} c^{-8}, \dots, c^8 w_m c^{-8}, c^7 w_1 c^{-7}, \dots, c^7 w_{\frac{m}{2}} c^{-7}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^8 w_{\frac{m}{2}+i} c^{-8}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.43a}$$

$$w_i = c^7 w_{i-\frac{m}{2}} c^{-7}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.43b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1.$$

Writing t_1, t_2 for $w_i, w_{i+\frac{m}{2}}$, respectively, the equations (6.43) reduce to

$$t_1 = c^8 t_2 c^{-8}, \tag{6.44a}$$

$$t_2 = c^7 t_1 c^{-7}. \tag{6.44b}$$

One of these equations is in fact superfluous: if we substitute (6.44b) in (6.44a), then we obtain $t_1 = c^{15} t_1 c^{-15}$ which is automatically satisfied due to Lemma 29 with $d = 2$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{27})$, we must have $t_1 t_2 \le_T c$. Combining this with (6.44), we infer that

$$t_1(c^7 t_1 c^{-7}) \le_T c. \tag{6.45}$$

Using that $c^5 t_1 c^{-5} = t_1$, due to Lemma 29 with $d = 6$, we see that this equation is equivalent with (6.42). Therefore, we are facing exactly the same enumeration problem here as for $p = 5m/2$, and, consequently, the number of solutions to (6.45) is the same, namely $\frac{5m+2}{2}$, as required.

Finally, we turn to (6.39d). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 6$ and $m_2 = 3$, $h_2 = 6$ and $m_2 = 2$, $h_2 = m_2 = 3$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 5m/3$, $5m/2$, $10m/3$, respectively $15m/2$, out of which only $p = 5m/3$ and $p = 10m/3$ have not yet been discussed and belong to the current case. If $p = 5m/3$, the corresponding action of ϕ^p is given by (6.6), so that we have to solve for t_1 with $\ell_T(t_1) = 1$ in the equation (6.9). A computation with the help of `CHEVIE` finds no solution. If $p = 10m/3$, the corresponding action of ϕ^p is given by (6.11), so that we have to solve for t_1 with $\ell_T(t_1)$ in the equation (6.14). Using that $c^5 t_1 c^{-5} = t_1$, due to Lemma 29 with $d = 6$, we see that this equation is equivalent with the one in (6.9). Hence, in both cases, the left-hand side of (3.3) is equal to 1, as required.

Case $G_{28} = F_4$. The degrees are $2, 6, 8, 12$, and hence we have

$$\operatorname{Cat}^m(F_4; q) = \frac{[12m+12]_q\,[12m+8]_q\,[12m+6]_q\,[12m+2]_q}{[12]_q\,[8]_q\,[6]_q\,[2]_q}.$$

Let ζ be a $12m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4; q) = m+1, \quad \text{if } \zeta = \zeta_{12}, \tag{6.46a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4; q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta = \zeta_8,\ 2 \mid m, \tag{6.46b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4; q) = (m+1)(2m+1), \quad \text{if } \zeta = \zeta_6, \zeta_3, \tag{6.46c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4; q) = \tfrac{(m+1)(3m+2)}{2}, \quad \text{if } \zeta = \zeta_4, \tag{6.46d}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4; q) = \operatorname{Cat}^m(F_4), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{6.46e}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4; q) = 1, \quad \text{otherwise.} \tag{6.46f}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.46). The only cases not covered by Lemmas 27 and 28 are the ones in (6.46b) and (6.46f). By Lemma 26, we are free to choose $p = 3m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{2}+1} c^{-2}, c^2 w_{\frac{m}{2}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{2}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{2}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.47a}$$

$$w_i = c w_{i-\frac{m}{2}} c^{-1}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.47b}$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \tag{6.48a}$$

and all other w_j's are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \tag{6.48b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.48c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(F_4)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} = c$. Together with equations (6.47)–(6.48), this implies that

$$w_i = c^3 w_i c^{-3} \quad \text{and} \quad w_i(c w_i c^{-1}) = c, \tag{6.49}$$

respectively that

$$w_i = c^3 w_i c^{-3}, \quad w_i(c w_i c^{-1}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.50}$$

respectively that

$$w_{i_1} = c^3 w_{i_1} c^{-3}, \quad w_{i_1}(c w_{i_1} c^{-1}) \le_T c, \quad \text{and} \quad \ell_T(w_{i_1}) = 1. \tag{6.51}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains three solutions for w_i in (6.49):

$$w_i \in \big\{[1,2,3,4,3,2],\ [2,3],\ [1,3,2,1,3,4]\big\},$$

where we have again used the short notation of `coxeter`, $\{s_1, s_2, s_3, s_4\}$ being a simple system of generators of F_4, corresponding to the Dynkin diagram displayed in Figure 2. Each of the above solutions for w_i gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(F_4)}(\phi^p)$ since i ranges from 1 to $m/2$.

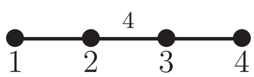

FIGURE 2. The Dynkin diagram for F_4

There are no solutions for w_i in (6.50) and for w_{i_1} in (6.51).

In total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(F_4)}(\phi^p)$, which agrees with the limit in (6.46b).

Finally, we turn to (6.46f). By Remark 3, the are no choices for h_2 and m_2 to be considered. Hence, the left-hand side of (3.3) is equal to 1, as required.

CASE G_{29}. The degrees are $4, 8, 12, 20$, and hence we have

$$\mathrm{Cat}^m(G_{29}; q) = \frac{[20m+20]_q\,[20m+12]_q\,[20m+8]_q\,[20m+4]_q}{[20]_q\,[12]_q\,[8]_q\,[4]_q}.$$

Let ζ be a $20m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{29};q)=m+1,\quad \text{if } \zeta=\zeta_{20},\zeta_{10},\zeta_5, \tag{6.52a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{29};q)=\tfrac{5m+3}{3},\quad \text{if } \zeta=\zeta_{12},\zeta_6,\zeta_3,\ 3\mid m, \tag{6.52b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{29};q)=\tfrac{5m+2}{2},\quad \text{if } \zeta=\zeta_8,\ 2\mid m, \tag{6.52c}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{29};q)=\operatorname{Cat}^m(G_{29}),\quad \text{if } \zeta=\zeta_4,-1,1, \tag{6.52d}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{29};q)=1,\quad \text{otherwise.} \tag{6.52e}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.52). The only cases not covered by Lemmas 27 and 28 are the ones in (6.52b), (6.52c), and (6.52e).

We begin with the case in (6.52b). By Lemma 26, we are free to choose $p=5m/3$ if $\zeta=\zeta_{12}$, we are free to choose $p=10m/3$ if $\zeta=\zeta_6$, we are free to choose $p=20m/3$ if $\zeta=\zeta_3$. In particular, in all three cases, m must be divisible by 3.

We start with the case that $p=5m/3$. From (5.1), we infer

$$\phi^p\big((w_0;w_1,\dots,w_m)\big)=(*;c^2w_{\frac{m}{3}+1}c^{-2},c^2w_{\frac{m}{3}+2}c^{-2},\dots,c^2w_mc^{-2},cw_1c^{-1},\dots,cw_{\frac{m}{3}}c^{-1}\big).$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^2w_{\frac{m}{3}+i}c^{-2},\quad i=1,2,\dots,\tfrac{2m}{3}, \tag{6.53a}$$

$$w_i=cw_{i-\frac{2m}{3}}c^{-1},\quad i=\tfrac{2m}{3}+1,\tfrac{2m}{3}+2,\dots,m. \tag{6.53b}$$

There are two distinct possibilities for choosing the w_i's, $1\le i\le m$:

(i) all the w_i's are equal to ε (and $w_0=c$),

(ii) there is an i with $1\le i\le \frac{m}{3}$ such that

$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{3}})=\ell_T(w_{i+\frac{2m}{3}})=1, \tag{6.54a}$$

and the other w_j's, $1\le j\le m$, are equal to ε.

Moreover, since $(w_0;w_1,\dots,w_m)\in NC^m(G_{29})$, we must have $w_iw_{i+\frac{m}{3}}w_{i+\frac{2m}{3}}\le_T c$. Together with equations (6.53)–(6.54), this implies that

$$w_i=c^5w_ic^{-5}\quad\text{and}\quad w_i(c^3w_ic^{-3})(cw_ic^{-1})=c. \tag{6.55}$$

With the help of `CHEVIE`, one obtains five solutions for w_i in (6.55):

$$w_i\in\big\{[1],\,[2],\,[8],\,[25],\,[31]\big\},$$

where we have again used the short notation of `CHEVIE` referring to the internal ordering of the roots of G_{29} in `CHEVIE`. Each of the above solutions for w_i gives rise to $m/3$ elements of $\operatorname{Fix}_{NC^m(G_{29})}(\phi^p)$ since i ranges from 1 to $m/3$.

In total, we obtain $1+5\frac{m}{3}=\frac{5m+3}{3}$ elements in $\operatorname{Fix}_{NC^m(G_{29})}(\phi^p)$, which agrees with the limit in (6.52b).

In the case that $p=10m/3$, we infer from (5.1) that

$$\phi^p\big((w_0;w_1,\dots,w_m)\big)\\=(*;c^4w_{\frac{2m}{3}+1}c^{-4},c^4w_{\frac{2m}{3}+2}c^{-4},\dots,c^4w_mc^{-4},c^3w_1c^{-3},\dots,c^3w_{\frac{2m}{3}}c^{-3}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^4 w_{\frac{2m}{3}+i} c^{-4}, \quad i = 1, 2, \dots, \tfrac{m}{3}, \tag{6.56a}$$

$$w_i = c^3 w_{i-\frac{m}{3}} c^{-3}, \quad i = \tfrac{m}{3}+1, \tfrac{m}{3}+2, \dots, m. \tag{6.56b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{3}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 1.$$

Writing t_1, t_2, t_3 for $w_i, w_{i+\frac{m}{3}}, w_{i+\frac{2m}{3}}$, respectively, the equations (6.56) reduce to

$$t_1 = c^4 t_3 c^{-4}, \tag{6.57a}$$

$$t_2 = c^3 t_1 c^{-3}, \tag{6.57b}$$

$$t_3 = c^3 t_2 c^{-3}. \tag{6.57c}$$

One of these equations is in fact superfluous: if we substitute (6.57b) and (6.57c) in (6.57a), then we obtain $t_1 = c^{10} t_1 c^{-10}$ which is automatically satisfied due to Lemma 29 with $d = 2$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{29})$, we must have $t_1 t_2 t_3 \le_T c$. Combining this with (6.57), we infer that

$$t_1 (c^3 t_1 c^{-3})(c^6 t_1 c^{-6}) \le_T c. \tag{6.58}$$

Using that $c^5 t_1 c^{-5} = t_1$, due to Lemma 29 with $d = 4$, we see that this equation is equivalent with (6.55). Therefore, we are facing exactly the same enumeration problem here as for $p = 5m/3$, and, consequently, the number of solutions to (6.58) is the same, namely $\frac{5m+3}{3}$, as required.

In the case that $p = 20m/3$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^7 w_{\frac{m}{3}+1} c^{-7}, c^7 w_{\frac{m}{3}+2} c^{-7}, \dots, c^7 w_m c^{-7}, c^6 w_1 c^{-6}, \dots, c^6 w_{\frac{m}{3}} c^{-6}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^7 w_{\frac{m}{3}+i} c^{-7}, \quad i = 1, 2, \dots, \tfrac{2m}{3}, \tag{6.59a}$$

$$w_i = c^6 w_{i-\frac{2m}{3}} c^{-6}, \quad i = \tfrac{2m}{3}+1, \tfrac{2m}{3}+2, \dots, m. \tag{6.59b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{3}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 1.$$

Writing t_1, t_2, t_3 for $w_i, w_{i+\frac{m}{3}}, w_{i+\frac{2m}{3}}$, respectively, the equations (6.59) reduce to

$$t_1 = c^7 t_2 c^{-7}, \tag{6.60a}$$

$$t_2 = c^7 t_3 c^{-7}, \tag{6.60b}$$

$$t_3 = c^6 t_1 c^{-6}. \tag{6.60c}$$

One of these equations is in fact superfluous: if we substitute (6.60b) and (6.60c) in (6.60a), then we obtain $t_1 = c^{20} t_1 c^{-20}$ which is automatically satisfied since $c^{20} = \varepsilon$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{29})$, we must have $t_1t_2t_3 \le_T c$. Combining this with (6.60), we infer that

$$t_1(c^{13}t_1c^{-13})(c^6t_1c^{-6}) \le_T c. \tag{6.61}$$

Using that $c^5t_1c^{-5} = t_1$, due to Lemma 29 with $d = 4$, we see that this equation is equivalent with (6.55). Therefore, we are facing exactly the same enumeration problem here as for $p = 5m/3$, and, consequently, the number of solutions to (6.61) is the same, namely $\frac{5m+3}{3}$, as required.

Next we discuss the case in (6.52c). By Lemma 26, we are free to choose $p = 5m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = (*; c^3w_{\frac{m}{2}+1}c^{-3}, c^3w_{\frac{m}{2}+2}c^{-3}, \dots, c^3w_mc^{-3}, c^2w_1c^{-2}, \dots, c^2w_{\frac{m}{2}}c^{-2}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3w_{\frac{m}{2}+i}c^{-3}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.62a}$$

$$w_i = c^2w_{i-\frac{m}{2}}c^{-2}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.62b}$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \tag{6.63a}$$

and all other w_j's are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \tag{6.63b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.63c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{29})$, we must have $w_iw_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1}w_{i_2}w_{i_1+\frac{m}{2}}w_{i_2+\frac{m}{2}} = c$. Together with equations (6.62)–(6.63), this implies that

$$w_i = c^3w_ic^{-3} \quad \text{and} \quad w_i(c^2w_ic^{-2}) = c, \tag{6.64}$$

respectively that

$$w_i = c^3w_ic^{-3}, \quad w_i(c^2w_ic^{-2}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.65}$$

respectively that

$$w_{i_1} = c^3w_{i_1}c^{-3}, \quad w_{i_2} = c^3w_{i_2}c^{-3}, \\ w_{i_1}w_{i_2}(c^2w_{i_1}c^{-2})(c^2w_{i_2}c^{-2}) = c, \quad \text{and} \quad \ell_T(w_{i_1}) = \ell_T(w_{i_2}) = 1. \tag{6.66}$$

With the help of `CHEVIE`, one obtains five solutions for w_i in (6.65):

$$w_i \in \big\{[4], [9], [14], [27], [32]\big\},$$

where we have again used the short notation of `CHEVIE` referring to the internal ordering of the roots of G_{29} in `CHEVIE`. Each of these solutions for w_i gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(G_{29})}(\phi^p)$ since i ranges from 1 to $m/2$.

There are no solutions for w_i in (6.64) and for (w_{i_1}, w_{i_2}) in (6.66).

In total, we obtain $1 + 5\frac{m}{2} = \frac{5m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(G_{29})}(\phi^p)$, which agrees with the limit in (6.52c).

Finally, we turn to (6.52e). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 1$ and $m_2 = 3$, $h_2 = 2$ and $m_2 = 3$, $h_2 = 4$ and $m_2 = 2$, $h_2 = 4$ and $m_2 = 3$, respectively $h_2 = m_2 = 4$. These correspond to the choices $p = 20m/3$, $p = 10m/3$, $p = 5m/2$, $p = 5m/3$, respectively $p = 5m/4$, out of which only $p = 5m/4$ has not yet been discussed and belongs to the current case. The corresponding action of ϕ^p is given by

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^2 w_{\frac{3m}{4}+1} c^{-2}, c^2 w_{\frac{3m}{4}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{3m}{4}} c^{-1}\big),$$

so that we have to solve

$$t_1 (c t_1 c^{-1})(c^2 t_1 c^{-2})(c^3 t_1 c^{-3}) = c$$

for t_1 with $\ell_T(t_1)$. A computation with the help of `CHEVIE` finds no solution. Hence, the left-hand side of (3.3) is equal to 1, as required.

Case $G_{30} = H_4$. The degrees are $2, 12, 20, 30$, and hence we have

$$\mathrm{Cat}^m(H_4; q) = \frac{[30m+30]_q\,[30m+20]_q\,[30m+12]_q\,[30m+2]_q}{[30]_q\,[20]_q\,[12]_q\,[2]_q}.$$

Let ζ be a $30m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_4; q) = m+1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \tag{6.67a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_4; q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta = \zeta_{20},\ 2 \mid m, \tag{6.67b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_4; q) = \tfrac{5m+2}{2}, \quad \text{if } \zeta = \zeta_{12},\ 2 \mid m, \tag{6.67c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_4; q) = \tfrac{(m+1)(3m+2)}{2}, \quad \text{if } \zeta = \zeta_{10}, \zeta_5, \tag{6.67d}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_4; q) = \tfrac{(m+1)(5m+2)}{2}, \quad \text{if } \zeta = \zeta_6, \zeta_3, \tag{6.67e}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_4; q) = \tfrac{(3m+2)(5m+2)}{4}, \quad \text{if } \zeta = \zeta_4,\ 2 \mid m, \tag{6.67f}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_4; q) = \mathrm{Cat}^m(H_4), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{6.67g}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_4; q) = 1, \quad \text{otherwise.} \tag{6.67h}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.67). The only cases not covered by Lemmas 27 and 28 are the ones in (6.67b), (6.67c), (6.67f), and (6.67h).

We begin with the case in (6.67b). By Lemma 26, we are free to choose $p = 3m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{2}+1} c^{-2}, c^2 w_{\frac{m}{2}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{2}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{2}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.68a}$$

$$w_i = c w_{i-\frac{m}{2}} c^{-1}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.68b}$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \tag{6.69a}$$

and all other w_j's are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \tag{6.69b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.69c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(H_4)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} = c$. Together with equations (6.68)–(6.69), this implies that

$$w_i = c^3 w_i c^{-3} \quad \text{and} \quad w_i(c w_i c^{-1}) = c, \tag{6.70}$$

respectively that

$$w_i = c^3 w_i c^{-3}, \quad w_i(c w_i c^{-1}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.71}$$

respectively that

$$w_{i_1} = c^3 w_{i_1} c^{-3}, \quad w_{i_1}(c w_{i_1} c^{-1}) \le_T c, \quad \text{and} \quad \ell_T(w_{i_1}) = 1. \tag{6.72}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains three solutions for w_i in (6.70):

$$w_i \in \big\{[1,2,3,4,3,2,1,2], [2,1,2,3], [1,3,2,1,2,1,3,4]\big\},$$

where we have again used the short notation of `coxeter`, $\{s_1, s_2, s_3, s_4\}$ being a simple system of generators of H_4, corresponding to the Dynkin diagram displayed in Figure 3. Each of the above solutions for w_i gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(H_4)}(\phi^p)$ since i ranges from 1 to $m/2$.

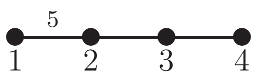

FIGURE 3. The Dynkin diagram for H_4

There are no solutions for w_i in (6.71) and for w_{i_1} in (6.72).

In total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(H_4)}(\phi^p)$, which agrees with the limit in (6.67b).

Next we discuss the case in (6.67c). By Lemma 26, we are free to choose $p = 5m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^3 w_{\frac{m}{2}+1} c^{-3}, c^3 w_{\frac{m}{2}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{m}{2}} c^{-2}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{m}{2}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.73a}$$

$$w_i = c^2 w_{i-\frac{m}{2}} c^{-2}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.73b}$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \tag{6.74a}$$

and all other w_j's are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \tag{6.74b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.74c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(H_4)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} = c$. Together with equations (6.73)–(6.74), this implies that

$$w_i = c^3 w_i c^{-3} \quad \text{and} \quad w_i(c^2 w_i c^{-2}) = c, \tag{6.75}$$

respectively that

$$w_i = c^3 w_i c^{-3}, \quad w_i(c^2 w_i c^{-2}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.76}$$

respectively that

$$w_{i_1} = c^3 w_{i_1} c^{-3}, \quad w_{i_1}(c^2 w_{i_1} c^{-2}) \le_T c, \quad \text{and} \quad \ell_T(w_{i_1}) = 1. \tag{6.77}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains five solutions for w_i in (6.75):

$$w_i \in \big\{[1,3,2,1,2,1,3,2], [1,2,1,4,3,2,1,2,1,3,2,1,4,3], \\ [2,1,2,3,2,1,2,4], [2,1,2,1,4,3,2,1,2,1,3,4], [1,2,3,2,1,4,3,2,1,2,1,3]\big\},$$

where we used again `coxeter`'s short notation, $\{s_1, s_2, s_3, s_4\}$ being a simple system of generators of H_4, corresponding to the Dynkin diagram displayed in Figure 3. Each of these solutions for w_i gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(H_4)}(\phi^p)$ since i ranges from 1 to $m/2$.

There are no solutions for w_i in (6.76) and for w_{i_1} in (6.77).

In total, we obtain $1 + 5\frac{m}{2} = \frac{5m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(H_4)}(\phi^p)$, which agrees with the limit in (6.67c).

Finally we discuss the case in (6.67f). By Lemma 26, we are free to choose $p = 15m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^8 w_{\frac{m}{2}+1} c^{-8}, c^8 w_{\frac{m}{2}+2} c^{-8}, \dots, c^8 w_m c^{-8}, c^7 w_1 c^{-7}, \dots, c^7 w_{\frac{m}{2}} c^{-7}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^8 w_{\frac{m}{2}+i} c^{-8}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.78a}$$

$$w_i = c^7 w_{i-\frac{m}{2}} c^{-7}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.78b}$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \tag{6.79a}$$

and all other w_j's are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \tag{6.79b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.79c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(H_4)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} = c$. Together with equations (6.78)–(6.79), this implies that

$$w_i = c^{15} w_i c^{-15} \quad \text{and} \quad w_i(c^7 w_i c^{-7}) = c, \tag{6.80}$$

respectively that

$$w_i = c^{15} w_i c^{-15}, \quad w_i(c^7 w_i c^{-7}) = c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.81}$$

respectively that

$$w_{i_1} = c^{15} w_{i_1} c^{-15}, \quad w_{i_2} = c^{15} w_{i_2} c^{-15}, \quad w_{i_1} w_{i_2} (c^7 w_{i_1} c^{-7})(c^7 w_{i_2} c^{-7}) = c. \tag{6.82}$$

Here, the first equations in both (6.80) and (6.81), and the first two equations in (6.82) are automatically satisfied due to Lemma 29 with $d = 2$.

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains eight solutions for w_i in (6.80):

$$w_i \in \big\{[1,3,2,1,2,1,3,2],\ [1,2,3,4,3,2,1,2],\ [2,1,2,3], \\ [1,2,1,4,3,2,1,2,1,3,2,1,4,3],\ [1,3,2,1,2,1,3,4],\ [2,1,2,3,2,1,2,4], \\ [2,1,2,1,4,3,2,1,2,1,3,4],\ [1,2,3,2,1,4,3,2,1,2,1,3]\big\}, \tag{6.83}$$

where $\{s_1, s_2, s_3, s_4\}$ is a simple system of generators of H_4, corresponding to the Dynkin diagram displayed in Figure 3, and each of them gives rise to $m/2$ elements of

$\mathrm{Fix}_{NC^m(H_4)}(\phi^p)$ since i ranges from 1 to $m/2$. Furthermore, one obtains 15 solutions for w_i in (6.81):

$$\begin{aligned}w_i \in \big\{&[2],\ [3],\ [4],\ [2,1,2],\ [3,2,1,2,3],\ [1,3,2,1,2,1,3],\ [2,1,2,3,2,1,2],\ [1,2,3,4,3,2,1],\\ &[2,1,3,2,1,2,1,3,2],\ [1,4,3,2,1,2,1,3,4],\ [2,1,2,3,4,3,2,1,2],\\ &[1,2,1,4,3,2,1,2,1,3,2,1,4],\ [1,3,2,1,2,3,4,3,2,1,2,1,3],\\ &[2,1,2,1,4,3,2,1,2,1,3,2,1,2,4],\ [1,3,2,1,4,3,2,1,2,1,3,2,1,4,3]\big\},\end{aligned}$$

each of them giving rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(H_4)}(\phi^p)$ since i ranges from 1 to $m/2$, and one obtains 30 pairs (w_{i_1}, w_{i_2}) of solutions in (6.82):

$$\begin{aligned}(w_{i_1}, w_{i_2}) \in \big\{&([2],\ [2,1,3,2,1,2,1,3,2]),\ ([2],\ [2,1,2,3,4,3,2,1,2]),\ ([3],\ [3,2,1,2,3]),\\ &([3],\ [1,3,2,1,4,3,2,1,2,1,3,2,1,4,3]),\ ([4],\ [1,4,3,2,1,2,1,3,4]),\\ &([4],\ [2,1,2,3,4,3,2,1,2]),\ ([2,1,2],\ [3]),\ ([2,1,2],\ [1,4,3,2,1,2,1,3,4]),\\ &([3,2,1,2,3],\ [2,1,3,2,1,2,1,3,2]),\ ([3,2,1,2,3],\ [1,3,2,1,2,3,4,3,2,1,2,1,3]),\\ &([1,3,2,1,2,1,3],\ [2]),\ ([1,3,2,1,2,1,3],\ [4]),\ ([2,1,2,3,2,1,2],\ [4]),\\ &([2,1,2,3,2,1,2],\ [2,1,2]),\ ([1,2,3,4,3,2,1],\ [2]),\ ([1,2,3,4,3,2,1],\ [3,2,1,2,3]),\\ &([2,1,3,2,1,2,1,3,2],\ [1,3,2,1,2,1,3]),\ ([2,1,3,2,1,2,1,3,2],\ [2,1,2,3,2,1,2]),\\ &([1,4,3,2,1,2,1,3,4],\ [2,1,2,1,4,3,2,1,2,1,3,2,1,2,4]),\\ &([1,4,3,2,1,2,1,3,4],\ [1,3,2,1,4,3,2,1,2,1,3,2,1,4,3]),\\ &([2,1,2,3,4,3,2,1,2],\ [2,1,2,3,2,1,2]),\\ &([2,1,2,3,4,3,2,1,2],\ [2,1,2,1,4,3,2,1,2,1,3,2,1,2,4]),\\ &([1,2,1,4,3,2,1,2,1,3,2,1,4],\ [3]),\\ &([1,2,1,4,3,2,1,2,1,3,2,1,4],\ [1,2,3,4,3,2,1]),\\ &([1,3,2,1,2,3,4,3,2,1,2,1,3],\ [1,3,2,1,2,1,3]),\\ &([1,3,2,1,2,3,4,3,2,1,2,1,3],\ [1,2,3,4,3,2,1]),\\ &([2,1,2,1,4,3,2,1,2,1,3,2,1,2,4],\ [2,1,2]),\\ &([2,1,2,1,4,3,2,1,2,1,3,2,1,2,4],\ [1,2,1,4,3,2,1,2,1,3,2,1,4]),\\ &([1,3,2,1,4,3,2,1,2,1,3,2,1,4,3],\ [1,2,1,4,3,2,1,2,1,3,2,1,4]),\\ &([1,3,2,1,4,3,2,1,2,1,3,2,1,4,3],\ [1,3,2,1,2,3,4,3,2,1,2,1,3])\big\},\end{aligned}\tag{6.84}$$

each of them giving rise to $\binom{m/2}{2}$ elements of $\mathrm{Fix}_{NC^m(H_4)}(\phi^p)$ since $1 \le i_1 < i_2 \le \frac{m}{2}$.

In total, we obtain $1+(15+8)\frac{m}{2}+30\binom{m/2}{2} = \frac{(3m+2)(5m+2)}{4}$ elements in $\mathrm{Fix}_{NC^m(H_4)}(\phi^p)$, which agrees with the limit in (6.67f).

Finally, we turn to (6.67h). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 2$ and $m_2 = 4$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 15m/2$, respectively $p = 15m/4$, out of which only $p = 15m/4$ has not yet been discussed and belongs to the current case. The corresponding action of ϕ^p is given by

$$\begin{aligned}&\phi^p\big((w_0; w_1, \dots, w_m)\big)\\ &\qquad = \big(*; c^4 w_{\frac{m}{4}+1} c^{-4}, c^4 w_{\frac{m}{4}+2} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_1 c^{-3}, \dots, c^3 w_{\frac{m}{4}} c^{-3}\big),\end{aligned}$$

so that we have to solve

$$t_1(c^{11}t_1c^{-11})(c^7t_1c^{-7})(c^3t_1c^{-3}) = c$$

for t_1 with $\ell_T(t_1)$. A computation with Stembridge's *Maple* package `coxeter` [35] finds no solution. Hence, the left-hand side of (3.3) is equal to 1, as required.

Case G_{32}. The degrees are $12, 18, 24, 30$, and hence we have

$$\operatorname{Cat}^m(G_{32}; q) = \frac{[30m+30]_q\,[30m+24]_q\,[30m+18]_q\,[30m+12]_q}{[30]_q\,[24]_q\,[18]_q\,[12]_q}.$$

Let ζ be a $30m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{32}; q) = m+1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \zeta_{10}, \zeta_5, \tag{6.85a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{32}; q) = \tfrac{5m+4}{4}, \quad \text{if } \zeta = \zeta_{24}, \zeta_8,\ 4 \mid m, \tag{6.85b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{32}; q) = \tfrac{5m+3}{3}, \quad \text{if } \zeta = \zeta_{18}, \zeta_9,\ 3 \mid m, \tag{6.85c}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{32}; q) = \tfrac{(5m+4)(5m+2)}{8}, \quad \text{if } \zeta = \zeta_{12}, \zeta_4,\ 2 \mid m, \tag{6.85d}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{32}; q) = \operatorname{Cat}^m(G_{32}), \quad \text{if } \zeta = \zeta_6, \zeta_3, -1, 1, \tag{6.85e}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{32}; q) = 1, \quad \text{otherwise.} \tag{6.85f}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.85). The only cases not covered by Lemmas 27 and 28 are the ones in (6.85b), (6.85c), (6.85d), and (6.85f).

We begin with the case in (6.85b). By Lemma 26, we are free to choose $p = 5m/4$ if $\zeta = \zeta_{24}$, and we are free to choose $p = 15m/4$ if $\zeta = \zeta_8$. In particular, in all both cases, m must be divisible by 4.

We start with the case that $p = 5m/4$. From (5.1), we infer

$$\begin{aligned}\phi^p\big((w_0; w_1, \dots, w_m)\big)\\ &= \big(*; c^2w_{\frac{3m}{4}+1}c^{-2}, c^2w_{\frac{3m}{4}+2}c^{-2}, \dots, c^2w_mc^{-2}, cw_1c^{-1}, \dots, cw_{\frac{3m}{4}}c^{-1}\big).\end{aligned}$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2w_{\frac{3m}{4}+i}c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{4}, \tag{6.86a}$$

$$w_i = cw_{i-\frac{m}{4}}c^{-1}, \quad i = \tfrac{m}{4}+1, \tfrac{m}{4}+2, \dots, m. \tag{6.86b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{4}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{4}}) = \ell_T(w_{i+\frac{2m}{4}}) = \ell_T(w_{i+\frac{3m}{4}}) = 1, \tag{6.87a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{32})$, we must have

$$w_iw_{i+\frac{m}{4}}w_{i+\frac{2m}{4}}w_{i+\frac{3m}{4}} = c.$$

Together with equations (6.86)–(6.87), this implies that

$$w_i = c^5 w_i c^{-5} \quad \text{and} \quad w_i(cw_ic^{-1})(c^2w_ic^{-2})(c^3w_ic^{-3}) = c. \tag{6.88}$$

With the help of CHEVIE, one obtains five solutions for w_i in (6.88):

$$w_i \in \{[1],\ [2],\ [3],\ [4],\ [27]\}, \tag{6.89}$$

where we have again used the short notation of CHEVIE referring to the internal ordering of the roots of G_{32} in CHEVIE. Each of the above solutions for w_i gives rise to $m/4$ elements of $\mathrm{Fix}_{NC^m(G_{32})}(\phi^p)$ since i ranges from 1 to $m/4$.

In total, we obtain $1 + 5\frac{m}{4} = \frac{5m+4}{4}$ elements in $\mathrm{Fix}_{NC^m(G_{32})}(\phi^p)$, which agrees with the limit in (6.85b).

In the case that $p = 15m/4$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^4 w_{\frac{m}{4}+1} c^{-4}, c^4 w_{\frac{m}{4}+2} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_1 c^{-3}, \dots, c^3 w_{\frac{m}{4}} c^{-3}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^4 w_{\frac{m}{4}+i} c^{-4}, \quad i = 1, 2, \dots, \tfrac{3m}{4}, \tag{6.90a}$$

$$w_i = c^3 w_{i-\frac{3m}{4}} c^{-3}, \quad i = \tfrac{3m}{4}+1, \tfrac{3m}{4}+2, \dots, m. \tag{6.90b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{4}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{4}}) = \ell_T(w_{i+\frac{2m}{4}}) = \ell_T(w_{i+\frac{3m}{4}}) = 1.$$

Writing t_1, t_2, t_3, t_4 for $w_i, w_{i+\frac{m}{4}}, w_{i+\frac{2m}{4}}, w_{i+\frac{3m}{4}}$, respectively, the equations (6.90) reduce to

$$t_1 = c^4 t_2 c^{-4}, \tag{6.91a}$$

$$t_2 = c^4 t_3 c^{-4}, \tag{6.91b}$$

$$t_3 = c^4 t_4 c^{-4}, \tag{6.91c}$$

$$t_4 = c^3 t_1 c^{-3}. \tag{6.91d}$$

One of these equations is in fact superfluous: if we substitute (6.91b)–(6.91d) in (6.91a), then we obtain $t_1 = c^{15} t_1 c^{-15}$ which is automatically satisfied due to Lemma 29 with $d = 2$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{32})$, we must have $t_1t_2t_3t_4 = c$. Combining this with (6.91), we infer that

$$t_1(c^{11}t_1c^{-11})(c^7t_1c^{-7})(c^3t_1c^{-3}) = c. \tag{6.92}$$

Using that $c^5 t_1 c^{-5} = t_1$, due to Lemma 29 with $d = 6$, we see that this equation is equivalent with (6.88). Therefore, we are facing exactly the same enumeration problem here as for $p = 5m/4$, and, consequently, the number of solutions to (6.92) is the same, namely $\frac{5m+4}{4}$, as required.

Next we consider the case in (6.85c). By Lemma 26, we are free to choose $p = 5m/3$ if $\zeta = \zeta_{18}$, and we are free to choose $p = 10m/3$ if $\zeta = \zeta_9$. In particular, in both cases, m must be divisible by 3.

We start with the case that $p = 5m/3$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = \big(*; c^2 w_{\frac{m}{3}+1} c^{-2}, c^2 w_{\frac{m}{3}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{3}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{3}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{2m}{3}, \tag{6.93a}$$

$$w_i = c w_{i-\frac{2m}{3}} c^{-1}, \quad i = \tfrac{2m}{3}+1, \tfrac{2m}{3}+2, \dots, m. \tag{6.93b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 1, \tag{6.94a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{32})$, we must have $w_i w_{i+\frac{m}{3}} w_{i+\frac{2m}{3}} \le_T c$. Together with equations (6.93)–(6.94), this implies that

$$w_i = c^5 w_i c^{-5} \quad \text{and} \quad w_i (c^3 w_i c^{-3})(c w_i c^{-1}) \le_T c. \tag{6.95}$$

With the help of `CHEVIE`, one obtains three solutions for w_i in (6.95):

$$w_i \in \{[1], [2], [3], [4], [27]\},$$

where we have again used the short notation of `CHEVIE` referring to the internal ordering of the roots of G_{32} in `CHEVIE`. Each of the above solutions for w_i gives rise to $m/3$ elements of $\mathrm{Fix}_{NC^m(G_{32})}(\phi^p)$ since i ranges from 1 to $m/3$.

In total, we obtain $1 + 5\frac{m}{3} = \frac{5m+3}{3}$ elements in $\mathrm{Fix}_{NC^m(G_{32})}(\phi^p)$, which agrees with the limit in (6.85c).

In the case that $p = 10m/3$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = \big(*; c^4 w_{\frac{2m}{3}+1} c^{-4}, c^4 w_{\frac{2m}{3}+2} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_1 c^{-3}, \dots, c^3 w_{\frac{2m}{3}} c^{-3}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^4 w_{\frac{2m}{3}+i} c^{-4}, \quad i = 1, 2, \dots, \tfrac{m}{3}, \tag{6.96a}$$

$$w_i = c^3 w_{i-\frac{m}{3}} c^{-3}, \quad i = \tfrac{m}{3}+1, \tfrac{m}{3}+2, \dots, m. \tag{6.96b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{3}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 1.$$

Writing t_1, t_2, t_3 for $w_i, w_{i+\frac{m}{3}}, w_{i+\frac{2m}{3}}$, respectively, the equations (6.96) reduce to

$$t_1 = c^4 t_3 c^{-4}, \tag{6.97a}$$

$$t_2 = c^3 t_1 c^{-3}, \tag{6.97b}$$

$$t_3 = c^3 t_2 c^{-3}. \tag{6.97c}$$

One of these equations is in fact superfluous: if we substitute (6.97b) and (6.97c) in (6.97a), then we obtain $t_1 = c^{10} t_1 c^{-10}$ which is automatically satisfied due to Lemma 29 with $d = 2$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{32})$, we must have $t_1 t_2 t_3 \le_T c$. Combining this with (6.97), we infer that

$$t_1 (c^3 t_1 c^{-3})(c^6 t_1 c^{-6}) \le_T c. \tag{6.98}$$

Using that $c^5 t_1 c^{-5} = t_1$, due to Lemma 29 with $d = 4$, we see that this equation is equivalent with (6.95). Therefore, we are facing exactly the same enumeration problem here as for $p = 5m/3$, and, consequently, the number of solutions to (6.98) is the same, namely $\frac{5m+3}{3}$, as required.

Next we discuss the case in (6.85d). By Lemma 26, we are free to choose $p = 5m/2$ if $\zeta = \zeta_{12}$, and we are free to choose $p = 15m/2$ if $\zeta = \zeta_4$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^3 w_{\frac{m}{2}+1} c^{-3}, c^3 w_{\frac{m}{2}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{m}{2}} c^{-2}\big). \quad (6.99)$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{m}{2}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \quad (6.100a)$$

$$w_i = c^2 w_{i-\frac{m}{2}} c^{-2}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \quad (6.100b)$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \quad (6.101a)$$

and all other w_j's are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \quad (6.101b)$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \quad (6.101c)$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{32})$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} = c$. Together with equations (6.100)–(6.101), this implies that

$$w_i = c^3 w_i c^{-3} \quad \text{and} \quad w_i(c^2 w_i c^{-2}) = c, \quad (6.102)$$

respectively that

$$w_i = c^3 w_i c^{-3}, \quad w_i(c^2 w_i c^{-2}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \quad (6.103)$$

respectively that

$$w_{i_1} = c^3 w_{i_1} c^{-3}, \quad w_{i_2} = c^3 w_{i_2} c^{-3}, \\ w_{i_1} w_{i_2} (c^2 w_{i_1} c^{-2})(c^2 w_{i_2} c^{-2}) = c, \quad \text{and} \quad \ell_T(w_{i_1}) = \ell_T(w_{i_2}) = 1. \quad (6.104)$$

With the help of CHEVIE, one obtains ten solutions for w_i in (6.102):

$$w_i \in \big\{[3,4],\ [1,20],\ [4,7],\ [1,2],\ [2,3],\ [4,27],\ [3,23],\ [1,23],\ [2,16],\ [12,27]\big\}, \quad (6.105)$$

where we have again used the short notation of CHEVIE referring to the internal ordering of the roots of G_{32} in CHEVIE, one obtains solutions for w_i in (6.103):

$$w_i \in \big\{[4]\ \ [3]\ \ [20]\ \ [1]\ \ [7]\ \ [2]\ \ [16]\ \ [12]\ \ [27]\ \ [23]\big\},$$

each of them giving rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(G_{32})}(\phi^p)$ since i ranges from 1 to $m/2$, and one obtains 25 pairs (w_{i_1}, w_{i_2}) satisfying (6.104):

$$\begin{multline}(w_{i_1}, w_{i_2}) \in \big\{([4], [20]),\ ([4], [7]),\ ([4], [27]),\ ([3], [4]),\ ([3], [12]),\ ([3], [23]),\ ([20], [3]),\\ ([20], [1]),\ ([1], [20]),\ ([1], [2]),\ ([1], [23]),\ ([7], [4]),\ ([7], [1]),\ ([2], [3]),\ ([2], [7]),\\ ([2], [16]),\ ([16], [4]),\ ([16], [2]),\ ([12], [2]),\ ([12], [27]),\ ([27], [1]),\ ([27], [16]),\\ ([27], [12]),\ ([23], [3]),\ ([23], [27])\big\}, \quad (6.106)\end{multline}$$

each of them giving rise to $\binom{m/2}{2}$ elements of $\mathrm{Fix}_{NC^m(G_{32})}(\phi^p)$ since $1 \le i_1 < i_2 \le \frac{m}{2}$.

In total, we obtain $1 + 20\frac{m}{2} + 25\binom{m/2}{2} = \frac{(5m+4)(5m+2)}{8}$ elements in $\mathrm{Fix}_{NC^m(G_{32})}(\phi^p)$, which agrees with the limit in (6.85d).

If $p = 15m/2$, then, from (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^8 w_{\frac{m}{2}+1} c^{-8}, c^8 w_{\frac{m}{2}+2} c^{-8}, \dots, c^8 w_m c^{-8}, c^7 w_1 c^{-7}, \dots, c^7 w_{\frac{m}{2}} c^{-7}\big).$$

Using that $c^5 w c^{-5} = w$ for all $w \in NC(G_{32}$, due to Lemma 29 with $d = 6$, we see that this action is identical with the one in (6.99). Therefore, we are facing exactly the same enumeration problem here as for $p = 5m/2$, and, consequently, the number of elements in $\mathrm{Fix}_{NC^m(G_{32})}(\phi^p)$ is the same, namely $\frac{(5m+4)(5m+2)}{8}$, as required.

Finally, we turn to (6.85f). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 2$ and $m_2 = 4$, $h_2 = m_2 = 2$, $h_2 = m_2 = 3$, $h_2 = 6$ and $m_2 = 4$, $h_2 = 6$ and $m_2 = 3$, respectively $h_2 = 6$ and $m_2 = 2$. These correspond to the choices $p = 15m/4$, $p = 15m/2$, $p = 10m/3$, $p = 5m/4$, $p = 5m/3$, respectively $p = 5m/2$, all of which have already been discussed as they do not belong to (6.85f). Hence, (3.3) must necessarily hold, as required.

Case G_{33}. The degrees are $4, 6, 10, 12, 18$, and hence we have

$$\mathrm{Cat}^m(G_{33}; q) = \frac{[18m+18]_q\,[18m+12]_q\,[18m+10]_q\,[18m+6]_q\,[18m+4]_q}{[18]_q\,[12]_q\,[10]_q\,[6]_q\,[4]_q}.$$

Let ζ be a $18m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = m+1, \quad \text{if } \zeta = \zeta_{18}, \zeta_9, \tag{6.107a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta = \zeta_{12},\ 2 \mid m, \tag{6.107b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = \tfrac{9m+5}{5}, \quad \text{if } \zeta = \zeta_{10}, \zeta_5,\ 5 \mid m, \tag{6.107c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = \tfrac{(m+1)(3m+2)(3m+1)}{2}, \quad \text{if } \zeta = \zeta_6, \zeta_3, \tag{6.107d}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = \tfrac{(3m+2)(9m+2)}{4}, \quad \text{if } \zeta = \zeta_4,\ 2 \mid m, \tag{6.107e}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = \mathrm{Cat}^m(G_{33}), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{6.107f}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = 1, \quad \text{otherwise.} \tag{6.107g}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.107). The only cases not covered by Lemmas 27 and 28 are the ones in (6.107b), (6.107c), (6.107e), and (6.107g).

We begin with the case in (6.107b). By Lemma 26, we are free to choose $p = 3m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{2}+1} c^{-2}, c^2 w_{\frac{m}{2}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{2}} c^{-1}).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{2}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.108a}$$

$$w_i = c w_{i-\frac{m}{2}} c^{-1}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.108b}$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \tag{6.109a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \tag{6.109b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.109c}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{33})$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} \le_T c$. Together with equations (6.108)–(6.109), this implies that

$$w_i = c^3 w_i c^{-3} \quad \text{and} \quad w_i(c w_i c^{-1}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 2, \tag{6.110}$$

respectively that

$$w_i = c^3 w_i c^{-3}, \quad w_i(c w_i c^{-1}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.111}$$

respectively that

$$w_{i_1} = c^3 w_{i_1} c^{-3}, \quad w_{i_1}(c w_{i_1} c^{-1}) \le_T c, \quad \text{and} \quad \ell_T(w_{i_1}) = 1. \tag{6.112}$$

With the help of CHEVIE, one obtains three solutions for w_i in (6.110):

$$w_i \in \big\{[1,44],\ [2,4],\ [5,42]\big\},$$

where we have again used the short notation of CHEVIE referring to the internal ordering of the roots of G_{33} in CHEVIE. Each of them gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(G_{33})}(\phi^p)$ since i ranges from 1 to $m/2$.

There are no solutions to (6.111) and to (6.112).

In total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(G_{33})}(\phi^p)$, which agrees with the limit in (6.107b).

Next we turn to the case in (6.107c). By Lemma 26, we are free to choose $p = 9m/5$ if $\zeta = \zeta_{10}$, and we are free to choose $p = 18m/5$ if $\zeta = \zeta_5$. In particular, in all both cases, m must be divisible by 5.

We start with the case that $p = 9m/5$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = (*; c^2 w_{\frac{m}{5}+1} c^{-2}, c^2 w_{\frac{m}{5}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{5}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{5}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{4m}{5}, \tag{6.113a}$$

$$w_i = c w_{i-\frac{4m}{5}} c^{-1}, \quad i = \tfrac{4m}{5}+1, \tfrac{4m}{5}+2, \dots, m. \tag{6.113b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{5}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{5}}) = \ell_T(w_{i+\frac{2m}{5}}) = \ell_T(w_{i+\frac{3m}{5}}) = \ell_T(w_{i+\frac{4m}{5}}) = 1, \tag{6.114a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{33})$, we must have

$$w_i w_{i+\frac{m}{5}} w_{i+\frac{2m}{5}} w_{i+\frac{3m}{5}} w_{i+\frac{4m}{5}} = c.$$

Together with equations (6.113)–(6.114), this implies that

$$w_i = c^9 w_i c^{-9} \quad \text{and} \quad w_i (c^7 w_i c^{-7})(c^5 w_i c^{-5})(c^3 w_i c^{-3})(c w_i c^{-1}) = c. \tag{6.115}$$

With the help of CHEVIE, one obtains nine solutions for w_i in (6.115):

$$w_i \in \big\{[5], [4], [1], [2], [10], [27], [42], [44], [52]\big\}, \tag{6.116}$$

where we have again used the short notation of CHEVIE referring to the internal ordering of the roots of G_{33} in CHEVIE. Each of the above solutions for w_i gives rise to $m/5$ elements of $\mathrm{Fix}_{NC^m(G_{33})}(\phi^p)$ since i ranges from 1 to $m/5$.

In total, we obtain $1 + 9\frac{m}{5} = \frac{9m+5}{5}$ elements in $\mathrm{Fix}_{NC^m(G_{33})}(\phi^p)$, which agrees with the limit in (6.107c).

In the case that $p = 18m/5$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = (*; c^4 w_{\frac{2m}{5}+1} c^{-4}, c^4 w_{\frac{2m}{5}+2} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_1 c^{-3}, \dots, c^3 w_{\frac{2m}{5}} c^{-3}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^4 w_{\frac{2m}{5}+i} c^{-4}, \quad i = 1, 2, \dots, \tfrac{3m}{5}, \tag{6.117a}$$

$$w_i = c^3 w_{i-\frac{3m}{5}} c^{-3}, \quad i = \tfrac{3m}{5}+1, \tfrac{3m}{5}+2, \dots, m. \tag{6.117b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{5}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{5}}) = \ell_T(w_{i+\frac{2m}{5}}) = \ell_T(w_{i+\frac{3m}{5}}) = \ell_T(w_{i+\frac{4m}{5}}) = 1.$$

Writing t_1, t_2, t_3, t_4, t_5 for $w_i, w_{i+\frac{m}{5}}, w_{i+\frac{2m}{5}}, w_{i+\frac{3m}{5}}, w_{i+\frac{4m}{5}}$, respectively, the equations (6.117) reduce to

$$t_1 = c^4 t_3 c^{-4}, \tag{6.118a}$$
$$t_2 = c^4 t_4 c^{-4}, \tag{6.118b}$$
$$t_3 = c^4 t_5 c^{-4}, \tag{6.118c}$$
$$t_4 = c^3 t_1 c^{-3}, \tag{6.118d}$$
$$t_5 = c^3 t_2 c^{-3}. \tag{6.118e}$$

One of these equations is in fact superfluous: if we substitute (6.118b)–(6.118e) in (6.118a), then we obtain $t_1 = c^{18} t_1 c^{-18}$ which is automatically satisfied since $c^{18} = \varepsilon$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{33})$, we must have $t_1 t_2 t_3 t_4 t_5 = c$. Combining this with (6.118), we infer that

$$t_1 (c^7 t_1 c^{-7})(c^{14} t_1 c^{-14})(c^3 t_1 c^{-3})(c^{10} t_1 c^{-10}) = c. \tag{6.119}$$

Using that $c^9 t_1 c^{-9} = t_1$, due to Lemma 29 with $d = 2$, we see that this equation is equivalent with (6.115). Therefore, we are facing exactly the same enumeration problem here as for $p = 9m/5$, and, consequently, the number of solutions to (6.119) is the same, namely $\frac{9m+5}{5}$, as required.

Next we consider the case in (6.107e). By Lemma 26, we are free to choose $p = 9m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big)$$
$$= \big(*; c^5 w_{\frac{m}{2}+1} c^{-5}, c^5 w_{\frac{m}{2}+2} c^{-5}, \dots, c^5 w_m c^{-5}, c^4 w_1 c^{-4}, \dots, c^4 w_{\frac{m}{2}} c^{-4}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^5 w_{\frac{m}{2}+i} c^{-5}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.120a}$$
$$w_i = c^4 w_{i-\frac{m}{2}} c^{-4}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.120b}$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that
$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \tag{6.121a}$$
and the other w_j's, $1 \le j \le m$, are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that
$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \tag{6.121b}$$
and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that
$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.121c}$$
and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{33})$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} \le_T c$. Together with equations (6.120)–(6.121), this implies that

$$w_i = c^9 w_i c^{-9} \quad \text{and} \quad w_i (c^4 w_i c^{-4}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 2, \tag{6.122}$$

respectively that

$$w_i = c^9 w_i c^{-9}, \quad w_i(c^4 w_i c^{-4}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.123}$$

respectively that

$$\begin{aligned} w_{i_1} = c^9 w_{i_1} c^{-9}, \quad w_{i_2} = c^9 w_{i_2} c^{-9},& \\ w_{i_1} w_{i_2} (c^4 w_{i_1} c^{-4})(c^4 w_{i_2} c^{-4}) \le_T c, \quad \text{and} \quad \ell_T(w_{i_1}) = \ell_T(w_{i_2}) = 1.& \end{aligned} \tag{6.124}$$

With the help of CHEVIE, one obtains 21 solutions for w_i in (6.122):

$$\begin{aligned} w_i \in \big\{&[4,5],\ [1,20],\ [5,7],\ [1,2],\ [2,4],\ [10,259],\ [27,208],\ [27,44],\ [4,39],\ [5,42],\ [1,39],\\ &[5,52],\ [10,208],\ [1,44],\ [42,113],\ [4,113],\ [2,27],\ [10,42],\ [2,49],\ [52,61],\ [44,123]\big\}, \end{aligned}$$

where we have again used the short notation of CHEVIE referring to the internal ordering of the roots of G_{33} in CHEVIE, one obtains 18 solutions for w_i in (6.123):

$$\begin{aligned} w_i \in \big\{&[5],\ [4],\ [20],\ [208],\ [259],\ [1],\ [2],\ [7],\ [10],\ [27],\\ &[39],\ [42],\ [49],\ [44],\ [52],\ [113],\ [61],\ [123]\big\}, \end{aligned}$$

each of them giving rise to $m/2$ elements of $\operatorname{Fix}_{NC^m(G_{33})}(\phi^p)$ since i ranges from 1 to $m/2$, and one obtains 54 pairs (w_{i_1}, w_{i_2}) satisfying (6.124):

$$\begin{aligned} (w_{i_1}, w_{i_2}) \in \big\{&([5],[20]),\ ([5],[7]),\ ([5],[42]),\ ([5],[52]),\ ([4],[5]),\ ([4],[10]),\ ([4],[39]),\ ([4],[113]),\ ([20],[4]),\\ &([20],[1]),\ ([208],[27]),\ ([208],[52]),\ ([259],[10]),\ ([259],[27]),\ ([1],[20]),\ ([1],[2]),\ ([1],[39]),\ ([1],[44]),\ ([2],[4]),\\ &([2],[7]),\ ([2],[27]),\ ([2],[49]),\ ([7],[5]),\ ([7],[1]),\ ([10],[208]),\ ([10],[259]),\ ([10],[2]),\ ([10],[42]),\ ([27],[5]),\\ &([27],[208]),\ ([27],[44]),\ ([27],[61]),\ ([39],[4]),\ ([39],[42]),\ ([42],[1]),\ ([42],[27]),\ ([42],[113]),\ ([42],[123]),\\ &([49],[5]),\ ([49],[2]),\ ([44],[4]),\ ([44],[259]),\ ([44],[52]),\ ([44],[123]),\ ([52],[1]),\ ([52],[10]),\ ([52],[49]),\\ &([52],[61]),\ ([113],[42]),\ ([113],[44]),\ ([61],[2]),\ ([61],[52]),\ ([123],[10]),\ ([123],[44])\big\}, \end{aligned}$$

each of them giving rise to $\binom{m/2}{2}$ elements of $\operatorname{Fix}_{NC^m(G_{33})}(\phi^p)$ since $1 \le i_1 < i_2 \le m$.

In total, we obtain $1+(21+18)\frac{m}{2}+54\binom{m/2}{2} = \frac{(3m+2)(9m+2)}{4}$ elements in $\operatorname{Fix}_{NC^m(G_{33})}(\phi^p)$, which agrees with the limit in (6.107e).

Finally, we turn to (6.107g). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 1$ and $m_2 = 5$, $h_2 = 2$ and $m_2 = 5$, $h_2 = 2$ and $m_2 = 4$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 18m/5$, $p = 9m/5$, $p = 9m/4$, respectively $p = 9m/2$, out of which only $p = 9m/4$ has not yet been discussed and belongs to the current case. The corresponding action of ϕ^p is given by

$$\begin{aligned} &\phi^p\big((w_0; w_1, \dots, w_m)\big)\\ &\qquad = \big(*; c^3 w_{\frac{3m}{4}+1} c^{-3}, c^3 w_{\frac{3m}{4}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{3m}{4}} c^{-2}\big), \end{aligned}$$

so that we have to solve

$$t_1(c^2 t_1 c^{-2})(c^4 t_1 c^{-4})(c^6 t_1 c^{-6}) \le_T c$$

for t_1 with $\ell_T(t_1)$. A computation with the help of CHEVIE finds no solution. Hence, the left-hand side of (3.3) is equal to 1, as required.

Case G_{34}. The degrees are $6, 12, 18, 24, 30, 42$, and hence we have

$$\operatorname{Cat}^m(G_{34};q)=\frac{[42m+42]_q\,[42m+30]_q\,[42m+24]_q}{[42]_q\,[30]_q\,[24]_q}\times\frac{[42m+18]_q\,[42m+12]_q\,[42m+6]_q}{[18]_q\,[12]_q\,[6]_q}.$$

Let ζ be a $42m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=m+1,\quad \text{if } \zeta=\zeta_{42},\zeta_{21},\zeta_{14},\zeta_7, \tag{6.125a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=\tfrac{7m+5}{5},\quad \text{if } \zeta=\zeta_{30},\zeta_{15},\zeta_{10},\zeta_5,\ 5\mid m, \tag{6.125b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=\tfrac{7m+4}{4},\quad \text{if } \zeta=\zeta_{24},\zeta_8,\ 4\mid m, \tag{6.125c}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=\tfrac{7m+3}{3},\quad \text{if } \zeta=\zeta_{18},\zeta_9,\ 3\mid m, \tag{6.125d}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=\tfrac{(7m+4)(7m+2)}{8},\quad \text{if } \zeta=\zeta_{12},\zeta_4,\ 2\mid m, \tag{6.125e}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=\operatorname{Cat}^m(G_{34}),\quad \text{if } \zeta=\zeta_6,\zeta_3,-1,1, \tag{6.125f}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=1,\quad \text{otherwise.} \tag{6.125g}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.125). The only cases not covered by Lemmas 27 and 28 are the ones in (6.125b), (6.125c), (6.125d), (6.125e), and (6.125g).

We begin with the case in (6.125b). By Lemma 26, we are free to choose $p=7m/5$ if $\zeta=\zeta_{30}$, we are free to choose $p=14m/5$ if $\zeta=\zeta_{15}$, we are free to choose $p=21m/5$ if $\zeta=\zeta_{10}$, and we are free to choose $p=42m/5$ if $\zeta=\zeta_5$. In particular, in all cases, m must be divisible by 5.

We start with the case that $p=7m/5$. From (5.1), we infer

$$\phi^p\big((w_0;w_1,\dots,w_m)\big)=\big(*;c^2w_{\frac{3m}{5}+1}c^{-2},c^2w_{\frac{3m}{5}+2}c^{-2},\dots,c^2w_mc^{-2},cw_1c^{-1},\dots,cw_{\frac{3m}{5}}c^{-1}\big).$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^2w_{\frac{3m}{5}+i}c^{-2},\quad i=1,2,\dots,\tfrac{2m}{5}, \tag{6.126a}$$

$$w_i=cw_{i-\frac{2m}{5}}c^{-1},\quad i=\tfrac{2m}{5}+1,\tfrac{2m}{5}+2,\dots,m. \tag{6.126b}$$

There are two distinct possibilities for choosing the w_i's, $1\le i\le m$:

(i) all the w_i's are equal to ε (and $w_0=c$),

(ii) there is an i with $1\le i\le\frac{m}{5}$ such that

$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{5}})=\ell_T(w_{i+\frac{2m}{5}})=\ell_T(w_{i+\frac{3m}{5}})=\ell_T(w_{i+\frac{4m}{5}})=1, \tag{6.127a}$$

and the other w_j's, $1\le j\le m$, are equal to ε.

Moreover, since $(w_0;w_1,\dots,w_m)\in NC^m(G_{34})$, we must have

$$w_iw_{i+\frac{m}{5}}w_{i+\frac{2m}{5}}w_{i+\frac{3m}{5}}w_{i+\frac{4m}{5}}\le_T c.$$

Together with equations (6.126)–(6.127), this implies that

$$w_i = c^7 w_i c^{-7} \quad \text{and} \quad w_i(c^4 w_i c^{-4})(c w_i c^{-1})(c^5 w_i c^{-5})(c^2 w_i c^{-2}) \le_T c. \tag{6.128}$$

With the help of `CHEVIE`, one obtains seven solutions for w_i in (6.128):

$$w_i \in \big\{[4], [5], [6], [2], [11], [44], [63], [74]\big\},$$

where we have again used the short notation of `CHEVIE` referring to the internal ordering of the roots of G_{34} in `CHEVIE`. Each of the above solutions for w_i gives rise to $m/5$ elements of $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$ since i ranges from 1 to $m/5$.

In total, we obtain $1 + 7\frac{m}{5} = \frac{7m+5}{5}$ elements in $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$, which agrees with the limit in (6.125b).

In the case that $p = 14m/5$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^3 w_{\frac{m}{5}+1} c^{-3}, c^3 w_{\frac{m}{5}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{m}{5}} c^{-2}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{m}{5}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{4m}{5}, \tag{6.129a}$$
$$w_i = c^2 w_{i-\frac{4m}{5}} c^{-2}, \quad i = \tfrac{4m}{5}+1, \tfrac{4m}{5}+2, \dots, m. \tag{6.129b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{5}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{5}}) = \ell_T(w_{i+\frac{2m}{5}}) = \ell_T(w_{i+\frac{3m}{5}}) = \ell_T(w_{i+\frac{4m}{5}}) = 1.$$

Writing t_1, t_2, t_3, t_4, t_5 for $w_i, w_{i+\frac{m}{5}}, w_{i+\frac{2m}{5}}, w_{i+\frac{3m}{5}}, w_{i+\frac{4m}{5}}$, respectively, the equations (6.129) reduce to

$$t_1 = c^3 t_2 c^{-3}, \tag{6.130a}$$
$$t_2 = c^3 t_3 c^{-3}, \tag{6.130b}$$
$$t_3 = c^3 t_4 c^{-3}, \tag{6.130c}$$
$$t_4 = c^3 t_5 c^{-3}, \tag{6.130d}$$
$$t_5 = c^2 t_1 c^{-2}. \tag{6.130e}$$

One of these equations is in fact superfluous: if we substitute (6.130b)–(6.130e) in (6.130a), then we obtain $t_1 = c^{14} t_1 c^{-14}$ which is automatically satisfied due to Lemma 29 with $d = 3$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{34})$, we must have $t_1 t_2 t_3 t_4 t_5 \le_T c$. Combining this with (6.130), we infer that

$$t_1(c^{11} t_1 c^{-11})(c^8 t_1 c^{-8})(c^5 t_1 c^{-5})(c^2 t_1 c^{-2}) \le_T c. \tag{6.131}$$

Using that $c^7 t_1 c^{-7} = t_1$, due to Lemma 29 with $d = 6$, we see that this equation is equivalent with (6.128). Therefore, we are facing exactly the same enumeration problem here as for $p = 7m/5$, and, consequently, the number of solutions to (6.131) is the same, namely $\frac{7m+5}{5}$, as required.

In the case that $p = 21m/5$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = \big(*; c^5 w_{\frac{4m}{5}+1} c^{-5}, c^5 w_{\frac{4m}{5}+2} c^{-5}, \dots, c^5 w_m c^{-5}, c^4 w_1 c^{-4}, \dots, c^4 w_{\frac{4m}{5}} c^{-4}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^5 w_{\frac{4m}{5}+i} c^{-5}, \quad i = 1, 2, \dots, \tfrac{m}{5}, \tag{6.132a}$$

$$w_i = c^4 w_{i-\frac{m}{5}} c^{-4}, \quad i = \tfrac{m}{5}+1, \tfrac{m}{5}+2, \dots, m. \tag{6.132b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{5}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{5}}) = \ell_T(w_{i+\frac{2m}{5}}) = \ell_T(w_{i+\frac{3m}{5}}) = \ell_T(w_{i+\frac{4m}{5}}) = 1.$$

Writing t_1, t_2, t_3, t_4, t_5 for $w_i, w_{i+\frac{m}{5}}, w_{i+\frac{2m}{5}}, w_{i+\frac{3m}{5}}, w_{i+\frac{4m}{5}}$, respectively, the equations (6.132) reduce to

$$t_1 = c^5 t_5 c^{-5}, \tag{6.133a}$$

$$t_2 = c^4 t_1 c^{-4}, \tag{6.133b}$$

$$t_3 = c^4 t_2 c^{-4}, \tag{6.133c}$$

$$t_4 = c^4 t_3 c^{-4}, \tag{6.133d}$$

$$t_5 = c^4 t_4 c^{-4}. \tag{6.133e}$$

One of these equations is in fact superfluous: if we substitute (6.133b)–(6.133e) in (6.133a), then we obtain $t_1 = c^{21} t_1 c^{-21}$ which is automatically satisfied due to Lemma 29 with $d = 6$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{34})$, we must have $t_1 t_2 t_3 t_4 t_5 \le_T c$. Combining this with (6.133), we infer that

$$t_1 (c^4 t_1 c^{-4})(c^8 t_1 c^{-8})(c^{12} t_1 c^{-12})(c^{16} t_1 c^{-16}) \le_T c. \tag{6.134}$$

Using that $c^7 t_1 c^{-7} = t_1$, due to Lemma 29 with $d = 6$, we see that this equation is equivalent with (6.128). Therefore, we are facing exactly the same enumeration problem here as for $p = 7m/5$, and, consequently, the number of solutions to (6.134) is the same, namely $\frac{7m+5}{5}$, as required.

In the case that $p = 42m/5$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = \big(*; c^9 w_{\frac{3m}{5}+1} c^{-9}, c^9 w_{\frac{3m}{5}+2} c^{-9}, \dots, c^9 w_m c^{-9}, c^8 w_1 c^{-8}, \dots, c^8 w_{\frac{3m}{5}} c^{-8}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^9 w_{\frac{3m}{5}+i} c^{-9}, \quad i = 1, 2, \dots, \tfrac{2m}{5}, \tag{6.135a}$$

$$w_i = c^8 w_{i-\frac{2m}{5}} c^{-8}, \quad i = \tfrac{2m}{5}+1, \tfrac{2m}{5}+2, \dots, m. \tag{6.135b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$: either all the w_i's are equal to ε, or there is an i with $1 \le i \le \frac{m}{5}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{5}}) = \ell_T(w_{i+\frac{2m}{5}}) = \ell_T(w_{i+\frac{3m}{5}}) = \ell_T(w_{i+\frac{4m}{5}}) = 1.$$

Writing t_1, t_2, t_3, t_4, t_5 for $w_i, w_{i+\frac{m}{5}}, w_{i+\frac{2m}{5}}, w_{i+\frac{3m}{5}}, w_{i+\frac{4m}{5}}$, respectively, the equations (6.135) reduce to

$$t_1 = c^9 t_4 c^{-9}, \tag{6.136a}$$
$$t_2 = c^9 t_5 c^{-9}, \tag{6.136b}$$
$$t_3 = c^8 t_1 c^{-8}, \tag{6.136c}$$
$$t_4 = c^8 t_2 c^{-8}, \tag{6.136d}$$
$$t_5 = c^8 t_3 c^{-8}. \tag{6.136e}$$

One of these equations is in fact superfluous: if we substitute (6.136b)–(6.136e) in (6.136a), then we obtain $t_1 = c^{42} t_1 c^{-42}$ which is automatically satisfied since $c^{42} = \varepsilon$.

Since $(w_0; w_1, \dots, w_m) \in NC^m(G_{34})$, we must have $t_1 t_2 t_3 t_4 t_5 \le_T c$. Combining this with (6.136), we infer that

$$t_1 (c^{25} t_1 c^{-25})(c^8 t_1 c^{-8})(c^{33} t_1 c^{-33})(c^{16} t_1 c^{-16}) \le_T c. \tag{6.137}$$

Using that $c^7 t_1 c^{-7} = t_1$, due to Lemma 29 with $d = 6$, we see that this equation is equivalent with (6.128). Therefore, we are facing exactly the same enumeration problem here as for $p = 7m/5$, and, consequently, the number of solutions to (6.137) is the same, namely $\frac{7m+5}{5}$, as required.

Next we consider the case in (6.125c). By Lemma 26, we are free to choose $p = 7m/4$ if $\zeta = \zeta_{24}$, and we are free to choose $p = 21m/4$ if $\zeta = \zeta_8$. In both cases, m must be divisible by 4.

We start with the case that $p = 7m/4$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{4}+1} c^{-2}, c^2 w_{\frac{m}{4}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{4}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{4}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{3m}{4}, \tag{6.138a}$$
$$w_i = c w_{i-\frac{3m}{4}} c^{-1}, \quad i = \tfrac{3m}{4}+1, \tfrac{3m}{4}+2, \dots, m. \tag{6.138b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{4}}) = \ell_T(w_{i+\frac{2m}{4}}) = \ell_T(w_{i+\frac{3m}{4}}) = 1, \tag{6.139a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{34})$, we must have

$$w_i w_{i+\frac{m}{4}} w_{i+\frac{2m}{4}} w_{i+\frac{3m}{4}} \le_T c.$$

Together with equations (6.138)–(6.139), this implies that

$$w_i = c^7 w_i c^{-7} \quad \text{and} \quad w_i (c^5 w_i c^{-5})(c^3 w_i c^{-3})(c w_i c^{-1}) \le_T c. \tag{6.140}$$

With the help of `CHEVIE`, one obtains seven solutions for w_i in (6.140):

$$w_i \in \big\{[1],\ [2],\ [28],\ [34],\ [61],\ [46],\ [168]\big\},$$

where we have again used the short notation of `CHEVIE` referring to the internal ordering of the roots of G_{34} in `CHEVIE`. Each of them gives rise to $m/4$ elements of $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$ since i ranges from 1 to $m/4$.

In total, we obtain $1+7\frac{m}{4}=\frac{7m+4}{4}$ elements in $\operatorname{Fix}_{NC^m(G_{34})}(\phi^p)$, which agrees with the limit in (6.125c).

If $p=21m/4$, then, from (5.1), we infer

$$\phi^p\big((w_0;w_1,\dots,w_m)\big)\\ =\big(*;c^6w_{\frac{3m}{4}+1}c^{-6},c^6w_{\frac{3m}{4}+2}c^{-6},\dots,c^6w_mc^{-6},c^5w_1c^{-5},\dots,c^5w_{\frac{3m}{4}}c^{-5}\big).$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^6w_{\frac{3m}{4}+i}c^{-6},\quad i=1,2,\dots,\tfrac{m}{4}, \tag{6.141a}$$
$$w_i=c^5w_{i-\frac{3m}{4}}c^{-5},\quad i=\tfrac{m}{4}+1,\tfrac{m}{4}+2,\dots,m. \tag{6.141b}$$

There are two distinct possibilities for choosing the w_i's, $1\le i\le m$:

(i) all the w_i's are equal to ε (and $w_0=c$),

(ii) there is an i with $1\le i\le\frac{m}{2}$ such that

$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{4}})=\ell_T(w_{i+\frac{2m}{4}})=\ell_T(w_{i+\frac{3m}{4}})=1, \tag{6.142a}$$

and the other w_j's, $1\le j\le m$, are equal to ε.

Moreover, since $(w_0;w_1,\dots,w_m)\in NC^m(G_{34})$, we must have

$$w_iw_{i+\frac{m}{4}}w_{i+\frac{2m}{4}}w_{i+\frac{3m}{4}}\le_T c.$$

Together with equations (6.141)–(6.142), this implies that

$$w_i=c^{21}w_ic^{-21}\quad\text{and}\quad w_i(c^5w_ic^{-5})(c^{10}w_ic^{-10})(c^{15}w_ic^{-15})\le_T c. \tag{6.143}$$

Using that $c^7t_1c^{-7}=t_1$, due to Lemma 29 with $d=6$, we see that this equation is equivalent with (6.140). Therefore, we are facing exactly the same enumeration problem here as for $p=7m/4$, and, consequently, the number of solutions to (6.137) is the same, namely $\frac{7m+4}{4}$, as required.

Our next case is the case in (6.125d). By Lemma 26, we are free to choose $p=7m/3$ if $\zeta=\zeta_{18}$, and we are free to choose $p=14m/3$ if $\zeta=\zeta_9$. In both cases, m must be divisible by 3.

We start with the case that $p=7m/3$. From (5.1), we infer

$$\phi^p\big((w_0;w_1,\dots,w_m)\big)\\ =\big(*;c^3w_{\frac{2m}{3}+1}c^{-3},c^3w_{\frac{2m}{3}+2}c^{-3},\dots,c^3w_mc^{-3},c^2w_1c^{-2},\dots,c^2w_{\frac{2m}{3}}c^{-2}\big).$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^3w_{\frac{2m}{3}+i}c^{-3},\quad i=1,2,\dots,\tfrac{m}{3}, \tag{6.144a}$$
$$w_i=c^2w_{i-\frac{m}{3}}c^{-2},\quad i=\tfrac{m}{3}+1,\tfrac{m}{3}+2,\dots,m. \tag{6.144b}$$

There are four distinct possibilities for choosing the w_i's, $1\le i\le m$:

(i) all the w_i's are equal to ε (and $w_0=c$),

(ii) there is an i with $1\le i\le\frac{m}{2}$ such that

$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{3}})=\ell_T(w_{i+\frac{2m}{3}})=2, \tag{6.145a}$$

and all other w_j's are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 1, \tag{6.145b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{3}}) = \ell_T(w_{i_2+\frac{m}{3}}) = \ell_T(w_{i_1+\frac{2m}{3}}) = \ell_T(w_{i_2+\frac{2m}{3}}) = 1, \tag{6.145c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{34})$, we must have $w_i w_{i+\frac{m}{3}} w_{i+\frac{2m}{3}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{3}} w_{i_2+\frac{m}{3}} w_{i_1+\frac{2m}{3}} w_{i_2+\frac{2m}{3}} = c$. Together with equations (6.144)–(6.145), this implies that

$$w_i = c^7 w_i c^{-7} \quad \text{and} \quad w_i (c^2 w_i c^{-2})(c^4 w_i c^{-4}) = c, \tag{6.146}$$

respectively that

$$w_i = c^7 w_i c^{-7}, \quad w_i (c^2 w_i c^{-2})(c^4 w_i c^{-4}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.147}$$

respectively that

$$\begin{aligned} w_{i_1} = c^7 w_{i_1} c^{-7}, \quad & w_{i_2} = c^7 w_{i_2} c^{-7}, \\ & \text{and} \quad w_{i_1} w_{i_2} (c^2 w_{i_1} c^{-2})(c^2 w_{i_2} c^{-2})(c^4 w_{i_1} c^{-4})(c^4 w_{i_2} c^{-4}) = c. \end{aligned} \tag{6.148}$$

With the help of CHEVIE, one obtains 21 solutions for w_i in (6.147):

$$w_i \in \{[4]\ \ [5]\ \ [6]\ \ [11]\ \ [44]\ \ [63]\ \ [74]\},$$

where we have again used the short notation of CHEVIE referring to the internal ordering of the roots of G_{34} in CHEVIE. Each of them gives rise to $m/3$ elements of $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$ since i ranges from 1 to $m/3$.

There are no solutions w_i in (6.146) and for (w_{i_1}, w_{i_2}) in (6.148).

In total, we obtain $1 + 7\frac{m}{3} = \frac{7m+3}{3}$ elements in $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$, which agrees with the limit in (6.125d).

In the case that $p = 14m/3$, we infer from (5.1) that

$$\begin{aligned} &\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ &\qquad = \big(*; c^5 w_{\frac{m}{3}+1} c^{-5}, c^5 w_{\frac{m}{3}+2} c^{-5}, \dots, c^5 w_m c^{-5}, c^4 w_1 c^{-4}, \dots, c^4 w_{\frac{m}{3}} c^{-4}\big). \end{aligned}$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^5 w_{\frac{m}{3}+i} c^{-5}, \quad i = 1, 2, \dots, \tfrac{2m}{3}, \tag{6.149a}$$

$$w_i = c^4 w_{i-\frac{2m}{3}} c^{-4}, \quad i = \tfrac{2m}{3}+1, \tfrac{2m}{3}+2, \dots, m. \tag{6.149b}$$

There are four distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 2, \tag{6.150a}$$

and all other w_j's are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) = 1, \tag{6.150b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{3}}) = \ell_T(w_{i_2+\frac{m}{3}}) = \ell_T(w_{i_1+\frac{2m}{3}}) = \ell_T(w_{i_2+\frac{2m}{3}}) = 1, \tag{6.150c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{34})$, we must have $w_i w_{i+\frac{m}{3}} w_{i+\frac{2m}{3}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{3}} w_{i_2+\frac{m}{3}} w_{i_1+\frac{2m}{3}} w_{i_2+\frac{2m}{3}} = c$. Together with equations (6.149)–(6.150), this implies that

$$w_i = c^{14} w_i c^{-14} \quad \text{and} \quad w_i (c^9 w_i c^{-9})(c^4 w_i c^{-4}) = c, \tag{6.151}$$

respectively that

$$w_i = c^{14} w_i c^{-14}, \quad w_i (c^9 w_i c^{-9})(c^4 w_i c^{-4}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.152}$$

respectively that

$$\begin{aligned} w_{i_1} = c^{14} w_{i_1} c^{-14}, \quad w_{i_2} = c^{14} w_{i_2} c^{-14},& \\ \text{and} \quad w_{i_1} w_{i_2} (c^9 w_{i_1} c^{-9})(c^9 w_{i_2} c^{-9})(c^4 w_{i_1} c^{-4})(c^4 w_{i_2} c^{-4}) = c.& \end{aligned} \tag{6.153}$$

Using that $c^7 w c^{-7} = w$ for all $w \in NC(G_{34}$, due to Lemma 29 with $d = 6$, we see that this equation is equivalent with (6.146). Therefore, we are facing exactly the same enumeration problem here as for $p = 7m/3$, and, consequently, the number of solutions to (6.151) is the same, namely $\frac{7m+3}{3}$, as required.

Next we consider the case in (6.125e). By Lemma 26, we are free to choose $p = 7m/2$ if $\zeta = \zeta_{12}$, and we are free to choose $p = 21m/2$ if $\zeta = \zeta_4$. In both cases, m must be divisible by 2.

We begin with the case that $p = 7m/2$. From (5.1), we infer

$$\begin{aligned} &\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ &\quad = \big(*; c^4 w_{\frac{m}{2}+1} c^{-4}, c^4 w_{\frac{m}{2}+2} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_1 c^{-3}, \dots, c^3 w_{\frac{m}{2}} c^{-3}\big). \end{aligned} \tag{6.154}$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^4 w_{\frac{m}{2}+i} c^{-4}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.155a}$$

$$w_i = c^3 w_{i-\frac{m}{2}} c^{-3}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.155b}$$

There are several distinct possibilities for choosing the w_i's, $1 \le i \le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$1 \le \ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) \le 3, \tag{6.156a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iii) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_1 = \ell_T(w_{i_1}) = \ell_T(w_{i_1+\frac{m}{2}}) \ge 1, \quad \ell_2 = \ell_T(w_{i_2}) == \ell_T(w_{i_2+\frac{m}{2}}) \ge 1, \quad \ell_1 + \ell_2 \le 3, \tag{6.156b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1, i_2, i_3 with $1 \le i_1 < i_2 < i_3 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_3}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = \ell_T(w_{i_3+\frac{m}{2}}) = 1, \tag{6.156c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(G_{34})$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} \le_T c$, respectively

$$w_{i_1} w_{i_2} w_{i_3} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} w_{i_3+\frac{m}{2}} = c$$

. Together with equations (6.155)–(6.156), this implies that

$$w_i = c^7 w_i c^{-7} \quad \text{and} \quad w_i(c^4 w_i c^{-4}) \le_T c, \tag{6.157}$$

respectively that

$$w_{i_1} = c^7 w_{i_1} c^{-7}, \quad w_{i_2} = c^7 w_{i_2} c^{-7}, \quad \text{and} \quad w_{i_1} w_{i_2} (c^4 w_{i_1} c^{-4})(c^4 w_{i_2} c^{-4}) \le_T c, \tag{6.158}$$

respectively that

$$\begin{aligned} w_{i_1} = c^7 w_{i_1} c^{-7}, \quad w_{i_2} = c^7 w_{i_2} c^{-7}, \quad w_{i_3} = c^7 w_{i_3} c^{-7},& \\ \text{and} \quad w_{i_1} w_{i_2} w_{i_3} (c^4 w_{i_1} c^{-4})(c^4 w_{i_2} c^{-4})(c^4 w_{i_3} c^{-4}) = c.& \end{aligned} \tag{6.159}$$

With the help of `CHEVIE`, one obtains 14 solutions for w_i in (6.157) with $\ell_T(w_i) = 1$:

$$w_i \in \{[14], [35], [1], [2], [10], [24], [28], [34], [39], [56], [61], [46], [168], [105]\},$$

where we have again used the short notation of `CHEVIE` referring to the internal ordering of the roots of G_{34} in `CHEVIE`, one obtains 21 solutions for w_i in (6.157) with $\ell_T(w_i) = 2$:

$$\begin{aligned} w_i \in \{&[1,14], [1,35], [1,24], [1,34], [14,61], [2,39], [10,34], [2,56], [2,35], [14,46], [35,168], \\ &[34,56], [2,61], [28,56], [10,28], [10,61], [34,105], [28,105], [24,61], [39,168], [24,46]\}, \end{aligned}$$

each of them giving rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$ since i ranges from 1 to $m/2$, and one obtains 49 pairs (w_{i_1}, w_{i_2}) satisfying (6.158):

$$\begin{aligned} (w_{i_1}, w_{i_2}) \in \{&([14],[1]), ([14],[61]), ([14],[46]), ([35],[1]), ([35],[46]), ([35],[168]), ([1],[14]), ([1],[35]), ([1],[24]), \\ &([1],[34]), ([2],[35]), ([2],[39]), ([2],[56]), ([2],[61]), ([10],[28]), ([10],[34]), ([10],[61]), ([24],[28]), ([24],[61]), \\ &([24],[46]), ([28],[1]), ([28],[10]), ([28],[56]), ([28],[105]), ([34],[39]), ([34],[56]), ([34],[46]), ([34],[105]), \\ &([39],[1]), ([39],[2]), ([39],[168]), ([56],[2]), ([56],[34]), ([56],[168]), ([61],[10]), ([61],[24]), ([61],[168]), \\ &([61],[105]), ([46],[14]), ([46],[2]), ([46],[10]), ([46],[24]), ([168],[14]), ([168],[35]), ([168],[28]), \\ &([168],[39]), ([105],[2]), ([105],[28]), ([105],[34])\}, \end{aligned}$$

each of them giving rise to $\binom{m/2}{2}$ elements of $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$ since $1 \le i_1 < i_2 \le m$.

There are no solutions for w_i with $\ell_T(w_i) = 3$ in (6.157), and hence no solutions for (w_{i_1}, w_{i_2}) with $\ell_T(w_{i_1}) + \ell_T(w_{i_2}) = 3$ in (6.158), and no solutions for $(w_{i_1}, w_{i_2}, w_{i_3})$ in (6.159).

In total, we obtain $1+(14+21)\frac{m}{2}+49\binom{m/2}{2} = \frac{(7m+2)(7m+4)}{8}$ elements in $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$, which agrees with the limit in (6.125e).

If $p = 21m/2$, from (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^{11} w_{\frac{m}{2}+1} c^{-11}, c^{11} w_{\frac{m}{2}+2} c^{-11}, \dots, c^{11} w_m c^{-11}, c^{10} w_1 c^{-10}, \dots, c^{10} w_{\frac{m}{2}} c^{-10}\big).$$

Using that $c^7 w c^{-7} = w$ for all $w \in NC(G_{34}$, due to Lemma 29 with $d = 6$, we see that this action is identical with the one in (6.154). Therefore, we are facing exactly the same enumeration problem here as for $p = 7m/2$, and, consequently, the number of elements in $\mathrm{Fix}_{NC^m(G_{34})}(\phi^p)$ is the same, namely $\frac{(7m+2)(7m+4)}{8}$, as required.

Finally, we turn to (6.125g). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 1$ and $m_2 = 5$, $h_2 = 2$ and $m_2 = 5$, $h_2 = 2$ and $m_2 = 4$, $h_2 = m_2 = 2$, $h_2 = 3$ and $m_2 = 5$, $h_2 = m_2 = 3$, $h_2 = 6$ and $m_2 = 6$, $h_2 = 6$ and $m_2 = 5$, $h_2 = 6$ and $m_2 = 4$, $h_2 = 6$ and $m_2 = 3$, respectively $h_2 = 6$ and $m_2 = 2$, . These correspond to the choices $p = 42m/5$, $p = 21m/5$, $p = 21m/4$, $p = 21m/2$, $p = 14m/3$, $p = 14m/5$, $p = 7m/6$, $p = 7m/5$, $p = 7m/4$, $p = 7m/3$, respectively $p = 7m/2$, out of which only $p = 7m/6$ has not yet been discussed and belongs to the current case. The corresponding action of ϕ^p is given by

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^2 w_{\frac{5m}{6}+1} c^{-2}, c^2 w_{\frac{5m}{6}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{5m}{6}} c^{-1}\big),$$

so that we have to solve

$$t_1 (c t_1 c^{-1})(c^2 t_1 c^{-2})(c^3 t_1 c^{-3})(c^4 t_1 c^{-4})(c^5 t_1 c^{-5}) = c.$$

A computation with the help of `CHEVIE` finds no solution. Hence, the left-hand side of (3.3) is equal to 1, as required.

Case $G_{35} = E_6$. The degrees are $2, 5, 6, 8, 9, 12$, and hence we have

$$\mathrm{Cat}^m(E_6; q) = \frac{[12m+12]_q\,[12m+9]_q\,[12m+8]_q\,[12m+6]_q\,[12m+5]_q\,[12m+2]_q}{[12]_q\,[9]_q\,[8]_q\,[6]_q\,[5]_q\,[2]_q}.$$

Let ζ be a $12m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = m+1, \quad \text{if } \zeta=\zeta_{12}, \tag{6.160a}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{4m+3}{3}, \quad \text{if } \zeta=\zeta_{9},\ 3\mid m, \tag{6.160b}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta=\zeta_{8},\ 2\mid m, \tag{6.160c}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = (m+1)(2m+1), \quad \text{if } \zeta=\zeta_{6}, \tag{6.160d}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{12m+5}{5}, \quad \text{if } \zeta=\zeta_{5},\ 5\mid m, \tag{6.160e}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{(m+1)(3m+2)}{2}, \quad \text{if } \zeta=\zeta_{4}, \tag{6.160f}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{(m+1)(4m+3)(2m+1)}{3}, \quad \text{if } \zeta=\zeta_{3}, \tag{6.160g}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{(m+1)(3m+2)(2m+1)(6m+1)}{2}, \quad \text{if } \zeta=-1, \tag{6.160h}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \operatorname{Cat}^m(E_6), \quad \text{if } \zeta=1, \tag{6.160i}$$
$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = 1, \quad \text{otherwise.} \tag{6.160j}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.160). The only cases not covered by Lemmas 27 and 28 are the ones in (6.160b), (6.160c), (6.160e), and (6.160j).

We begin with the case in (6.160b). By Lemma 26, we are free to choose $p=4m/3$. In particular, m must be divisible by 3. From (5.1), we infer

$$\phi^p\big((w_0;w_1,\dots,w_m)\big) \\ = (*; c^2w_{\frac{2m}{3}+1}c^{-2}, c^2w_{\frac{2m}{3}+2}c^{-2},\dots, c^2w_mc^{-2}, cw_1c^{-1},\dots, cw_{\frac{2m}{3}}c^{-1}\big).$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2w_{\frac{2m}{3}+i}c^{-2}, \quad i=1,2,\dots,\tfrac{m}{3}, \tag{6.161a}$$
$$w_i = cw_{i-\frac{m}{3}}c^{-1}, \quad i=\tfrac{m}{3}+1,\tfrac{m}{3}+2,\dots,m. \tag{6.161b}$$

There are four distinct possibilities for choosing the w_i's, $1\le i\le m$:

(i) all the w_i's are equal to ε (and $w_0=c$),

(ii) there is an i with $1\le i\le \frac{m}{3}$ such that
$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{3}})=\ell_T(w_{i+\frac{2m}{3}})=2, \tag{6.162a}$$
and all other w_j's are equal to ε,

(iii) there is an i with $1\le i\le \frac{m}{3}$ such that
$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{3}})=\ell_T(w_{i+\frac{2m}{3}})=1, \tag{6.162b}$$
and the other w_j's, $1\le j\le m$, are equal to ε,

(iv) there are i_1 and i_2 with $1\le i_1<i_2\le \frac{m}{3}$ such that
$$\ell_T(w_{i_1})=\ell_T(w_{i_2})=\ell_T(w_{i_1+\frac{m}{3}})=\ell_T(w_{i_2+\frac{m}{3}})=\ell_T(w_{i_1+\frac{2m}{3}})=\ell_T(w_{i_2+\frac{2m}{3}})=1, \tag{6.162c}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_6)$, we must have $w_i w_{i+\frac{m}{3}} w_{i+\frac{2m}{3}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{3}} w_{i_2+\frac{m}{3}} w_{i_1+\frac{2m}{3}} w_{i_2+\frac{2m}{3}} = c$. Together with equations (6.161)–(6.162), this implies that

$$w_i = c^4 w_i c^{-4} \quad \text{and} \quad w_i (c w_i c^{-1})(c^2 w_i c^{-2}) = c, \tag{6.163}$$

respectively that

$$w_i = c^4 w_i c^{-4}, \quad w_i (c w_i c^{-1})(c^2 w_i c^{-2}) \le_T c, \quad \text{and} \quad \ell_T(w_i) = 1, \tag{6.164}$$

respectively that

$$w_{i_1} = c^4 w_{i_1} c^{-4}, \quad w_{i_1} (c w_{i_1} c^{-1})(c^2 w_{i_1} c^{-2}) \le_T c, \quad \text{and} \quad \ell_T(w_{i_1}) = 1. \tag{6.165}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains four solutions for w_i in (6.163):

$$w_i \in \big\{[1,2,3,4,5,6,5,4,2,3],\ [3,4],\ [2,4,2,5],\ [1,3,4,5,4,3,1,6]\big\},$$

where we have again used the short notation of `coxeter`, $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ being a simple system of generators of E_6, corresponding to the Dynkin diagram displayed in Figure 4. Each of the above solutions for w_i gives rise to $m/3$ elements of $\mathrm{Fix}_{NC^m(E_6)}(\phi^p)$ since i ranges from 1 to $m/3$.

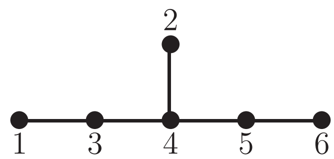

FIGURE 4. The Dynkin diagram for E_6

There are no solutions for w_i in (6.164) and for w_{i_1} in (6.165).

In total, we obtain $1 + 4\frac{m}{3} = \frac{4m+3}{3}$ elements in $\mathrm{Fix}_{NC^m(E_6)}(\phi^p)$, which agrees with the limit in (6.160b).

Next we discuss the case in (6.160c). By Lemma 26, we are free to choose $p = 3m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{2}+1} c^{-2}, c^2 w_{\frac{m}{2}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{2}} c^{-1}).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{2}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.166a}$$

$$w_i = c w_{i-\frac{m}{2}} c^{-1}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.166b}$$

There are several distinct possibilities for choosing the w_i's, $1 \le i \le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 3, \tag{6.167a}$$

and all other w_j's are equal to ε,

(iii) there is a j with $1 \le j \le \frac{m}{2}$ such that

$$1 \le \ell_T(w_j) = \ell_T(w_{j+\frac{m}{2}}) \le 2. \tag{6.167b}$$

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_6)$, we must have $w_i w_{i+\frac{m}{2}} = c$, respectively $w_j w_{j+\frac{m}{2}} \le_T c$. Together with equations (6.166)–(6.167), this implies that

$$w_i = c^3 w_i c^{-3} \quad \text{and} \quad w_i(c w_i c^{-1}) = c, \tag{6.168}$$

respectively that

$$w_j = c^3 w_j c^{-3}, \quad w_j(c w_j c^{-1}) \le_T c, \quad \text{and} \quad 1 \le \ell_T(w_j) \le 2. \tag{6.169}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains three solutions for w_i in (6.168):

$$w_i \in \big\{[1,3,4,3,5],\ [1,2,3,4,3,5,6,5,4,3,1],\ [2,3,4,5,4,2,6]\big\},$$

where we used again `coxeter`'s short notation, $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ being a simple system of generators of E_6, corresponding to the Dynkin diagram displayed in Figure 4. Each of these solutions for w_i gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(E_6)}(\phi^p)$ since i ranges from 1 to $m/2$.

There are no solutions for w_j in (6.169).

In total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(E_6)}(\phi^p)$, which agrees with the limit in (6.160c).

Finally we discuss the case in (6.160e). By Lemma 26, we are free to choose $p = 12m/5$. In particular, m must be divisible by 5. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = \big(*; c^3 w_{\frac{3m}{5}+1} c^{-3}, c^3 w_{\frac{3m}{5}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{3m}{5}} c^{-2}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{3m}{5}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{2m}{5}, \tag{6.170a}$$

$$w_i = c^2 w_{i-\frac{2m}{5}} c^{-2}, \quad i = \tfrac{2m}{5}+1, \tfrac{2m}{5}+2, \dots, m. \tag{6.170b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{5}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{5}}) = \ell_T(w_{i+\frac{2m}{5}}) = \ell_T(w_{i+\frac{3m}{5}}) = \ell_T(w_{i+\frac{4m}{5}}) = 1, \tag{6.171}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_6)$, we must have

$$w_i w_{i+\frac{m}{5}} w_{i+\frac{2m}{5}} w_{i+\frac{3m}{5}} w_{i+\frac{4m}{5}} \le_T c.$$

Together with equations (6.170)–(6.171), this implies that

$$w_i = c^{12} w_i c^{-12} \quad \text{and} \quad w_i(c^7 w_i c^{-7})(c^2 w_i c^{-2})(c^9 w_i c^{-9})(c^4 w_i c^{-4}) \le_T c. \tag{6.172}$$

Here, the first equation is automatically satisfied since $c^{12} = \varepsilon$.

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 12 solutions for w_i in (6.172):

$$w_i \in \big\{[1],\ [3],\ [4],\ [5],\ [6],\ [2,4,2],\ [3,4,3],\ [2,4,5,4,2],\ [1,3,4,5,4,3,1],\\ [1,3,4,5,6,5,4,3,1],\ [2,3,4,5,6,5,4,2,3],\ [1,2,3,4,5,6,5,4,2,3,1]\big\},$$

where $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ is a simple system of generators of E_6, corresponding to the Dynkin diagram displayed in Figure 4, and each of them gives rise to $m/5$ elements of $\mathrm{Fix}_{NC^m(E_6)}(\phi^p)$ since i ranges from 1 to $m/5$.

In total, we obtain $1 + 12\frac{m}{5} = \frac{12m+5}{5}$ elements in $\mathrm{Fix}_{NC^m(E_6)}(\phi^p)$, which agrees with the limit in (6.160e).

Finally, we turn to (6.160j). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 1$ and $m_2 = 5$ and $h_2 = 2$ and $m_2 = 5$. These correspond to the choices $p = 12m/5$, respectively $p = 6m/5$, out which only $p = 6m/5$ has not yet been discussed and belongs to the current case. The corresponding action of ϕ^p is given by

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^2 w_{\frac{4m}{5}+1} c^{-2}, c^2 w_{\frac{4m}{5}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{4m}{5}} c^{-1}\big),$$

so that we have to solve

$$t_1 (c t_1 c^{-1})(c^2 t_1 c^{-2})(c^3 t_1 c^{-3})(c^4 t_1 c^{-4}) \le_T c$$

for t_1 with $\ell_T(t_1) = 1$. A computation with the help of Stembridge's *Maple* package `coxeter` [35] finds no solution. Hence, the left-hand side of (3.3) is equal to 1, as required.

CASE $G_{36} = E_7$. The degrees are $2, 6, 8, 10, 12, 14, 18$, and hence we have

$$\mathrm{Cat}^m(E_7; q) = \frac{[18m+18]_q\,[18m+14]_q\,[18m+12]_q}{[18]_q\,[14]_q\,[12]_q} \\ \times \frac{[18m+10]_q\,[18m+8]_q\,[18m+6]_q\,[18m+2]_q}{[10]_q\,[8]_q\,[6]_q\,[2]_q}.$$

Let ζ be a $18m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = m+1, \quad \text{if } \zeta = \zeta_{18}, \zeta_9, \tag{6.173a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = \tfrac{9m+7}{7}, \quad \text{if } \zeta = \zeta_{14}, \zeta_7,\ 7 \mid m, \tag{6.173b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta = \zeta_{12},\ 2 \mid m, \tag{6.173c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = \tfrac{9m+5}{5}, \quad \text{if } \zeta = \zeta_{10}, \zeta_5,\ 5 \mid m, \tag{6.173d}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = \tfrac{9m+4}{4}, \quad \text{if } \zeta = \zeta_8,\ 4 \mid m, \tag{6.173e}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = \frac{(m+1)(3m+2)(3m+1)}{2}, \quad \text{if } \zeta = \zeta_6, \zeta_3, \tag{6.173f}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = \frac{(3m+2)(9m+4)}{8}, \quad \text{if } \zeta = \zeta_4,\ 2 \mid m, \tag{6.173g}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = \mathrm{Cat}^m(E_7), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{6.173h}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(E_7; q) = 1, \quad \text{otherwise.} \tag{6.173i}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.173). The only cases not covered by Lemmas 27 and 28 are the ones in (6.173b), (6.173c), (6.173d), (6.173e), (6.173g), and (6.173i).

We begin with the case in (6.173b). By Lemma 26, we are free to choose $p = 9m/7$ if $\zeta = \zeta_{14}$, respectively $p = 18m/7$ if $\zeta = \zeta_7$. In both cases, m must be divisible by 7.

We start with the case that $p = 9m/7$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^2 w_{\frac{5m}{7}+1} c^{-2}, c^2 w_{\frac{5m}{7}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{5m}{7}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{5m}{7}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{2m}{7}, \tag{6.174a}$$

$$w_i = c w_{i-\frac{2m}{7}} c^{-1}, \quad i = \tfrac{2m}{7}+1, \tfrac{2m}{7}+2, \dots, m. \tag{6.174b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{7}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{7}}) = \ell_T(w_{i+\frac{2m}{7}}) = \ell_T(w_{i+\frac{3m}{7}}) \\ = \ell_T(w_{i+\frac{4m}{7}}) = \ell_T(w_{i+\frac{5m}{7}}) = \ell_T(w_{i+\frac{6m}{7}}) = 1, \tag{6.175}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_7)$, we must have

$$w_i w_{i+\frac{m}{7}} w_{i+\frac{2m}{7}} w_{i+\frac{3m}{7}} w_{i+\frac{4m}{7}} w_{i+\frac{5m}{7}} w_{i+\frac{6m}{7}} = c.$$

Together with equations (6.174)–(6.175), this implies that

$$w_i = c^9 w_i c^{-9} \quad \text{and} \quad w_i(c^5 w_i c^{-5})(c w_i c^{-1})(c^6 w_i c^{-6})(c^2 w_i c^{-2})(c^7 w_i c^{-7})(c^3 w_i c^{-3}) = c. \tag{6.176}$$

Here, the first equation is automatically satisfied due to Lemma 29 with $d = 2$.

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 9 solutions for w_i in (6.176):

$$w_i \in \big\{[4], [5], [6], [7], [3,4,3], [2,4,5,4,2], [1,3,4,5,6,5,4,3,1], \\ [2,3,4,5,6,7,6,5,4,2,3], [1,2,3,4,5,6,7,6,5,4,2,3,1]\big\}, \tag{6.177}$$

where we have again used the short notation of `coxeter`, $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ being a simple system of generators of E_7, corresponding to the Dynkin diagram displayed in Figure 5. Each of the above solutions for w_i gives rise to $m/7$ elements of $\mathrm{Fix}_{NC^m(E_7)}(\phi^p)$ since i ranges from 1 to $m/7$.

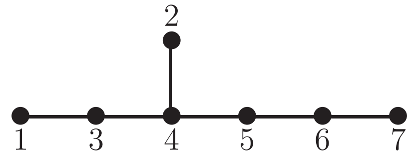

FIGURE 5. The Dynkin diagram for E_7

In total, we obtain $1 + 9\frac{m}{7} = \frac{9m+7}{7}$ elements in $\mathrm{Fix}_{NC^m(E_7)}(\phi^p)$, which agrees with the limit in (6.173b).

In the case that $p = 18m/7$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^3 w_{\frac{3m}{7}+1} c^{-3}, c^3 w_{\frac{3m}{7}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{3m}{7}} c^{-2}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{3m}{7}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{4m}{7}, \tag{6.178a}$$

$$w_i = c^2 w_{i-\frac{4m}{7}} c^{-2}, \quad i = \tfrac{4m}{7}+1, \tfrac{4m}{7}+2, \dots, m. \tag{6.178b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{7}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{7}}) = \ell_T(w_{i+\frac{2m}{7}}) = \ell_T(w_{i+\frac{3m}{7}}) \\ = \ell_T(w_{i+\frac{4m}{7}}) = \ell_T(w_{i+\frac{5m}{7}}) = \ell_T(w_{i+\frac{6m}{7}}) = 1, \tag{6.179}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_7)$, we must have

$$w_i w_{i+\frac{m}{7}} w_{i+\frac{2m}{7}} w_{i+\frac{3m}{7}} w_{i+\frac{4m}{7}} w_{i+\frac{5m}{7}} w_{i+\frac{6m}{7}} = c.$$

Together with equations (6.178)–(6.179), this implies that

$$w_i = c^{18} w_i c^{-18} \\ \text{and} \quad w_i (c^5 w_i c^{-5})(c^{10} w_i c^{-10})(c^{15} w_i c^{-15})(c^2 w_i c^{-2})(c^7 w_i c^{-7})(c^{12} w_i c^{-12}) = c. \tag{6.180}$$

Here, the first equation is automatically satisfied since $c^{18} = \varepsilon$. Due to Lemma 29 with $d = 2$, we have $c^9 w_i c^{-9} = w_i$, hence also $c^{10} w_i c^{-10} = c w_i c^{-1}$, etc., so that (6.180) reduces to (6.176). Therefore, we are facing exactly the same enumeration problem here as for $p = 9m/7$, and, consequently, the number of solutions to (6.180) is the same, namely $\frac{9m+7}{7}$, as required.

Next we discuss the case in (6.173c). By Lemma 26, we are free to choose $p = 3m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = \big(*; c^2 w_{\frac{m}{2}+1} c^{-2}, c^2 w_{\frac{m}{2}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{2}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{2}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.181a}$$

$$w_i = c w_{i-\frac{m}{2}} c^{-1}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.181b}$$

There are several distinct possibilities for choosing the w_i's, $1 \le i \le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$1 \le \ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) \le 3. \tag{6.182}$$

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_7)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$. Together with equations (6.181)–(6.182), this implies that

$$w_i = c^3 w_i c^{-3}, \quad w_i(cw_ic^{-1}) \le_T c, \quad \text{and} \quad 1 \le \ell_T(w_i) \le 3. \tag{6.183}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains three solutions for w_i in (6.183) with $\ell_T(w_i) = 2$:

$$w_i \in \big\{[1,3,4,5,4,3],\ [2,3,4,5,6,5,4,2],\ [1,2,4,2,3,4,5,6,7,6,5,4,3,1]\big\},$$

where we used again `coxeter`'s short notation, $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ being a simple system of generators of E_7, corresponding to the Dynkin diagram displayed in Figure 5, and none if $\ell_T(w_i) = 1$ or $\ell_T(w_i) = 3$. Each of the solutions for w_i gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(E_7)}(\phi^p)$ since i ranges from 1 to $m/2$.

In total, we obtain $1 + 3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(E_7)}(\phi^p)$, which agrees with the limit in (6.173c).

Next we consider the case in (6.173d). By Lemma 26, we are free to choose $p = 9m/5$ if $\zeta = \zeta_{10}$, respectively $p = 18m/5$ if $\zeta = \zeta_5$. In both cases, m must be divisible by 5.

We start with the case that $p = 9m/5$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{5}+1} c^{-2}, c^2 w_{\frac{m}{5}+2} c^{-2}, \dots, c^2 w_m c^{-2}, cw_1c^{-1}, \dots, cw_{\frac{m}{5}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{5}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{4m}{5}, \tag{6.184a}$$

$$w_i = cw_{i-\frac{4m}{5}} c^{-1}, \quad i = \tfrac{4m}{5}+1, \tfrac{4m}{5}+2, \dots, m. \tag{6.184b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{5}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{5}}) = \ell_T(w_{i+\frac{2m}{5}}) = \ell_T(w_{i+\frac{3m}{5}}) = \ell_T(w_{i+\frac{4m}{5}}) = 1, \tag{6.185a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_7)$, we must have

$$w_i w_{i+\frac{m}{5}} w_{i+\frac{2m}{5}} w_{i+\frac{3m}{5}} w_{i+\frac{4m}{5}} \le_T c.$$

Together with equations (6.184)–(6.185), this implies that

$$w_i = c^9 w_i c^{-9} \quad \text{and} \quad w_i(c^7w_ic^{-7})(c^5w_ic^{-5})(c^3w_ic^{-3})(cw_ic^{-1}) \le_T c. \tag{6.186}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 9 solutions for w_i in (6.186):

$$w_i \in \big\{[4],\ [5],\ [6],\ [7],\ [3,4,3],\ [2,4,5,4,2],\ [1,3,4,5,6,5,4,3,1],\\ [2,3,4,5,6,7,6,5,4,2,3],\ [1,2,3,4,5,6,7,6,5,4,2,3,1]\big\},$$

where $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ is a simple system of generators of E_7, corresponding to the Dynkin diagram displayed in Figure 5, and each of them gives rise to $m/5$ elements of $\mathrm{Fix}_{NC^m(E_7)}(\phi^p)$ since i ranges from 1 to $m/5$.[4]

In total, we obtain $1 + 9\frac{m}{5} = \frac{9m+5}{5}$ elements in $\mathrm{Fix}_{NC^m(E_7)}(\phi^p)$, which agrees with the limit in (6.173d).

[4]Miraculously, these are exactly the same solutions as in the case of (6.173b). We have no explanation for this phenomenon.

In the case that $p = 18m/5$, we infer from (5.1) that

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^4 w_{\frac{2m}{5}+1} c^{-4}, c^4 w_{\frac{2m}{5}+2} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_1 c^{-3}, \dots, c^3 w_{\frac{2m}{5}} c^{-3}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^4 w_{\frac{2m}{5}+i} c^{-4}, \quad i = 1, 2, \dots, \tfrac{3m}{5}, \tag{6.187a}$$

$$w_i = c^3 w_{i-\frac{3m}{5}} c^{-3}, \quad i = \tfrac{3m}{5}+1, \tfrac{3m}{5}+2, \dots, m. \tag{6.187b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{5}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{5}}) = \ell_T(w_{i+\frac{2m}{5}}) = \ell_T(w_{i+\frac{3m}{5}}) = \ell_T(w_{i+\frac{4m}{5}}) = 1, \tag{6.188a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_7)$, we must have

$$w_i w_{i+\frac{m}{5}} w_{i+\frac{2m}{5}} w_{i+\frac{3m}{5}} w_{i+\frac{4m}{5}} \le_T c.$$

Together with equations (6.187)–(6.188), this implies that

$$w_i = c^{18} w_i c^{-18} \quad \text{and} \quad w_i (c^7 w_i c^{-7})(c^{14} w_i c^{-14})(c^3 w_i c^{-3})(c^{10} w_i c^{-10}) \le_T c. \tag{6.189}$$

Here, the first equation is automatically satisfied since $c^{18} = \varepsilon$. Due to Lemma 29 with $d = 2$, we have $c^9 w_i c^{-9} = w_i$, hence also $c^{14} w_i c^{-14} = c^5 w_i c^{-5}$, etc., so that (6.189) reduces to (6.186). Therefore, we are facing exactly the same enumeration problem here as for $p = 9m/5$, and, consequently, the number of solutions to (6.189) is the same, namely $\frac{9m+5}{5}$, as required.

Our next case is the case in (6.173e). By Lemma 26, we are free to choose $p = 9m/4$. In particular, m must be divisible by 4. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^3 w_{\frac{3m}{4}+1} c^{-3}, c^3 w_{\frac{3m}{4}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{3m}{4}} c^{-2}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{3m}{4}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{m}{4}, \tag{6.190a}$$

$$w_i = c^2 w_{i-\frac{m}{4}} c^{-2}, \quad i = \tfrac{m}{4}+1, \tfrac{m}{4}+2, \dots, m. \tag{6.190b}$$

There are several distinct possibilities for choosing the w_i's, $1 \le i \le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{4}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{4}}) = \ell_T(w_{i+\frac{2m}{4}}) = \ell_T(w_{i+\frac{3m}{4}}) = 1, \tag{6.191}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_7)$, we must have $w_i w_{i+\frac{m}{4}} w_{i+\frac{2m}{4}} w_{i+\frac{3m}{4}} \le_T c$. Together with equations (6.190)–(6.191), this implies that

$$w_i = c^9 w_i c^{-9} \quad \text{and} \quad w_i(c^2 w_i c^{-2})(c^4 w_i c^{-4})(c^6 w_i c^{-6}) \le_T c. \tag{6.192}$$

Here, the first equation in (6.192) is automatically satisfied due to Lemma 29 with $d = 2$.

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 9 solutions for w_i in (6.192) with $\ell_T(w_i) = 1$:

$$w_i \in \big\{[1],\ [3],\ [2,4,2],\ [3,4,5,4,3],\ [1,3,4,5,4,3,1],\ [2,4,5,6,5,4,2],\ [2,3,4,5,6,5,4,2,3],\\ [1,3,4,5,6,7,6,5,4,3,1],\ [1,4,2,3,4,5,6,7,6,5,4,2,3,1,4]\big\},$$

where $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ is a simple system of generators of E_7, corresponding to the Dynkin diagram displayed in Figure 5, and each of them gives rise to $m/4$ elements of $\operatorname{Fix}_{NC^m(E_7)}(\phi^p)$ since i ranges from 1 to $m/4$. Hence, we obtain $1 + 9\frac{m}{4} = \frac{9m+4}{4}$ elements in $\operatorname{Fix}_{NC^m(E_7)}(\phi^p)$, which agrees with the limit in (6.173e).

Finally we discuss the case in (6.173g). By Lemma 26, we are free to choose $p = 9m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big)\\ = \big(*; c^5 w_{\frac{m}{2}+1} c^{-5}, c^5 w_{\frac{m}{2}+2} c^{-5}, \dots, c^5 w_m c^{-5}, c^4 w_1 c^{-4}, \dots, c^4 w_{\frac{m}{2}} c^{-4}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^5 w_{\frac{m}{2}+i} c^{-5}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.193a}$$

$$w_i = c^4 w_{i-\frac{m}{2}} c^{-4}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.193b}$$

There are several distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 3, \tag{6.194a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 2, \tag{6.194b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$\ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) = 1, \tag{6.194c}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(v) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.194d}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(vi) there are i_1 and i_2 with $1 \le i_1, i_2 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_1+\frac{m}{2}}) = 2 \quad \text{and} \quad \ell_T(w_{i_2}) = \ell_T(w_{i_2+\frac{m}{2}}) = 1, \tag{6.194e}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(vii) there are i_1, i_2, i_3 with $1 \le i_1 < i_2 < i_3 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_3}) = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = \ell_T(w_{i_3+\frac{m}{2}}) = 1, \qquad (6.194\text{f})$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_7)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} \le_T c$, respectively

$$w_{i_1} w_{i_2} w_{i_3} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} w_{i_3+\frac{m}{2}} \le_T c.$$

Together with equations (6.193)–(6.194), this implies that

$$w_i = c^9 w_i c^{-9} \quad \text{and} \quad w_i (c^4 w_i c^{-4}) \le_T c, \qquad (6.195)$$

respectively that

$$w_{i_1} = c^9 w_{i_1} c^{-9}, \quad w_{i_2} = c^9 w_{i_2} c^{-9}, \quad \text{and} \quad w_{i_1} w_{i_2} (c^4 w_{i_1} c^{-4})(c^4 w_{i_2} c^{-4}) \le_T c, \qquad (6.196)$$

respectively that

$$\begin{aligned} w_{i_1} = c^9 w_{i_1} c^{-9}, \quad w_{i_2} = c^9 w_{i_2} c^{-9}, \quad w_{i_3} = c^9 w_{i_3} c^{-9},& \\ \text{and} \quad w_{i_1} w_{i_2} w_{i_3} (c^4 w_{i_1} c^{-4})(c^4 w_{i_2} c^{-4})(c^4 w_{i_3} c^{-4}) \le_T c.& \qquad (6.197) \end{aligned}$$

Here, the first equation in (6.195), the first two in (6.196), and the first three in (6.197), are all automatically satisfied due to Lemma 29 with $d = 2$.

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 9 solutions for w_i in (6.195) with $\ell_T(w_i) = 1$:

$$\begin{aligned} w_i \in \big\{&[1],\ [3],\ [2,4,2],\ [3,4,5,4,3],\ [1,3,4,5,4,3,1],\ [2,4,5,6,5,4,2],\ [2,3,4,5,6,5,4,2,3], \\ &[1,3,4,5,6,7,6,5,4,3,1],\ [1,4,2,3,4,5,6,7,6,5,4,2,3,1,4]\big\}, \end{aligned}$$

where $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ is a simple system of generators of E_7, corresponding to the Dynkin diagram displayed in Figure 5, and each of them gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(E_7)}(\phi^p)$ since i ranges from 1 to $m/2$. Furthermore, one obtains 12 solutions for w_i in (6.195) with $\ell_T(w_i) = 2$:

$$\begin{aligned} w_i \in \big\{&[1,2,4,2],\ [1,3,4,5,4,3],\ [1,2,4,5,6,5,4,2],\ [1,3,1,4,5,4,3,1],\ [2,3,4,5,6,5,4,2], \\ &[1,3,1,4,5,6,7,6,5,4,3,1],\ [2,3,4,3,5,6,5,4,2,3],\ [1,2,4,2,3,4,5,6,7,6,5,4,3,1], \\ &[2,3,4,2,5,4,2,6,5,4,2,3],\ [1,3,1,4,3,5,4,3,1,6,7,6,5,4,3,1], \\ &[1,3,4,2,3,4,5,4,2,6,7,6,5,4,2,3,1,4],\ [1,2,4,2,3,4,5,4,6,5,4,3,7,6,5,4,2,3,1,4]\big\}, \end{aligned}$$

each of them giving rise to $m/2$ elements of $\operatorname{Fix}_{NC^m(E_7)}(\phi^p)$ since i ranges from 1 to $m/2$, and one obtains 27 pairs (w_{i_1}, w_{i_2}) of solutions in (6.196) with $\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = 1$:

$$\begin{aligned} w_i \in \big\{&([1],[2,4,2]),\ ([1],[3,4,5,4,3]),\ ([1],[2,4,5,6,5,4,2]),\ ([3],[1,3,4,5,4,3,1]),\\ &([3],[2,4,5,6,5,4,2]),\ ([3],[1,3,4,5,6,7,6,5,4,3,1]),\ ([2,4,2],[1]),\\ &([2,4,2],[2,3,4,5,6,5,4,2,3]),\ ([2,4,2],[1,3,4,5,6,7,6,5,4,3,1]),\\ &([3,4,5,4,3],[1,3,4,5,4,3,1]),\ ([3,4,5,4,3],[2,3,4,5,6,5,4,2,3]),\\ &([3,4,5,4,3],[1,3,4,5,6,7,6,5,4,3,1]),\ ([1,3,4,5,4,3,1],[1]),\\ &([1,3,4,5,4,3,1],[3]),\ ([1,3,4,5,4,3,1],[1,4,2,3,4,5,6,7,6,5,4,2,3,1,4]),\\ &([2,4,5,6,5,4,2],[1]),\ ([2,4,5,6,5,4,2],[2,3,4,5,6,5,4,2,3]),\\ &([2,4,5,6,5,4,2],[1,4,2,3,4,5,6,7,6,5,4,2,3,1,4]),\ ([2,3,4,5,6,5,4,2,3],[3]),\\ &([2,3,4,5,6,5,4,2,3],[2,4,2]),\ ([2,3,4,5,6,5,4,2,3],[3,4,5,4,3]),\\ &([1,3,4,5,6,7,6,5,4,3,1],[3]),\ ([1,3,4,5,6,7,6,5,4,3,1],[3,4,5,4,3]),\\ &([1,3,4,5,6,7,6,5,4,3,1],[1,4,2,3,4,5,6,7,6,5,4,2,3,1,4]),\\ &([1,4,2,3,4,5,6,7,6,5,4,2,3,1,4],[2,4,2]),\\ &([1,4,2,3,4,5,6,7,6,5,4,2,3,1,4],[1,3,4,5,4,3,1]),\\ &([1,4,2,3,4,5,6,7,6,5,4,2,3,1,4],[2,4,5,6,5,4,2])\big\}, \end{aligned}$$

each of them giving rise to $\binom{m/2}{2}$ elements of $\operatorname{Fix}_{NC^m(E_7)}(\phi^p)$ since $1 \le i_1 < i_2 \le \frac{m}{2}$.

There are no solutions for w_i in (6.195) with $\ell_T(w_i) = 3$, and hence there are no solutions for w_{i_1}, w_{i_2} in (6.196) if we are in case (vi), or for $w_{i_1}, w_{i_2}, w_{i_3}$ in (6.197).

In total, we obtain $1+(9+12)\frac{m}{2}+27\binom{m/2}{2} = \frac{(3m+2)(9m+4)}{8}$ elements in $\operatorname{Fix}_{NC^m(E_7)}(\phi^p)$, which agrees with the limit in (6.173g).

Finally, we turn to (6.173i). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 1$ and $m_2 = 7$, $h_2 = 1$ and $m_2 = 5$, $h_2 = 2$ and $m_2 = 7$, $h_2 = 2$ and $m_2 = 5$, $h_2 = 2$ and $m_2 = 4$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 18m/7$, $p = 18m/5$, $p = 9m/7$, $p = 9m/5$, $p = 9m/4$, respectively $p = 9m/2$, all of which have already been discussed as they do not belong to (6.173i). Hence, (3.3) must necessarily hold, as required.

Case $G_{37} = E_8$. The degrees are $2, 8, 12, 14, 18, 20, 24, 30$, and hence we have

$$\begin{aligned} \operatorname{Cat}^m(E_8; q) = \frac{[30m+30]_q\,[30m+24]_q\,[30m+20]_q\,[30m+18]_q}{[30]_q\,[24]_q\,[20]_q\,[18]_q}\\ \times \frac{[30m+14]_q\,[30m+12]_q\,[30m+8]_q\,[30m+2]_q}{[14]_q\,[12]_q\,[8]_q\,[2]_q}. \end{aligned}$$

Let ζ be a $30m$-th root of unity. The following cases on the right-hand side of (3.3) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = m+1, \quad \text{if } \zeta=\zeta_{30},\zeta_{15}, \tag{6.198a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \tfrac{5m+4}{4}, \quad \text{if } \zeta=\zeta_{24},\ 4\mid m, \tag{6.198b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \tfrac{3m+2}{2}, \quad \text{if } \zeta=\zeta_{20},\ 2\mid m, \tag{6.198c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \tfrac{5m+3}{3}, \quad \text{if } \zeta=\zeta_{18},\zeta_9,\ 3\mid m, \tag{6.198d}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \tfrac{15m+7}{7}, \quad \text{if } \zeta=\zeta_{14},\zeta_7,\ 7\mid m, \tag{6.198e}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \tfrac{(5m+4)(5m+2)}{8}, \quad \text{if } \zeta=\zeta_{12},\ 2\mid m, \tag{6.198f}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \tfrac{(m+1)(3m+2)}{2}, \quad \text{if } \zeta=\zeta_{10},\zeta_5, \tag{6.198g}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \frac{(5m+4)(15m+4)}{16}, \quad \text{if } \zeta=\zeta_8,\ 4\mid m, \tag{6.198h}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \frac{(m+1)(5m+4)(5m+3)(5m+2)}{24}, \quad \text{if } \zeta=\zeta_6,\zeta_3, \tag{6.198i}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \frac{(5m+4)(3m+2)(5m+2)(15m+4)}{64}, \quad \text{if } \zeta=\zeta_4,\ 2\mid m, \tag{6.198j}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = \operatorname{Cat}^m(E_8), \quad \text{if } \zeta=-1 \text{ or } \zeta=1, \tag{6.198k}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_8;q) = 1, \quad \text{otherwise.} \tag{6.198l}$$

We must now prove that the left-hand side of (3.3) in each case agrees with the values exhibited in (6.198). The only cases not covered by Lemmas 27 and 28 are the ones in (6.198b), (6.198c), (6.198d), (6.198e), (6.198f), (6.198h), (6.198j), and (6.198l).

We begin with the case in (6.198b). By Lemma 26, we are free to choose $p=5m/4$. In particular, m must be divisible by 4. From (5.1), we infer

$$\begin{aligned}&\phi^p\big((w_0;w_1,\dots,w_m)\big)\\ &\qquad= \big(*;c^2w_{\frac{3m}{4}+1}c^{-2},c^2w_{\frac{3m}{4}+2}c^{-2},\dots,c^2w_mc^{-2},cw_1c^{-1},\dots,cw_{\frac{3m}{4}}c^{-1}\big).\end{aligned}$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2w_{\frac{3m}{4}+i}c^{-2}, \quad i=1,2,\dots,\tfrac{m}{4}, \tag{6.199a}$$

$$w_i = cw_{i-\frac{m}{4}}c^{-1}, \quad i=\tfrac{m}{4}+1,\tfrac{m}{4}+2,\dots,m. \tag{6.199b}$$

There are several distinct possibilities for choosing the w_i's, $1\le i\le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0=c$),
(ii) there is an i with $1\le i\le \frac{m}{4}$ such that

$$1\le \ell_T(w_i)=\ell_T(w_{i+\frac{m}{4}})=\ell_T(w_{i+\frac{2m}{4}})=\ell_T(w_{i+\frac{3m}{4}})\le 2. \tag{6.200}$$

Moreover, since $(w_0;w_1,\dots,w_m)\in NC^m(E_8)$, we must have

$$w_iw_{i+\frac{m}{4}}w_{i+\frac{2m}{4}}w_{i+\frac{3m}{4}} \le_T c.$$

Together with equations (6.199)–(6.200), this implies that

$$w_i = c^5 w_i c^{-5} \quad \text{and} \quad w_i(cw_ic^{-1})(c^2w_ic^{-2})(c^3w_ic^{-3}) \le_T c. \tag{6.201}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 5 solutions for w_i in (6.201) with $\ell_T(w_i) = 2$:

$$w_i \in \big\{[1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],\ [3,4,3,5],\ [2,4,5,4,2,6],\\ [1,3,4,5,6,5,4,3,1,7],\ [2,3,4,5,6,7,6,5,4,2,3,8]\big\},$$

where we have again used the short notation of `coxeter`, $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ being a simple system of generators of E_8, corresponding to the Dynkin diagram displayed in Figure 6. Each of the above solutions for w_i gives rise to $m/4$ elements of $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$ since i ranges from 1 to $m/4$.

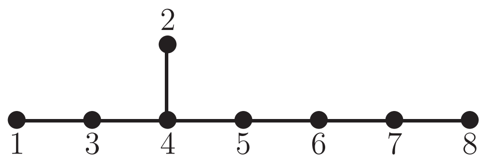

FIGURE 6. The Dynkin diagram for E_8

There are no solutions for w_i in (6.201) with $\ell_T(w_i) = 1$.

In total, we obtain $1 + 5\frac{m}{4} = \frac{5m+4}{4}$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$, which agrees with the limit in (6.198b).

Next we discuss the case in (6.198c). By Lemma 26, we are free to choose $p = 3m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{2}+1} c^{-2}, c^2 w_{\frac{m}{2}+2} c^{-2}, \dots, c^2 w_m c^{-2}, cw_1c^{-1}, \dots, cw_{\frac{m}{2}}c^{-1}).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{2}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.202a}$$

$$w_i = c w_{i-\frac{m}{2}} c^{-1}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.202b}$$

There are several distinct possibilities for choosing the w_i's, $1 \le i \le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$1 \le \ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) \le 4. \tag{6.203}$$

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_8)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$. Together with equations (6.202)–(6.203), this implies that

$$w_i = c^3 w_i c^{-3}, \quad w_i(cw_ic^{-1}) \le_T c, \quad \text{and} \quad 1 \le \ell_T(w_i) \le 4. \tag{6.204}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains three solutions for w_i in (6.204) with $\ell_T(w_i) = 4$:

$$w_i \in \big\{[1,2,3,4,2,5,4,2,6,5,7,8,7,6,5,4,2,3],\ [2,3,4,2,5,6,5,4,2,7],\\ [1,2,4,2,3,4,5,4,6,7,6,5,4,3,1,8]\big\},$$

where we used again `coxeter`'s short notation, $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ being a simple system of generators of E_8, corresponding to the Dynkin diagram displayed in Figure 6.

Each of these solutions for w_i gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$ since i ranges from 1 to $m/2$. There are no solutions for w_i in (6.204) with $1 \le \ell_T(w_i) \le 3$.

In total, we obtain $1+3\frac{m}{2} = \frac{3m+2}{2}$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$, which agrees with the limit in (6.198c).

Next we consider the case in (6.198d). If $\zeta = \zeta_{18}$, then, by Lemma 26, we are free to choose $p = 5m/3$, whereas, for $\zeta = \zeta_9$, we can choose $p = 10m/3$. In particular, in both cases m must be divisible by 3.

First, let $p = 5m/3$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) = (*; c^2 w_{\frac{m}{3}+1} c^{-2}, c^2 w_{\frac{m}{3}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{3}} c^{-1}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^2 w_{\frac{m}{3}+i} c^{-2}, \quad i = 1, 2, \dots, \tfrac{2m}{3}, \tag{6.205a}$$

$$w_i = c w_{i-\frac{2m}{3}} c^{-1}, \quad i = \tfrac{2m}{3}+1, \tfrac{2m}{3}+2, \dots, m. \tag{6.205b}$$

There several distinct possibilities for choosing the w_i's, $1 \le i \le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{3}$ such that

$$1 \le \ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) \le 2. \tag{6.206}$$

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_8)$, we must have

$$w_i w_{i+\frac{m}{3}} w_{i+\frac{2m}{3}} \le_T c.$$

Together with equations (6.205)–(6.206), this implies that

$$w_i = c^5 w_i c^{-5} \quad \text{and} \quad w_i(c^3 w_i c^{-3})(c w_i c^{-1}) \le_T c. \tag{6.207}$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains five solutions for w_i in (6.207) with $\ell_T(w_i) = 2$:

$$w_i \in \big\{[1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],\ [3,4,3,5],\ [2,4,5,4,2,6],\\ [1,3,4,5,6,5,4,3,1,7],\ [2,3,4,5,6,7,6,5,4,2,3,8]\big\},$$

where $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ is a simple system of generators of E_8, corresponding to the Dynkin diagram displayed in Figure 6, and each of them gives rise to $m/3$ elements of $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$ since i ranges from 1 to $m/3$. There are no solutions for w_i in (6.207) with $\ell_T(w_i) = 1$.

In total, we obtain $1+5\frac{m}{3} = \frac{5m+3}{3}$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$, which agrees with the limit in (6.198d).

Now let $p = 10m/3$. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big)\\ = (*; c^4 w_{\frac{2m}{3}+1} c^{-4}, c^4 w_{\frac{2m}{3}+2} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_1 c^{-3}, \dots, c^3 w_{\frac{2m}{3}} c^{-3}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^4 w_{\frac{2m}{3}+i} c^{-4}, \quad i = 1, 2, \dots, \tfrac{m}{3}, \tag{6.208a}$$

$$w_i = c^3 w_{i-\frac{m}{3}} c^{-3}, \quad i = \tfrac{m}{3}+1, \tfrac{m}{3}+2, \dots, m. \tag{6.208b}$$

There several distinct possibilities for choosing the w_i's, $1 \le i \le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{3}$ such that

$$1 \le \ell_T(w_i) = \ell_T(w_{i+\frac{m}{3}}) = \ell_T(w_{i+\frac{2m}{3}}) \le 2. \tag{6.209}$$

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_8)$, we must have

$$w_i w_{i+\frac{m}{3}} w_{i+\frac{2m}{3}} \le_T c.$$

Together with equations (6.208)–(6.209), this implies that

$$w_i = c^{10} w_i c^{-10} \quad \text{and} \quad w_i(c^3 w_i c^{-3})(c^6 w_i c^{-6}) \le_T c. \tag{6.210}$$

Due to Lemma 29 with $d = 2$, we have $c^{15} w_i c^{-15} = w_i$, hence also $c^5 w_i c^{-5} = w_i$, so that (6.210) reduces to (6.207). Therefore, we are facing exactly the same enumeration problem here as for $p = 5m/3$, and, consequently, the number of solutions to (6.210) is the same, namely $\frac{5m+3}{3}$, as required.

Our next case is the case in (6.198e). If $\zeta = \zeta_{14}$, then, by Lemma 26, we are free to choose $p = 15m/7$, whereas, for $\zeta = \zeta_7$, we can choose $p = 30m/7$. In particular, in both cases m must be divisible by 7.

First, let $p = 15m/7$. From (5.1), we infer

$$\begin{aligned}\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^3 w_{\frac{6m}{7}+1} c^{-3}, c^3 w_{\frac{6m}{7}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{6m}{7}} c^{-2}\big).\end{aligned}$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{6m}{7}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{m}{7}, \tag{6.211a}$$

$$w_i = c^2 w_{i-\frac{m}{7}} c^{-2}, \quad i = \tfrac{m}{7}+1, \tfrac{m}{7}+2, \dots, m. \tag{6.211b}$$

There are two distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),
(ii) there is an i with $1 \le i \le \frac{m}{7}$ such that

$$\begin{aligned}\ell_T(w_i) = \ell_T(w_{i+\frac{m}{7}}) = \ell_T(w_{i+\frac{2m}{7}}) = \ell_T(w_{i+\frac{3m}{7}}) \\ = \ell_T(w_{i+\frac{4m}{7}}) = \ell_T(w_{i+\frac{5m}{7}}) = \ell_T(w_{i+\frac{6m}{7}}) = 1,\end{aligned} \tag{6.212}$$

and the other w_j's, $1 \le j \le m$, are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_8)$, we must have

$$w_i w_{i+\frac{m}{7}} w_{i+\frac{2m}{7}} w_{i+\frac{3m}{7}} w_{i+\frac{4m}{7}} w_{i+\frac{5m}{7}} w_{i+\frac{6m}{7}} \le_T c.$$

Together with the equations (6.211)–(6.212), this implies that

$$\begin{aligned}w_i = c^{15} w_i c^{-15} \\ \text{and} \quad w_i(c^2 w_i c^{-2})(c^4 w_i c^{-4})(c^6 w_i c^{-6})(c^8 w_i c^{-8})(c^{10} w_i c^{-10})(c^{12} w_i c^{-12}) \le_T c.\end{aligned} \tag{6.213}$$

Here, the first equation is automatically satisfied due to Lemma 29 with $d = 2$.

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 15 solutions for w_i in (6.213):

$$w_i \in \big\{[4],\ [5],\ [6],\ [7],\ [8],\ [3,4,3],\ [2,4,5,4,2],\ [3,4,5,4,3],\ [2,4,5,6,5,4,2],\\ [1,3,4,5,6,5,4,3,1],\ [1,3,4,5,6,7,6,5,4,3,1],\\ [2,3,4,5,6,7,6,5,4,2,3],\ [2,3,4,5,6,7,8,7,6,5,4,2,3],\\ [1,2,3,4,5,6,7,8,7,6,5,4,2,3,1],\ [1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4]\big\},$$

where we have again used the short notation of `coxeter`, $\{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}$ being a simple system of generators of E_8, corresponding to the Dynkin diagram displayed in Figure 6, and each of them gives rise to $m/7$ elements of $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$ since i ranges from 1 to $m/7$.

In total, we obtain $1+15\frac{m}{7}=\frac{15m+7}{7}$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$, which agrees with the limit in (6.198e).

Now let $p=30m/7$. From (5.1), we infer

$$\phi^p\big((w_0;w_1,\dots,w_m)\big)\\ =\big(*;c^5w_{\frac{5m}{7}+1}c^{-5},c^5w_{\frac{5m}{7}+2}c^{-5},\dots,c^5w_mc^{-5},c^4w_1c^{-4},\dots,c^4w_{\frac{5m}{7}}c^{-4}\big).$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^5w_{\frac{5m}{7}+i}c^{-5},\quad i=1,2,\dots,\tfrac{2m}{7}, \tag{6.214a}$$

$$w_i=c^4w_{i-\frac{2m}{7}}c^{-4},\quad i=\tfrac{2m}{7}+1,\tfrac{2m}{7}+2,\dots,m. \tag{6.214b}$$

There are two distinct possibilities for choosing the w_i's, $1\le i\le m$:

(i) all the w_i's are equal to ε (and $w_0=c$),
(ii) there is an i with $1\le i\le\frac{m}{7}$ such that

$$\ell_T(w_i)=\ell_T(w_{i+\frac{m}{7}})=\ell_T(w_{i+\frac{2m}{7}})=\ell_T(w_{i+\frac{3m}{7}})\\ =\ell_T(w_{i+\frac{4m}{7}})=\ell_T(w_{i+\frac{5m}{7}})=\ell_T(w_{i+\frac{6m}{7}})=1, \tag{6.215}$$

and the other w_j's, $1\le j\le m$, are equal to ε.

Moreover, since $(w_0;w_1,\dots,w_m)\in NC^m(E_8)$, we must have

$$w_iw_{i+\frac{m}{7}}w_{i+\frac{2m}{7}}w_{i+\frac{3m}{7}}w_{i+\frac{4m}{7}}w_{i+\frac{5m}{7}}w_{i+\frac{6m}{7}}\le_T c.$$

Together with the equations (6.214)–(6.215), this implies that

$$w_i=c^{30}w_ic^{-30}\\ \text{and}\quad w_i(c^{17}w_ic^{-17})(c^4w_ic^{-4})(c^{21}w_ic^{-21})(c^8w_ic^{-8})(c^{25}w_ic^{-25})(c^{12}w_ic^{-12})\le_T c. \tag{6.216}$$

Here, the first equation is automatically satisfied since $c^{30}=\varepsilon$. Moreover, due to Lemma 29 with $d=2$, we have $c^{17}w_ic^{-17}=c^2w_ic^{-2}$, etc., so that (6.216) reduces to (6.213). Therefore, we are facing exactly the same enumeration problem here as for $p=15m/7$, and, consequently, the number of solutions to (6.216) is the same, namely $\frac{15m+7}{7}$, as required.

We now turn to the case in (6.198f). By Lemma 26, we are free to choose $p = 5m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(*; c^3 w_{\frac{m}{2}+1} c^{-3}, c^3 w_{\frac{m}{2}+2} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_1 c^{-2}, \dots, c^2 w_{\frac{m}{2}} c^{-2}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^3 w_{\frac{m}{2}+i} c^{-3}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.217a}$$

$$w_i = c^2 w_{i-\frac{m}{2}} c^{-2}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.217b}$$

There are several distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$1 \le \ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) \le 4, \tag{6.218a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iii) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_1 := \ell_T(w_{i_1}) = \ell_T(w_{i_1+\frac{m}{2}}) \ge 1, \quad \ell_2 := \ell_T(w_{i_2}) = \ell_T(w_{i_2+\frac{m}{2}}) \ge 1, \quad \text{and} \quad \ell_1 + \ell_2 \le 4, \tag{6.218b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1, i_2, i_3 with $1 \le i_1 < i_2 < i_3 \le \frac{m}{2}$ such that

$$\ell_1 := \ell_T(w_{i_1}) = \ell_T(w_{i_1+\frac{m}{2}}) \ge 1, \quad \ell_2 := \ell_T(w_{i_2}) = \ell_T(w_{i_2+\frac{m}{2}}) \ge 1, \\ \ell_3 := \ell_T(w_{i_3}) = \ell_T(w_{i_3+\frac{m}{2}}) \ge 1, \quad \text{and} \quad \ell_1 + \ell_2 + \ell_3 \le 4, \tag{6.218c}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(v) there are i_1, i_2, i_3, i_4 with $1 \le i_1 < i_2 < i_3 < i_4 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_3}) = \ell_T(w_{i_4}) \\ = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = \ell_T(w_{i_3+\frac{m}{2}}) = \ell_T(w_{i_4+\frac{m}{2}}) = 1, \tag{6.218d}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_8)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} \le_T c$, respectively

$$w_{i_1} w_{i_2} w_{i_3} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} w_{i_3+\frac{m}{2}} \le_T c,$$

respectively

$$w_{i_1} w_{i_2} w_{i_3} w_{i_4} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} w_{i_3+\frac{m}{2}} w_{i_4+\frac{m}{2}} = c.$$

Together with the equations (6.217)–(6.218), this implies that

$$w_i = c^5 w_i c^{-5} \quad \text{and} \quad w_i (c^2 w_i c^{-2}) \le_T c, \tag{6.219}$$

respectively that

$$w_{i_1} = c^5 w_{i_1} c^{-5}, \quad w_{i_2} = c^5 w_{i_2} c^{-5}, \quad \text{and} \quad w_{i_1} w_{i_2} (c^2 w_{i_1} c^{-2})(c^2 w_{i_2} c^{-2}) \le_T c, \tag{6.220}$$

respectively that

$$w_{i_1} = c^5 w_{i_1} c^{-5}, \quad w_{i_2} = c^5 w_{i_2} c^{-5}, \quad w_{i_3} = c^5 w_{i_3} c^{-5}, \\ \text{and} \quad w_{i_1} w_{i_2} w_{i_3} (c^2 w_{i_1} c^{-2})(c^2 w_{i_2} c^{-2})(c^2 w_{i_3} c^{-2}) \le_T c, \tag{6.221}$$

respectively that

$$w_{i_1} = c^5 w_{i_1} c^{-5}, \quad w_{i_2} = c^5 w_{i_2} c^{-5}, \quad w_{i_3} = c^5 w_{i_3} c^{-5}, \quad w_{i_4} = c^5 w_{i_4} c^{-5},$$
$$\text{and} \quad w_{i_1} w_{i_2} w_{i_3} w_{i_4} (c^2 w_{i_1} c^{-2})(c^2 w_{i_2} c^{-2})(c^2 w_{i_3} c^{-2})(c^2 w_{i_4} c^{-2}) = c. \quad (6.222)$$

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 10 solutions for w_i in (6.219) with $\ell_T(w_i) = 2$:

$$w_i \in \big\{[1,4,2,3,4,5,6,7,6,5,4,2,3,4],\ [3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5],$$
$$[1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],\ [3,4,3,5],\ [2,4,5,4,2,6],\ [1,3,4,5,6,5,4,3,1,7],$$
$$[2,3,4,5,6,7,6,5,4,2,3,8],\ [2,4,2,3,4,5,6,5,4,3],\ [1,3,4,5,4,2,3,1,4,5,6,7,6,5,4,2],$$
$$[2,3,1,4,5,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1]\big\}, \quad (6.223)$$

and one obtains 10 solutions for w_i in (6.219) with $\ell_T(w_i) = 4$:

$$w_i \in \big\{[1,2,3,4,5,6,7,6,5,4,2,3,4,8],\ [1,2,4,2,3,4,5,4,3,6,5,7,6,5,4,2,3,4],$$
$$[1,2,3,1,4,5,6,7,8,7,6,5,4,2,3,1,4,5],$$
$$[1,3,1,4,5,4,2,3,4,5,6,5,4,2,7,6,8,7,6,5,4,2,3,1,4,5],$$
$$[1,2,3,4,2,5,4,2,3,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1,4],\ [4,2,3,4,5,6],$$
$$[2,3,1,4,2,5,4,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1],\ [1,5,4,2,3,1,4,5,6,7],$$
$$[3,1,6,5,4,2,3,1,4,3,5,6,7,8],\ [2,3,4,2,3,5,4,6,5,4,2,7,6,5,4,2,3,8]\big\}, \quad (6.224)$$

where we have again used the short notation of `coxeter`, $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ being a simple system of generators of E_8, corresponding to the Dynkin diagram displayed in Figure 6, and each of them gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$ since i ranges from 1 to $m/2$. There are no solutions for w_i in (6.219) with $\ell_T(w_i) = 1$ or $\ell_T(w_i) = 3$.

Consequently, there are no solutions in Cases (iv) and (v), and the only possible solutions occurring in Case (iii) are pairs (w_{i_1}, w_{i_2}) of elements of (6.223) whose product is in (6.224). Another computation using Stembridge's *Maple* package `coxeter` finds the following 25 pairs meeting that description:

$$w_i \in \big\{([1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],\ [1,4,2,3,4,5,6,7,6,5,4,2,3,4]),$$
$$([2,4,5,4,2,6],\ [1,4,2,3,4,5,6,7,6,5,4,2,3,4]),$$
$$([3,4,3,5],\ [3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5]),$$
$$([1,3,4,5,6,5,4,3,1,7],\ [3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5]),$$
$$([3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5],\ [1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4]),$$
$$([2,3,4,5,6,7,6,5,4,2,3,8],\ [1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4]),$$
$$([1,3,4,5,4,2,3,1,4,5,6,7,6,5,4,2],\ [1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4]),$$
$$([1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],[3,4,3,5]),\ ([2,4,2,3,4,5,6,5,4,3],\ [3,4,3,5]),$$
$$([2,3,1,4,5,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1],\ [3,4,3,5]),$$
$$([1,4,2,3,4,5,6,7,6,5,4,2,3,4],\ [2,4,5,4,2,6]),\ ([3,4,3,5],\ [2,4,5,4,2,6]),$$
$$([1,3,4,5,4,2,3,1,4,5,6,7,6,5,4,2],\ [2,4,5,4,2,6]),$$
$$([3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5],\ [1,3,4,5,6,5,4,3,1,7]),$$
$$([2,4,5,4,2,6],\ [1,3,4,5,6,5,4,3,1,7]),$$
$$([2,3,1,4,5,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1],\ [1,3,4,5,6,5,4,3,1,7]),$$

$$([1,4,2,3,4,5,6,7,6,5,4,2,3,4],\ [2,3,4,5,6,7,6,5,4,2,3,8]),$$
$$([1,3,4,5,6,5,4,3,1,7],\ [2,3,4,5,6,7,6,5,4,2,3,8]),$$
$$([2,4,2,3,4,5,6,5,4,3],\ [2,3,4,5,6,7,6,5,4,2,3,8]),\ ([2,4,5,4,2,6],\ [2,4,2,3,4,5,6,5,4,3]),$$
$$([2,3,4,5,6,7,6,5,4,2,3,8],\ [2,4,2,3,4,5,6,5,4,3]),$$
$$([1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],\ [1,3,4,5,4,2,3,1,4,5,6,7,6,5,4,2]),$$
$$([1,3,4,5,6,5,4,3,1,7],\ [1,3,4,5,4,2,3,1,4,5,6,7,6,5,4,2]),$$
$$([3,4,3,5],\ [2,3,1,4,5,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1]),$$
$$([2,3,4,5,6,7,6,5,4,2,3,8],\ [2,3,1,4,5,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1])\big\},$$

where $\{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}$ is a simple system of generators of E_8, corresponding to the Dynkin diagram displayed in Figure 6, and each of them gives rise to $\binom{m/2}{2}$ elements of $\operatorname{Fix}_{NC^m(E_8)}(\phi^p)$ since $1\le i_1<i_2\le \frac{m}{2}$.

In total, we obtain

$$1+20\frac{m}{2}+25\binom{m/2}{2}=\frac{(5m+4)(5m+2)}{8}$$

elements in $\operatorname{Fix}_{NC^m(E_8)}(\phi^p)$, which agrees with the limit in (6.198f).

Next we consider the case in (6.198h). By Lemma 26, we are free to choose $p=15m/4$. In particular, m must be divisible by 4. From (5.1), we infer

$$\phi^p\big((w_0;w_1,\dots,w_m)\big)\\ =\big(*;c^4w_{\frac{m}{4}+1}c^{-4},c^4w_{\frac{m}{4}+2}c^{-4},\dots,c^4w_mc^{-4},c^3w_1c^{-3},\dots,c^3w_{\frac{m}{4}}c^{-3}\big).$$

Supposing that $(w_0;w_1,\dots,w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i=c^4w_{\frac{m}{4}+i}c^{-4},\quad i=1,2,\dots,\tfrac{3m}{4},\tag{6.225a}$$
$$w_i=c^3w_{i-\frac{3m}{4}}c^{-3},\quad i=\tfrac{3m}{4}+1,\tfrac{3m}{4}+2,\dots,m.\tag{6.225b}$$

There are several distinct possibilities for choosing the w_i's, $1\le i\le m$, which we summarise as follows:

(i) all the w_i's are equal to ε (and $w_0=c$),

(ii) there is an i with $1\le i\le \frac{m}{4}$ such that

$$1\le \ell_T(w_i)=\ell_T(w_{i+\frac{m}{4}})=\ell_T(w_{i+\frac{2m}{4}})=\ell_T(w_{i+\frac{3m}{4}})\le 2,\tag{6.226a}$$

and the other w_j's, $1\le j\le m$, are equal to ε,

(iii) there are i_1 and i_2 with $1\le i_1<i_2\le\frac{m}{4}$ such that

$$\ell_T(w_{i_1})=\ell_T(w_{i_2})=\ell_T(w_{i_1+\frac{m}{4}})=\ell_T(w_{i_2+\frac{m}{4}})\\ =\ell_T(w_{i_1+\frac{2m}{4}})=\ell_T(w_{i_2+\frac{2m}{4}})=\ell_T(w_{i_1+\frac{3m}{4}})=\ell_T(w_{i_2+\frac{3m}{4}})=1,\tag{6.226b}$$

and all other w_j are equal to ε.

Moreover, since $(w_0;w_1,\dots,w_m)\in NC^m(E_8)$, we must have

$$w_iw_{i+\frac{m}{4}}w_{i+\frac{2m}{4}}w_{i+\frac{3m}{4}}\le_T c,$$

or

$$w_{i_1}w_{i_2}w_{i_1+\frac{m}{4}}w_{i_2+\frac{m}{4}}w_{i_1+\frac{2m}{4}}w_{i_2+\frac{2m}{4}}w_{i_1+\frac{3m}{4}}w_{i_2+\frac{3m}{4}}=c.$$

Together with equations (6.225)–(6.226), this implies that

$$w_i = c^{15} w_i c^{-15} \quad \text{and} \quad w_i (c^{11} w_i c^{-11})(c^7 w_i c^{-7})(c^3 w_i c^{-3}) \le_T c, \tag{6.227}$$

or that

$$\begin{aligned} &w_{i_1} = c^{15} w_{i_1} c^{-15}, \quad w_{i_1} = c^{15} w_{i_2} c^{-15}, \\ &\quad \text{and} \quad w_{i_1} w_{i_2} (c^{11} w_{i_1} c^{-11})(c^{11} w_{i_2} c^{-11})(c^7 w_{i_1} c^{-7})(c^7 w_{i_2} c^{-7})(c^3 w_{i_1} c^{-3})(c^3 w_{i_2} c^{-3}) = c. \end{aligned} \tag{6.228}$$

Here, the first equation in (6.227) and the first two equations in (6.228) are automatically satisfied due to Lemma 29 with $d = 2$.

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 30 solutions for w_i in (6.227) with $\ell_T(w_i) = 1$:

$$\begin{gathered} w_i \in \big\{[4],\ [5],\ [6],\ [7],\ [8],\ [3,4,3],\ [4,5,4],\ [5,6,5],\ [6,7,6],\ [7,8,7],\ [2,4,5,4,2],\ [3,4,5,4,3], \\ [2,4,5,6,5,4,2],\ [1,3,4,5,6,5,4,3,1],\ [4,2,3,4,5,4,2,3,4],\ [1,3,4,5,6,7,6,5,4,3,1], \\ [2,3,4,5,6,7,6,5,4,2,3],\ [1,2,3,4,5,6,7,6,5,4,2,3,1],\ [5,4,2,3,4,5,6,5,4,2,3,4,5], \\ [2,3,4,5,6,7,8,7,6,5,4,2,3],\ [1,5,4,2,3,4,5,6,5,4,2,3,1,4,5], \\ [1,2,3,4,5,6,7,8,7,6,5,4,2,3,1],\ [4,2,3,4,5,6,7,8,7,6,5,4,2,3,4], \\ [1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],\ [3,1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3], \\ [1,6,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,5,6],\ [3,1,6,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,3,5,6], \\ [4,3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4], \\ [3,1,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7], \\ [2,4,3,1,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7]\big\}, \end{gathered}$$

one obtains 45 solutions for w_i in (6.227) with $\ell_T(w_i) = 2$ and w_i of type A_1^2 (as a parabolic Coxeter element; see the end of Section 2):

$$\begin{gathered} w_i \in \big\{[1,2,3,1,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3,1,4,5,8,7,6,5,4,2,3,1], \\ [1,2,3,4,2,3,1,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7], \\ [1,2,3,4,2,5,4,6,5,4,2,7,6,5,4,2,3,1],\ [1,2,3,4,3,5,4,6,5,4,3,7,6,5,4,2,3,1], \\ [1,2,3,4,5,4,6,5,4,7,8,7,6,5,4,2,3,1],\ [1,2,3,4,5,4,6,7,6,5,4,2,3,1], \\ [1,2,3,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3,4,5,8,7,6,5,4,2,3,1],\ [1,2,6,5,4,2,3,1,7,6,5,4,2,3,1,4,3,5,4,2,6,5,7,6], \\ [1,3,1,4,2,3,4,5,4,6,5,7,6,5,4,2,8,7,6,5,4,2,3,1,4,3], \\ [1,3,1,4,2,7,6,5,4,2,3,1,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1,7,6,8,7], \\ [1,3,4,3,5,4,3,6,5,4,3,1],\ [1,3,4,5,4,6,5,4,7,6,5,4,3,1],\ [1,4,2,3,1,4,3,5,4,3,1,6,7,8,7,6,5,4,2,3,1,4], \\ [1,4,2,3,4,5,6,5,7,6,5,8,7,6,5,4,2,3,1,4],\ [1,4,5,4,2,3,4,5,6,5,4,2,3,1,4,5], \\ [1,5,4,2,3,1,4,5,6,5,4,2,3,1,4,5],\ [1,5,4,2,3,4,5,6,5,4,2,3,1,4,5,8], \\ [1,5,6,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,5,6],\ [2,3,1,4,2,3,4,5,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1,4,3], \\ [2,3,1,6,5,4,2,3,1,4,3,5,6,7,6,5,4,2,3,1,4,3,5,6], \\ [2,3,4,2,3,1,4,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7], \\ [2,3,4,2,3,5,4,2,3,6,7,8,7,6,5,4,2,3],\ [2,3,4,5,4,6,5,4,7,6,5,4,2,3],\ [2,3,4,5,6,5,7,6,5,8,7,6,5,4,2,3], \\ [2,4,2,3,4,5,4,3,6,7,8,7,6,5,4,2,3,4],\ [2,4,2,5,4,2,6,5,4,2], \\ [2,4,3,1,6,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7],\ [2,4,5,4,2,7,8,7], \\ [3,1,4,2,3,4,5,6,5,7,8,7,6,5,4,2,3,1,4,3],\ [3,1,4,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7], \\ [3,1,5,6,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,3,5,6],\ [3,1,6,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7], \\ [3,4,2,3,1,5,4,2,3,1,4,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,7,8],\ [3,4,2,3,4,5,4,2,6,7,8,7,6,5,4,2,3,4], \end{gathered}$$

$$[3,4,3,6,7,6],\ [3,4,5,4,3,7,8,7],\ [4,2,3,1,5,4,2,3,1,4,3,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4],\\ [4,2,3,4,5,4,2,3,4,7],\ [4,2,3,4,5,6,7,6,8,7,6,5,4,2,3,4],\ [4,3,1,5,4,2,3,4,5,6,7,6,8,7,6,5,4,2,3,1,4,3,5,4],\\ [4,5,4,2,3,4,5,6,5,4,2,3,4,5],\ [4,5,4,8],\ [4,7,8,7],\\ [5,4,2,3,4,5,6,5,4,2,3,4,5,8],\ [5,4,3,1,6,5,4,2,3,1,4,3,5,4,6,5]\big\}, \tag{6.229}$$

and one obtains 20 solutions for w_i in (6.227) with $\ell_T(w_i) = 2$ and w_i of type A_2:

$$w_i \in \big\{[1,2,3,1,4,5,6,7,8,7,6,5,4,2,3,1,4,3],\ [1,2,3,4,5,6,7,6,5,4,2,3,1,8],\\ [1,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],\ [1,2,4,5,4,2,3,4,5,6,5,4,3,1],\\ [1,2,4,5,6,5,4,2,3,4,5,6,7,6,5,4,3,1],\ [1,3,1,4,5,6,5,4,2,3,1,4,5,6,7,6,5,4,2,3],\\ [1,3,1,4,5,6,7,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,2,3],\ [1,3,4,5,6,5,4,3,1,7],\\ [1,4,2,3,1,4,5,4,6,7,8,7,6,5,4,2,3,1,4,3,5,4],\\ [2,3,4,3,1,5,6,7,6,5,4,2,3,1,4,3,5,6,7,8,7,6,5,4,2,3,1,4],\ [2,3,4,5,6,7,6,5,4,2,3,8],\\ [2,3,4,5,6,7,8,7,6,5,4,2,3,4],\ [2,4,5,4,2,6],\ [3,4,2,3,4,5,4,2],\ [3,4,3,5],\\ [3,4,5,4,2,3,4,5,6,5,4,2],\ [4,5],\ [5,6],\ [6,7],\ [7,8]\big\}, \tag{6.230}$$

where $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ is a simple system of generators of E_8, corresponding to the Dynkin diagram displayed in Figure 6, and each of them gives rise to $m/4$ elements of $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$ since i ranges from 1 to $m/4$.

The number of solutions in Case (iii) can be computed from our knowledge of the solutions in Case (ii) according to type, using some elementary counting arguments. Namely, the number of solutions of (6.228) is equal to

$$45 \cdot 2 + 20 \cdot 3 = 150,$$

since an element of type A_1^2 can be decomposed in two ways into a product of two elements of absolute length 1, while for an element of type A_2 this can be done in 3 ways.

In total, we obtain $1 + (30 + 45 + 20)\frac{m}{4} + 150\binom{m/4}{2} = \frac{(5m+4)(15m+4)}{16}$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$, which agrees with the limit in (6.198h).

Finally we discuss the case in (6.198j). By Lemma 26, we are free to choose $p = 15m/2$. In particular, m must be divisible by 2. From (5.1), we infer

$$\phi^p\big((w_0; w_1, \dots, w_m)\big)\\ = \big(*; c^8 w_{\frac{m}{2}+1} c^{-8}, c^8 w_{\frac{m}{2}+2} c^{-8}, \dots, c^8 w_m c^{-8}, c^7 w_1 c^{-7}, \dots, c^7 w_{\frac{m}{2}} c^{-7}\big).$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ϕ^p, we obtain the system of equations

$$w_i = c^8 w_{\frac{m}{2}+i} c^{-8}, \quad i = 1, 2, \dots, \tfrac{m}{2}, \tag{6.231a}$$

$$w_i = c^7 w_{i-\frac{m}{2}} c^{-7}, \quad i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \dots, m. \tag{6.231b}$$

There are several distinct possibilities for choosing the w_i's, $1 \le i \le m$:

(i) all the w_i's are equal to ε (and $w_0 = c$),

(ii) there is an i with $1 \le i \le \frac{m}{2}$ such that

$$1 \le \ell_T(w_i) = \ell_T(w_{i+\frac{m}{2}}) \le 4, \tag{6.232a}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iii) there are i_1 and i_2 with $1 \le i_1 < i_2 \le \frac{m}{2}$ such that

$$\ell_1 := \ell_T(w_{i_1}) = \ell_T(w_{i_1+\frac{m}{2}}) \ge 1, \quad \ell_2 := \ell_T(w_{i_2}) = \ell_T(w_{i_2+\frac{m}{2}}) \ge 1, \quad \text{and} \quad \ell_1 + \ell_2 \le 4, \tag{6.232b}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(iv) there are i_1, i_2, i_3 with $1 \le i_1 < i_2 < i_3 \le \frac{m}{2}$ such that

$$\ell_1 := \ell_T(w_{i_1}) = \ell_T(w_{i_1+\frac{m}{2}}) \ge 1, \quad \ell_2 := \ell_T(w_{i_2}) = \ell_T(w_{i_2+\frac{m}{2}}) \ge 1,\\ \ell_3 := \ell_T(w_{i_3}) = \ell_T(w_{i_3+\frac{m}{2}}) \ge 1, \quad \text{and} \quad \ell_1 + \ell_2 + \ell_3 \le 4, \tag{6.232c}$$

and the other w_j's, $1 \le j \le m$, are equal to ε,

(v) there are i_1, i_2, i_3, i_4 with $1 \le i_1 < i_2 < i_3 < i_4 \le \frac{m}{2}$ such that

$$\ell_T(w_{i_1}) = \ell_T(w_{i_2}) = \ell_T(w_{i_3}) = \ell_T(w_{i_4})\\ = \ell_T(w_{i_1+\frac{m}{2}}) = \ell_T(w_{i_2+\frac{m}{2}}) = \ell_T(w_{i_3+\frac{m}{2}}) = \ell_T(w_{i_4+\frac{m}{2}}) = 1, \tag{6.232d}$$

and all other w_j's are equal to ε.

Moreover, since $(w_0; w_1, \dots, w_m) \in NC^m(E_8)$, we must have $w_i w_{i+\frac{m}{2}} \le_T c$, respectively $w_{i_1} w_{i_2} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} \le_T c$, respectively

$$w_{i_1} w_{i_2} w_{i_3} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} w_{i_3+\frac{m}{2}} \le_T c,$$

respectively

$$w_{i_1} w_{i_2} w_{i_3} w_{i_4} w_{i_1+\frac{m}{2}} w_{i_2+\frac{m}{2}} w_{i_3+\frac{m}{2}} w_{i_4+\frac{m}{2}} = c.$$

Together with equations (6.231)–(6.232), this implies that

$$w_i = c^{15} w_i c^{-15} \quad \text{and} \quad w_i (c^7 w_i c^{-7}) \le_T c, \tag{6.233}$$

respectively that

$$w_{i_1} = c^{15} w_{i_1} c^{-15}, \quad w_{i_2} = c^{15} w_{i_2} c^{-15}, \quad \text{and} \quad w_{i_1} w_{i_2} (c^7 w_{i_1} c^{-7})(c^7 w_{i_2} c^{-7}) \le_T c, \tag{6.234}$$

respectively that

$$w_{i_1} = c^{15} w_{i_1} c^{-15}, \quad w_{i_2} = c^{15} w_{i_2} c^{-15}, \quad w_{i_3} = c^{15} w_{i_3} c^{-15},\\ \text{and} \quad w_{i_1} w_{i_2} w_{i_3} (c^7 w_{i_1} c^{-7})(c^7 w_{i_2} c^{-7})(c^7 w_{i_3} c^{-7}) \le_T c, \tag{6.235}$$

respectively that

$$w_{i_1} = c^{15} w_{i_1} c^{-15}, \quad w_{i_2} = c^{15} w_{i_2} c^{-15}, \quad w_{i_3} = c^{15} w_{i_3} c^{-15}, \quad w_{i_4} = c^{15} w_{i_4} c^{-15},\\ \text{and} \quad w_{i_1} w_{i_2} w_{i_3} w_{i_4} (c^7 w_{i_1} c^{-7})(c^7 w_{i_2} c^{-7})(c^7 w_{i_3} c^{-7})(c^7 w_{i_4} c^{-7}) = c. \tag{6.236}$$

Here, the first equation in (6.233), the first two in (6.234), the first three in (6.235), and the first four in (6.236), are all automatically satisfied due to Lemma 29 with $d = 2$.

With the help of Stembridge's *Maple* package `coxeter` [35], one obtains 45 solutions for w_i in (6.233) with $\ell_T(w_i) = 1$:

$$\begin{gathered}
w_i \in \big\{[1],\ [3],\ [4],\ [5],\ [6],\ [7],\ [8],\ [2,4,2],\ [3,4,3],\ [4,5,4],\ [5,6,5],\ [6,7,6],\ [7,8,7],\\
[2,4,5,4,2],\ [3,4,5,4,3],\ [1,3,4,5,4,3,1],\ [2,4,5,6,5,4,2],\ [3,4,5,6,5,4,3],\\
[1,3,4,5,6,5,4,3,1],\ [4,2,3,4,5,4,2,3,4],\ [2,3,4,5,6,5,4,2,3],\ [2,4,5,6,7,6,5,4,2],\\
[1,3,4,5,6,7,6,5,4,3,1],\ [4,2,3,4,5,6,5,4,2,3,4],\ [2,3,4,5,6,7,6,5,4,2,3],\\
[1,2,3,4,5,6,7,6,5,4,2,3,1],\ [1,3,4,5,6,7,8,7,6,5,4,3,1],\ [5,4,2,3,4,5,6,5,4,2,3,4,5],\\
[4,2,3,4,5,6,7,6,5,4,2,3,4],\ [2,3,4,5,6,7,8,7,6,5,4,2,3],\ [1,5,4,2,3,4,5,6,5,4,2,3,1,4,5],\\
[1,4,2,3,4,5,6,7,6,5,4,2,3,1,4],\ [1,2,3,4,5,6,7,8,7,6,5,4,2,3,1],\\
[4,2,3,4,5,6,7,8,7,6,5,4,2,3,4],\ [1,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,5],\\
[1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],\ [3,1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3],\\
[1,6,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,5,6],\ [1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5],\\
[3,1,6,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,3,5,6],\\
[3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5],\\
[4,3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4],\\
[3,1,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6],\\
[3,1,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7],\\
[2,4,3,1,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7]\big\},
\end{gathered}$$

one obtains 150 solutions for w_i in (6.233) with $\ell_T(w_i) = 2$ and w_i of type A_1^2:

$$\begin{gathered}
w_i \in \big\{[1,2,3,1,4,3,5,4,6,5,7,6,5,4,3,1,8,7,6,5,4,2,3,1],\\
[1,2,3,1,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3,1,4,5,8,7,6,5,4,2,3,1],\ [1,2,3,1,4,5,6,7,8,7,6,5,4,2,3,1],\\
[1,2,3,4,2,3,1,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7],\\
[1,2,3,4,2,3,5,4,6,5,7,6,5,4,2,3,8,7,6,5,4,2,3,1],\ [1,2,3,4,2,5,4,2,6,7,8,7,6,5,4,2,3,1],\\
[1,2,3,4,2,5,4,6,5,4,2,7,6,5,4,2,3,1],\ [1,2,3,4,2,5,4,6,5,4,2,7,8,7,6,5,4,2,3,1],\\
[1,2,3,4,3,5,4,6,5,4,3,7,6,5,4,2,3,1],\ [1,2,3,4,3,5,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1],\\
[1,2,3,4,5,4,2,3,4,6,5,7,6,5,4,2,3,4,8,7,6,5,4,2,3,1],\ [1,2,3,4,5,4,6,5,4,7,8,7,6,5,4,2,3,1],\\
[1,2,3,4,5,4,6,7,6,5,4,2,3,1],\ [1,2,3,4,5,4,6,7,8,7,6,5,4,2,3,1],\\
[1,2,3,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3,4,5,8,7,6,5,4,2,3,1],\ [1,2,3,4,5,6,5,7,8,7,6,5,4,2,3,1],\\
[1,2,4,2,3,1,7,6,5,4,2,3,1,4,3,5,4,2,6,7],\ [1,2,4,2,3,1,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8],\\
[1,2,4,2,3,4,5,4,3,6,7,6,5,4,2,3,1,4],\ [1,2,4,2,3,4,5,4,3,6,7,8,7,6,5,4,2,3,1,4],\\
[1,2,4,2,3,4,5,4,6,5,4,3,7,6,5,4,2,3,1,4],\ [1,2,4,2,3,4,5,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1,4],\\
[1,2,4,5,4,2,3,4,5,6,5,4,3,7,6,5,4,2,3,1,4,5],\ [1,2,4,5,4,2],\ [1,2,4,5,6,5,4,2],\\
[1,2,6,5,4,2,3,1,7,6,5,4,2,3,1,4,3,5,4,2,6,5,7,6],\ [1,3,1,4,2,3,4,5,4,6,5,7,6,5,4,2,8,7,6,5,4,2,3,1,4,3],\\
[1,3,1,4,2,5,4,2,3,1,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1,7,8],\\
[1,3,1,4,2,7,6,5,4,2,3,1,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1,7,6,8,7],\\
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[1,3,1,4,5,4,2,3,4,5,6,5,4,2,7,8,7,6,5,4,2,3,1,4,3,5],\\
[1,3,1,4,5,4,2,3,4,5,6,5,7,6,5,4,2,8,7,6,5,4,2,3,1,4,3,5],\\
[1,3,1,4,5,6,5,4,2,3,4,5,6,7,6,5,4,2,8,7,6,5,4,2,3,1,4,3,5,6],\\
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[1,3,4,2,3,4,5,4,6,5,7,6,5,4,2,8,7,6,5,4,2,3,1,4],\ [1,3,4,3,5,4,3,1],\\
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\end{gathered}$$

[1, 3, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[1, 3, 4, 5, 4, 2, 3, 4, 5, 6, 5, 7, 6, 5, 4, 2, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5], [1, 3, 4, 5, 4, 3, 1, 7],
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[1, 4], [1, 5, 4, 2, 3, 1, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5], [1, 5, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[1, 5, 4, 2, 3, 1, 4, 5], [1, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 8], [1, 5, 4, 2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
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[2, 3, 1, 4, 2, 3, 4, 5, 4, 6, 5, 7, 6, 5, 4, 3, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3],
[2, 3, 1, 4, 5, 6, 5, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6], [2, 3, 1, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
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[2, 4, 2, 3, 4, 5, 4, 3, 6, 7, 6, 5, 4, 2, 3, 4], [2, 4, 2, 3, 4, 5, 4, 3, 6, 7, 8, 7, 6, 5, 4, 2, 3, 4], [2, 4, 2, 3, 4, 5, 4, 6, 5, 4, 3, 7, 6, 5, 4, 2, 3, 4],
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[3, 1, 4, 6, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6], [3, 1, 4, 7, 6, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7],
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[3, 1, 5, 6, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6], [3, 1, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
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[3, 1, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 8], [3, 4, 2, 3, 1, 5, 4, 2, 3, 1, 4, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 7, 8],
[3, 4, 2, 3, 4, 5, 4, 2, 6, 5, 4, 2, 3, 4], [3, 4, 2, 3, 4, 5, 4, 2, 6, 7, 8, 7, 6, 5, 4, 2, 3, 4], [3, 4, 3, 5, 4, 3], [3, 4, 3, 6, 7, 6],
[3, 4, 3, 6], [3, 4, 3, 7], [3, 4, 5, 4, 3, 7, 8, 7], [3, 4, 5, 4, 6, 5, 4, 3], [3, 4, 5, 6, 5, 4, 3, 8], [3, 5], [3, 7],
[4, 2, 3, 1, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4], [4, 2, 3, 4, 5, 4, 2, 3, 4, 7], [4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 8],
[4, 2, 3, 4, 5, 6, 5, 7, 6, 5, 4, 2, 3, 4], [4, 2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 4], [4, 2, 3, 4],
[4, 3, 1, 5, 4, 2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4], [4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 5], [4, 5, 4, 8], [4, 7, 8, 7],
[4, 7], [4, 8], [5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 5, 8], [5, 4, 2, 3, 4, 5],
[5, 4, 3, 1, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 6, 5], [5, 8], [6, 5, 4, 2, 3, 4, 5, 6]$\}$,

one obtains 100 solutions for w_i in (6.233) with $\ell_T(w_i) = 2$ and w_i of type A_2:

$w_i \in \{$[1, 2, 3, 1, 4, 5, 4, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5], [1, 2, 3, 1, 4, 5, 6, 5, 4, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
[1, 2, 3, 1, 4, 5, 6, 5, 4, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6], [1, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3],
[1, 2, 3, 4, 3, 1, 5, 6, 7, 6, 5, 4, 3, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 7], [1, 2, 3, 4, 5, 4, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[1, 2, 3, 4, 5, 4, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5], [1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4],
[1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 8], [1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3], [1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4],
[1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3], [1, 2, 4, 2, 3, 1, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3],
[1, 2, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 1], [1, 2, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 1], [1, 2, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 3, 1],

$$[1,2,4,5,4,2,3,4,5,6,7,6,5,4,3,1],\ [1,2,4,5,4,2,3,4,5,6,7,8,7,6,5,4,3,1],\ [1,2,4,5,6,5,4,2,3,4,5,6,7,6,5,4,3,1],$$
$$[1,3,1,4,5,4,2,3,1,4,5,6,7,8,7,6,5,4,2,3],\ [1,3,1,4,5,6,5,4,2,3,1,4,5,6,7,6,5,4,2,3],$$
$$[1,3,1,4,5,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,2,3],\ [1,3,1,4,5,6,7,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,2,3],$$
$$[1,3,4,3,1,5,4,2,3,1,4,5,6,7,8,7,6,5,4,2,3,4],\ [1,3,4,5,4,2,3,1,4,5,6,5,4,2],$$
$$[1,3,4,5,4,2,3,1,4,5,6,7,6,5,4,2],\ [1,3,4,5,4,3,1,6,7,6],\ [1,3,4,5,4,3,1,6],\ [1,3,4,5,4,3],$$
$$[1,3,4,5,6,5,4,2,3,1,4,5,6,7,6,5,4,2],\ [1,3,4,5,6,5,4,3,1,7,8,7],\ [1,3,4,5,6,5,4,3,1,7],$$
$$[1,3,4,5,6,5,4,3],\ [1,3,4,5,6,7,6,5,4,3,1,8],\ [1,4,2,3,1,4,5,4,6,7,8,7,6,5,4,2,3,1,4,3,5,4],$$
$$[1,4,2,3,4,5,6,5,7,6,5,4,2,3,1,4,5,6],\ [1,4,2,3,4,5,6,7,6,5,4,2,3,1,4,5],\ [1,4,2,3,4,5,6,7,6,5,4,2,3,1,4,8],$$
$$[1,4,2,3,4,5,6,7,6,5,4,2,3,4],\ [1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5],\ [1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,4],$$
$$[1,5,4,2,3,4,5,6,5,4,2,3,1,4,5,7,8,7],\ [1,5,4,2,3,4,5,6,5,4,2,3,1,4,5,7],\ [1,5,4,2,3,4,5,6,5,4,2,3,4,5],$$
$$[1,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,5,6],\ [1,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,5,8],$$
$$[2,3,1,4,2,3,1,4,5,6,7,6,5,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7],\ [2,3,1,4,5,6,5,4,2,3,1,4,5,6,7,6,5,4,3,1],$$
$$[2,3,1,4,5,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1],\ [2,3,1,4,5,6,7,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1],$$
$$[2,3,4,3,1,5,6,7,6,5,4,2,3,1,4,3,5,6,7,8,7,6,5,4,2,3,1,4],\ [2,3,4,5,4,6,5,4,2,3,4,5],$$
$$[2,3,4,5,6,5,4,2,3,4],\ [2,3,4,5,6,5,4,2,3,7,8,7],\ [2,3,4,5,6,5,4,2,3,7],$$
$$[2,3,4,5,6,5,4,2],\ [2,3,4,5,6,7,6,5,4,2,3,4],\ [2,3,4,5,6,7,6,5,4,2,3,8],\ [2,3,4,5,6,7,6,5,4,2],$$
$$[2,3,4,5,6,7,8,7,6,5,4,2,3,4],\ [2,4,2,3,1,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7],$$
$$[2,4,2,3,4,5,4,3],\ [2,4,2,3,4,5,6,5,4,3],\ [2,4,2,5,6,5],\ [2,4,2,5],\ [2,4,5,4,2,3,4,5,6,5,4,3],$$
$$[2,4,5,4,2,6,7,6],\ [2,4,5,4,2,6],\ [2,4,5,6,5,4,2,7],\ [3,1,4,2,3,4,5,6,5,7,8,7,6,5,4,2,3,1,4,3,5,6],$$
$$[3,1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5],\ [3,1,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4],$$
$$[3,1,5,4,2,3,4,5,6,7,6,8,7,6,5,4,2,3,1,4,3,5,6,7],\ [3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4],$$
$$[3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6],\ [3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5],$$
$$[3,1,6,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,3,5,6,8],\ [3,1,6,5,4,2,3,4,5,6,7,6,5,4,2,3,1,4,5,6],$$
$$[3,1,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7],\ [3,4,2,3,4,5,4,2],\ [3,4,2,3,4,5,6,5,4,2],$$
$$[3,4,2,3,4,5,6,7,6,5,4,2],\ [3,4,3,1,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,5],\ [3,4,3,5,6,5],$$
$$[3,4,3,5],\ [3,4,5,4,2,3,4,5,6,5,4,2],\ [3,4,5,4,3,6],\ [3,4,5,4],\ [3,4],$$
$$[4,2,3,1,4,5,4,2,3,1,4,3,5,6,7,8,7,6,5,4,3,1],\ [4,2,3,4,5,4,2,3,4,6,7,6],\ [4,2,3,4,5,4,2,3,4,6],$$
$$[4,2,3,4,5,6,5,4,2,3,4,5],\ [4,2,3,4,5,6,5,4,2,3,4,7,8,7],\ [4,2,3,4,5,6,5,4,2,3,4,7],$$
$$[4,2,3,4,5,6,7,6,5,4,2,3,4,8],\ [4,5],\ [5,6],\ [6,7],\ [7,8]\big\},$$

one obtains 75 solutions for w_i in (6.233) with $\ell_T(w_i)=3$ and w_i of type A_1^3:

$$w_i \in \big\{[1,2,3,1,4,2,5,4,6,5,4,2,7,8,7,6,5,4,2,3,1],\ [1,2,3,1,4,3,5,4,3,6,5,4,3,7,6,5,4,3,1,8,7,6,5,4,2,3,1],$$
$$[1,2,3,1,4,5,4,6,5,4,2,3,4,5,7,6,5,4,2,3,1,4,5,8,7,6,5,4,2,3,1],\ [1,2,3,1,4,5,4,6,7,8,7,6,5,4,2,3,1],$$
$$[1,2,3,4,2,3,1,6,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7],$$
$$[1,2,3,4,2,3,5,4,6,5,4,7,6,5,4,2,3,8,7,6,5,4,2,3,1],$$
$$[1,2,3,4,2,5,4,2,3,4,6,5,4,3,7,6,5,4,2,3,4,8,7,6,5,4,2,3,1],$$
$$[1,2,3,4,3,1,5,6,7,6,5,4,2,3,1,4,3,5,6,8,7,6,5,4,2,3,1],$$
$$[1,2,3,4,3,5,4,3,6,5,4,3,7,6,5,4,2,3,1],\ [1,2,3,4,3,5,4,6,5,4,7,6,5,4,3,8,7,6,5,4,2,3,1],$$
$$[1,2,3,4,5,4,6,5,4,2,3,4,5,7,6,5,4,2,3,4,5,8,7,6,5,4,2,3,1],\ [1,2,3,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3,1],$$
$$[1,2,3,4,5,6,5,4,3,1,7,6,5,4,2,3,1,4,3,5,4,6,5,8,7,6,5,4,2,3,1],$$
$$[1,2,4,2,3,1,4,3,5,4,3,1,6,5,7,6,5,4,3,1,8,7,6,5,4,2,3,1,4],$$
$$[1,2,4,2,3,1,5,6,5,7,6,5,4,2,3,1,4,3,5,4,2,6,7],\ [1,2,4,2,3,1,6,7,6,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8],$$
$$[1,2,4,2,5,4,2,3,4,5,6,5,4,3,7,6,5,4,2,3,1,4,5],\ [1,2,4,2,5,6,5,4,2],\ [1,2,4,5,4,2,8],$$
$$[1,3,1,4,2,3,4,5,4,6,5,4,7,6,5,4,2,8,7,6,5,4,2,3,1,4,3],$$
$$[1,3,1,4,2,3,5,4,2,3,1,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1,7,8],$$
$$[1,3,1,4,2,3,7,6,5,4,2,3,1,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1,7,6,8,7],$$
$$[1,3,1,4,2,5,4,2,3,1,6,7,6,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1,7,8],$$

$[1,3,1,4,3,1,5,6,7,6,8,7,6,5,4,3,1]$, $[1,3,1,4,5,4,6,5,4,2,3,4,5,6,7,6,5,4,2,8,7,6,5,4,2,3,1,4,3,5,6]$,
$[1,3,1,4,5,4,6,7,6,5,4,3,1]$, $[1,3,4,3,5,4,3,1,7]$, $[1,3,4,3,5,4,3,6,5,4,3,1,8]$,
$[1,3,4,5,4,2,3,1,4,5,6,5,4,2,7,8,7,6,5,4,2,3,1,4,5]$, $[1,4,2,3,1,4,5,6,5,7,6,5,8,7,6,5,4,2,3,1,4]$,
$[1,4,2,3,4,5,4,2,3,4,6,7,8,7,6,5,4,2,3,1,4]$, $[1,4,2,3,4,5,6,7,6,5,4,2,3,4,5,6,8,7,6,5,4,2,3,1,4]$,
$[1,4,5,4,2,3,1,4,5,6,5,4,2,3,1,4,5]$, $[1,4,5,4,2,3,1,4,5,6,7,8,7,6,5,4,2,3,1,4,5]$,
$[1,4,5,4,2,3,4,5,6,5,4,2,3,1,4,5,8]$, $[1,4,5,4,2,3,4,5,6,7,6,8,7,6,5,4,2,3,1,4,5]$,
$[1,4,5,4,6,5,4,2,3,1,4,5,6]$, $[1,4,8]$, $[1,5,4,2,3,1,4,5,7]$,
$[1,5,4,2,3,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3,1,4,5]$, $[1,6,7,6,5,4,2,3,1,4,5,6,7]$,
$[2,3,1,4,2,5,6,5,4,2,3,4,5,6,7,6,5,4,3,8,7,6,5,4,2,3,1,4,3,5,6]$,
$[2,3,1,5,6,5,4,2,3,1,4,3,5,6,7,6,5,4,2,3,1,4,3,5,6]$,
$[2,3,4,2,3,1,4,6,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7]$, $[2,3,4,2,3,5,4,2,3,6,7,6,8,7,6,5,4,2,3]$,
$[2,3,4,2,3,5,4,3,6,5,4,2,3,7,8,7,6,5,4,2,3]$, $[2,3,4,2,3,5,4,6,5,4,7,6,5,4,2,3,8,7,6,5,4,2,3]$,
$[2,3,4,2,3,5,6,7,6,5,4,2,3]$, $[2,3,4,3,5,6,5,7,8,7,6,5,4,2,3]$, $[2,3,4,5,4,6,5,4,2,3,8]$,
$[2,3,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3]$, $[2,4,2,3,1,4,5,4,3,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8]$,
$[2,4,2,3,1,4,5,6,7,6,5,4,3,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8]$, $[2,4,2,3,4,5,4,3,6,5,4,2,3,4,8]$,
$[2,4,2,3,4,5,4,3,6,7,6,8,7,6,5,4,2,3,4]$, $[3,1,4,2,3,4,5,6,7,6,5,4,2,3,4,5,6,8,7,6,5,4,2,3,1,4,3]$,
$[3,1,4,5,4,2,3,4,5,6,7,6,8,7,6,5,4,2,3,1,4,3,5]$, $[3,1,4,5,4,6,5,4,2,3,1,4,3,5,6]$,
$[3,1,4,5,4,8,7,6,5,4,2,3,1,4,3,5,6,7,8]$, $[3,1,4,6,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7]$,
$[3,1,5,6,5,7,6,5,4,2,3,1,4,3,5,6,7]$, $[3,1,6,5,4,2,3,1,4,3,5,6,8]$,
$[3,1,6,5,4,2,3,4,5,6,7,6,5,4,2,3,4,5,6,8,7,6,5,4,2,3,1,4,3,5,6]$, $[3,1,7,8,7,6,5,4,2,3,1,4,3,5,6,7,8]$,
$[3,4,2,3,1,5,4,2,3,1,4,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,7,8]$, $[3,4,3,5,4,3,7,8,7]$,
$[3,4,5,4,6,5,4,3,8]$, $[4,2,3,1,5,4,2,3,1,4,3,5,6,7,6,8,7,6,5,4,2,3,1,4,3,5,4]$,
$[4,2,3,4,5,4,2,3,4,6,5,4,2,3,4]$, $[4,2,3,4,5,4,2,3,4,6,7,8,7,6,5,4,2,3,4]$, $[4,2,3,4,6]$,
$[4,5,4,2,3,4,5,6,5,4,2,3,4,5,8]$, $[5,4,2,3,4,5,7,8,7]$, $[5,4,3,1,6,5,4,2,3,1,4,3,5,4,6,5,8]$, $[5,6,5,4,2,3,4,5,6]\}$,

one obtains 165 solutions for w_i in (6.233) with $\ell_T(w_i)=3$ and w_i of type $A_1 * A_2$:

$w_i \in \{[1,3,1,4,5,4,2,3,4,5,6,5,4,2,7,8,7,6,5,4,2,3,1,4,5]$,
$[1,2,3,1,4,2,5,4,2,6,5,4,2,7,8,7,6,5,4,2,3,1,4,3,5]$,
$[1,2,3,1,4,2,5,4,6,5,4,2,3,4,5,7,6,5,4,3,1,8,7,6,5,4,2,3,1]$,
$[1,2,3,1,4,3,5,4,6,5,4,3,7,6,5,4,3,8,7,6,5,4,2,3,1,4,3,5,6]$,
$[1,2,3,1,4,3,5,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1,4,3]$, $[1,2,3,1,4,3,5,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1]$,
$[1,2,3,1,4,5,4,6,5,4,7,6,5,4,2,3,1,4,3,5,6]$, $[1,2,3,1,4,5,4,6,5,4,7,8,7,6,5,4,2,3,1,4,3,5,6]$,
$[1,2,3,1,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3,4,5,8,7,6,5,4,2,3,1]$, $[1,2,3,1,4,5,6,5,7,8,7,6,5,4,2,3,1,4,3]$,
$[1,2,3,4,2,3,5,4,6,5,4,7,6,5,4,2,3,4,5,8,7,6,5,4,2,3,1]$, $[1,2,3,4,2,3,5,4,6,5,7,6,5,4,2,3,4,8,7,6,5,4,2,3,1]$,
$[1,2,3,4,2,5,4,2,3,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1]$, $[1,2,3,4,2,5,4,2,6,5,4,2,7,6,5,4,2,3,1,4,5]$,
$[1,2,3,4,2,5,4,2,6,5,4,2,7,8,7,6,5,4,2,3,1,4,5]$, $[1,2,3,4,2,5,4,2,6,5,7,8,7,6,5,4,2,3,1]$,
$[1,2,3,4,2,5,4,2,6,7,8,7,6,5,4,2,3,1,4]$, $[1,2,3,4,2,5,4,6,5,4,2,3,4,5,7,6,5,4,3,8,7,6,5,4,2,3,1]$,
$[1,2,3,4,2,5,4,6,5,4,2,7,6,5,4,2,3,1,8]$, $[1,2,3,4,3,1,5,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7]$,
$[1,2,3,4,3,5,4,3,6,5,4,3,7,6,5,4,2,3,1,4,5]$, $[1,2,3,4,3,5,4,6,5,4,3,7,6,5,4,2,3,1,4]$,
$[1,2,3,4,3,5,4,6,5,4,3,7,6,5,4,2,3]$, $[1,2,3,4,3,5,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1,4]$,
$[1,2,3,4,5,4,2,3,4,6,5,7,6,5,4,2,3,4,5,8,7,6,5,4,2,3,1]$,
$[1,2,3,4,5,4,2,3,4,6,5,7,6,5,4,2,3,4,8,7,6,5,4,2,3,1,4]$,
$[1,2,3,4,5,4,6,5,7,8,7,6,5,4,2,3,1]$, $[1,2,3,4,5,4,6,7,6,5,4,2,3,1,8]$,
$[1,2,3,4,5,6,5,4,3,1,7,6,5,4,2,3,1,4,3,5,6]$, $[1,2,3,4,5,6,5,7,8,7,6,5,4,2,3]$,
$[1,2,4,2,3,1,4,3,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,7,8]$,
$[1,2,4,2,3,1,4,5,6,5,7,6,5,4,2,3,1,4,3,5,6,7,8,7,6,5,4,2,3]$,
$[1,2,4,2,3,1,6,7,6,5,4,2,3,1,4,3,5,4,2,6,5,7,6]$, $[1,2,4,2,3,4,5,4,2,3,4,6,5,7,6,5,4,2,8,7,6,5,4,2,3,1,4]$,

[1, 2, 4, 2, 3, 4, 5, 4, 3, 6, 5, 7, 6, 5, 4, 2, 3, 1, 4], [1, 2, 4, 2, 3, 4, 5, 4, 3, 6, 5, 7, 6, 5, 8, 7, 6, 5, 4, 2, 3, 1, 4],
[1, 2, 4, 2, 3, 4, 5, 4, 3, 6, 7, 6, 5, 4, 2, 3, 1, 4, 8], [1, 2, 4, 2, 3, 4, 5, 4, 3, 6, 7, 6, 5, 4, 2, 3, 4],
[1, 2, 4, 2, 3, 4, 5, 4, 3, 6, 7, 8, 7, 6, 5, 4, 2, 3, 4], [1, 2, 4, 2, 3, 4, 5, 4, 6, 5, 4, 3, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[1, 2, 4, 2, 3, 4, 5, 4, 6, 5, 4, 3, 7, 6, 5, 4, 2, 3, 4], [1, 2, 4, 2, 3, 4, 5, 6, 5, 7, 6, 5, 4, 3, 1], [1, 2, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 3, 1, 8],
[1, 2, 4, 5, 4, 2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 3, 1], [1, 2, 4, 5, 4, 2, 6], [1, 2, 4, 5, 4, 6, 5, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 1],
[1, 3, 1, 4, 2, 3, 4, 5, 4, 6, 5, 4, 7, 6, 5, 4, 2, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
[1, 3, 1, 4, 2, 3, 4, 5, 4, 6, 5, 7, 6, 5, 4, 2, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5],
[1, 3, 1, 4, 2, 3, 4, 5, 4, 6, 5, 7, 6, 5, 4, 2, 8, 7, 6, 5, 4, 2, 3, 1, 4],
[1, 3, 1, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 7],
[1, 3, 1, 4, 2, 5, 4, 2, 3, 1, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 1, 7, 6, 8, 7], [1, 3, 1, 4, 3, 1, 5, 4, 6, 7, 8, 7, 6, 5, 4, 3, 1],
[1, 3, 1, 4, 3, 5, 4, 3, 1, 6, 5, 4, 3, 1, 7, 8, 7], [1, 3, 1, 4, 3, 5, 4, 3, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3],
[1, 3, 1, 4, 3, 5, 4, 3, 6, 5, 4, 3, 1], [1, 3, 1, 4, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3],
[1, 3, 1, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5],
[1, 3, 1, 4, 5, 4, 2, 3, 4, 5, 6, 5, 7, 6, 5, 4, 2, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
[1, 3, 1, 4, 5, 4, 2, 3, 4, 5, 6, 5, 7, 6, 5, 4, 2, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5], [1, 3, 1, 4, 5, 6, 5, 4, 3, 1, 7],
[1, 3, 1, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 6, 5, 7, 6, 5, 4, 2, 3], [1, 3, 4, 2, 3, 4, 5, 4, 2, 3, 4, 6, 5, 7, 6, 5, 4, 3, 8, 7, 6, 5, 4, 2, 3, 1, 4],
[1, 3, 4, 2, 3, 4, 5, 4, 2, 6, 5, 7, 6, 5, 8, 7, 6, 5, 4, 2, 3, 1, 4], [1, 3, 4, 2, 3, 4, 5, 4, 6, 5, 7, 6, 5, 4, 2, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[1, 3, 4, 2, 3, 5, 4, 2, 3, 1, 4, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 7, 8],
[1, 3, 4, 2, 3, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 7, 8],
[1, 3, 4, 3, 1, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 4], [1, 3, 4, 3, 5, 4, 2, 3, 1, 4, 5, 6, 5, 4, 2],
[1, 3, 4, 3, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 5, 4, 2], [1, 3, 4, 3, 5, 4, 3], [1, 3, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 7, 6, 5, 4, 2, 3, 1, 4, 5, 8],
[1, 3, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5], [1, 3, 4, 5, 4, 6, 5, 7, 6, 5, 4, 3, 1],
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[1, 4, 2, 3, 1, 4, 3, 5, 4, 3, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4], [1, 4, 2, 3, 1, 4, 3, 5, 4, 6, 5, 7, 6, 5, 4, 3, 8, 7, 6, 5, 4, 2, 3, 1, 4],
[1, 4, 2, 3, 1, 4, 3, 5, 6, 7, 8, 7, 6, 5, 4, 3, 1], [1, 4, 2, 3, 4, 5, 4, 3, 1, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4],
[1, 4, 2, 3, 4, 5, 6, 5, 7, 6, 5, 4, 2, 3, 4], [1, 4, 3, 1, 5, 4, 2, 3, 1, 4, 3, 5, 4, 6, 7, 8, 7, 6, 5, 4, 2, 3, 4],
[1, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 7, 8, 7], [1, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 7],
[1, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 5], [1, 4, 5, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5, 8],
[1, 5, 4, 2, 3, 1, 4, 3, 5, 4, 6, 5, 4, 3, 1], [1, 5, 4, 2, 3, 1, 4, 3, 5, 4, 6, 7, 8, 7, 6, 5, 4, 3, 1], [1, 5, 4, 2, 3, 1, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 7, 8, 7],
[1, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 5, 8], [1, 5, 4, 2, 3, 4, 5], [1, 6, 5, 4, 2, 3, 4, 5, 6],
[2, 3, 1, 4, 2, 3, 1, 4, 5, 6, 5, 7, 6, 5, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 7],
[2, 3, 1, 4, 2, 3, 4, 5, 4, 6, 5, 4, 7, 6, 5, 4, 3, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
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[2, 3, 1, 4, 2, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 1],
[2, 3, 1, 4, 3, 1, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4],
[2, 3, 1, 4, 5, 4, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 5, 4, 3, 1], [2, 3, 1, 4, 5, 4, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 1],
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[2, 3, 4, 2, 3, 5, 4, 2, 3, 6, 5, 7, 8, 7, 6, 5, 4, 2, 3], [2, 3, 4, 2, 3, 5, 4, 2, 6, 7, 8, 7, 6, 5, 4, 2, 3],
[2, 3, 4, 2, 3, 5, 4, 6, 5, 4, 2, 3, 7, 6, 5, 4, 2, 3, 8], [2, 3, 4, 2, 3, 5, 4, 6, 5, 4, 2, 3, 7, 6, 8, 7, 6, 5, 4, 2, 3],
[2, 3, 4, 2, 3, 5, 4, 6, 5, 4, 2, 7, 6, 5, 4, 2, 3], [2, 3, 4, 2, 3, 5, 4, 6, 5, 4, 2, 7, 8, 7, 6, 5, 4, 2, 3], [2, 3, 4, 2, 3, 5, 4, 6, 5, 4, 3, 7, 6, 5, 4, 2, 3],
[2, 3, 4, 2, 5, 4, 2, 6, 5, 4, 2, 3, 4], [2, 3, 4, 2, 5, 4, 2, 6, 5, 4, 2, 3, 7, 8, 7], [2, 3, 4, 2, 5, 4, 2, 6, 5, 7, 8, 7, 6, 5, 4, 2, 3],
[2, 3, 4, 2, 5, 4, 2, 6, 7, 8, 7, 6, 5, 4, 2, 3, 4], [2, 3, 4, 2, 5, 4, 6, 5, 4, 2, 3], [2, 3, 4, 2, 5, 4, 6, 5, 4, 2, 7, 6, 5, 4, 2, 3, 8],
[2, 3, 4, 2, 5, 4, 6, 5, 4, 7, 6, 5, 4, 2, 3], [2, 3, 4, 2, 5, 6, 5, 4, 2, 3, 7], [2, 3, 4, 3, 5, 4, 6, 5, 4, 3, 7, 6, 5, 4, 2, 3, 4],
[2, 3, 4, 3, 5, 4, 6, 5, 4, 7, 6, 5, 4, 2, 3], [2, 3, 4, 3, 5, 6, 7, 6, 5, 4, 2, 3, 8], [2, 3, 4, 5, 4, 6, 5, 4, 2, 3, 4, 5, 8],
[2, 3, 4, 5, 4, 6, 5, 4, 2], [2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 8], [2, 3, 4, 5, 6, 5, 7, 6, 8, 7, 6, 5, 4, 2, 3],
[2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 4], [2, 4, 2, 3, 1, 4, 7, 6, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7],
[2, 4, 2, 3, 1, 6, 7, 6, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7], [2, 4, 2, 3, 1, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7],

[2, 4, 2, 3, 1, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 8], [2, 4, 2, 3, 4, 5, 4, 3, 6, 5, 4, 2, 3, 4, 7, 8, 7], [2, 4, 2, 3, 4, 5, 4, 3, 6, 5, 4, 2, 3, 4, 7],
[2, 4, 2, 3, 4, 5, 4, 3, 6, 5, 7, 6, 5, 4, 2, 3, 4], [2, 4, 2, 3, 4, 5, 4, 3, 6, 7, 6, 5, 4, 2, 3, 4, 8], [2, 4, 2, 3, 4, 5, 6, 5, 4, 3, 8],
[2, 4, 2, 5, 4, 6, 5, 4, 2], [2, 4, 2, 5, 6, 5, 4, 2, 7], [2, 4, 2, 5, 8], [2, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 3, 8],
[2, 4, 5, 4, 2, 7, 8], [2, 4, 5, 6, 5, 4, 2, 3, 1, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7], [3, 1, 4, 5, 4, 2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7],
[3, 1, 4, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6], [3, 1, 4, 5, 6, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
[3, 1, 4, 6, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7], [3, 1, 5, 4, 2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4],
[3, 1, 5, 4, 2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5], [3, 1, 5, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 6, 5],
[3, 1, 5, 6, 5, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 8], [3, 1, 5, 6, 5, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5, 6],
[3, 1, 6, 5, 4, 2, 3, 1, 4, 5, 6], [3, 1, 7, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7],
[3, 4, 2, 3, 4, 5, 4, 2, 3, 1, 4, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 8], [3, 4, 2, 3, 4, 5, 4, 2, 6, 5, 4, 2, 3, 4, 7, 8, 7],
[3, 4, 2, 3, 4, 5, 4, 2, 7], [3, 4, 2, 3, 4, 5, 6, 5, 7, 6, 5, 4, 2], [3, 4, 3, 1, 5, 4, 2, 3, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[3, 4, 3, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2], [3, 4, 3, 6, 7], [3, 4, 5, 4, 3, 1, 6, 5, 4, 2, 3, 1, 4, 5, 6], [3, 4, 7],
[4, 2, 3, 1, 4, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 6, 8, 7, 6, 5, 4, 3, 1], [4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 5, 8], [4, 5, 8], [4, 7, 8],
[1, 3, 1, 4, 5, 6, 5, 7, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3]},

one obtains 90 solutions for w_i in (6.233) with $\ell_T(w_i) = 3$ and w_i of type A_3:

$w_i \in \{$[1, 2, 3, 1, 4, 3, 1, 5, 6, 7, 6, 5, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 7],
[1, 2, 3, 1, 4, 3, 5, 4, 6, 5, 7, 6, 5, 4, 3, 1, 8, 7, 6, 5, 4, 2, 3, 1, 4], [1, 2, 3, 1, 4, 5, 4, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4],
[1, 2, 3, 1, 4, 5, 4, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6], [1, 2, 3, 1, 4, 5, 4, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[1, 2, 3, 1, 4, 5, 6, 5, 4, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 8], [1, 2, 3, 1, 4, 5, 6, 5, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
[1, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5], [1, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4],
[1, 2, 3, 4, 2, 3, 5, 4, 6, 5, 7, 6, 5, 4, 2, 3, 8, 7, 6, 5, 4, 2, 3], [1, 2, 3, 4, 2, 5, 4, 2, 6, 7, 8, 7, 6, 5, 4, 2, 3],
[1, 2, 3, 4, 2, 5, 4, 6, 5, 4, 2, 7, 6, 5, 4, 2, 3], [1, 2, 3, 4, 2, 5, 4, 6, 5, 4, 2, 7, 8, 7, 6, 5, 4, 2, 3], [1, 2, 3, 4, 5, 4, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5, 8],
[1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5], [1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 8], [1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 4],
[1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 8], [1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 4], [1, 2, 4, 2, 3, 1, 4, 3, 5, 4, 3, 1, 6, 7, 8, 7, 6, 5, 4, 3, 1],
[1, 2, 4, 2, 3, 1, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3], [1, 2, 4, 2, 3, 1, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 7, 8],
[1, 2, 4, 2, 3, 4, 5, 4, 6, 5, 4, 7, 6, 5, 4, 3, 1], [1, 2, 4, 2, 3, 4, 5, 4, 6, 7, 6, 5, 4, 3, 1], [1, 2, 4, 2, 3, 4, 5, 4, 6, 7, 8, 7, 6, 5, 4, 3, 1],
[1, 2, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 1, 8], [1, 2, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 3, 1, 7, 8, 7], [1, 2, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 3, 1, 7],
[1, 2, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 3], [1, 2, 4, 5, 4, 2, 3, 4, 5, 6, 5, 7, 6, 5, 4, 3, 1], [1, 2, 4, 5, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 1, 8],
[1, 3, 1, 4, 5, 4, 2, 3, 1, 4, 5, 6, 5, 7, 6, 5, 8, 7, 6, 5, 4, 2, 3], [1, 3, 1, 4, 5, 4, 2, 3, 1, 4, 5, 6, 5, 7, 8, 7, 6, 5, 4, 2, 3],
[1, 3, 1, 4, 5, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 4], [1, 3, 1, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 5, 4, 2, 3, 8],
[1, 3, 1, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 5, 4, 2], [1, 3, 1, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 2, 3],
[1, 3, 4, 2, 3, 4, 5, 4, 2, 6, 7, 8, 7, 6, 5, 4, 2, 3, 4], [1, 3, 4, 5, 4, 2, 3, 1, 4, 5, 6, 5, 4, 2, 7],
[1, 3, 4, 5, 4, 2, 3, 1, 4, 5, 6, 5, 7, 6, 5, 4, 2], [1, 3, 4, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2],
[1, 3, 4, 5, 4, 3, 1, 6, 7], [1, 3, 4, 5, 4, 3, 6], [1, 3, 4, 5, 6, 5, 4, 3, 1, 7, 8], [1, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[1, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5, 6], [1, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5, 8], [1, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 4, 8],
[1, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6], [1, 5, 4, 2, 3, 1, 4, 5, 6], [1, 5, 4, 2, 3, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 7, 8], [1, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7],
[2, 3, 1, 4, 2, 3, 4, 5, 4, 6, 5, 7, 6, 5, 4, 3, 8, 7, 6, 5, 4, 2, 3, 1, 4], [2, 3, 1, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 5, 4, 3, 1, 8],
[2, 3, 1, 4, 5, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 6, 8, 7, 6, 5, 4, 3, 1], [2, 3, 1, 4, 5, 6, 5, 4, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 1], [2, 3, 4, 2, 5, 4, 2, 6, 5, 4, 2],
[2, 3, 4, 2, 5, 6, 5, 4, 2], [2, 3, 4, 2, 5, 6, 7, 6, 5, 4, 2], [2, 3, 4, 3, 1, 5, 6, 7, 6, 5, 4, 2, 3, 1, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 1],
[2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 5], [2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 7, 8, 7], [2, 3, 4, 5, 6, 5, 4, 2, 3, 4, 7],
[2, 3, 4, 5, 6, 5, 4, 2, 3, 7, 8], [2, 3, 4, 5, 6, 5, 4, 2, 7], [2, 3, 4, 5, 6, 7, 6, 5, 4, 2, 3, 4, 8], [2, 4, 2, 3, 4, 5, 4, 3, 6],
[2, 4, 2, 3, 4, 5, 4, 6, 5, 4, 3], [2, 4, 2, 5, 6], [2, 4, 5, 4, 2, 6, 7], [3, 1, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6],
[3, 1, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5], [3, 1, 4, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 5],
[3, 1, 5, 4, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7], [3, 1, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 8, 7],
[3, 1, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7], [3, 1, 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 6, 7, 8],

$$[3,4,2,3,1,4,5,4,2,3,4,5,6,7,8,7,6,5,4,3,1],\ [3,4,2,3,4,5,4,2,6,7,6],\ [3,4,2,3,4,5,4,2,6],\ [3,4,2,3,4,5,4,6,5,4,2],\\ [3,4,2,3,4,5,6,5,4,2,7],\ [3,4,3,5,6],\ [3,4,5],\ [4,2,3,4,5,4,2,3,4,6,7],\\ [4,2,3,4,5,6,5,4,2,3,4,7,8],\ [4,2,3,4,5,6,5],\ [4,2,3,4,5],\ [5,4,2,3,4,5,6]\big\},$$

one obtains 15 solutions for w_i in (6.233) with $\ell_T(w_i)=4$ and w_i of type $A_1^2*A_2$:

$$\begin{gathered} w_i\in\big\{[1,2,3,1,4,5,4,6,5,4,2,3,4,5,7,6,5,4,2,3,4,5,8,7,6,5,4,2,3,1],\\ [1,2,3,4,3,1,5,6,5,7,6,5,4,2,3,1,4,3,5,4,6,5,8,7,6,5,4,2,3,1],\\ [1,2,3,4,5,4,6,5,4,2,3,4,5,7,6,5,4,2,3,1,4,5],\\ [1,2,3,4,5,4,6,5,4,3,1,7,6,5,4,2,3,1,4,3,5,6],\\ [1,3,1,4,2,3,4,5,6,5,7,6,5,4,2,3,1,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7],\\ [1,3,1,4,2,3,5,4,2,3,1,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1,7,6,8,7],\\ [1,4,3,1,5,4,2,3,1,4,3,5,4,6,7,6,8,7,6,5,4,2,3,4],\\ [1,4,5,4,2,3,1,4,5,6,5,4,2,3,1,4,5,7,8,7],\\ [1,4,5,4,2,3,4,5,6,5,4,2,3,4,5,8],\\ [2,4,2,3,1,4,5,4,3,6,7,6,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8],\\ [2,4,2,3,1,4,6,7,6,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7],\\ [3,1,4,2,3,4,5,6,5,7,6,5,4,2,3,4,5,6,8,7,6,5,4,2,3,1,4,3,5,6],\\ [3,1,5,6,5,4,2,3,1,4,3,5,4,6,5,8],\\ [3,4,2,3,4,5,4,2,3,1,4,7,8,7,6,5,4,2,3,1,4,3,5,6,7,8],\\ [4,2,3,4,5,4,2,3,4,6,5,4,2,3,4,7,8,7]\big\}, \end{gathered}\tag{6.237}$$

one obtains 45 solutions for w_i in (6.233) with $\ell_T(w_i)=4$ and w_i of type A_1*A_3:

$$\begin{gathered} w_i\in\big\{[1,2,3,1,4,2,5,4,2,6,5,4,2,7,8,7,6,5,4,2,3,1,4,5],\\ [1,2,3,1,4,2,5,4,6,5,4,2,3,4,5,7,6,5,4,3,8,7,6,5,4,2,3,1],\\ [1,2,3,1,4,3,1,5,6,5,7,6,5,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7],\\ [1,2,3,1,4,3,5,4,6,5,4,7,6,5,4,3,8,7,6,5,4,2,3,1,4,3,5,6],\\ [1,2,3,1,4,5,4,6,5,4,7,6,5,4,2,3,1,4,3,5,6,8],\ [1,2,3,4,2,3,5,4,6,5,4,7,6,5,4,2,3,8,7,6,5,4,2,3],\\ [1,2,3,4,2,3,5,4,6,5,7,6,5,4,2,3,4,5,8,7,6,5,4,2,3,1],\ [1,2,3,4,2,5,4,2,3,4,6,5,4,7,6,5,4,3,8,7,6,5,4,2,3,1],\\ [1,2,3,4,2,5,4,2,6,5,4,2,7,6,5,4,2,3,1,4,5,8],\ [1,2,3,4,3,5,4,6,5,4,3,7,6,5,4,2,3,1,4,5],\\ [1,2,3,4,3,5,4,6,5,4,3,7,6,5,4,2,3,4],\ [1,2,3,4,5,6,5,4,2,3,4,5,7,6,5,4,2,3],\\ [1,2,4,2,3,1,4,5,6,5,7,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,2,3],\ [1,2,4,2,3,4,5,4,3,6,7,6,5,4,2,3,4,8],\\ [1,2,4,5,4,2,3,4,5,6,5,4,3,8],\ [1,3,1,4,2,3,4,5,4,6,5,7,6,5,4,2,8,7,6,5,4,2,3,1,4,3,5,6],\\ [1,3,1,4,2,3,4,5,4,6,5,7,6,5,4,2,8,7,6,5,4,2,3,1,4,5],\ [1,3,1,4,5,4,2,3,1,4,5,6,7,6,8,7,6,5,4,2,3,4],\\ [1,3,4,2,3,5,4,2,3,1,4,6,7,6,8,7,6,5,4,2,3,1,4,3,5,4,2,6,7,8],\ [1,3,4,3,5,4,2,3,1,4,5,6,5,4,2,7],\\ [1,3,4,3,5,4,2,3,4,5,6,5,4,2],\ [1,4,2,3,1,4,3,5,4,3,6,5,7,6,5,8,7,6,5,4,2,3,1,4],\\ [1,4,2,3,4,5,4,2,3,4,6,5,7,6,5,8,7,6,5,4,2,3,1,4],\ [1,4,2,3,4,5,4,2,3,4,6,7,8,7,6,5,4,2,3,4],\\ [1,4,5,4,2,3,4,5,6,5,4,2,3,1,4,5,7,8],\ [1,5,4,2,3,1,4,3,5,4,6,5,4,3,1,7,8,7],\\ [2,3,1,4,2,5,6,5,4,2,3,1,4,5,6,7,6,8,7,6,5,4,3,1],\ [2,3,1,4,5,4,6,5,4,2,3,1,4,5,6,7,6,5,4,3,1,8],\\ [2,3,1,4,5,4,6,5,4,2,3,4,5,6,7,6,5,4,3,1],\ [2,3,4,2,3,1,5,6,7,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1],\\ [2,3,4,2,3,5,4,2,3,6,5,7,6,8,7,6,5,4,2,3],\ [2,3,4,2,3,5,4,2,6,5,7,8,7,6,5,4,2,3],\\ [2,3,4,2,3,5,4,6,5,4,7,6,5,4,2,3],\ [2,3,4,2,5,4,2,6,5,4,2,3,4,7,8,7],\\ [2,3,4,5,6,5,4,2,3,4,5,8],\ [2,4,2,3,1,4,5,4,8,7,6,5,4,2,3,1,4,3,5,6,7,8],\\ [2,4,2,3,1,5,6,5,7,6,5,4,2,3,1,4,3,5,6,7],\ [2,4,2,3,4,5,4,3,6,5,4,2,3,4,7,8],\ [2,4,2,3,4,5,4,6,5,4,3,8],\\ [3,1,4,2,3,4,5,6,7,6,5,4,2,3,4,5,6,8,7,6,5,4,2,3,1,4],\ [3,1,4,5,4,2,3,4,5,6,7,6,8,7,6,5,4,2,3,1,4,5], \end{gathered}$$

$$[3,1,4,5,4,2,3,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6,7],\ [3,1,4,5,4,6,5,4,2,3,1,4,3,5,6,7,8,7],\\ [3,1,4,5,4,6,5,4,2,3,1,4,5,6],\ [3,4,2,3,1,4,5,4,2,3,4,5,6,7,6,8,7,6,5,4,3,1]\big\}, \tag{6.238}$$

one obtains 5 solutions for w_i in (6.233) with $\ell_T(w_i) = 4$ and w_i of type A_2^2:

$$w_i \in \big\{[1,2,3,4,2,5,4,2,3,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1,4],\\ [1,2,4,2,3,4,5,4,3,6,5,7,6,5,4,2,3,4],\\ [1,3,1,4,5,4,2,3,4,5,6,5,4,2,7,6,8,7,6,5,4,2,3,1,4,5],\\ [2,3,1,4,2,5,4,6,5,4,2,3,1,4,5,6,7,8,7,6,5,4,3,1],\\ [2,3,4,2,3,5,4,6,5,4,2,7,6,5,4,2,3,8]\big\}, \tag{6.239}$$

one obtains 18 solutions for w_i in (6.233) with $\ell_T(w_i) = 4$ and w_i of type A_4:

$$w_i \in \big\{[1,2,3,1,4,3,5,4,6,5,7,6,5,4,3,8,7,6,5,4,2,3,1,4],\ [1,2,3,1,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,6],\\ [1,2,3,4,2,5,4,2,6,5,7,8,7,6,5,4,2,3],\ [1,2,3,4,2,5,4,2,6,7,8,7,6,5,4,2,3,4],\\ [1,2,3,4,2,5,4,6,5,4,2,7,6,5,4,2,3,8],\ [1,2,3,4,5,6,7,6,5,4,2,3,1,4,5,8],\ [1,2,4,2,3,4,5,4,6,5,7,6,5,4,3,1],\\ [1,2,4,2,3,4,5,4,6,7,6,5,4,3,1,8],\ [1,2,4,5,4,2,3,4,5,6,5,4,3,1,7,8],\\ [1,3,1,4,5,4,2,3,1,4,5,6,5,7,6,8,7,6,5,4,2,3],\ [1,4,2,3,1,4,3,5,4,6,7,8,7,6,5,4,3,1],\\ [1,5,4,2,3,4,5,6],\ [2,3,4,2,5,4,6,5,4,2],\ [2,3,4,2,5,6,5,4,2,7],\ [2,3,4,5,6,5,4,2,3,4,7,8],\\ [2,4,2,3,1,7,6,5,4,2,3,1,4,3,5,6,7,8],\ [3,1,6,5,4,2,3,1,4,5,6,7],\ [3,4,2,3,4,5,4,2,6,7]\big\}, \tag{6.240}$$

and one obtains 5 solutions for w_i in (6.233) with $\ell_T(w_i) = 4$ and w_i of type D_4:

$$w_i \in \big\{[1,2,3,1,4,5,6,7,8,7,6,5,4,2,3,1,4,5],\ [1,2,3,4,5,6,7,6,5,4,2,3,4,8],\\ [1,5,4,2,3,1,4,5,6,7],\ [3,1,6,5,4,2,3,1,4,3,5,6,7,8],\ [4,2,3,4,5,6]\big\}, \tag{6.241}$$

where $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ is a simple system of generators of E_8, corresponding to the Dynkin diagram displayed in Figure 6, and each of them gives rise to $m/2$ elements of $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$ since i ranges from 1 to $m/2$. There are no solutions for w_i in (6.233) with w_i of type A_1^4.

Letting the computer find all solutions in cases (iii)–(v) would take years. However, the number of these solutions can be computed from our knowledge of the solutions in Case (ii) according to type, if this information is combined with the decomposition numbers in the sense of [17, 18, 20] (see the end of Section 2) and some elementary (multiset) permutation counting. The decomposition numbers for A_2, A_3, A_4, and D_4 of which we make use can be found in the appendix of [18].

To begin with, the number of solutions of (6.234) with $\ell_1 = \ell_2 = 1$ is equal to

$$n_{1,1} := 150 \cdot 2 + 100 \cdot N_{A_2}(A_1, A_1) = 600,$$

since an element of type A_1^2 can be decomposed in two ways into a product of two elements of absolute length 1, while for an element of type A_2 this can be done in $N_{A_2}(A_1, A_1) = 3$ ways. Similarly, the number of solutions of (6.234) with $\ell_1 = 2$ and $\ell_2 = 1$ is equal to

$$n_{2,1} := 75 \cdot 3 + 165 \cdot (1 + N_{A_2}(A_1, A_1)) + 90 \cdot N_{A_3}(A_2, A_1) = 1425,$$

the number of solutions of (6.234) with $\ell_1 = 3$ and $\ell_2 = 1$ is equal to

$$n_{3,1} := 15 \cdot (2 + N_{A_2}(A_1, A_1)) + 45 \cdot (1 + N_{A_3}(A_2, A_1)) + 5 \cdot (2N_{A_2}(A_1, A_1))\\ + 18 \cdot (N_{A_4}(A_3, A_1) + N_{A_4}(A_1 * A_2, A_1)) + 5 \cdot (N_{D_4}(A_3, A_1) + N_{D_4}(A_1^3, A_1)) = 660,$$

the number of solutions of (6.234) with $\ell_1 = \ell_2 = 2$ is equal to

$$\begin{aligned} n_{2,2} := 15 \cdot (2 + 2N_{A_2}(A_1, A_1)) + 45 \cdot (2N_{A_3}(A_2, A_1)) + 5 \cdot (2 + N_{A_2}(A_1, A_1)^2) \\ + 18 \cdot (N_{A_4}(A_2, A_2) + N_{A_4}(A_1^2, A_1^2) + 2N_{A_4}(A_2, A_1^2)) \\ + 5 \cdot (N_{D_4}(A_2, A_2) + 2N_{D_4}(A_2, A_1^2)) = 1195, \end{aligned}$$

the number of solutions of (6.235) with $\ell_1 = \ell_2 = \ell_3 = 1$ is equal to

$$n_{1,1,1} := 75 \cdot 3! + 165 \cdot (3N_{A_2}(A_1, A_1)) + 90N_{A_3}(A_1, A_1, A_1) = 3375,$$

the number of solutions of (6.235) with $\ell_1 = 2$ and $\ell_2 = \ell_3 = 1$ is equal to

$$\begin{aligned} n_{2,1,1} := 15\cdot(2+N_{A_2}(A_1, A_1)+2\cdot 2\cdot N_{A_2}(A_1, A_1))+45\cdot(2N_{A_3}(A_2, A_1)+N_{A_3}(A_1, A_1, A_1)) \\ + 5 \cdot (2N_{A_2}(A_1, A_1) + 2N_{A_2}(A_1, A_1)^2) + 18 \cdot (N_{A_4}(A_2, A_1, A_1) + N_{A_4}(A_1^2, A_1, A_1)) \\ + 5 \cdot (N_{D_4}(A_2, A_1, A_1) + N_{D_4}(A_1^2, A_1, A_1)) = 2850, \end{aligned}$$

and the number of solutions of (6.236) is equal to

$$\begin{aligned} n_{1,1,1,1} := 15 \cdot (12N_{A_2}(A_1, A_1)) + 45 \cdot (4N_{A_3}(A_1, A_1, A_1)) + 5 \cdot (6N_{A_2}(A_1, A_1)^2) \\ + 18 \cdot N_{A_4}(A_1, A_1, A_1, A_1) + 5 \cdot N_{D_4}(A_1, A_1, A_1, A_1) = 6750. \end{aligned}$$

In total, we obtain

$$\begin{aligned} 1+(45+150+100+75+165+90+15+45+5+18+5)\frac{m}{2}+(n_{1,1}+2n_{2,1}+2n_{3,1}+n_{2,2})\binom{m/2}{2} \\ + (n_{1,1,1} + 3n_{2,1,1})\binom{m/2}{3} + n_{1,1,1,1}\binom{m/2}{4} = \frac{(5m+4)(3m+2)(5m+2)(15m+4)}{64} \end{aligned}$$

elements in $\mathrm{Fix}_{NC^m(E_8)}(\phi^p)$, which agrees with the limit in (6.198j).

Finally, we turn to (6.198l). By Remark 3, the only choices for h_2 and m_2 to be considered are $h_2 = 1$ and $m_2 = 7$, $h_2 = 2$ and $m_2 = 8$, $h_2 = 2$ and $m_2 = 7$, $h_2 = 2$ and $m_2 = 4$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 30m/7$, $p = 15m/8$, $p = 15m/7$, $p = 15m/4$, respectively $15m/2$, out of which only $p = 15m/8$ has not yet been discussed and belongs to the current case. The corresponding action of ϕ^p is given by

$$\begin{aligned} \phi^p\big((w_0; w_1, \dots, w_m)\big) \\ = (*; c^2 w_{\frac{m}{8}+1} c^{-2}, c^2 w_{\frac{m}{8}+2} c^{-2}, \dots, c^2 w_m c^{-2}, c w_1 c^{-1}, \dots, c w_{\frac{m}{8}} c^{-1}\big), \end{aligned}$$

so that we have to solve

$$t_1(c^{13}t_1c^{-13})(c^{11}t_1c^{-11})(c^9t_1c^{-9})(c^7t_1c^{-7})(c^5t_1c^{-5})(c^3t_1c^{-3})(ct_1c^{-1}) = c \tag{6.242}$$

for t_1 with $\ell_T(t_1) = 1$. A computation with the help of Stembridge's *Maple* package `coxeter` [35] finds no solution. Hence, the left-hand side of (3.3) is equal to 1, as required.

7. Cyclic sieving II

In this section we present the second cyclic sieving conjecture due to Bessis and Reiner [10, Conj. 6.5].

Let $\psi : NC^m(W) \to NC^m(W)$ be the map defined by

$$(w_0; w_1, \dots, w_m) \mapsto \big(c w_m c^{-1}; w_0, w_1, \dots, w_{m-1}\big). \tag{7.1}$$

For $m = 1$, we have $w_0 = c w_1^{-1}$, so that this action reduces to the inverse of the *Kreweras complement* K_{id}^c as defined by Armstrong [3, Def. 2.5.3].

It is easy to see that $\psi^{(m+1)h}$ acts as the identity, where h is the Coxeter number of W (see (8.1) below). By slight abuse of notation as before, let C_2 be the cyclic group of order $(m+1)h$ generated by ψ.

Given these definitions, we are now in the position to state the second cyclic sieving conjecture of Bessis and Reiner. By the results of [19] and of this paper, it becomes the following theorem.

Theorem 33. *For an irreducible well-generated complex reflection group W and any $m \ge 1$, the triple $(NC^m(W), \mathrm{Cat}^m(W;q), C_2)$, where $\mathrm{Cat}^m(W;q)$ is the q-analogue of the Fuß–Catalan number defined in* (3.2)*, exhibits the cyclic sieving phenomenon.*

By definition of the cyclic sieving phenomenon, we have to prove that

$$|\operatorname{Fix}_{NC^m(W)}(\psi^p)| = \mathrm{Cat}^m(W;q)\big|_{q=e^{2\pi i p/(m+1)h}}, \tag{7.2}$$

for all p in the range $0 \le p < (m+1)h$.

8. Auxiliary results II

This section collects several auxiliary results which allow us to reduce the problem of proving Theorem 33, respectively the equivalent statement (7.2), for the 26 exceptional groups listed in Section 2 to a finite problem. While Lemmas 35 and 36 cover special choices of the parameters, Lemmas 34 and 37 afford an inductive procedure. More precisely, if we assume that we have already verified Theorem 33 for all groups of smaller rank, then Lemmas 34 and 37, together with Lemmas 35 and 38, reduce the verification of Theorem 33 for the group that we are currently considering to a finite problem; see Remark 4. The final lemma of this section, Lemma 39, disposes of complex reflection groups with a special property satisfied by their degrees.

Let $p = a(m+1) + b$, $0 \le b < m+1$. We have

$$\begin{multline}\psi^p\big((w_0; w_1, \dots, w_m)\big)\\ = (c^{a+1} w_{m-b+1} c^{-a-1}; c^{a+1} w_{m-b+2} c^{-a-1}, \dots, c^{a+1} w_m c^{-a-1},\\ c^a w_0 c^{-a}, \dots, c^a w_{m-b} c^{-a}\big).\end{multline} \tag{8.1}$$

Lemma 34. *It suffices to check* (7.2) *for p a divisor of $(m+1)h$. More precisely, let p be a divisor of $(m+1)h$, and let k be another positive integer with $\gcd(k, (m+1)h/p) = 1$, then we have*

$$\mathrm{Cat}^m(W;q)\big|_{q=e^{2\pi i p/(m+1)h}} = \mathrm{Cat}^m(W;q)\big|_{q=e^{2\pi i k p/(m+1)h}} \tag{8.2}$$

and

$$|\operatorname{Fix}_{NC^m(W)}(\psi^p)| = |\operatorname{Fix}_{NC^m(W)}(\psi^{kp})|. \tag{8.3}$$

Proof. For (8.3), this follows in the same way as (5.3) in Lemma 26.

For (8.2), we must argue differently than in Lemma 26. Let us write $\zeta = e^{2\pi i p/(m+1)h}$. For a given group W, we write $S_1(W)$ for the set of all indices i such that $\zeta^{d_i-h} = 1$, and we write $S_2(W)$ for the set of all indices i such that $\zeta^{d_i} = 1$. By the rule of de l'Hospital, we have

$$\operatorname{Cat}^m(W;q)\big|_{q=e^{2\pi i p/(m+1)h}} = \begin{cases} 0 & \text{if } |S_1(W)| > |S_2(W)|, \\ \frac{\prod_{i\in S_1(W)}(mh+d_i)}{\prod_{i\in S_2(W)} d_i} \frac{\prod_{i\notin S_1(W)}(1-\zeta^{d_i-h})}{\prod_{i\notin S_2(W)}(1-\zeta^{d_i})}, & \text{if } |S_1(W)| = |S_2(W)|. \end{cases} \tag{8.4}$$

Since, by Theorem 25, $\operatorname{Cat}^m(W;q)$ is a polynomial in q, the case $|S_1(W)| < |S_2(W)|$ cannot occur.

We claim that, for the case where $|S_1(W)| = |S_2(W)|$, the factors in the quotient of products

$$\frac{\prod_{i\notin S_1(W)}(1-\zeta^{d_i-h})}{\prod_{i\notin S_2(W)}(1-\zeta^{d_i})}$$

cancel pairwise. If we assume the correctness of the claim, it is obvious that we get the same result if we replace ζ by ζ^k, where $\gcd(k,(m+1)h/p) = 1$, hence establishing (8.2).

In order to see that our claim is indeed valid, we proceed in a case-by-case fashion, making appeal to the classification of irreducible well-generated complex reflection groups, which we recalled in Section 2. First of all, since $d_n = h$, the set $S_1(W)$ is always non-empty as it contains the element n. Hence, if we want to have $|S_1(W)| = |S_2(W)|$, the set $S_2(W)$ must be non-empty as well. In other words, the integer $(m+1)h/p$ must divide at least one of the degrees $d_1, d_2, \dots, d_n$. In particular, this implies that, for each fixed reflection group W of exceptional type, only a finite number of values of $(m+1)h/p$ has to be checked. Writing M for $(m+1)h/p$, what needs to be checked is whether the *multi*sets (that is, multiplicities of elements must be taken into account)

$$\{(d_i - h) \bmod M : i \notin S_1(W)\} \quad \text{and} \quad \{d_i \bmod M : i \notin S_2(W)\}$$

are the same. Since, for a fixed irreducible well-generated complex reflection group, there is only a finite number of possibilities for M, this amounts to a routine verification. □

Lemma 35. *Let p be a divisor of $(m+1)h$. If p is divisible by $m+1$, then* (7.2) *is true.*

Proof. According to (8.1), the action of ψ^p on $NC^m(W)$ is described by

$$\psi^p\big((w_0; w_1, \dots, w_m)\big) = \big(c^{p/(m+1)} w_0 c^{-p/(m+1)}; c^{p/(m+1)} w_1 c^{-p/(m+1)}, \dots, c^{p/(m+1)} w_m c^{-p/(m+1)}\big).$$

Hence, if $(w_0; w_1, \dots, w_m)$ is fixed by ψ^p, then each individual w_i must be fixed under conjugation by $c^{p/(m+1)}$.

Using the notation $W' = \operatorname{Cent}_W(c^{p/(m+1)})$, the previous observation means that $w_i \in W'$, $i = 1, 2, \dots, m$. By the theorem of Springer cited in the proof of Lemma 27 and by (5.5), the tuples $(w_0; w_1, \dots, w_m)$ fixed by ψ^p are in fact identical with the elements of

$NC^m(W')$, which implies that

$$|\operatorname{Fix}_{NC^m(W)}(\psi^p)| = |NC^m(W')|. \tag{8.5}$$

Application of Theorem 1 with W replaced by W' and of the "limit rule" (5.4) then yields that

$$|NC^m(W')| = \prod_{\substack{1\le i\le n\\ \frac{(m+1)h}{p} \mid d_i}} \frac{mh+d_i}{d_i} = \operatorname{Cat}^m(W;q)\big|_{q=e^{2\pi i p/(m+1)h}}. \tag{8.6}$$

Combining (8.5) and (8.6), we obtain (7.2). This finishes the proof of the lemma. $\square$

Lemma 36. *Equation* (7.2) *holds for all divisors p of $m+1$.*

Proof. We have

$$\operatorname{Cat}^m(W;q)\big|_{q=e^{2\pi i p/(m+1)h}} = \begin{cases} 0 & \text{if } p < m+1, \\ m+1 & \text{if } p = m+1. \end{cases}$$

Here, the first case follows from (8.4) and the fact that we have $S_1(W) \supseteq \{n\}$ and $S_2(W) = \emptyset$ if $p \mid (m+1)$ and $p < m+1$.

On the other hand, if $(w_0; w_1, \dots, w_m)$ is fixed by ψ^p, then, because of the action (8.1), we must have $w_0 = w_p = \cdots = w_{m-p+1}$ and $w_0 = c w_{m-p+1} c^{-1}$. In particular, $w_0 \in \operatorname{Cent}_W(c)$. By the theorem of Springer cited in the proof of Lemma 27, the subgroup $\operatorname{Cent}_W(c)$ is itself a complex reflection group whose degrees are those degrees of W that are divisible by h. The only such degree is h itself, hence $\operatorname{Cent}_W(c)$ is the cyclic group generated by c. Moreover, by (5.5), we obtain that $w_0 = \varepsilon$ or $w_0 = c$. If $p = m+1$, the set $\operatorname{Fix}_{NC^m(W)}(\psi^p)$ consists of the $m+1$ elements $(w_0; w_1, \dots, w_m)$ obtained by choosing $w_i = c$ for a particular i between 0 and m, all other w_j's being equal to ε. If $p < m+1$, then there is no element in $\operatorname{Fix}_{NC^m(W)}(\psi^p)$. $\square$

Lemma 37. *Let W be an irreducible well-generated complex reflection group of rank n, and let $p = m_1 h_1$ be a divisor of $(m+1)h$, where $m+1 = m_1 m_2$ and $h = h_1 h_2$. We assume that* $\gcd(h_1, m_2) = 1$. *Suppose that Theorem 33 has already been verified for all irreducible well-generated complex reflection groups with rank $< n$. If h_2 does not divide all degrees d_i, then equation* (7.2) *is satisfied.*

Proof. Let us write $h_1 = am_2 + b$, with $0 \le b < m_2$. The condition $\gcd(h_1, m_2) = 1$ translates into $\gcd(b, m_2) = 1$. From (8.1), we infer that

$$\begin{multline}\psi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(c^{a+1} w_{m-m_1 b+1} c^{-a-1}; c^{a+1} w_{m-m_1 b+2} c^{-a-1}, \dots, c^{a+1} w_m c^{-a-1}, \\ c^a w_0 c^{-a}, \dots, c^a w_{m-m_1 b} c^{-a}\big). \tag{8.7}\end{multline}$$

Supposing that $(w_0; w_1, \dots, w_m)$ is fixed by ψ^p, we obtain the system of equations

$$\begin{aligned} w_i &= c^{a+1} w_{i+m-m_1 b+1} c^{-a-1}, \quad i = 0, 1, \dots, m_1 b - 1, \\ w_i &= c^a w_{i-m_1 b} c^{-a}, \quad i = m_1 b, m_1 b+1, \dots, m, \end{aligned}$$

which, after iteration, implies in particular that

$$w_i = c^{b(a+1)+(m_2-b)a} w_i c^{-b(a+1)-(m_2-b)a} = c^{h_1} w_i c^{-h_1}, \quad i = 0, 1, \dots, m.$$

It is at this point where we need $\gcd(b, m_2) = 1$. The last equation shows that each w_i, $i = 0, 1, \ldots, m$, lies in $\mathrm{Cent}_W(c^{h_1})$. By the theorem of Springer cited in the proof of Lemma 27, this centraliser subgroup is itself a complex reflection group, W' say, whose degrees are those degrees of W that are divisible by $h/h_1 = h_2$. Since, by assumption, h_2 does not divide *all* degrees, W' has rank strictly less than n. Again by assumption, we know that Theorem 33 is true for W', so that in particular,

$$|\operatorname{Fix}_{NC^m(W')}(\psi^p)| = \mathrm{Cat}^m(W'; q)\big|_{q=e^{2\pi i p/(m+1)h}}.$$

The arguments above together with (5.5) show that $\operatorname{Fix}_{NC^m(W)}(\psi^p) = \operatorname{Fix}_{NC^m(W')}(\psi^p)$. On the other hand, it is straightforward to see that

$$\mathrm{Cat}^m(W; q)\big|_{q=e^{2\pi i p/(m+1)h}} = \mathrm{Cat}^m(W'; q)\big|_{q=e^{2\pi i p/(m+1)h}}.$$

This proves (7.2) for our particular p, as required. $\square$

Lemma 38. *Let W be an irreducible well-generated complex reflection group of rank n, and let $p = m_1h_1$ be a divisor of $(m+1)h$, where $m+1 = m_1m_2$ and $h = h_1h_2$. We assume that* $\gcd(h_1, m_2) = 1$. *If $m_2 > n$ then*

$$\operatorname{Fix}_{NC^m(W)}(\psi^p) = \emptyset.$$

Proof. Let us suppose that $(w_0; w_1, \ldots, w_m) \in \operatorname{Fix}_{NC^m(W)}(\psi^p)$ and that there exists a $j \ge 1$ such that $w_j \ne \varepsilon$. By (8.7), it then follows for such a j that also $w_k \ne \varepsilon$ for all $k \equiv j - lm_1b \pmod{m+1}$, where, as before, b is defined as the unique integer with $h_1 = am_2 + b$ and $0 \le b < m_2$. Since, by assumption, $\gcd(b, m_2) = 1$, there are exactly m_2 such k's which are distinct mod $m+1$. However, this implies that the sum of the absolute lengths of the w_i's, $0 \le i \le m$, is at least $m_2 > n$, a contradiction. This leaves as only possibility $(w_0; w_1, \ldots, w_m) = (c; \varepsilon, \ldots, \varepsilon)$. However, this is clearly not an element of $\operatorname{Fix}_{NC^m(W)}(\psi^p)$ unless p is divisible by $m+1$. This is impossible since

$$\frac{p}{m+1} = \frac{m_1h_1}{m_1m_2} = \frac{h_1}{m_2}$$

is not an integer by our hypotheses. $\square$

Remark 4. (1) If we put ourselves in the situation of the assumptions of Lemma 37, then we may conclude that equation (7.2) only needs to be checked for pairs (m_2, h_2) subject to the following restrictions:

$$m_2 \ge 2, \quad \gcd(h_1, m_2) = 1, \quad \text{and } h_2 \text{ divides all degrees of } W. \tag{8.8}$$

Indeed, Lemmas 35 and 37 together imply that equation (7.2) is always satisfied except if $m_2 \ge 2$, h_2 divides all degrees of W, and $\gcd(h_1, m_2) = 1$.

(2) Still putting ourselves in the situation of Lemma 37, if $m_2 > n$ and m_2h_2 does not divide any of the degrees of W, then equation (7.2) is satisfied. Indeed, Lemma 38 says that in this case the left-hand side of (7.2) equals 0, while it is obvious that in this case the right-hand side of (7.2) equals 0 as well.

(3) It should be observed that this leaves a finite number of choices for m_2 to consider, whence a finite number of choices for (m_1, m_2, h_1, h_2). Altogether, there remains a finite number of choices for $p = h_1m_1$ to be checked.

Lemma 39. *Let W be an irreducible well-generated complex reflection group of rank n with the property that $d_i \mid h$ for $i = 1, 2, \dots, n$. Then Theorem 33 is true for this group W.*

Proof. By Lemma 34, we may restrict ourselves to divisors p of $(m+1)h$.

Suppose that $e^{2\pi i p/(m+1)h}$ is a d_i-th root of unity for some i. In other words, $(m+1)h/p$ divides d_i. Since d_i is a divisor of h by assumption, the integer $(m+1)h/p$ also divides h. But this is equivalent to saying that $m+1$ divides p, and equation (7.2) holds by Lemma 35.

Now assume that $(m+1)h/p$ does not divide any of the d_i's. In this case, it follows from (8.4) and the fact that we have $S_1(W) \supseteq \{n\}$ and $S_2(W) = \emptyset$ that the right-hand side of (7.2) equals 0. Inspection of the classification of all irreducible well-generated complex reflection groups, which we recalled in Section 2, reveals that all groups satisfying the hypotheses of the lemma have rank $n \le 2$. Except for the groups contained in the infinite series $G(d, 1, n)$ and $G(e, e, n)$ for which Theorem 2 has been established in [19], these are the groups $G_5, G_6, G_9, G_{10}, G_{14}, G_{17}, G_{18}, G_{21}$. We now discuss these groups case by case, keeping the notation of Lemma 37. In order to simplify the argument, we note that Lemma 38 implies that equation (7.2) holds if $m_2 > 2$, so that in the following arguments we always may assume that $m_2 = 2$.

Case G_5. The degrees are $6, 12$, and therefore Remark 4.(1) implies that equation (7.2) is always satisfied.

Case G_6. The degrees are $4, 12$, and therefore, according to Remark 4.(1), we need only consider the case where $h_2 = 4$ and $m_2 = 2$, that is, $p = 3(m+1)/2$. Then (8.7) becomes

$$\psi^p\big((w_0; w_1, \dots, w_m)\big) = \big(c^2 w_{\frac{m+1}{2}} c^{-2}; c^2 w_{\frac{m+3}{2}} c^{-2}, \dots, c^2 w_m c^{-2}, c w_0 c^{-1}, \dots, c w_{\frac{m-1}{2}} c^{-1}\big). \tag{8.9}$$

If $(w_0; w_1, \dots, w_m)$ is fixed by ψ^p, there must exist an i with $0 \le i \le \frac{m-1}{2}$ such that $\ell_T(w_i) = 1$, $w_i c w_i c^{-1} = c$, and all w_j, $j \ne i, \frac{m+1}{2} + i$, equal ε. However, with the help of `CHEVIE`, one verifies that there is no such solution to this equation. Hence, the left-hand side of (7.2) is equal to 0, as required.

Case G_9. The degrees are $8, 24$, and therefore, according to Remark 4.(1), we need only consider the case where $h_2 = 8$ and $m_2 = 2$, that is, $p = 3(m+1)/2$. This is the same p as for G_6. Again, `CHEVIE` finds no solution. Hence, the left-hand side of (7.2) is equal to 0, as required.

Case G_{10}. The degrees are $12, 24$, and therefore, according to Remark 4.(1), we need only consider the case where $h_2 = 12$ and $m_2 = 2$, that is, $p = 3(m+1)/2$. This is the same p as for G_6. Again, `CHEVIE` finds no solution. Hence, the left-hand side of (7.2) is equal to 0, as required.

Case G_{14}. The degrees are $6, 24$, and therefore Remark 4.(1) implies that equation (7.2) is always satisfied.

Case G_{17}. The degrees are $20, 60$, and therefore, according to Remark 4.(1), we need only consider the cases where $h_2 = 20$ and $m_2 = 2$, respectively that $h_2 = 4$ and $m_2 = 2$. In the first case, $p = 3(m+1)/2$, which is the same p as for G_6. Again, `CHEVIE` finds no solution. In the second case, $p = 15(m+1)/2$. Then (8.7) becomes

$$\psi^p\big((w_0; w_1, \dots, w_m)\big)\\ = (c^8 w_{\frac{m+1}{2}} c^{-8}; c^8 w_{\frac{m+3}{2}} c^{-8}, \dots, c^8 w_m c^{-8}, c^7 w_0 c^{-7}, \dots, c^7 w_{\frac{m-1}{2}} c^{-7}\big). \quad (8.10)$$

By Lemma 29, every element of $NC(W)$ is fixed under conjugation by c^3, and, thus, on elements fixed by ψ^p, the above action of ψ^p reduces to the one in (8.9). This action was already discussed in the first case. Hence, in both cases, the left-hand side of (7.2) is equal to 0, as required.

Case G_{18}. The degrees are $30, 60$, and therefore Remark 4.(1) implies that equation (7.2) is always satisfied.

Case G_{21}. The degrees are $12, 60$, and therefore, according to Remark 4.(1), we need only consider the cases where $h_2 = 5$ and $m_2 = 2$, respectively that $h_2 = 15$ and $m_2 = 2$. In the first case, $p = 5(m+1)/2$, so that (8.7) becomes

$$\psi^p\big((w_0; w_1, \dots, w_m)\big)\\ = (c^3 w_{\frac{m+1}{2}} c^{-3}; c^3 w_{\frac{m+3}{2}} c^{-3}, \dots, c^3 w_m c^{-3}, c^2 w_0 c^{-2}, \dots, c^2 w_{\frac{m-1}{2}} c^{-2}\big). \quad (8.11)$$

If $(w_0; w_1, \dots, w_m)$ is fixed by ψ^p, there must exist an i with $0 \le i \le \frac{m-1}{2}$ such that $\ell_T(w_i) = 1$ and $w_i c^2 w_i c^{-2} = c$. However, with the help of `CHEVIE`, one verifies that there is no such solution to this equation. In the second case, $p = 15(m+1)/2$. Then (8.7) becomes the action in (8.10). By Lemma 29, every element of $NC(W)$ is fixed under conjugation by c^5, and, thus, on elements fixed by ψ^p, the action of ψ^p in (8.10) reduces to the one in the first case. Hence, in both cases, the left-hand side of (7.2) is equal to 0, as required.

This completes the proof of the lemma. □

9. Case-by-case verification of Theorem 33

We now perform a case-by-case verification of Theorem 33. It should be observed that the action of ψ (given in (7.1)) is exactly the same as the action of ϕ (given in (3.1)) with m replaced by $m+1$ *on the components* $w_1, w_2, \dots, w_{m+1}$, that is, if we disregard the 0-th component of the elements of the generalised non-crossing partitions involved. The only difference which arises is that, while the $(m+1)$-tuples $(w_0; w_1, \dots, w_m)$ in (7.1) must satisfy $w_0 w_1 \cdots w_m = c$, for $w_1, w_2, \dots, w_{m+1}$ in (3.1) we only must have $w_1 w_2 \cdots w_{m+1} \le_T c$. The condition for $(w_0; w_1, \dots, w_m)$ of being in $\mathrm{Fix}_{NC^m(W)}(\psi^p)$ is therefore exactly the same as the condition on $w_1, w_2, \dots, w_{m+1}$ for the element $(\varepsilon; w_1, \dots, w_m, w_{m+1})$ being in $\mathrm{Fix}_{NC^{m+1}(W)}(\phi^p)$. Consequently, we may use the counting results from Section 6, except that we have to restrict our attention to those elements $(w_0; w_1, \dots, w_m, w_{m+1}) \in NC^{m+1}(W)$ for which $w_1 w_2 \cdots w_{m+1} = c$, or, equivalently, $w_0 = \varepsilon$.

As before, we write ζ_d for a primitive d-th root of unity.

Case G_4. The degrees are $4, 6$, and hence we have

$$\mathrm{Cat}^m(G_4; q) = \frac{[6m+6]_q\,[6m+4]_q}{[6]_q\,[4]_q}.$$

Let ζ be a $6(m+1)$-th root of unity. As before, in what follows we abbreviate the assertion that "ζ is a primitive d-th root of unity" as "$\zeta=\zeta_d$." The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_4;q) = m+1, \quad \text{if } \zeta=\zeta_6,\zeta_3, \tag{9.1a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_4;q) = \tfrac{3m+3}{2}, \quad \text{if } \zeta=\zeta_4,\ 2\mid(m+1), \tag{9.1b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_4;q) = \operatorname{Cat}^m(G_4), \quad \text{if } \zeta=-1 \text{ or } \zeta=1, \tag{9.1c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_4;q) = 0, \quad \text{otherwise.} \tag{9.1d}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.1). The only cases not covered by Lemmas 35 and 36 are the ones in (9.1b) and (9.1d). On the other hand, the only case left to consider according to Remark 4 is the case where $h_2=m_2=2$, that is the case (9.1b) where $p=3(m+1)/2$. In particular, $m+1$ must be divisible by 2. The action of ψ^p is the same as the one in (8.9). Hence, the counting problem is the same as there, except that the underlying group now is G_4. With the help of `CHEVIE`, one finds that each of the 3 (complex) reflections in G_4 which are less than the (chosen) Coxeter element is a valid choice for w_i, and each of these choices gives rise to $(m+1)/2$ elements in $\operatorname{Fix}_{NC^m(G_4)}(\psi^p)$ since the index i ranges from 0 to $(m-1)/2$.

Hence, in total, we obtain $3\frac{m+1}{2}=\frac{3m+3}{2}$ elements in $\operatorname{Fix}_{NC^m(G_4)}(\psi^p)$, which agrees with the limit in (9.1b).

Case G_8. The degrees are $8,12$, and hence we have

$$\operatorname{Cat}^m(G_8;q)=\frac{[12m+12]_q\,[12m+8]_q}{[12]_q\,[8]_q}.$$

Let ζ be a $12(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_8;q) = m+1, \quad \text{if } \zeta=\zeta_{12},\zeta_6,\zeta_3, \tag{9.2a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_8;q) = \tfrac{3m+3}{2}, \quad \text{if } \zeta=\zeta_8,\ 2\mid(m+1), \tag{9.2b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_8;q) = \operatorname{Cat}^m(G_8), \quad \text{if } \zeta=\zeta_4,-1,1, \tag{9.2c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_8;q) = 0, \quad \text{otherwise.} \tag{9.2d}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.2). The only cases not covered by Lemmas 35 and 36 are the ones in (9.2b) and (9.2d). On the other hand, the only case left to consider according to Remark 4 is the case where $h_2=4$ and $m_2=2$, that is the case (9.2b) where $p=3(m+1)/2$. In particular, $m+1$ must be divisible by 2. The action of ψ^p is the same as the one in (8.9). Hence, the counting problem is the same as there, except that the underlying group now is G_8. With the help of `CHEVIE`, one finds that each of the 3 (complex) reflections in G_8 which are less than the (chosen) Coxeter element is a valid choice for w_i, and each of these choices gives rise to $(m+1)/2$ elements in $\operatorname{Fix}_{NC^m(G_8)}(\psi^p)$ since the index i ranges from 0 to $(m-1)/2$.

Hence, in total, we obtain $3\frac{m+1}{2} = \frac{3m+3}{2}$ elements in $\operatorname{Fix}_{NC^m(G_8)}(\psi^p)$, which agrees with the limit in (9.2b).

Case G_{16}. The degrees are $20, 30$, and hence we have

$$\operatorname{Cat}^m(G_{16}; q) = \frac{[30m + 30]_q \, [30m + 20]_q}{[30]_q \, [20]_q}.$$

Let ζ be a $30(m + 1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{16}; q) = m + 1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \zeta_6, \zeta_3, \tag{9.3a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{16}; q) = \tfrac{3m+3}{2}, \quad \text{if } \zeta = \zeta_{20}, \zeta_4,\ 2 \mid (m + 1), \tag{9.3b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{16}; q) = \operatorname{Cat}^m(G_{16}), \quad \text{if } \zeta = \zeta_{10}, \zeta_5, -1, 1, \tag{9.3c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{16}; q) = 0, \quad \text{otherwise.} \tag{9.3d}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.3). The only cases not covered by Lemmas 35 and 36 are the ones in (9.3b) and (9.3d). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2 = 10$ and $m_2 = 2$, respectively $h_2 = m_2 = 2$. Both cases belong to (9.3b). In the first case, we have $p = 3(m + 1)/2$, while in the second case we have $p = 15(m + 1)/2$. In particular, $m + 1$ must be divisible by 2. In the first case, the action of ψ^p is the same as the one in (8.9). Hence, the counting problem is the same as there, except that the underlying group now is G_{16}. With the help of `CHEVIE`, one finds that each of the 3 (complex) reflections in G_{16} which are less than the (chosen) Coxeter element is a valid choice for w_i, and each of these choices gives rise to $(m + 1)/2$ elements in $\operatorname{Fix}_{NC^m(G_{16})}(\psi^p)$ since the index i ranges from 0 to $(m - 1)/2$. On the other hand, if $p = 15(m + 1)/2$, then the action of ψ^p is the same as the one in (8.10). By Lemma 29, every element of $NC(G_{16})$ is fixed under conjugation by c^3, and, thus, on elements fixed by ψ^p, the action of ψ^p reduces to the one in the first case.

Hence, in total, we obtain $3\frac{m+1}{2} = \frac{3m+3}{2}$ elements in $\operatorname{Fix}_{NC^m(G_{16})}(\psi^p)$, which agrees with the limit in (9.3b).

Case G_{20}. The degrees are $12, 30$, and hence we have

$$\operatorname{Cat}^m(G_{20}; q) = \frac{[30m + 30]_q \, [30m + 12]_q}{[30]_q \, [12]_q}.$$

Let ζ be a $30(m + 1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{20}; q) = m + 1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \zeta_{10}, \zeta_5, \tag{9.4a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{20}; q) = \tfrac{5m+5}{2}, \quad \text{if } \zeta = \zeta_{12}, \zeta_4,\ 2 \mid (m + 1), \tag{9.4b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{20}; q) = \operatorname{Cat}^m(G_{20}), \quad \text{if } \zeta = \zeta_6, \zeta_3, -1, 1, \tag{9.4c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{20}; q) = 0, \quad \text{otherwise.} \tag{9.4d}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.4). The only cases not covered by Lemmas 35 and 36 are the ones in (9.4b) and (9.4d). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2 = 6$ and $m_2 = 2$, respectively $h_2 = m_2 = 2$. Both cases belong to (9.4b). In the first case, we have $p = 5(m+1)/2$, while in the second case we have $p = 15(m+1)/2$. In particular, $m+1$ must be divisible by 2. In the first case, the action of ψ^p is the same as the one in (8.11). Hence, the counting problem is the same as there, except that the underlying group now is G_{20}. With the help of `CHEVIE`, one finds that each of the 5 (complex) reflections in G_{20} which are less than the (chosen) Coxeter element is a valid choice for w_i, and each of these choices gives rise to $(m+1)/2$ elements in $\mathrm{Fix}_{NC^m(G_{20})}(\psi^p)$ since the index i ranges from 0 to $(m-1)/2$. On the other hand, if $p = 15(m+1)/2$, then the action of ψ^p is the same as the one in (8.10). By Lemma 29, every element of $NC(G_{20})$ is fixed under conjugation by c^5, and, thus, on elements fixed by ψ^p, the action of ψ^p reduces to the one in the first case.

Hence, in total, we obtain $5\frac{m+1}{2} = \frac{5m+5}{2}$ elements in $\mathrm{Fix}_{NC^m(G_{20})}(\psi^p)$, which agrees with the limit in (9.4b).

Case $G_{23} = H_3$. The degrees are $2, 6, 10$, and hence we have

$$\mathrm{Cat}^m(H_3; q) = \frac{[10m+10]_q\,[10m+6]_q\,[10m+2]_q}{[10]_q\,[6]_q\,[2]_q}.$$

Let ζ be a $10(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_3; q) = m+1, \quad \text{if } \zeta = \zeta_{10}, \zeta_5, \tag{9.5a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_3; q) = \tfrac{5m+5}{3}, \quad \text{if } \zeta = \zeta_6, \zeta_3,\ 3 \mid (m+1), \tag{9.5b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_3; q) = \mathrm{Cat}^m(H_3), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{9.5c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(H_3; q) = 0, \quad \text{otherwise.} \tag{9.5d}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.5). The only cases not covered by Lemmas 35 and 36 are the ones in (9.5b) and (9.5d). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2 = 1$ and $m_2 = 3$, $h_2 = 2$ and $m_2 = 3$, and $h_2 = m_2 = 2$. These correspond to the choices $p = 10(m+1)/3$, $p = 5(m+1)/3$, respectively $p = 5(m+1)/2$. The first two cases belong to (9.5b), while $p = 5(m+1)/2$ belongs to (9.5d).

In the case that $p = 5(m+1)/3$, the action of ψ^p is given by

$$\psi^p\big((w_0; w_1, \dots, w_m)\big) = \big(c^2 w_{\frac{m+1}{3}} c^{-2}; c^2 w_{\frac{m+4}{3}} c^{-2}, \dots, c^2 w_m c^{-2}, c w_0 c^{-1}, \dots, c w_{\frac{m-2}{3}} c^{-1}\big).$$

Hence, for an i with $0 \le i \le \frac{m-2}{3}$, we must find an element $w_i = t_1$, where t_1 satisfies (6.9), and all other w_j, $j \notin \big\{i, i+\frac{m+1}{3}, i+\frac{2(m+1)}{3}\big\}$, are set equal to ε. We have found five solutions to the counting problem (6.9) in (6.10). Each of them gives rise to $(m+1)/3$ elements in $\mathrm{Fix}_{NC^m(H_3)}(\psi^p)$ since the index i ranges from 0 to $(m-2)/3$. On the other hand, if $p = 10(m+1)/3$, then the action of ψ^p is given by

$$\psi^p\big((w_0; w_1, \dots, w_m)\big) \\ = \big(c^4 w_{\frac{2m+2}{3}} c^{-4}; c^4 w_{\frac{2m+5}{3}} c^{-4}, \dots, c^4 w_m c^{-4}, c^3 w_0 c^{-3}, \dots, c^3 w_{\frac{2m-1}{3}} c^{-3}\big).$$

By Lemma 29, every element of $NC(H_3)$ is fixed under conjugation by c^5, and, thus, on elements fixed by ψ^p, the action of ψ^p reduces to the one in the first case.

Hence, in total, we obtain $5\frac{m+1}{3} = \frac{5m+5}{3}$ elements in $\mathrm{Fix}_{NC^m(H_3)}(\psi^p)$, which agrees with the limit in (9.5b).

If $p = 5(m+1)/2$, then the action of ψ^p is the same as the one in (8.11). The computation at the end of Case H_3 in Section 6 did not find any solutions, which is in agreement with (9.5d).

Case G_{24}. The degrees are $4, 6, 14$, and hence we have

$$\mathrm{Cat}^m(G_{24}; q) = \frac{[14m+14]_q\,[14m+6]_q\,[14m+4]_q}{[14]_q\,[6]_q\,[4]_q}.$$

Let ζ be a $14(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{24}; q) = m+1, \quad \text{if } \zeta = \zeta_{14}, \zeta_7, \tag{9.6a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{24}; q) = \tfrac{7m+7}{3}, \quad \text{if } \zeta = \zeta_6, \zeta_3,\ 3 \mid (m+1), \tag{9.6b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{24}; q) = \mathrm{Cat}^m(G_{24}), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{9.6c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{24}; q) = 0, \quad \text{otherwise}. \tag{9.6d}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.6). The only cases not covered by Lemmas 35 and 36 are the ones in (9.6b) and (9.6d). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2 = 1$ and $m_2 = 3$, $h_2 = 2$ and $m_2 = 3$, and $h_2 = m_2 = 2$. These correspond to the choices $p = 14(m+1)/3$, $p = 7(m+1)/3$, respectively $p = 7(m+1)/2$. The first two cases belong to (9.6b), while $p = 7(m+1)/2$ belongs to (9.6d).

In the case that $p = 14(m+1)/3$ or $p = 7(m+1)/3$, we have found seven solutions to the counting problem (6.18) in (6.19), and each of them gives rise to $(m+1)/3$ elements in $\mathrm{Fix}_{NC^m(G_{24})}(\psi^p)$ (in the style as discussed in Case H_3). Hence, in total, we obtain $7\frac{m+1}{3} = \frac{7m+7}{3}$ elements in $\mathrm{Fix}_{NC^m(G_{24})}(\psi^p)$, which agrees with the limit in (9.6b).

If $p = 7(m+1)/2$, the relevant counting problem is (6.25). However, no element $(w_0; w_1, \dots, w_m) \in \mathrm{Fix}_{NC^m(G_{24})}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m)$ be even, which contradicts the fact that $\ell_T(c) = n = 3$. This is in agreement with the limit in (9.6d).

Case G_{25}. The degrees are $6, 9, 12$, and hence we have

$$\mathrm{Cat}^m(G_{25}; q) = \frac{[12m+12]_q\,[12m+9]_q\,[12m+6]_q}{[12]_q\,[9]_q\,[6]_q}.$$

Let ζ be a $12(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{25};q) = m+1, \quad \text{if } \zeta=\zeta_{12},\zeta_4, \tag{9.7a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{25};q) = \tfrac{4m+4}{3}, \quad \text{if } \zeta=\zeta_9,\ 3\mid(m+1), \tag{9.7b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{25};q) = (m+1)(2m+1), \quad \text{if } \zeta=\zeta_6,-1 \tag{9.7c}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{25};q) = \operatorname{Cat}^m(G_{25}), \quad \text{if } \zeta=\zeta_3,1, \tag{9.7d}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{25};q) = 0, \quad \text{otherwise.} \tag{9.7e}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.7). The only cases not covered by Lemmas 35 and 36 are the ones in (9.7b) and (9.7e). On the other hand, the only case left to consider according to Remark 4 is the case where $h_2=m_2=3$. This corresponds to the choice $p=4(m+1)/3$, which belongs to (9.7b). We have found four solutions to the counting problem (6.30) in (6.31), and each of them gives rise to $(m+1)/3$ elements in $\operatorname{Fix}_{NC^m(G_{25})}(\psi^p)$ (in the style as discussed in Case H_3). Hence, in total, we obtain $4\frac{m+1}{3}=\frac{4m+4}{3}$ elements in $\operatorname{Fix}_{NC^m(G_{25})}(\psi^p)$, which agrees with the limit in (9.7b).

Case G_{26}. The degrees are $6,12,18$, and hence we have

$$\operatorname{Cat}^m(G_{26};q)=\frac{[18m+18]_q\,[18m+12]_q\,[18m+6]_q}{[18]_q\,[12]_q\,[6]_q}.$$

Let ζ be a $14(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{26};q) = m+1, \quad \text{if } \zeta=\zeta_{18},\zeta_9, \tag{9.8a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{26};q) = \operatorname{Cat}^m(G_{26}), \quad \text{if } \zeta=\zeta_6,\zeta_3,-1,1, \tag{9.8b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{26};q) = 0, \quad \text{otherwise.} \tag{9.8c}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.8). The only case not covered by Lemmas 35 and 36 is the one in (9.8c). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2=6$ and $m_2=2$, respectively $h_2=m_2=2$. These correspond to the choices $p=3(m+1)/2$, respectively $p=9(m+1)/2$, both of which belong to (9.8c). The relevant counting problem is (6.35). However, no element $(w_0;w_1,\dots,w_m)\in\operatorname{Fix}_{NC^m(G_{26})}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0)+\ell_T(w_1)+\dots+\ell_T(w_m)$ be even, which is absurd. This is in agreement with the limit in (9.8c).

Case G_{27}. The degrees are $6,12,30$, and hence we have

$$\operatorname{Cat}^m(G_{27};q)=\frac{[30m+30]_q\,[30m+12]_q\,[30m+6]_q}{[30]_q\,[12]_q\,[6]_q}.$$

Let ζ be a $14(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{27};q) = m+1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \zeta_{10}, \zeta_5, \tag{9.9a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{27};q) = \operatorname{Cat}^m(G_{27}), \quad \text{if } \zeta = \zeta_6, \zeta_3, -1, 1, \tag{9.9b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{27};q) = 0, \quad \text{otherwise.} \tag{9.9c}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.9). The only case not covered by Lemmas 35 and 36 is the one in (9.9c). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2 = 6$ and $m_2 = 3$, $h_2 = m_2 = 3$, $h_2 = 6$ and $m_2 = 2$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 5(m+1)/3$, $10(m+1)/3$, $5(m+1)/2$, respectively $15(m+1)/2$, all of which belong to (9.9c).

If $p = 5(m+1)/3$ or $p = 10(m+1)/3$, the computation with the help of `CHEVIE` at the end of Case G_{27} in Section 6 did not find any solutions for the corresponding counting problem. This is in agreement with the limit in (9.9c).

In the case that $5(m+1)/2$ or $15(m+1)/2$, the relevant counting problem is (6.42). However, no element $(w_0; w_1, \dots, w_m) \in \operatorname{Fix}_{NC^m(G_{27})}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m)$ be even, which is absurd. This is again in agreement with the limit in (9.9c).

Case $G_{28} = F_4$. The degrees are $2, 6, 8, 12$, and hence we have

$$\operatorname{Cat}^m(F_4;q) = \frac{[12m+12]_q\,[12m+8]_q\,[12m+6]_q\,[12m+2]_q}{[12]_q\,[8]_q\,[6]_q\,[2]_q}.$$

Let ζ be a $12(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4;q) = m+1, \quad \text{if } \zeta = \zeta_{12}, \tag{9.10a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4;q) = \tfrac{3m+3}{2}, \quad \text{if } \zeta = \zeta_8,\ 2 \mid (m+1), \tag{9.10b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4;q) = (m+1)(2m+1), \quad \text{if } \zeta = \zeta_6, \zeta_3, \tag{9.10c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4;q) = \tfrac{(m+1)(3m+2)}{2}, \quad \text{if } \zeta = \zeta_4, \tag{9.10d}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4;q) = \operatorname{Cat}^m(F_4), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{9.10e}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(F_4;q) = 0, \quad \text{otherwise.} \tag{9.10f}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.10). The only cases not covered by Lemmas 35 and 36 are the ones in (9.10b) and (9.10f). On the other hand, according to Remark 4, the are no choices for h_2 and m_2 left to be considered.

Case G_{29}. The degrees are $4, 8, 12, 20$, and hence we have

$$\operatorname{Cat}^m(G_{29};q) = \frac{[20m+20]_q\,[20m+12]_q\,[20m+8]_q\,[20m+4]_q}{[20]_q\,[12]_q\,[8]_q\,[4]_q}.$$

Let ζ be a $20(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{29};q) = m+1, \quad \text{if } \zeta = \zeta_{20}, \zeta_{10}, \zeta_5, \tag{9.11a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{29};q) = \operatorname{Cat}^m(G_{29}), \quad \text{if } \zeta = \zeta_4, -1, 1, \tag{9.11b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(G_{29};q) = 0, \quad \text{otherwise.} \tag{9.11c}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.11). The only case not covered by Lemmas 35 and 36 is the one in (9.11c). On the other hand, the only cases left to consider according to Remark 4, the only choices for h_2 and m_2 to be considered are $h_2 = 1$ and $m_2 = 3$, $h_2 = 2$ and $m_2 = 3$, $h_2 = 4$ and $m_2 = 3$, $h_2 = 4$ and $m_2 = 2$, respectively $h_2 = m_2 = 4$. These correspond to the choices $p = 20(m+1)/3$, $p = 10(m+1)/3$, $p = 5(m+1)/3$, $p = 5(m+1)/2$, respectively $p = 5(m+1)/4$, all of which belong to (9.11c).

In the case that $p = 20(m+1)/3$, $p = 10(m+1)/3$, or $p = 5(m+1)/3$, the relevant counting problem is (6.55). However, no element $(w_0; w_1, \dots, w_m) \in \operatorname{Fix}_{NC^m(G_{27})}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m)$ be divisible by 3, which is absurd. This is in agreement with the limit in (9.11c).

In the case that $p = 5(m+1)/2$, the relevant counting problem is (6.64), for which we did not find any solutions. This is again in agreement with the limit in (9.11c).

In the case that $p = 5(m+1)/4$, the computation at the end of Case G_{29} in Section 6 did not find any solutions, which is as well in agreement with the limit in (9.11c).

Case $G_{30} = H_4$. The degrees are $2, 12, 20, 30$, and hence we have

$$\operatorname{Cat}^m(H_4;q) = \frac{[30m+30]_q\,[30m+20]_q\,[30m+12]_q\,[30m+2]_q}{[30]_q\,[20]_q\,[12]_q\,[2]_q}.$$

Let ζ be a $30(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(H_4;q) = m+1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \tag{9.12a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(H_4;q) = \tfrac{3m+3}{2}, \quad \text{if } \zeta = \zeta_{20},\ 2 \mid (m+1), \tag{9.12b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(H_4;q) = \tfrac{5m+5}{2}, \quad \text{if } \zeta = \zeta_{12},\ 2 \mid (m+1), \tag{9.12c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(H_4;q) = \tfrac{(m+1)(3m+2)}{2}, \quad \text{if } \zeta = \zeta_{10}, \zeta_5, \tag{9.12d}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(H_4;q) = \tfrac{(m+1)(5m+2)}{2}, \quad \text{if } \zeta = \zeta_6, \zeta_3, \tag{9.12e}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(H_4;q) = \tfrac{(m+1)(15m+1)}{4}, \quad \text{if } \zeta = \zeta_4,\ 2 \mid (m+1), \tag{9.12f}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(H_4;q) = \operatorname{Cat}^m(H_4), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{9.12g}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(H_4;q) = 0, \quad \text{otherwise.} \tag{9.12h}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.12). The only cases not covered by Lemmas 35 and 36 are the ones in (9.12b), (9.12c), (9.12f), and (9.12h). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2 = 2$ and $m_2 = 4$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 15(m+1)/2$, respectively $p = 15(m+1)/4$, out of which the first belongs to (9.12f), while the second belongs to (9.12h).

In the case that $p = 15(m+1)/2$, the action of ψ^p is the same as the one in (8.10). We have found eight solutions to the counting problem (6.80) in (6.83), each of them giving rise to $(m+1)/2$ elements in $\mathrm{Fix}_{NC^m(H_4)}(\psi^p)$ since the index i (in (6.80)) ranges from 0 to $(m-1)/2$, and we have found 30 solutions to the counting problem (6.82) in (6.84), each of them giving rise to $\binom{(m+1)/2}{2}$ elements in $\mathrm{Fix}_{NC^m(H_4)}(\psi^p)$ since $0 \le i_1 < i_2 \le (m-1)/2$ (in (6.82)). Hence, we obtain $8\frac{m+1}{2} + 30\binom{(m+1)/2}{2} = \frac{(m+1)(15m+1)}{4}$ elements in $\mathrm{Fix}_{NC^m(H_4)}(\psi^p)$, which agrees with the limit in (9.12f).

If $p = 15(m+1)/4$, the computation at the end of Case H_4 in Section 6 did not find any solutions, which is in agreement with the limit in (9.12h).

Case G_{32}. The degrees are $12, 18, 24, 30$, and hence we have

$$\mathrm{Cat}^m(G_{32}; q) = \frac{[30m+30]_q\,[30m+24]_q\,[30m+18]_q\,[30m+12]_q}{[30]_q\,[24]_q\,[18]_q\,[12]_q}.$$

Let ζ be a $30(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta}\mathrm{Cat}^m(G_{32}; q) = m+1, \quad \text{if } \zeta = \zeta_{30}, \zeta_{15}, \zeta_{10}, \zeta_5, \tag{9.13a}$$

$$\lim_{q\to\zeta}\mathrm{Cat}^m(G_{32}; q) = \tfrac{5m+5}{4}, \quad \text{if } \zeta = \zeta_{24}, \zeta_8,\ 4 \mid (m+1), \tag{9.13b}$$

$$\lim_{q\to\zeta}\mathrm{Cat}^m(G_{32}; q) = \tfrac{(5m+5)(5m+3)}{8}, \quad \text{if } \zeta = \zeta_{12}, \zeta_4,\ 2 \mid (m+1), \tag{9.13c}$$

$$\lim_{q\to\zeta}\mathrm{Cat}^m(G_{32}; q) = \mathrm{Cat}^m(G_{32}), \quad \text{if } \zeta = \zeta_6, \zeta_3, -1, 1, \tag{9.13d}$$

$$\lim_{q\to\zeta}\mathrm{Cat}^m(G_{32}; q) = 0, \quad \text{otherwise.} \tag{9.13e}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.13). The only cases not covered by Lemmas 35 and 36 are the ones in (9.13b), (9.13c), and (9.13e). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2 = 2$ and $m_2 = 4$, $h_2 = 6$ and $m_2 = 4$, $h_2 = m_2 = 3$, $h_2 = 6$ and $m_2 = 3$, $h_2 = m_2 = 2$, respectively $h_2 = 6$ and $m_2 = 2$. These correspond to the choices $p = 15(m+1)/4$, $p = 5(m+1)/4$, $p = 10(m+1)/3$, $p = 5(m+1)/3$, $p = 15(m+1)/2$, respectively $p = 5(m+1)/2$, out of which the first two belong to (9.13b), the next two belong to (9.13e), and the last two belong to (9.13c).

In the case that $p = 15(m+1)/4$ or $p = 5(m+1)/4$, we have found five solutions to the counting problem (6.88) in (6.89), each of them giving rise to $(m+1)/4$ elements in $\mathrm{Fix}_{NC^m(G_{32})}(\psi^p)$. Hence, we obtain $5\frac{m+1}{4} = \frac{5m+5}{4}$ elements in $\mathrm{Fix}_{NC^m(G_{32})}(\psi^p)$, which agrees with the limit in (9.13b).

In the case that $p = 10(m+1)/3$ or $p = 5(m+1)/3$, the relevant counting problem is (6.95). However, no element $(w_0; w_1, \dots, w_m) \in \mathrm{Fix}_{NC^m(G_{32})}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m)$ be divisible by 3, which is absurd. This is in agreement with the limit in (9.13e).

In the case that $p = 15(m+1)/2$ or $p = 5(m+1)/2$, we have found ten solutions to the counting problem (6.102) in (6.105), each of them giving rise to $(m+1)/2$ elements in $\mathrm{Fix}_{NC^m(G_{32})}(\psi^p)$, and we have found 25 solutions to the counting problem (6.104) in (6.106), each of them giving rise to $\binom{(m+1)/2}{2}$ elements in $\mathrm{Fix}_{NC^m(G_{32})}(\psi^p)$. Hence, we obtain $10\frac{m+1}{2} + 25\binom{(m+1)/2}{2} = \frac{(5m+5)(5m+3)}{8}$ elements in $\mathrm{Fix}_{NC^m(G_{32})}(\psi^p)$, which agrees with the limit in (9.13c).

CASE G_{33}. The degrees are $4, 6, 10, 12, 18$, and hence we have

$$\mathrm{Cat}^m(G_{33}; q) = \frac{[18m+18]_q\,[18m+12]_q\,[18m+10]_q\,[18m+6]_q\,[18m+4]_q}{[18]_q\,[12]_q\,[10]_q\,[6]_q\,[4]_q}.$$

Let ζ be a $18(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = m+1, \quad \text{if } \zeta = \zeta_{18}, \zeta_9, \tag{9.14a}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = \tfrac{9m+9}{5}, \quad \text{if } \zeta = \zeta_{10}, \zeta_5,\ 5 \mid (m+1), \tag{9.14b}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = \tfrac{(m+1)(3m+2)(3m+1)}{2}, \quad \text{if } \zeta = \zeta_6, \zeta_3, \tag{9.14c}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = \mathrm{Cat}^m(G_{33}), \quad \text{if } \zeta = -1 \text{ or } \zeta = 1, \tag{9.14d}$$

$$\lim_{q\to\zeta} \mathrm{Cat}^m(G_{33}; q) = 0, \quad \text{otherwise.} \tag{9.14e}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.14). The only cases not covered by Lemmas 35 and 36 are the ones in (9.14b) and (9.14e). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2 = 1$ and $m_2 = 5$, $h_2 = 2$ and $m_2 = 5$, $h_2 = 2$ and $m_2 = 4$, respectively $h_2 = m_2 = 2$. These correspond to the choices $p = 18(m+1)/5$, $p = 9(m+1)/5$, $p = 9(m+1)/4$, respectively $p = 9(m+1)/2$, out of which the first two belong to (9.14b), while the others belong to (9.14e).

In the case that $p = 18(m+1)/5$ or $p = 9(m+1)/5$, we have found nine solutions to the counting problem (6.115) in (6.116). Hence, we obtain $9\frac{m+1}{5} = \frac{9m+9}{5}$ elements in $\mathrm{Fix}_{NC^m(G_{33})}(\psi^p)$, which agrees with the limit in (9.14b).

If $p = 9(m+1)/4$, the computation at the end of Case G_{33} in Section 6 did not find any solutions, which is again in agreement with the limit in (9.13e).

In the case that $p = 9(m+1)/2$, the relevant counting problems are (6.122) and (6.124). However, no element $(w_0; w_1, \dots, w_m) \in \mathrm{Fix}_{NC^m(G_{33})}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m)$ be even, which is absurd. This is in agreement with the limit in (9.14e).

CASE G_{34}. The degrees are $6, 12, 18, 24, 30, 42$, and hence we have

$$\mathrm{Cat}^m(G_{34}; q) = \frac{[42m+42]_q\,[42m+30]_q\,[42m+24]_q}{[42]_q\,[30]_q\,[24]_q} \times \frac{[42m+18]_q\,[42m+12]_q\,[42m+6]_q}{[18]_q\,[12]_q\,[6]_q}.$$

Let ζ be a $42(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=m+1,\quad \text{if } \zeta=\zeta_{42},\zeta_{21},\zeta_{14},\zeta_7, \tag{9.15a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=\operatorname{Cat}^m(G_{34}),\quad \text{if } \zeta=\zeta_6,\zeta_3,-1,1, \tag{9.15b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(G_{34};q)=0,\quad \text{otherwise.} \tag{9.15c}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.15). The only case not covered by Lemmas 35 and 36 is the one in (9.15c). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2=1$ and $m_2=5$, $h_2=2$ and $m_2=5$, $h_2=3$ and $m_2=5$, $h_2=6$ and $m_2=5$, $h_2=2$ and $m_2=4$, $h_2=6$ and $m_2=4$, $h_2=m_2=3$, $h_2=6$ and $m_2=3$, $h_2=m_2=2$, $h_2=6$ and $m_2=2$, respectively $h_2=6$ and $m_2=6$. These correspond to the choices $p=42(m+1)/5$, $p=21(m+1)/5$, $p=14(m+1)/5$, $p=7(m+1)/5$, $p=21(m+1)/4$, $p=7(m+1)/4$, $p=14(m+1)/3$, $p=7(m+1)/3$, $p=21(m+1)/2$, $p=7(m+1)/2$, respectively $p=7(m+1)/6$, all of which belong to (9.15c).

In the case that $p=42(m+1)/5$, $p=21(m+1)/5$, $p=14(m+1)/5$, or $p=7(m+1)/5$, the relevant counting problem is (6.128). However, no element $(w_0;w_1,\dots,w_m)\in \operatorname{Fix}_{NC^m(G_{34})}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0)+\ell_T(w_1)+\dots+\ell_T(w_m)$ be divisible by 5, which is absurd. This is in agreement with the limit in (9.15c).

In the case that $p=21(m+1)/4$ or $p=7(m+1)/4$, the relevant counting problem is (6.140). However, no element $(w_0;w_1,\dots,w_m)\in \operatorname{Fix}_{NC^m(G_{34})}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0)+\ell_T(w_1)+\dots+\ell_T(w_m)$ be divisible by 4, which is absurd. This is in agreement with the limit in (9.15c).

In the case that $p=14(m+1)/3$ or $p=7(m+1)/3$, the relevant counting problems are (6.146) and (6.148). However, the computations with the help of `CHEVIE` performed in Case G_{34} in Section 6 did not find any solutions for (6.146) or (6.148). This is in agreement with the limit in (9.15c).

In the case that $p=21(m+1)/2$, the relevant counting problems are (6.157), (6.158), and (6.159). However, the computations with the help of `CHEVIE` performed in Case G_{34} in Section 6 found no w_i with $\ell_T(w_i)=3$ in (6.157), and hence no solutions for (w_{i_1},w_{i_2}) with $\ell_T(w_{i_1})+\ell_T(w_{i_2})=3$ in (6.158), and no solutions for $(w_{i_1},w_{i_2},w_{i_3})$ in (6.159). This is in agreement with the limit in (9.15c).

If $p=7(m+1)/6$, the computation at the end of Case G_{34} in Section 6 did not find any solutions, which is also in agreement with the limit in (9.15c).

Case $G_{35}=E_6$. The degrees are $2,5,6,8,9,12$, and hence we have

$$\operatorname{Cat}^m(E_6;q)=\frac{[12m+12]_q\,[12m+9]_q\,[12m+8]_q\,[12m+6]_q\,[12m+5]_q\,[12m+2]_q}{[12]_q\,[9]_q\,[8]_q\,[6]_q\,[5]_q\,[2]_q}.$$

Let ζ be a $12(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = m+1, \quad \text{if } \zeta=\zeta_{12}, \tag{9.16a}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{4m+4}{3}, \quad \text{if } \zeta=\zeta_{9},\ 3 \mid (m+1), \tag{9.16b}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{3m+3}{2}, \quad \text{if } \zeta=\zeta_{8},\ 2 \mid (m+1), \tag{9.16c}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = (m+1)(2m+1), \quad \text{if } \zeta=\zeta_{6}, \tag{9.16d}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{(m+1)(3m+2)}{2}, \quad \text{if } \zeta=\zeta_{4}, \tag{9.16e}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{(m+1)(4m+3)(2m+1)}{3}, \quad \text{if } \zeta=\zeta_{3}, \tag{9.16f}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \tfrac{(m+1)(3m+2)(2m+1)(6m+1)}{2}, \quad \text{if } \zeta=-1, \tag{9.16g}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = \operatorname{Cat}^m(E_6), \quad \text{if } \zeta=1, \tag{9.16h}$$

$$\lim_{q\to\zeta} \operatorname{Cat}^m(E_6;q) = 0, \quad \text{otherwise.} \tag{9.16i}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.16). The only cases not covered by Lemmas 35 and 36 are the ones in (9.16b), (9.16c), and (9.16i). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2=1$ and $m_2=5$, respectively $h_2=2$ and $m_2=5$. These correspond to the choices $p=12(m+1)/5$, respectively $p=6(m+1)/5$, both of which belong to (9.16i).

In the case that $p=12(m+1)/5$, the relevant counting problem is (6.172). However, no element $(w_0;w_1,\dots,w_m)\in \operatorname{Fix}_{NC^m(E_6)}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0)+\ell_T(w_1)+\dots+\ell_T(w_m)$ be divisible by 5, which is absurd. This is in agreement with the limit in (9.16i).

If $p=6(m+1)/5$, the computation at the end of Case E_6 in Section 6 did not find any solutions, which is also in agreement with the limit in (9.16i).

Case $G_{36}=E_7$. The degrees are $2,6,8,10,12,14,18$, and hence we have

$$\operatorname{Cat}^m(E_7;q) = \frac{[18m+18]_q\,[18m+14]_q\,[18m+12]_q}{[18]_q\,[14]_q\,[12]_q} \times \frac{[18m+10]_q\,[18m+8]_q\,[18m+6]_q\,[18m+2]_q}{[10]_q\,[8]_q\,[6]_q\,[2]_q}.$$

Let ζ be a $18(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_7;q)=m+1,\quad \text{if } \zeta=\zeta_{18},\zeta_9, \tag{9.17a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_7;q)=\tfrac{9m+9}{7},\quad \text{if } \zeta=\zeta_{14},\zeta_7,\ 7\mid(m+1), \tag{9.17b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_7;q)=\frac{(m+1)(3m+2)(3m+1)}{2},\quad \text{if } \zeta=\zeta_6,\zeta_3, \tag{9.17c}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_7;q)=\operatorname{Cat}^m(E_7),\quad \text{if } \zeta=-1 \text{ or } \zeta=1, \tag{9.17d}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_7;q)=0,\quad \text{otherwise.} \tag{9.17e}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.17). The only cases not covered by Lemmas 35 and 36 are the ones in (9.17b) and (9.17e). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2=1$ and $m_2=7$, $h_2=2$ and $m_2=7$, $h_2=1$ and $m_2=5$, $h_2=2$ and $m_2=5$, $h_2=2$ and $m_2=4$, respectively $h_2=m_2=2$. These correspond to the choices $p=18(m+1)/7$, $p=9(m+1)/7$, $p=18(m+1)/5$, $p=9(m+1)/5$, $p=9(m+1)/4$, respectively $p=9(m+1)/2$, out of which the first two belong to (9.16b), and all others belong to (9.16i).

In the case that $p=18(m+1)/7$ or $p=9(m+1)/7$, we have found nine solutions to the counting problem (6.176) in (6.177). Hence, we obtain $9\frac{m+1}{7}=\frac{9m+9}{7}$ elements in $\operatorname{Fix}_{NC^m(E_7)}(\psi^p)$, which agrees with the limit in (9.17b).

In the case that $p=18(m+1)/5$ or $p=9(m+1)/5$, the relevant counting problem is (6.186). However, no element $(w_0;w_1,\dots,w_m)\in\operatorname{Fix}_{NC^m(E_7)}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0)+\ell_T(w_1)+\dots+\ell_T(w_m)$ be divisible by 5, which is absurd. This is in agreement with the limit in (9.17e).

In the case that $p=9(m+1)/4$, the relevant counting problem is (6.192). However, no element $(w_0;w_1,\dots,w_m)\in\operatorname{Fix}_{NC^m(E_7)}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0)+\ell_T(w_1)+\dots+\ell_T(w_m)$ be divisible by 4, which is absurd. This is again in agreement with the limit in (9.17e).

In the case that $p=9(m+1)/2$, the relevant counting problems are (6.195), (6.196), and (6.197). However, no element $(w_0;w_1,\dots,w_m)\in\operatorname{Fix}_{NC^m(E_7)}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0)+\ell_T(w_1)+\dots+\ell_T(w_m)$ be even, which is absurd. This is also in agreement with the limit in (9.17e).

Case $G_{37}=E_8$. The degrees are $2,8,12,14,18,20,24,30$, and hence we have

$$\operatorname{Cat}^m(E_8;q)=\frac{[30m+30]_q\,[30m+24]_q\,[30m+20]_q\,[30m+18]_q}{[30]_q\,[24]_q\,[20]_q\,[18]_q}\\ \times\frac{[30m+14]_q\,[30m+12]_q\,[30m+8]_q\,[30m+2]_q}{[14]_q\,[12]_q\,[8]_q\,[2]_q}.$$

Let ζ be a $30(m+1)$-th root of unity. The following cases on the right-hand side of (7.2) occur:

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = m+1, \quad \text{if } \zeta=\zeta_{30},\zeta_{15}, \tag{9.18a}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = \tfrac{5m+5}{4}, \quad \text{if } \zeta=\zeta_{24},\ 4\mid(m+1), \tag{9.18b}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = \tfrac{3m+3}{2}, \quad \text{if } \zeta=\zeta_{20},\ 2\mid(m+1), \tag{9.18c}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = \tfrac{(5m+5)(5m+3)}{8}, \quad \text{if } \zeta=\zeta_{12},\ 2\mid(m+1), \tag{9.18d}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = \tfrac{(m+1)(3m+2)}{2}, \quad \text{if } \zeta=\zeta_{10},\zeta_5, \tag{9.18e}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = \frac{(5m+5)(15m+7)}{16}, \quad \text{if } \zeta=\zeta_8,\ 4\mid(m+1), \tag{9.18f}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = \frac{(m+1)(5m+4)(5m+3)(5m+2)}{24}, \quad \text{if } \zeta=\zeta_6,\zeta_3, \tag{9.18g}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = \frac{(m+1)(5m+3)(15m+7)(15m+1)}{64}, \quad \text{if } \zeta=\zeta_4,\ 2\mid(m+1), \tag{9.18h}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = \operatorname{Cat}^m(E_8), \quad \text{if } \zeta=-1 \text{ or } \zeta=1, \tag{9.18i}$$

$$\lim_{q\to\zeta}\operatorname{Cat}^m(E_8;q) = 0, \quad \text{otherwise.} \tag{9.18j}$$

We must now prove that the left-hand side of (7.2) in each case agrees with the values exhibited in (9.18). The only cases not covered by Lemmas 35 and 36 are the ones in (9.18b), (9.18c), (9.18d), (9.18f), (9.18h), and (9.18j). On the other hand, the only cases left to consider according to Remark 4 are the cases where $h_2=2$ and $m_2=8$, $h_2=1$ and $m_2=7$, $h_2=2$ and $m_2=7$, $h_2=2$ and $m_2=4$, respectively $h_2=m_2=2$. These correspond to the choices $p=15(m+1)/8$, $p=30(m+1)/7$, $p=15(m+1)/7$, $p=15(m+1)/4$, respectively $15(m+1)/2$, out of which the first three belong to (9.18j), the fourth belongs to (9.18f), and the last belongs to (9.18h).

If $p=15(m+1)/8$, the relevant counting problem is (6.242). However, the computation at the end of Case E_8 in Section 6 did not find any solutions, which is in agreement with the limit in (9.18j). Hence, the left-hand side of (7.2) is equal to 0, as required.

In the case that $p=30(m+1)/7$ or $p=15(m+1)/7$, the relevant counting problem is (6.213). However, no element $(w_0;w_1,\dots,w_m)\in \operatorname{Fix}_{NC^m(E_8)}(\psi^p)$ can be produced in this way since the counting problem imposes the restriction that $\ell_T(w_0)+\ell_T(w_1)+\dots+\ell_T(w_m)$ be divisible by 7, which is absurd. This is also in agreement with the limit in (9.18j).

In the case that $p=15(m+1)/4$, the relevant counting problems are (6.227) and (6.228). We have found 45 solutions w_i to (6.227) of type A_1^2 in (6.229), and we have found 20 solutions w_i to (6.227) of type A_2 in (6.230), which implied 150 solutions for (w_{i_1},w_{i_2}) to (6.228). The first two give rise to to $(45+20)\frac{m+1}{4}=65\frac{m+1}{4}$ elements in $\operatorname{Fix}_{NC^m(E_8)}(\psi^p)$, while the third give rise to $150\binom{(m+1)/4}{2}$ elements in $\operatorname{Fix}_{NC^m(E_8)}(\psi^p)$. Hence, we obtain $65\frac{m+1}{4}+150\binom{(m+1)/4}{2}=\frac{(5m+5)(15m+7)}{16}$ elements in $\operatorname{Fix}_{NC^m(E_8)}(\psi^p)$, which agrees with the limit in (9.18f).

In the case that $p = 15(m+1)/2$, the relevant counting problems are (6.233), (6.234), (6.235), and (6.236). We have found 15 solutions w_i to (6.233) of type $A_1^2 * A_2$ in (6.237), we have found 45 solutions w_i to (6.233) of type $A_1 * A_3$ in (6.238), we have found 5 solutions w_i to (6.233) of type A_2^2 in (6.239), we have found 18 solutions w_i to (6.233) of type A_4 in (6.240), we have found 5 solutions w_i to (6.233) of type D_4 in (6.241), each giving rise to $(m+1)/2$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\psi^p)$. Using the notation from there, these imply $2n_{3,1} + n_{2,2} = 2 \cdot 660 + 1195 = 2515$ solutions for (w_{i_1}, w_{i_2}) to (6.234) with $\ell_T(w_{i_1}) + \ell_T(w_{i_2}) = 4$, each giving rise to $\binom{(m+1)/2}{2}$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\psi^p)$. They also imply $3n_{2,1,1} = 3 \cdot 2850 = 8550$ solutions for $(w_{i_1}, w_{i_2}, w_{i_3})$ to (6.235) with $\ell_T(w_{i_1}) + \ell_T(w_{i_2}) + \ell_T(w_{i_3}) = 4$, each giving rise to $\binom{(m+1)/2}{3}$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\psi^p)$. Finally, they imply as well $n_{1,1,1,1} = 6750$ solutions for $(w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4})$ to (6.236), each giving rise to $\binom{(m+1)/2}{4}$ elements in $\mathrm{Fix}_{NC^m(E_8)}(\psi^p)$.

In total, we obtain

$$(15+45+5+18+5)\frac{m+1}{2} + 2515\binom{(m+1)/2}{2} + 8550\binom{(m+1)/2}{3} + 6750\binom{(m+1)/2}{4} \\ = \frac{(m+1)(5m+3)(15m+7)(15m+1)}{64}$$

elements in $\mathrm{Fix}_{NC^m(E_8)}(\psi^p)$, which agrees with the limit in (9.18h).

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