CYCLIC SIEVING PHENOMENA FOR ROOTED PLANE TREES AND TREE-ROOTED PLANAR MAPS

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Abstract. We prove cyclic sieving phenomena satisfied by corner-rooted plane trees (alias ordered trees). The sets of rooted plane trees that we consider are: (1) all trees with n nodes; (2) all trees with n nodes and k leaves; (3) all trees with a given degree distribution of the nodes. Moreover, we consider four different cyclic group actions: (1) the root is moved to the next corner along a tour of the tree; (2) only trees in which the root is at a leaf are considered, and the action moves the root to the next leaf; (3) only trees in which the root is at a non-leaf are considered, and the action moves the root to the next non-leaf corner; (4) only trees in which the root is at a node of degree δ are considered, for a fixed δ, and the action moves the root to the next corner of this type.

We prove a cyclic sieving phenomenon for each meaningful combination of these sets and actions. As a bonus, we also establish corresponding cyclic sieving phenomena for tree-rooted planar maps.

1. Introduction

The *cyclic sieving phenomenon* was introduced by Reiner, Stanton and White [26] as the property of a set of combinatorial objects on which a cyclic group of order m acts that the number of objects that are invariant under a given power of a generator of the group is given by the evaluation of a fixed univariate polynomial at the corresponding power of a primitive m-th root of unity. (See Section 3 for the precise definition.) Reiner, Stanton and White already presented numerous instances of the cyclic sieving phenomenon in their original article [26]. Since then, the discovery of cyclic sieving phenomena has almost become an industry, with numerous more having been discovered and proved since the appearance of [26]. A survey of cyclic sieving phenomena published until 2011 can be found in [29]. It goes without saying that even more have been identified since then.

Inspired by these overwhelming developments, the second author, during his talk at the closing conference of the ANR project Cortipom in Le Croisic in June 2025 in which he presented further cyclic sieving phenomena [21], put forward a new principle of combinatorial mathematics:

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Every family of combinatorial objects satisfies
the cyclic sieving phenomenon!

If this is so, said the first author who was in the audience, then surely rooted plane trees satisfy the cyclic sieving phenomenon. The particular action of a cyclic group that she thought of was to move the root of the tree to the next *corner* (see Figure 2 for an example and Section 4 for precise definitions). Being a fan of maps, she had in mind a possible application to cyclic sieving for tree-rooted maps. As for evidence, she had already done some computations for small examples.

Surprisingly, a first literature search led to nothing, which let us believe that no investigations had been undertaken in this direction. As explained in Section 11, the suspected cyclic sieving result for trees *did* already exist in fact, but only in disguise, stated in terms of different objects[1].

Indeed, the cyclic sieving phenomenon for rooted plane trees is not very difficult to prove, see Theorem 5 in Section 5. In fact, there exists even a refinement in which one fixes the number of leaves, see Theorem 7, which is already more demanding to prove. In particular, to find the cyclic sieving polynomial was not obvious since the "q-ification" of the factor 2 in the counting formula (2.4) is not canonical (however, we realised later that this result too did already exist in a different language, cf. Subsection 11.1).

From here on, we became more adventurous (or one may also say: "greedy"). What if we modify the action so that the only allowed roots are leaves, and the group action moves the root from one leaf to the next? Or what happens if we do the analogous construction with non-leaves? What if we refine further and consider rooted plane trees in which the complete degree distribution of the nodes is prescribed? And finally, how about taking one step further and considering tree-rooted maps, as was the original motivation of the first author?

To give it away: the above principle did not let us down. In all instances and all (meaningful) combinations of the ingredients (objects and actions) we identified cyclic sieving phenomena, which we state and prove in this paper.

Outline of the paper. After a review of the relevant tree counting formulae in the next section, and a reminder of the definition of the cyclic sieving phenomenon in Section 3, in Section 4 we establish some preliminary results on the structure of trees having a rotational symmetry. In Section 5 we present the aforementioned cyclic sieving results for the rotation action that the first author had originally in mind, see Theorems 5 and 7. Section 6 is then devoted to the action of "external rotation", by which we mean that the root is moved from one leaf to the next. The main result in that section is Theorem 10. The action that we consider in Section 7 is what we call "internal rotation": here the root is moved from a non-leaf corner to the next. The corresponding cyclic sieving result is presented in Theorem 13.

The subsequent three sections concern cyclic sieving phenomena for trees in which the degrees of the nodes are prescribed. Section 8 presents a corresponding cyclic sieving phenomenon for the original rotation, see Theorem 16. In Section 9 we investigate a generalisation of the external rotation that we call "δ-rotation". Here, the trees that we consider have a root of degree δ, and δ-rotation moves the root to the next corner of

[1]We also found more recently a preprint where it *is* stated in terms of trees [16], albeit with a different proof than in our article.

degree δ. The corresponding cyclic sieving phenomenon, with prescribed vertex degrees, is presented in Theorem 19. Finally in Section 10 the action is that of internal rotation, and the corresponding cyclic sieving result is the subject of Theorem 22.

Each of the Sections 5–10 is built according to the same scheme: the cyclic sieving result of that particular section is presented and proved in a theorem. The proof of polynomiality and non-negativity of the coefficients of the cyclic sieving polynomial is usually delayed to the first lemma following the theorem. The evaluations of the cyclic sieving polynomial at the relevant roots of unity are done separately in the second lemma following the theorem. This restricts the proofs of the theorems to the enumeration of trees fixed by a prescribed rotation. Each of the three aspects of our proofs — enumeration, polynomiality, evaluation — always follows the same pattern, independently of the class of trees and of the rotation under consideration. We naturally provide more details in the early proofs than in the late ones.

In the subsequent section, Section 11, we take a temporary break from cyclic sieving proofs, and describe connections between some of our results and earlier ones, stated in terms of non-crossing matchings, non-crossing partitions, or dissections of a polygon. For instance, our first result, namely Theorem 5, and its refinement recording the number of leaves (Theorem 7) have actually appeared earlier in terms of non-crossing perfect matchings. Conversely, we explain how some of our results translate in these alternative languages. At the end of that section (see Subsection 11.4), we explain how the cyclic sieving phenomena for non-crossing matchings and non-crossing partitions can be embedded in a much more general cyclic sieving theorem for generalised non-crossing partitions associated with reflection groups.

Finally, in Section 12, we return to cyclic sieving proofs, this time for tree-rooted planar maps; see Theorems 30, 33, and 34. Tree-rooted maps can indeed be seen as amalgams of (slightly generalised) rooted plane trees and non-crossing perfect matchings. As we show, cyclic sieving phenomena result from this structure.

2. Tree enumeration

We consider *rooted plane trees* (also called *ordered trees*) with n edges, and thus $n+1$ vertices (or *nodes*); see Figure 1. The precise meaning of "rooted" is that there is a *pointer* to one of the *corners* of the tree, as is commonly done for rooted (combinatorial) maps (see e.g. [10]). It is *not* sufficient to just mark a node of a tree. The node incident to the root corner is called the *root node*. The edge that follows the root corner, in counterclockwise order around the root node, is called the *root edge*. The *degree* of a node is its graph-theoretic degree. A node of degree 1 is called a *leaf*. The degree of a corner is the degree of the incident node. All trees in this paper are plane rooted trees, and they will often be called just "trees" for short.

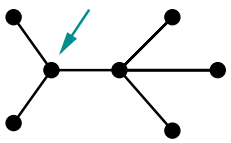


Figure 1. A rooted tree with $n = 6$ edges, hence 7 vertices and 12 corners.

We denote the set of all rooted plane trees with n edges by $\mathrm{T}(n)$. Such trees have $n+1$ nodes and $2n$ corners. It is well-known (see e.g. [32, Thm. 1.5.1(iii)]) that the cardinality of $\mathrm{T}(n)$ is given by the n-th *Catalan number* C_n; more precisely, we have

$$|\,\mathrm{T}(n)| = C_n := \frac{1}{n+1}\binom{2n}{n}. \tag{2.1}$$

Next we consider subsets of $\mathrm{T}(n)$ in which the number of leaves is specified. We begin with the set of all rooted plane trees with n edges and k leaves, where *the root is one of the leaves*. We denote the set of all these trees by $\mathrm{T}_l(n,k)$. (Here, the index "l" stands for "leaf-rooted".)

By deleting the root leaf and the edge emanating from it, and moving the root corner to the other endpoint of this edge, we obtain the set of rooted plane trees with $n-1$ edges (thus n nodes) and $k-1$ non-root leaves. Hence, by [12, Ex. III.13], the cardinality of $\mathrm{T}_l(n,k)$ is given by a *Narayana number*; more precisely, we have

$$|\,\mathrm{T}_l(n,k)| = \frac{1}{n}\binom{n-2}{k-2}\binom{n}{k-1} = \frac{1}{n-1}\binom{n-1}{k-2}\binom{n-1}{k-1}. \tag{2.2}$$

A further subset of $\mathrm{T}(n)$ which we consider in our article is the set of all rooted plane trees with n edges and k leaves, where *the root is one of the non-leaves*. We denote the set of all these trees by $\mathrm{T}_i(n,k)$. (Here, the index "i" stands for "internally-rooted".)

In order to determine the number of these trees, we recall that a rooted tree with n edges (and k leaves) has $2n$ corners. Exactly k of the corners are located at a leaf, and the other $2n-k$ at a non-leaf. A double rooting argument, where we count trees rooted at a non-leaf *and* having a marked leaf in two different ways, yields

$$k\,|\,\mathrm{T}_i(n,k)| = (2n-k)\,|\,\mathrm{T}_l(n,k)|,$$

so that

$$|\,\mathrm{T}_i(n,k)| = \frac{2n-k}{k}\,|\,\mathrm{T}_l(n,k)| = \frac{2n-k}{n(n-1)}\binom{n-1}{k-2}\binom{n}{k}. \tag{2.3}$$

A similar argument gives the number of rooted plane trees with n edges and k leaves without any restriction on the root degree. We denote this subset of $\mathrm{T}(n)$ by $\mathrm{T}(n,k)$. Indeed, by marking one of the k leaves in a tree of $\mathrm{T}(n,k)$, and by re-rooting the tree at this leaf, we obtain

$$|\,\mathrm{T}(n,k)| = \frac{2n}{k}\,|\,\mathrm{T}_l(n,k)| = \frac{2}{n-1}\binom{n-1}{k-2}\binom{n}{k}. \tag{2.4}$$

We now turn to families of trees with prescribed degree distribution. Let $\mathbf{n} = (n_1, n_2, \ldots)$ be a sequence of non-negative integers having finitely many non-zero entries. Let $\mathrm{T}(\mathbf{n})$ be the set of rooted trees having exactly n_i nodes of degree i, for $i \geq 1$. Such trees have $\sum_i n_i$ nodes, hence $n := -1 + \sum_i n_i$ edges. On the other hand, the edge number is also $\sum_i i n_i/2$, which means that such trees can only exist if

$$\sum_{i\geq 1}(i-2)n_i = -2. \tag{2.5}$$

Note also that n_i necessarily vanishes for $i > n$. As above, we can rely on classical results to count leaf-rooted trees of degree distribution $\mathbf{n}$, and then count other types

of trees by a double rooting argument. Indeed, it was proved by Tutte in 1964 [33, Eq. (8)] that the number of leaf-rooted trees with degree distribution $\mathbf{n}$ is

$$|\,\mathrm{T}_l(\mathbf{n})| = n_1 \frac{(n-1)!}{\prod_i n_i!} = \frac{1}{n}\binom{n}{n_1-1, n_2, n_3, \ldots}. \tag{2.6}$$

From this, we derive the number of trees with degree distribution $\mathbf{n}$ rooted at a non-leaf, as we did in (2.3). Namely, we have

$$|\,\mathrm{T}_i(\mathbf{n})| = \frac{2n-n_1}{n_1}\,|\,\mathrm{T}_l(\mathbf{n})| = (2n-n_1)\frac{(n-1)!}{\prod_i n_i!} = \frac{2n-n_1}{(n+1)n}\binom{n+1}{n_1, n_2, \ldots}. \tag{2.7}$$

Analogously, the total number of rooted trees with degree distribution $\mathbf{n}$ is then

$$|\,\mathrm{T}(\mathbf{n})| = \frac{2n}{n_1}\,|\,\mathrm{T}_l(\mathbf{n})| = 2\,\frac{n!}{\prod_i n_i!} = \frac{2}{n+1}\binom{n+1}{n_1, n_2, \ldots}. \tag{2.8}$$

Finally, we will also consider rooted trees in which the root node has a prescribed degree, say δ. We denote the set of such trees having degree distribution $\mathbf{n}$ by $\mathrm{T}_\delta(\mathbf{n})$. In these trees, the number of corners that are incident to a node of degree δ is δn_δ. Hence the number of such trees is

$$|\,\mathrm{T}_\delta(\mathbf{n})| = \frac{\delta n_\delta}{n_1}\,|\,\mathrm{T}_l(\mathbf{n})| = \delta n_\delta \frac{(n-1)!}{\prod_i n_i!} = \frac{\delta}{n}\binom{n}{n_1, \ldots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \ldots}, \tag{2.9}$$

generalising (2.6), which is the case $\delta = 1$.

3. The cyclic sieving phenomenon

Here we provide the definition of the *cyclic sieving phenomenon*. There are actually three equivalent ways to define this phenomenon (cf. [26, Prop. 2.1]). However, the one which gives the name is the following.

Definition 1 (Cyclic sieving phenomenon)**.** Given a set S of combinatorial objects, an action on S of a cyclic group G of order m with generator g , and a polynomial $P(q)$ in q with non-negative integer coefficients, we say that the triple (S, G, P) *exhibits the cyclic sieving phenomenon*, if the number of elements of S fixed by g^e equals $P(\omega^e)$ for all primitive m-th roots of unity ω and all integers e.

We shall denote the set of all elements of S that are fixed by g^e by $\mathrm{Fix}_{g^e}(S)$. Furthermore, we shall refer to the polynomial $P(q)$ in Definition 1 as the *cyclic sieving polynomial* of the cyclic sieving phenomenon.

A simple lemma states that, when proving a cyclic sieving phenomenon, one can restrict ω and e to fewer values. Note that there are two different letters "e" below, in two different fonts: the first one, e, is the number of times one applies the generator g, while the second, e, is Euler's constant in the exponential.

Lemma 2. *The triple (S, G, P), with $G = \langle g\rangle$ of order m, exhibits the cyclic sieving phenomenon if and only if the number of elements of S fixed by g^e equals $P(\omega_0^e)$, with $\omega_0 := \mathsf{e}^{2\pi i/m}$, for $e = 0$ and for divisors e of m with $e \in [\![1, m-1]\!]$. Here and subsequently, the symbol $[\![a, b]\!]$ stands for $[a, b] \cap \mathbb{Z}$, with $\mathbb{Z}$ denoting the set of integers.*

Proof. Our first argument is translation: an element is fixed by g^e if and only if it is fixed by g^{e+m}. However, we have $\omega^{e+m} = \omega^e$. So we can restrict our attention to $e \in [\![0, m-1]\!]$.

Let us now explain why it suffices to prove the condition of Definition 1 for $\omega = \omega_0$. All primitive m-th roots of unity are Galois conjugates of one another in the extension of $\mathbb{Q}$ by ω_0. Hence if a polynomial with rational coefficients vanishes at ω_0, it vanishes at all primitive m-th roots of unity ω. Applying this to the polynomial $x \mapsto P(x^e) - \mathrm{Fix}_{g^e}(S)$, for $e \in [\![0, m-1]\!]$, we obtain the result.

Finally, we explain why it suffices to focus on 0 and divisors of m. Let $e \in [\![0, m-1]\!]$, and let $d = \gcd(e, m)$. There exist integers j and k such that $d = je + km$. If an element is fixed by g^e, it is also fixed by g^{-e}, and of course by $g^m = g^{-m} = \mathrm{id}$. So it is fixed by g^d. Conversely, if it is fixed by g^d it is fixed by g^e, since e is a multiple of d. (The fact that the same elements are fixed by g^e and g^d appears already in [2, Lem. 2.7]). Assume that $P(\mathsf{e}^{2\pi i d/m}) = \mathrm{Fix}_{g^d}(S)$. We have to prove that $P(\mathsf{e}^{2\pi i e/m}) = P(\mathsf{e}^{2\pi i d/m})$. Write $e = de'$ and $m = dm'$. Now e' and m' are coprime, so that $\mathsf{e}^{\frac{2\pi i}{m'}e'}$ and $\mathsf{e}^{\frac{2\pi i}{m'}}$ are both primitive m'-th roots of unity. Thus, by the Galois argument used in the second part of the proof, we have

$$P(\mathsf{e}^{\frac{2\pi i}{m}e}) = P(\mathsf{e}^{\frac{2\pi i}{m'}e'}) = P(\mathsf{e}^{\frac{2\pi i}{m'}}) = P(\mathsf{e}^{\frac{2\pi i}{m}d}).$$

This concludes the proof. □

Example 1. Anticipating slightly the next section, let S be the set of rooted plane trees with 3 edges, on which a rotation R of order $m = 6$ acts by moving the root corner around the tree, as illustrated in Figure 2. Then, for $e = 0$ or e a divisor of 6, we find

$$\mathrm{Fix}_{R^e}(S) = \begin{cases} 5, & \text{if } e = 0, \\ 0, & \text{if } e = 1, \\ 2, & \text{if } e = 2, \\ 3, & \text{if } e = 3. \end{cases}$$

We observe that a cyclic sieving phenomenon holds with

$$P(q) = q^6 + q^4 + q^3 + q^2 + 1.$$

Indeed, for $\omega = \mathsf{e}^{\pi i/3}$ we check that

$$P(1) = 5, \quad P(\omega) = 0, \quad P(\omega^2) = 2 \quad \text{and} \quad P(\omega^3) = P(-1) = 3.$$

This will be generalised to any tree size in Theorem 5.

4. Tree rotations

In this paper, we consider three different rotations (plus a generalisation of one of them) that act on rooted trees by changing the root corner. We define them in this section, and characterise trees that are fixed by a given power of one of these rotations.

Our main rotation, denoted R and called *ordinary rotation* in what follows, acts on rooted trees by moving the root corner to the next corner in counterclockwise order around the tree. This is the rotation illustrated in Figure 2. Since a rooted tree with n edges has $2n$ corners, we obtain an action of the cyclic group of order $2n$ on $\mathrm{T}(n)$.

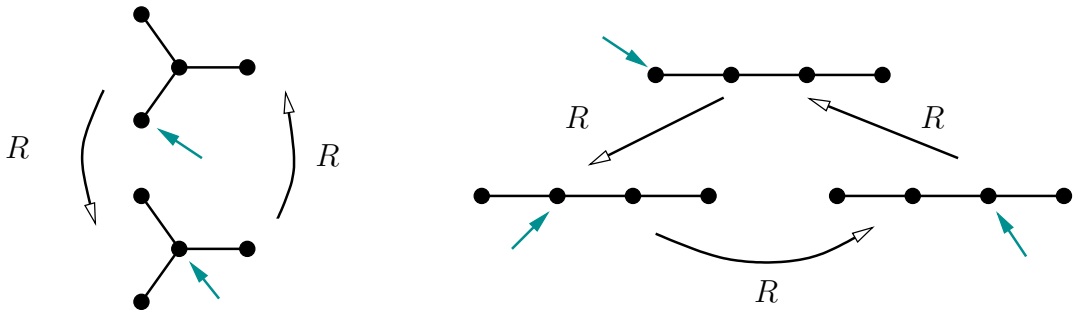


FIGURE 2. Rotation on rooted trees with 3 edges.

In the somewhat degenerate case where $n = 1$, there is a unique rooted tree in $\mathrm{T}(n)$, which is of course fixed by all powers of the rotation R. We could then consider that the natural cyclic group acting on $\mathrm{T}(n)$ has order 1 only. Nevertheless, even when $n = 1$, we study the action of the cyclic group of order $2n$ on $\mathrm{T}(n)$.

The *external rotation* R_l acts on trees rooted at a leaf by moving the root corner to the next leaf, still in counterclockwise order. This yields an action of the cyclic group of order k on trees with k leaves.

The *internal rotation* R_i acts on trees rooted at a non-leaf by moving the root corner to the next corner incident to a non-leaf, in counterclockwise order. This yields an action of the cyclic group of order $2n - k$ on trees with n edges and k leaves.

We will also consider a generalisation of the external rotation, for any integer $\delta \geq 1$. The *δ-rotation* R_δ acts on trees rooted at a vertex of degree δ by moving the root to the next corner of degree δ. This yields an action of the cyclic group of order δn_δ on trees with n_δ nodes of degree δ.

Recall from Lemma 2 that, in order to establish a cyclic sieving phenomenon for the triple (S, G, P), with $G = \langle g \rangle$, we only have to count elements of S fixed by g^e for $e = 0$ and for e a divisor of $|G|$. We now want to characterise trees of $\mathrm{T}(n)$ invariant by R^e, for e a divisor of $2n$. Let us first recall the definition of the *centre* of a tree τ: if τ is reduced to a node or to an edge then the centre is τ itself. Otherwise, it is determined recursively by deleting all leaves of τ, and by iterating this pruning until a tree with one or two nodes is obtained. For instance, pruning the tree of Figure 1 shows in one step that its centre is an edge.

We will also need some transformations Φ and Ψ_d, for $d \geq 2$, that extract a piece of (some) rooted trees τ. If the centre of τ is an edge, we define $\Phi(\tau)$ as the tree obtained by cutting the central edge c in its middle, retaining the piece that contains the root corner, and adding a marked leaf at the end of the half-edge coming from c (Figure 3). Note that Φ is only defined on edge-centred trees. For $d \geq 2$, and τ a rooted tree whose centre is a node c of degree ℓ divisible by d, we define $\Psi_d(\tau)$ as follows: let $\tau_1, \ldots, \tau_\ell$ be the trees incident to c, in counterclockwise order, with τ_1 containing the root *edge*. Then $\Psi_d(\tau)$ is obtained from τ by only retaining the trees $\tau_1, \ldots, \tau_{\ell/d}$ and marking the node c (Figure 4).

Proposition 3. *Let $n \geq 1$, and let e be a positive divisor of $2n$ with $e < 2n$. Write $d = 2n/e$. Note that $e \leq n$, so that $d \geq 2$.*

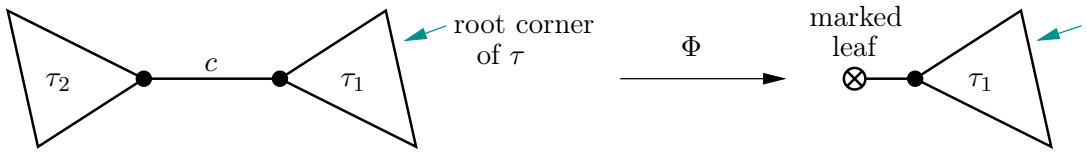


FIGURE 3. The transformation Φ, acting on an edge-centred tree τ.

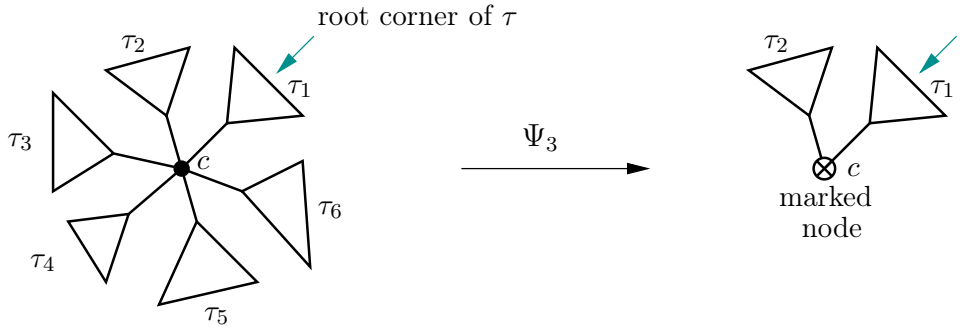


FIGURE 4. The transformation Ψ_d, with $d = 3$, acting on a node-centred tree τ of central degree $\ell = 6$.

If e is odd and $e = n$ (so that n is odd and $d = 2$), all trees of $\mathrm{T}(n)$ fixed by R^e have a central edge, and the above transformation Φ induces a bijection between these trees and trees of $\mathrm{T}\left(\frac{n+1}{2}\right)$ having a marked non-root leaf.

If e is odd and $e < n$, then no tree of $\mathrm{T}(n)$ is fixed by R^e.

If e is even (equivalently, $d \mid n$), then all trees of $\mathrm{T}(n)$ fixed by R^e have a central node, the degree of which is divisible by d, and the above transformation Ψ_d induces a bijection between these trees and trees of $\mathrm{T}\left(\frac{e}{2}\right) = \mathrm{T}\left(\frac{n}{d}\right)$ having a marked node.

Proof. We first observe that $\tau \in \mathrm{T}(n)$ is fixed by R^e if and only if any tree of the form $R^a(\tau)$ is fixed by R^e.

Let $\tau \in \mathrm{T}(n)$ be an edge-centred tree, as on the left of Figure 3, and assume that it is fixed by R^e for a positive divisor e of $2n$. We use the notation τ_1, τ_2 of this figure. Then the tree τ' obtained by re-rooting τ at the corner that follows τ_1 (in counterclockwise order) is also fixed by R^e. Since $e < 2n$, it must be that R^e moves the root corner of τ' to the corner that follows τ_2. This implies in turn that τ_1 and τ_2 are copies of the same tree, that n is odd, and that $e = n$, so that R^e performs a half-turn rotation. The tree $\Phi(\tau)$ then has $\frac{n+1}{2}$ edges. Conversely, any tree $\tau^\times$ of $\mathrm{T}\left(\frac{n+1}{2}\right)$ with a marked leaf allows us to reconstruct an edge-centred tree τ with n edges that is fixed by R^n: we glue two copies of $\tau^\times$ at their marked leaf, and then erase this leaf and one of the root corners.

Now let $\tau \in \mathrm{T}(n)$ be a node-centred tree, with a centre c of degree ℓ, as on the left of Figure 4, and assume that it is fixed by R^e. We use the notation $\tau_1, \dots, \tau_\ell$ of this figure. Then the tree τ' obtained by re-rooting τ at the corner that is incident to c and precedes τ_1 is also fixed by R^e. Hence R^e moves the root corner of τ' to another corner incident to c, and in particular e is even. Say that R^e moves the root corner of τ' to the corner that follows τ_i. Clearly, the $(2n/e)$-fold application of the transformation R^e to τ' sends the root back to its original position. Writing $d = 2n/e$ as in the statement

of the proposition, we see that this implies that $d \cdot i = \ell$ and that the trees τ_a and τ_{a+i} are copies of one another, for all a (taken modulo ℓ). Note that the tree $\Psi_d(\tau)$ then consists of $\tau_1, \ldots, \tau_i$ and has $n/d = e/2$ edges (and c as a marked node, by definition of Ψ_d). Conversely, starting from a tree $\tau^\times$ in $\mathrm{T}\left(\frac{e}{2}\right)$ with a marked node c, we reconstruct a tree τ fixed by R^e as follows: we label the subtrees of τ' attached to c by $\tau_1, \ldots, \tau_i$, with τ_1 containing the root edge. We then attach d copies of the i-tuple $\tau_1, \ldots, \tau_i$ to a central node, in this order. At the end, we only retain one of the root corners.

To summarise: for a positive divisor e of n, an edge-centred tree with n edges can only be invariant by R^e if $e = n$ and n is odd. A node-centred tree with n edges can only be invariant by R^e if e is even. The proposition now follows by putting all the above observations together. □

The following proposition relates trees fixed by a power of R_l, R_i or R_δ to trees fixed by a power of R.

Proposition 4. *Let τ be a rooted tree with n edges and k leaves, rooted at a leaf, and let e be a positive divisor of k with $e < k$. Write $d := k/e$. Then*

$$R_l^e(\tau) = \tau \quad \text{if and only if} \quad R^f(\tau) = \tau,$$

with $f = 2n/d$. In particular, such trees can only exist if $e = k/2$ with k and $n+1$ even, or if k/e divides n.

Let τ be a rooted tree with n edges and k leaves, rooted at a non-leaf, and let e be a positive divisor of $2n-k$ with $e < 2n-k$. Write $d = (2n-k)/e$. Then

$$R_i^e(\tau) = \tau \quad \text{if and only if} \quad R^f(\tau) = \tau,$$

with $f = 2n/d$. In particular, such trees can only exist if $e = (2n-k)/2$ with k and $n+1$ even, or if $(2n-k)/e$ divides n.

Let $\delta \geq 1$. Let τ be a rooted tree with n nodes and n_δ nodes of degree δ, rooted at a node of degree δ. Let e be a positive divisor of δn_δ with $e < \delta n_\delta$. Write $d = \delta n_\delta / e$. Then

$$R_\delta^e(\tau) = \tau \quad \text{if and only if} \quad R^f(\tau) = \tau,$$

with $f = 2n/d$. In particular, such trees can only exist if $e = \delta n_\delta/2$ with δn_δ and $n+1$ even, or if $\delta n_\delta / e$ divides n.

Proof. We only prove the first statement, since the other proofs are similar. Let $\ell_0, \ell_1, \ldots, \ell_{k-1}$ be the k leaves of τ, starting from the root leaf ℓ_0, in counterclockwise order around τ (cf. Figure 5). By definition, R_l^e moves the root of τ to ℓ_e. Given that $de = k$, the d-fold application of the transformation R_l^e to τ puts the root back in its original place. Let f be the length of the path that joins ℓ_0 to ℓ_e when walking around τ in counterclockwise order (we consider edges to have unit length). Then R^f moves ℓ_0 to ℓ_e, so that $R^f(\tau) = R_l^e(\tau) = \tau$. The d-fold repetition of the operation R^f, starting from τ, brings the root back to its original place, so $f = 2n/d$.

Conversely, a tree τ of $\mathrm{T}_l(n,k)$ that is fixed by $R^{2n/d}$ is fixed by R_l^e with $e = k/d$, since both transformations move the root to the same position.

The conditions on e come from the fact that trees of $\mathrm{T}(n)$ fixed by R^f can only exist if $f = n$ with n odd or if f is even, by Proposition 3. That is, with $d = 2n/f = k/e$, if $d = 2$ with n odd, or if d divides n. □

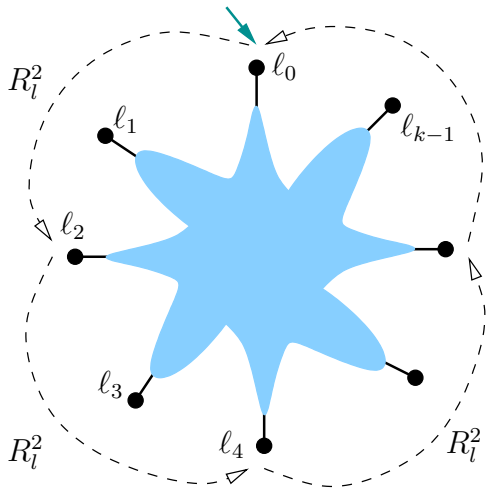


FIGURE 5. A tree of $\mathrm{T}_l(n,k)$ fixed by R_l^e, here with $k=8$, $e=2$ and $d=4$.

5. Ordinary tree rotation

In this section, we consider the ordinary rotation R, which moves the root corner of a rooted tree to the next corner in counterclockwise order (see Figure 2). This rotation yields an action of the cyclic group of order $2n$ on trees of $\mathrm{T}(n)$.

In the statement of the theorem below and as well in the rest of the article, we use the standard q-notations for *q-integers* and *q-binomial coefficients*:

$$[M]_q := \frac{1-q^M}{1-q},$$
$$[M]_q! := [1]_q\,[2]_q\cdots[M]_q \quad \text{and} \quad [0]_q! := 1,$$
$$\begin{bmatrix} M \\ N \end{bmatrix}_q := \frac{[M]_q!}{[N]_q!\,[M-N]_q!}, \quad \text{for } 0 \le N \le M.$$

By construction, the above quantities equal M, $M!$, and $\binom{M}{N}$, respectively, when $q=1$.

5.1. Trees with prescribed number of nodes

Theorem 5. *Let n be a positive integer. The cyclic group of order $2n$ acts on $\mathrm{T}(n)$ by the rotation R, and the triple*

$$\left(\mathrm{T}(n), \langle R\rangle, \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Example 2. For $n=3$, the above theorem gives the polynomial $P(q) = q^6+q^4+q^3+q^2+1$. This cyclic sieving phenomenon was already checked in Example 1.

Proof of Theorem 5. The fact that $\frac{1}{[n+1]_q}\left[\begin{smallmatrix} 2n \\ n \end{smallmatrix}\right]_q$ is a polynomial in q with non-negative coefficients goes back to MacMahon, who showed that it is the generating function for — what he calls — *lattice permutations*, objects that are in trivial bijection with Dyck paths, and which are enumerated with respect to — what MacMahon calls —

the *greater index*, which was later renamed the "major index" by Foata. We refer the interested reader to [13, p. 256].

Now let ω be a primitive $2n$-th root of unity. We have to prove

$$\mathrm{Fix}_{R^e}(\mathrm{T}(n)) = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q\Bigg|_{q=\omega^e} \tag{5.1}$$

for all integers e. By Lemma 2, it suffices to prove this for values of $e \in [\![0, 2n-1]\!]$ equal to 0 or dividing $2n$. Lemma 6 below gives the corresponding values of the right-hand side of (5.1).

We begin with $e = 0$. All trees in $\mathrm{T}(n)$ are invariant under $R^0 = \mathrm{id}$, and we have $|\mathrm{T}(n)| = \frac{1}{n+1}\binom{2n}{n}$ by (2.1). As observed above when defining q-integers, q-factorials and q-binomials, this is indeed the value of the right-hand side of (5.1) at $q = 1$.

We now assume that $0 < e < 2n$, with e a divisor of $2n$. We rely on Proposition 3, which describes the structure of trees fixed by R^e.

If e is odd and equals n, there are as many trees in $\mathrm{T}(n)$ fixed by R^e as trees of $\mathrm{T}\big(\frac{n+1}{2}\big)$ with a marked non-root leaf. The latter trees can be constructed as follows: we start with a tree of $\mathrm{T}\big(\frac{n-1}{2}\big)$, insert at the root corner a new edge ending with a marked leaf (the resulting tree has $\frac{n+1}{2}$ edges and $n+1$ corners), forget this root corner, and create a new root corner in one of the n corners of the tree that are not incident to the marked leaf. According to (2.1), this gives

$$\mathrm{Fix}_{R^e}(\mathrm{T}(n)) = n \times \left|\mathrm{T}\left(\frac{n-1}{2}\right)\right| = n \times \frac{1}{\frac{n+1}{2}}\binom{2\frac{n-1}{2}}{\frac{n-1}{2}}.$$

This is equivalent to the expression in the second alternative of (5.2), and, hence, proves (5.1) in this case.

For odd values of e distinct from n, no tree of $\mathrm{T}(n)$ is fixed by R^e. This proves (5.1) by the last alternative of (5.2) in that case.

Finally, if e is even and positive, Proposition 3 tells us that there are as many trees in $\mathrm{T}(n)$ fixed by R^e as trees of $\mathrm{T}(e/2)$ having a marked node. According to (2.1), with n replaced by $e/2$, we conclude that

$$\mathrm{Fix}_{R^e}(\mathrm{T}(n)) = \big(1+\tfrac{e}{2}\big) \times \frac{1}{1+\frac{e}{2}}\binom{2\frac{e}{2}}{\frac{e}{2}} = \binom{e}{\frac{e}{2}}.$$

This coincides with the expression in the third alternative of (5.2), and, hence, proves (5.1) also in this case.

The proof of the theorem is now complete (assuming Lemma 6 below). □

Lemma 6. *Let ω be a primitive $2n$-th root of unity, and let e be a non-negative integer $< 2n$ equal to 0 or dividing $2n$. Then we have*

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q\Bigg|_{q=\omega^e} = \begin{cases} \frac{1}{n+1}\binom{2n}{n}, & \text{if } e = 0,\\ \binom{n}{\frac{n+1}{2}}, & \text{if } e = n \text{ and } n \text{ is odd},\\ \binom{e}{\frac{e}{2}}, & \text{if } e \text{ is even and positive},\\ 0, & \text{otherwise.}\end{cases} \tag{5.2}$$

Remark. If we write $d = 2n/e$, the second case can be described by "$d = 2$ with n odd", and the third one by "$d \mid n$". This is closer to the descriptions used later in the numerous variants of this first cyclic sieving phenomenon (see for instance Lemmas 15 or 18).

Proof of Lemma 6. The result is obvious for $e = 0$, so let us assume $e > 0$. We will use two basic facts to evaluate the above q-Catalan polynomial at (some) roots of unity. Let $d \geq 2$ and let ρ be a primitive d-th root of unity. Let α, β be two positive integers. Then:

- $1 - q^\alpha$ vanishes at ρ if and only if α is a multiple of d; that is,

$$\lim_{q\to\rho} (1 - q^\alpha) = 0 \quad \text{if and only if} \quad \alpha \equiv 0 \pmod{d}, \tag{5.3}$$

  and in this case ρ is a simple root of this polynomial,
- if $\alpha \equiv \beta \pmod{d}$, then

$$\lim_{q\to\rho} \frac{[\alpha]_q}{[\beta]_q} = \lim_{q\to\rho} \frac{1-q^\alpha}{1-q^\beta} = \begin{cases} \frac{\alpha}{\beta}, & \text{if } \alpha \equiv \beta \equiv 0 \pmod{d}, \\ 1, & \text{otherwise.} \end{cases} \tag{5.4}$$

We leave the proof of these two facts to the reader; see also [29, Lem. 2.4]. The first point implies in particular that ρ is a root of multiplicity $\lfloor \alpha/d \rfloor$ of $[\alpha]_q!$.

Now let e be a divisor of $2n$ with $e \in [\![1, 2n-1]\!]$, so that in fact $e \in [\![1, n]\!]$. Then $\rho := \omega^e$ is a primitive d-th root of unity for $d = 2n/e$, with $d \geq 2$. The value of the q-Catalan polynomial at ρ is non-zero if and only if there is an equal number of factors of the form $[\alpha]_q$ with $d \mid \alpha$ in the numerator $[2n]_q!$ and in the denominator $[n]_q![n+1]_q!$; that is, if and only if

$$\left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n+1}{d} \right\rfloor = e - \left\lfloor \frac{e}{2} \right\rfloor - \left\lfloor \frac{e}{2} + \frac{1}{d} \right\rfloor = 0.$$

One readily checks that this holds if and only if e is even or if e is odd and equals n. This already proves the fourth case of the lemma.

Let us now examine more closely the first two cases. If $e = n$ is odd (so that $\rho = -1$), we write the q-Catalan polynomial as

$$\frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{1-q}{1-q^n} \cdot \frac{1-q^{2n}}{1-q^{n-1}} \cdot \frac{1-q^{2n-1}}{1-q^{n-2}} \cdots \frac{1-q^{n+2}}{1-q}. \tag{5.5}$$

Now we apply (5.4) to each of the above ratios and get

$$\lim_{q\to\rho} \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{2n}{n-1} \cdot \frac{2n-2}{n-3} \cdots \frac{n+3}{2}.$$

The second case of the lemma follows.

Finally, let e be even and divide $2n$. Then, with $d = 2n/e$, the number d divides n. We rewrite the q-Catalan polynomial as

$$\frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{1-q^{2n}}{1-q^n} \cdot \frac{1-q^{2n-1}}{1-q^{n-1}} \cdots \frac{1-q^{n+2}}{1-q^2}. \tag{5.6}$$

We now apply (5.4) to each of the above ratios and obtain

$$\lim_{q\to\rho}\frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q = \frac{2n}{n}\cdot\frac{2n-d}{n-d}\cdots\frac{n+d}{d}$$
$$= \frac{e}{e/2}\cdot\frac{e-1}{e/2-1}\cdots\frac{e/2+1}{1},$$

which proves the third case of the lemma. □

Remark. We have chosen to detail a typical proof of the evaluation of a polynomial involving q-factorials at roots of unity, but the q-Lucas theorem often offers an interesting shortcut for such evaluations [28, Thm. 2.2]. This theorem states that if $n = dn_1 + n_0$ and $k = dk_1 + k_0$ with $0 \le n_0, k_0 < d$, then

$$\begin{bmatrix}n\\ k\end{bmatrix}_q = \binom{n_1}{k_1}\begin{bmatrix}n_0\\ k_0\end{bmatrix}_q \mod \Phi_d,$$

where $\Phi_d(q)$ denotes the d-th cyclotomic polynomial in q. Recall that the roots of this polynomial are the primitive d-th roots of unity. Completed with the fact that $[n]_q = [n_0]_q \bmod \Phi_d$, this gives, for instance when $d = 2$ and n is odd:

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q\Bigg|_{q=-1} = \frac{1}{[2n+1]_q}\begin{bmatrix}2n+1\\ n\end{bmatrix}_q\Bigg|_{q=-1}$$
$$= \frac{1}{[1]_q}\binom{n}{\frac{n-1}{2}}\begin{bmatrix}1\\ 1\end{bmatrix}_q\Bigg|_{q=-1} = \binom{n}{\frac{n-1}{2}}.$$

5.2. Trees with prescribed number of nodes and leaves

Next we turn to the subset $\mathrm{T}(n,k)$ of $\mathrm{T}(n)$, consisting of all rooted plane trees with n edges and k leaves. Figure 2 illustrates the cases $n = 3$, $k = 3$ and $n = 3, k = 2$.

Theorem 7. *Let n and k be positive integers. The cyclic group of order $2n$ acts on $\mathrm{T}(n,k)$ by the rotation R, and the triple*

$$\left(\mathrm{T}(n,k), \langle R\rangle, \frac{1+q^n}{[n-1]_q}\begin{bmatrix}n-1\\ k-2\end{bmatrix}_q\begin{bmatrix}n\\ k\end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Example 3. The above theorem gives the polynomial $1+q^3$ for $\mathrm{T}(3,3)$ and $1+q^2+q^4$ for $\mathrm{T}(3,2)$. One can indeed check the cyclic sieving phenomenon with Figure 2.

Remark. We explain in Subsection 11.1 how Theorem 5 (dealing with $\mathrm{T}(n)$) can be derived from the above theorem by summation over k. The argument is borrowed from [2], where an equivalent cyclic sieving phenomenon is described in terms of non-crossing perfect matchings.

Proof of Theorem 7. Polynomiality and non-negativity of coefficients of the cyclic sieving polynomial are proved in Lemma 8 below.

Let ω be a primitive $2n$-th root of unity. We have to prove

$$\mathrm{Fix}_{R^e}(\mathrm{T}(n,k)) = \frac{1+q^n}{[n-1]_q}\begin{bmatrix}n-1\\ k-2\end{bmatrix}_q\begin{bmatrix}n\\ k\end{bmatrix}_q\Bigg|_{q=\omega^e} \tag{5.7}$$

for all integers e. By Lemma 2, it suffices to prove this for values of $e \in [\![0, 2n-1]\!]$ equal to 0 or dividing $2n$. Lemma 9 below gives the corresponding values of the right-hand side of (5.7).

We begin with $e = 0$. All trees in $\mathrm{T}(n,k)$ are invariant under $R^0 = \mathrm{id}$. By (2.4) we know the cardinality of $\mathrm{T}(n,k)$, which coincides with the value at $q = 1$ of the right-hand side of (5.7).

We now assume that $0 < e < 2n$, with e a divisor of $2n$. Again, we rely on Proposition 3, which describes the structure of trees fixed by R^e. We revisit the proof of Theorem 5 by keeping track of the number of leaves.

If e is odd and equals n, the map Φ sends trees of $\mathrm{T}(n,k)$ fixed by R^e bijectively to trees of $\mathrm{T}\big(\frac{n+1}{2}, 1+\frac{k}{2}\big)$ with a marked non-root leaf. In particular, k must be even. According to (2.4), with n replaced by $\frac{n+1}{2}$ and k replaced by $\frac{k}{2}+1$, there are

$$\big(\tfrac{k}{2}+1\big)\,\frac{2}{\frac{n+1}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}-1}\binom{\frac{n+1}{2}}{\frac{k}{2}+1}$$

ways of choosing a tree of $\mathrm{T}\big(\frac{n+1}{2}, 1+\frac{k}{2}\big)$ with a marked leaf. We must however subtract the number of those trees where the marked leaf is the root. By (2.2) with n replaced by $\frac{n+1}{2}$ and k replaced by $\frac{k}{2}+1$, this number equals

$$\frac{1}{\frac{n+1}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}}.$$

The difference of these two numbers simplifies to

$$\frac{2\frac{n+1}{2}-1}{\frac{n+1}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}},$$

which is equivalent to the expression in the second alternative of (5.13), and, hence, proves (5.7) in this case.

Finally, if e is even with $e = 2n/d$, the map Ψ_d sends trees of $\mathrm{T}(n,k)$ fixed by R^e bijectively to trees of $\mathrm{T}(e/2)$ having

- either $\frac{k}{d}$ leaves and a marked node that is not a leaf,
- or $1+\frac{k}{d}$ leaves, one of which is marked.

The second case occurs for trees of $\mathrm{T}(n,k)$ (fixed by R^e) in which the centre is a node of degree exactly d. Observe that in all cases d must divide k. According to (2.4), there are

$$\big(\tfrac{e}{2}+1-\tfrac{k}{d}\big)\cdot\frac{2}{\frac{e}{2}-1}\binom{\frac{e}{2}-1}{\frac{k}{d}-2}\binom{\frac{e}{2}}{\frac{k}{d}}$$

trees of the first type, and

$$\big(\tfrac{k}{d}+1\big)\cdot\frac{2}{\frac{e}{2}-1}\binom{\frac{e}{2}-1}{\frac{k}{d}-1}\binom{\frac{e}{2}}{\frac{k}{d}+1}$$

trees of the second type. The sum of these two numbers is equal to the expression in the third alternative of (5.13), and is thus proving (5.7) also in this case.

For all other triples (n,k,e), no tree of $\mathrm{T}(n,k)$ is fixed by R^e. This proves (5.7) in the last alternative of (5.13).

The proof of the theorem is now complete (assuming Lemmas 8 and 9 below). □

Lemma 8. *Let n and k be positive integers. Then the expression*

$$\frac{1+q^n}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{5.8}$$

is a polynomial in q with non-negative coefficients.

Proof. We start by recalling the well-known fact that

$$[n]_q = \frac{q^n-1}{q-1} = \prod_{d\geq 2,\ d\mid n} \Phi_d(q),$$

where $\Phi_d(q)$ denotes the d-th cyclotomic polynomial in q. Consequently,

$$\frac{1+q^n}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[2n]_q}{[n]_q\,[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{d\geq 2} \Phi_d(q)^{e_d}, \tag{5.9}$$

with

$$e_d = \chi\big(d \mid 2n\big) - \chi\big(d \mid n\big) + \left\lfloor \frac{n-2}{d} \right\rfloor + \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{k-2}{d} \right\rfloor - \left\lfloor \frac{k}{d} \right\rfloor - \left\lfloor \frac{n-k+1}{d} \right\rfloor - \left\lfloor \frac{n-k}{d} \right\rfloor, \tag{5.10}$$

where $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ otherwise. Note that e_d is the multiplicity of a primitive d-th root of unity as a root of (5.9) (where a negative multiplicity would indicate a pole). We want to prove that $e_d \geq 0$ for all $d \geq 2$.

From now on, let $d \geq 2$. Abusing notation, let us write $N = \{(n-2)/d\}$ and $K = \{(k-2)/d\}$, where $\{\alpha\} := \alpha - \lfloor\alpha\rfloor$ denotes the fractional part of α. Note that dN and dK are integers. We use N and K to rewrite all terms of (5.10) with the integer parts ("floors") of $(n-2)/d$ and $(k-2)/d$, plus N and K. For instance, we have

$$\left\lfloor \frac{n-k+1}{d} \right\rfloor = \left\lfloor \frac{n-2}{d} - \frac{k-2}{d} + \frac{1}{d} \right\rfloor = \left\lfloor \frac{n-2}{d} \right\rfloor - \left\lfloor \frac{k-2}{d} \right\rfloor + \left\lfloor N - K + \frac{1}{d} \right\rfloor.$$

We thus may rewrite Equation (5.10) as

$$e_d = \chi\big(d \mid 2n\big) - \chi\big(d \mid n\big) + \left\lfloor N + \frac{2}{d} \right\rfloor - \left\lfloor K + \frac{2}{d} \right\rfloor - \left\lfloor N - K + \frac{1}{d} \right\rfloor - \lfloor N - K \rfloor. \tag{5.11}$$

For the case where $d = 2$, this reduces to

$$\begin{aligned} e_2 &= 1 - \chi\big(2 \mid n\big) + \lfloor N+1 \rfloor - \lfloor K+1 \rfloor - \left\lfloor N - K + \frac{1}{2} \right\rfloor - \lfloor N - K \rfloor \\ &= 1 - \chi\big(2 \mid n\big) - \left\lfloor N - K + \frac{1}{2} \right\rfloor - \lfloor N - K \rfloor, \end{aligned}$$

because $\lfloor N \rfloor = \lfloor K \rfloor = 0$. Note that N and K can only take the values 0 and $1/2$ in the current case, depending on whether n and k are even or odd. One checks that the exponent e_2 is non-negative in all four cases. More precisely, $e_2 = 0$ if k is even, and $e_2 = 1$ otherwise.

Now let $d > 2$. By (5.11), the exponent e_d may be bounded from below as follows:

$$e_d \geq \left\lfloor N + \frac{2}{d} \right\rfloor - \left\lfloor K + \frac{2}{d} \right\rfloor - \left\lfloor N - K + \frac{1}{d} \right\rfloor - \lfloor N - K \rfloor . \tag{5.12}$$

Given that $N - K < 1$, the only terms that can be negative are $-\left\lfloor K + \frac{2}{d} \right\rfloor$ and $-\left\lfloor N - K + \frac{1}{d} \right\rfloor$, and if they are then they are equal to -1. If $N < K$, then the last term on the right-hand side equals $+1$. This is already sufficient to guarantee non-negativity of the right-hand side of (5.12) since under the condition $N < K$ the summand $-\left\lfloor N - K + \frac{1}{d} \right\rfloor$ cannot be negative. Consequently e_d is non-negative in this case. We may therefore assume from now on that $N \geq K$.

We have $-\left\lfloor N - K + \frac{1}{d} \right\rfloor = -1$ if and only if $N = 1 - \frac{1}{d}$ and $K = 0$. The inequality (5.12) then simplifies to $e_d \geq 1 - 0 - 1 - 0 = 0$, which indeed implies non-negativity of e_d.

On the other hand, we have $-\left\lfloor K + \frac{2}{d} \right\rfloor = -1$ if and only if $K = 1 - \frac{2}{d}$ or $K = 1 - \frac{1}{d}$. However, by our assumption $N \geq K$, this forces $\left\lfloor N + \frac{2}{d} \right\rfloor$ to equal $+1$. Consequently, the inequality (5.12) simplifies to $e_d \geq 1 - 1 - 0 - 0 = 0$, again implying non-negativity of e_d.

This proves the polynomiality part of the lemma.

We will now show that the coefficients of the polynomial in (5.8) are non-negative. In order to do this, we largely follow an argument that one finds in [4, Thm. 2] and [26, Prop. 10.1(iii)]. First, as an easy corollary of the classical result that q-binomial coefficients are reciprocal[2] and unimodal[3] polynomials, and the fact that the product of reciprocal and unimodal polynomials is also reciprocal and unimodal (cf. [3, Thm. 3.9]), one obtains that the product

$$\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q$$

is a reciprocal and unimodal polynomial in q. It is of degree A, say, where $A = (k-2)(n-k) + k(n-k+1)$ (the explicit value of A being however irrelevant). Hence, the coefficient of q^i in

$$(1-q) \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q$$

is non-negative for $0 \leq i \leq \lceil A/2 \rceil$. Therefore the same must be true for

$$\frac{1+q^n}{1-q^{n+1}} \times (1-q) \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \frac{1+q^n}{[n-1]_q} \begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q ,$$

which is a priori a *formal power series* in q. However, we know from above that this expression is actually a polynomial. Since it is reciprocal and has degree A, we infer that also the other coefficients of the polynomial must be non-negative.

This completes the proof of the lemma. □

[2] A polynomial $P(q)$ in q of degree m is called *reciprocal* if $P(q) = q^m P(1/q)$.

[3] A polynomial $P(q) = \sum_{i=0}^{m} p_i q^i$ is called *unimodal* if there is an integer r with $0 \leq r \leq m$ and $0 \leq p_0 \leq \cdots \leq p_r \geq \cdots \geq p_m \geq 0$. It is well-known that q-binomial coefficients are unimodal; see [31, Ex. 7.75.d].

Lemma 9. *Let ω be a primitive $2n$-th root of unity, and let e be a non-negative integer $< 2n$ equal to 0 or dividing $2n$. In the latter case, write $d := 2n/e$, so that $d \geq 2$. Then we have*

$$\frac{1+q^n}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=\omega^e} = \begin{cases} \frac{2}{n-1}\binom{n-1}{k-2}\binom{n}{k}, & \text{if } e=0, \\ \frac{n}{\frac{n+1}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}}, & \text{if } d=2 \text{ with } n+1 \text{ and } k \text{ even}, \\ 2\binom{\frac{e}{2}-1}{\frac{k}{d}-1}\binom{\frac{e}{2}}{\frac{k}{d}}, & \text{if } d \mid n \text{ and } d \mid k, \\ 0, & \text{otherwise.} \end{cases} \tag{5.13}$$

Proof. The case $e = 0$ being obvious, we assume that e is a positive divisor of $2n$ with $e < 2n$. Then ω^e is a primitive d-th root of unity, with $d = 2n/e$. Its minimal polynomial is the cyclotomic polynomial $\Phi_d(q)$, as defined in the proof of Lemma 8 just above.

We will first determine for which values of d the specialisation on the left-hand side of (5.13) is non-zero. This will settle the fourth case in (5.13). We return to the proof of Lemma 8, and use the same notation N, K, e_d. The specialisation of (5.13) at a primitive d-th root of unity is non-zero if and only if the exponent e_d is zero. In the proof of Lemma 8 we proved that $e_d \geq 0$ for all d, now we revisit this proof to decide when $e_d = 0$.

For $d = 2$, we already established that $e_d = 0$ if and only if k is even. This corresponds to the second case of (5.13) when n is odd, and to the third case of (5.13) (with $d = 2$, $e = n$) when n is even.

For $d > 2$, a comparison of (5.11) and (5.12) tells us that $e_d = 0$ if and only if $d \mid n$ (that is, e is even, or equivalently $N = \{(n-2)/d\} = 1 - \frac{2}{d}$) and the right-hand side of (5.12) vanishes. So let us focus on the right-hand side of (5.12). If $N < K$, that is, $K = 1 - \frac{1}{d}$, then this expression is $1-1-0-(-1) = 1 > 0$, so $e_d > 0$. If $K \leq N = 1 - \frac{2}{d}$, then this expression is

$$1 - \left\lfloor K + \frac{2}{d} \right\rfloor - 0 - 0.$$

It equals 0 if and only if $K = N = 1 - \frac{2}{d}$, or equivalently $d \mid k$ (since $K = \{(k-2)/d\}$). This corresponds to the third case of (5.13) with $d > 2$.

It remains to evaluate the left-hand side of (5.13) in the second and third case. This is an easy task based on the simple fact (5.4), as in the proof of Lemma 6. The key idea, again, is to form ratios of terms of the form $1 - q^a$ where the exponents of q differ by a multiple of d, as in (5.5) and (5.6). □

6. External tree rotation

In this section, we consider the set $\mathrm{T}_l(n,k)$ of all rooted plane trees with n edges and k leaves, where the root node is one of the leaves. The cardinality of $\mathrm{T}_l(n,k)$ is given in (2.2). The external rotation R_l defined in Section 4 acts on these trees by moving the root to the next leaf found in counterclockwise order around the tree.

Theorem 10. *Let n and k be positive integers. The cyclic group of order k acts on $\mathrm{T}_l(n,k)$ by the rotation R_l, and the triple*

$$\left(\mathrm{T}_l(n,k), \langle R_l\rangle, \frac{1}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2\end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1\end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Example 4. The sets $\mathrm{T}_l(3,3)$ and $\mathrm{T}_l(3,2)$ are singletons (see Figure 2), and the above theorem gives the polynomial $P(q)=1$ in both cases.

Proof of Theorem 10. Polynomiality and non-negativity of coefficients of the cyclic sieving polynomial are proved in Lemma 11 below.

Let ω be a primitive k-th root of unity. We have to prove

$$\mathrm{Fix}_{R_l^e}(\mathrm{T}_l(n,k)) = \frac{1}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2\end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1\end{bmatrix}_q \Bigg|_{q=\omega^e} \tag{6.1}$$

for all integers e. By Lemma 2, it suffices to prove this for values of $e \in [\![0, k-1]\!]$ equal to 0 or dividing k. Lemma 12 below gives the corresponding values of the right-hand side of (6.1).

We begin with $e=0$. Clearly, all trees in $\mathrm{T}_l(n,k)$ are invariant under $R_l^0 = \mathrm{id}$. By (2.2) we know the cardinality of $\mathrm{T}_l(n,k)$, which coincides with the value at $q=1$ of the right-hand side of (6.1).

We now assume that $0 < e < k$, with e a divisor of k. We rely in what follows on Proposition 4, which relates the rotations R_l and R, and on Proposition 3, which characterises trees fixed by a given power of R. Recall in particular that the only possible values of e are $e=k/2$ with $n+1$ and k even (corresponding to $f=n$ in the notation of Proposition 4), and those for which k/e divides n (corresponding to even values of f in the notation of Proposition 4).

If $e=k/2$, with k and $n+1$ even, Proposition 4 tells us that we just have to count trees of $\mathrm{T}_l(n,k)$ fixed by R^n. By Proposition 3, the map Φ sends them bijectively to trees of $\mathrm{T}_l(\frac{n+1}{2}, 1+\frac{k}{2})$ having a marked non-root leaf. By (2.2) with n replaced by $\frac{n+1}{2}$ and k replaced by $\frac{k}{2}+1$, we thus have:

$$\mathrm{Fix}_{R_l^e}(\mathrm{T}_l(n,k)) = \tfrac{k}{2}\cdot \frac{1}{\frac{n+1}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}}.$$

This is equivalent to the expression in the second alternative of (6.3). This proves (6.1) in the present case.

Now, if $d := k/e$ divides n, let us write $f = 2n/d$. By Proposition 4, we need to count trees of $\mathrm{T}_l(n,k)$ fixed by R^f. By Proposition 3, the map Ψ_d sends them bijectively to trees of $\mathrm{T}_l(\frac{f}{2})$ having

- either $\frac{k}{d} = e$ leaves and a marked node that is not a leaf,
- or $1+\frac{k}{d} = 1+e$ leaves, one of which being marked and distinct from the root.

According to (2.2), there are

$$\left(1+\tfrac{f}{2}-e\right)\frac{1}{\frac{f}{2}-1}\binom{\frac{f}{2}-1}{e-2}\binom{\frac{f}{2}-1}{e-1}$$

trees of the first type, and

$$e \cdot \frac{1}{\frac{f}{2}-1}\binom{\frac{f}{2}-1}{e-1}\binom{\frac{f}{2}-1}{e}$$

trees of the second type. The sum of these two numbers is equal to the expression in the third alternative of (6.3), and is thus proving (6.1) also in this case.

For all other triples (n,k,e), no tree of $\mathrm{T}_l(n,k)$ is fixed by R_l^e. This proves (6.1) in the last alternative of (6.3). □

Lemma 11. *Let n and k be positive integers. Then the expression*

$$\frac{1}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \tag{6.2}$$

is a polynomial in q with non-negative coefficients.

Proof. This follows immediately from the fact that the expression (6.2) has — up to a power of q — a combinatorial interpretation as generating function for certain lattice paths; see [13, Eq. (4.1)] or [19, Eq. (5.8) with $t=1$, n replaced by $n-1$, and k replaced by $k-1$]. □

Lemma 12. *Let $n \geq 1$ and $k \geq 2$. Let ω be a primitive k-th root of unity, and let e be a non-negative integer $< k$ being equal to 0 or dividing k. In the latter case, write $d := k/e$, so that $d \geq 2$. Then we have*

$$\frac{1}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \Bigg|_{q=\omega^e} = \begin{cases} \frac{1}{n-1}\binom{n-1}{k-2}\binom{n-1}{k-1}, & \text{if } e=0, \\ \binom{\frac{n+1}{2}-2}{\frac{k}{2}-1}\binom{\frac{n+1}{2}-1}{\frac{k}{2}-1}, & \text{if } d=2 \text{ with } k \text{ and } n+1 \text{ even}, \\ \binom{\frac{n}{d}-1}{e-1}^2, & \text{if } d \mid n, \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Proof. The case $e=0$ being obvious, we assume that e is a positive divisor of k with $e<k$. Then ω^e is a primitive d-th root of unity. By (5.3), its multiplicity, as a root of the conjectured cyclic sieving polynomial, is

$$f_d = \left\lfloor \frac{n-2}{d} \right\rfloor + \left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{k-2}{d} \right\rfloor - \left\lfloor \frac{k-1}{d} \right\rfloor - \left\lfloor \frac{n-k+1}{d} \right\rfloor - \left\lfloor \frac{n-k}{d} \right\rfloor.$$

We introduce again the fractional parts $N = \{(n-2)/d\}$ and $K = \{(k-2)/d\}$. Since by assumption $d \mid k$, we have $K = 1 - \frac{2}{d}$. Then the above equation for f_d can be rewritten as

$$\begin{aligned} f_d &= \left\lfloor N + \frac{1}{d} \right\rfloor - \left\lfloor K + \frac{1}{d} \right\rfloor - \left\lfloor N - K + \frac{1}{d} \right\rfloor - \lfloor N - K \rfloor \\ &= \left\lfloor N + \frac{1}{d} \right\rfloor - 0 - \left\lfloor N - \left(1 - \frac{3}{d}\right) \right\rfloor - \left\lfloor N - \left(1 - \frac{2}{d}\right) \right\rfloor. \end{aligned}$$

The left-hand side of (6.3) will be non-zero if and only if $f_d = 0$. If $N < K$ (which is only possible for $d > 2$), then f_d is 1 or 2. If $N = 1 - \frac{2}{d}$, then $f_d = 0$. By definition of N,

the condition $N = 1 - \frac{2}{d}$ means that $d \mid n$, so that we are in the third case of (6.3). Finally, if $N = 1 - \frac{1}{d}$ (equivalently, $d \mid (n-1)$), we find that $f_d = 0$ if and only if $d = 2$. This corresponds to the second case of (6.3). We have now isolated all values of d for which the left-hand side of (6.3) is non-zero. This already proves the fourth case of (6.3).

It is then an easy task to evaluate the left-hand side of (6.3) in the second and third case, using the simple fact (5.4). The key idea, again, is to form ratios of terms of the form $1-q^a$ where the exponents of q differ by a multiple of d, as in (5.5) and (5.6). □

7. Internal tree rotation

In this section, we consider the set $\mathrm{T}_i(n,k)$ of all rooted plane trees with n edges and k leaves, where the root is not a leaf. The cardinality of $\mathrm{T}_i(n,k)$ is given in (2.3). The internal rotation R_i defined in Section 4 acts on these trees by moving the root to the next corner not incident to a leaf found in counterclockwise order around the tree.

Theorem 13. *Let n and k be positive integers. The cyclic group of order $2n-k$ acts on $\mathrm{T}_i(n,k)$ by the internal rotation R_i, and the triple*

$$\left(\mathrm{T}_i(n,k), \langle R_i\rangle, \frac{[2n-k]_q}{[n]_q\,[n-1]_q}\begin{bmatrix} n-1 \\ k-2\end{bmatrix}_q \begin{bmatrix} n \\ k\end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Example 5. The set $\mathrm{T}_i(3,3)$ is a singleton, while $\mathrm{T}_i(3,2)$ has two elements that R_i exchanges (see Figure 2). The above theorem gives the polynomials 1 and $1+q^2$, respectively, and one checks that they indeed satisfy the cyclic sieving phenomenon.

Proof of Theorem 13. Polynomiality and non-negativity of coefficients of the cyclic sieving polynomial are proved in Lemma 14 below.

Let ω be a primitive $(2n-k)$-th root of unity. We have to prove

$$\mathrm{Fix}_{R_i^e}(\mathrm{T}_i(n,k)) = \frac{[2n-k]_q}{[n]_q\,[n-1]_q}\begin{bmatrix} n-1 \\ k-2\end{bmatrix}_q \begin{bmatrix} n \\ k\end{bmatrix}_q\Bigg|_{q=\omega^e} \tag{7.1}$$

for all integers e. It suffices to prove this for values of $e \in [\![0, 2n-1-k]\!]$ equal to 0 or dividing $2n-k$. Lemma 15 gives the corresponding values of the right-hand side of (7.1).

We begin with $e = 0$. Clearly, all trees in $\mathrm{T}_i(n,k)$ are invariant under $R_i^0 = \mathrm{id}$. By (2.3) we know the cardinality of $\mathrm{T}_i(n,k)$, which coincides with the value at $q=1$ of the right-hand side of (7.1).

We now assume $0 < e < 2n-k$, with e a divisor of $2n-k$. We rely in what follows on Proposition 4, which relates the rotations R_i and R, and on Proposition 3, which characterises trees fixed by a given power of R. Recall in particular that the only possible values of e are $e = n - \frac{k}{2}$, with $n+1$ and k even (corresponding to $f = n$ in the notation of Proposition 4) and those for which $d := (2n-k)/e$ divides n (corresponding to even values of f). Note that, in the latter case, d divides k.

If $e = n - \frac{k}{2}$, with k and $n+1$ even, Proposition 4 tells us that we have to count trees of $\mathrm{T}_i(n,k)$ fixed by R^n. By Proposition 3, the map Φ sends them bijectively to trees of

$\mathrm{T}_i(\frac{n+1}{2}, 1+\frac{k}{2})$ having a marked leaf. By (2.3), the number of such trees is

$$\left(\tfrac{k}{2}+1\right)\cdot\frac{2\,\frac{n+1}{2}-\left(\frac{k}{2}+1\right)}{\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}-1\right)}\binom{\frac{n+1}{2}-1}{\frac{k}{2}-1}\binom{\frac{n+1}{2}}{\frac{k}{2}+1}.$$

This coincides with the expression in the second alternative of (7.8). This proves (7.1) in the present case.

Now, if $d := (2n-k)/e$ divides n, let us write $f := 2n/d$. By Proposition 4, we need to count trees of $\mathrm{T}_i(n,k)$ fixed by R^f. By Proposition 3, the map Ψ_d sends them bijectively to trees τ' with $f/2$ edges and $\frac{k}{d}+\varepsilon$ leaves, where

- either $\varepsilon = 1$, and τ' is rooted at a (marked) leaf,
- or $\varepsilon = 1$, one leaf is marked, and τ' is rooted at a non-leaf,
- or $\varepsilon = 0$, one non-leaf is marked, and τ' is rooted at a non-leaf.

The first two cases occur for trees of $\mathrm{T}_i(n,k)$ (fixed by R^f) in which the central vertex c has degree d exactly. In the first (respectively second) case, the root node is c itself (respectively is different from c).

By (2.2), the number of trees of the first type is

$$\frac{1}{\frac{f}{2}-1}\binom{\frac{f}{2}-1}{\frac{k}{d}-1}\binom{\frac{f}{2}-1}{\frac{k}{d}}.$$

By (2.3), the number of trees of the second type is

$$\left(\tfrac{k}{d}+1\right)\cdot\frac{f-\frac{k}{d}-1}{\frac{f}{2}\left(\frac{f}{2}-1\right)}\binom{\frac{f}{2}-1}{\frac{k}{d}-1}\binom{\frac{f}{2}}{\frac{k}{d}+1},$$

while the number of trees of the third type is

$$\left(\tfrac{f}{2}+1-\tfrac{k}{d}\right)\cdot\frac{f-\frac{k}{d}}{\frac{f}{2}\left(\frac{f}{2}-1\right)}\binom{\frac{f}{2}-1}{\frac{k}{d}-2}\binom{\frac{f}{2}}{\frac{k}{d}}.$$

The sum of these three numbers is equal to the expression in the third alternative of (7.8) and hence proves (7.1) also in this case.

For all other triples (n,k,e), no tree of $\mathrm{T}_i(n,k)$ is fixed by R_i^e. This proves (7.1) in the last alternative of (7.8). □

Lemma 14. *Let $n \geq 1$ and $k \geq 2$. Then the expression*

$$\frac{[2n-k]_q}{[n]_q\,[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{7.2}$$

is a polynomial in q with non-negative coefficients.

Proof. We decompose the polynomial in (7.2) as

$$\frac{1}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k\frac{1+q^{n-k}}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{7.3}$$

We know from Lemma 11 that the first term in this expression is a polynomial in q with non-negative coefficients.

To see that the second term is a polynomial, we proceed as in the proof of Lemma 8. Denoting again by $\Phi_d(q)$ the d-th cyclotomic polynomial, we write

$$\frac{1+q^{n-k}}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \frac{[2n-2k]_q}{[n-k]_q\,[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \prod_{d\geq 2}\Phi_d(q)^{g_d}, \quad (7.4)$$

with

$$g_d = \chi\big(d \mid (2n-2k)\big) - \chi\big(d \mid (n-k)\big) + \left\lfloor \frac{n-2}{d} \right\rfloor + \left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{k-2}{d} \right\rfloor - \left\lfloor \frac{k}{d} \right\rfloor - \left\lfloor \frac{n-k+1}{d} \right\rfloor - \left\lfloor \frac{n-k-1}{d} \right\rfloor. \quad (7.5)$$

From now on, let $d \geq 2$. Abusing notation, here we introduce the fractional parts $N = \{(n-1)/d\}$ and $K = \{(k-2)/d\}$. Then we may rewrite Equation (7.5) as

$$g_d = \chi\big(d \mid (2n-2k)\big) - \chi\big(d \mid (n-k)\big) + \left\lfloor N - \frac{1}{d} \right\rfloor - \left\lfloor K + \frac{2}{d} \right\rfloor - \lfloor N-K \rfloor - \left\lfloor N - K - \frac{2}{d} \right\rfloor. \quad (7.6)$$

For the case where $d = 2$, this reduces to

$$\begin{aligned} g_2 &= 1 - \chi\big(2 \mid (n-k)\big) + \left\lfloor N - \frac{1}{2} \right\rfloor - \lfloor K+1 \rfloor - \lfloor N-K \rfloor - \lfloor N-K-1 \rfloor \\ &= 1 - \chi\big(2 \mid (n-k)\big) + \left\lfloor N - \frac{1}{2} \right\rfloor - 2\lfloor N-K \rfloor. \end{aligned}$$

A simple analysis of the four possible values of (N, K) shows that the exponent g_2 is always non-negative.

Now let $d > 2$. By (7.6), the exponent g_d may be bounded from below as follows:

$$g_d \geq \left\lfloor N - \frac{1}{d} \right\rfloor - \left\lfloor K + \frac{2}{d} \right\rfloor - \lfloor N-K \rfloor - \left\lfloor N - K - \frac{2}{d} \right\rfloor. \quad (7.7)$$

Here, the only terms that can be negative are $\lfloor N - \frac{1}{d} \rfloor$ and $-\lfloor K + \frac{2}{d} \rfloor$, and if they are then they are equal to -1. If $N < K$, then the last two terms on the right-hand side together produce at least $+2$, which is sufficient to guarantee non-negativity of the right-hand side of (7.7), and consequently of g_d. We may therefore assume from now on that $N \geq K$.

We have $\lfloor N - \frac{1}{d} \rfloor = -1$ if and only if $N = 0$. Since we have assumed $N \geq K$, this implies $K = 0$, and therefore the inequality (7.7) simplifies to $g_d \geq -1-0-0-(-1) = 0$, which indeed implies non-negativity of g_d.

On the other hand, we have $-\lfloor K + \frac{2}{d} \rfloor = -1$ if and only if $K = 1 - \frac{2}{d}$ or $K = 1 - \frac{1}{d}$. However, by our assumption $N \geq K$, this forces $-\lfloor N - K - \frac{2}{d} \rfloor$ to equal $+1$. Consequently, the inequality (7.7) simplifies to $g_d \geq 0-1-0-(-1) = 0$, again implying non-negativity of g_d.

This proves the polynomiality part of the lemma.

We will now show that the expression in (7.4) is in fact a polynomial with non-negative coefficients. In order to do this, we proceed as in the proof of Lemma 8. First, as an easy corollary of the earlier used result that q-binomial coefficients are reciprocal

and unimodal polynomials, and the fact that the product of reciprocal and unimodal polynomials is also reciprocal and unimodal, one obtains that the product

$$\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

is a reciprocal and unimodal polynomial in q. It is of degree B, say, where $B = (k-2)(n-k+1)+k(n-k-1)$ (the explicit value of B being irrelevant). Hence, the coefficient of q^i in

$$(1-q)\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

is non-negative for $0 \le i \le \lceil B/2 \rceil$. Therefore the same must be true for the series

$$\frac{1+q^{n-k}}{1-q^{n-1}} \times (1-q)\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \frac{1+q^{n-k}}{[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q,$$

However, we know from above that the last expression is a polynomial. Since it is reciprocal and has degree at most B as soon as $k \ge 2$, we infer that also the other coefficients of the polynomial must be non-negative.

This completes the proof of the lemma. □

Lemma 15. *Let ω be a primitive $(2n-k)$-th root of unity, and let e be a non-negative integer $< 2n-k$ being equal to 0 or dividing $2n-k$. In the latter case, write $d := (2n-k)/e$, so that $d \ge 2$. Then we have*

$$\begin{aligned} &\frac{[2n-k]_q}{[n]_q\,[n-1]_q}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=\omega^e} \\ &\qquad = \begin{cases} \frac{2n-k}{n(n-1)}\binom{n-1}{k-2}\binom{n}{k}, & \text{if } e=0, \\ \frac{2n-k}{n-1}\left(\frac{\frac{n-1}{2}}{\frac{k}{2}-1}\right)\left(\frac{\frac{n-1}{2}}{\frac{k}{2}}\right), & \text{if } d=2 \text{ with } n+1 \text{ and } k \text{ even}, \\ \frac{2n-k}{n}\left(\frac{\frac{n}{d}-1}{\frac{k}{d}-1}\right)\left(\frac{\frac{n}{d}}{\frac{k}{d}}\right), & \text{if } d \mid n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{7.8}$$

Proof. The case $e=0$ being obvious, we assume that e is a positive divisor of $2n-k$ with $e < 2n-k$. Then ω^e is a primitive d-th root of unity. By (5.3), its multiplicity, as a root of the conjectured cyclic sieving polynomial, is

$$h_d := 1 + \left\lfloor \frac{n-2}{d} \right\rfloor + \left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{k-2}{d} \right\rfloor - \left\lfloor \frac{k}{d} \right\rfloor - \left\lfloor \frac{n-k+1}{d} \right\rfloor - \left\lfloor \frac{n-k}{d} \right\rfloor.$$

Introducing the fractional parts $N := \{(n-2)/d\}$ and $K := \{(k-2)/d\}$, this can be rewritten as

$$h_d = 1 + \left\lfloor N + \frac{1}{d} \right\rfloor - \left\lfloor K + \frac{2}{d} \right\rfloor - \left\lfloor N - K + \frac{1}{d} \right\rfloor - \lfloor N - K \rfloor.$$

The left-hand side of (7.8) will be non-zero if and only if $h_d = 0$. The condition $d \mid (2n-k)$ means that $\Delta := 2N - K + \frac{2}{d}$ is an integer, and, given that N and K are in $[0, 1-\frac{1}{d}]$, this integer can only be 0, 1 or 2.

The simplest case is when $\Delta = 2$, which forces $K = 0$ and $N = 1 - \frac{1}{d}$. Then

$$h_d = 1 + 1 - \left\lfloor \frac{2}{d} \right\rfloor - 1 - 0 = 1 - \left\lfloor \frac{2}{d} \right\rfloor.$$

This vanishes only for $d = 2$. Combined with $K = 0$ and $N = 1 - \frac{1}{d} = 1 - \frac{1}{2}$, this implies that k and $n + 1$ are even. This is the second case in (7.8).

Now, if $\Delta = 0$ or 1, we have, by definition of Δ,

$$2N \le K + 1 - \frac{2}{d} \le 2 - \frac{3}{d},$$

so that $N \le 1 - \frac{2}{d}$. Hence

$$\begin{aligned} h_d &= 1 + 0 - \left\lfloor K + \frac{2}{d} \right\rfloor - \left\lfloor \Delta - N - \frac{1}{d} \right\rfloor - \left\lfloor \Delta - N - \frac{2}{d} \right\rfloor \\ &= 1 - \left\lfloor K + \frac{2}{d} \right\rfloor - 2\Delta + \left\lceil N + \frac{1}{d} \right\rceil + \left\lceil N + \frac{2}{d} \right\rceil \\ &= 3 - 2\Delta - \left\lfloor K + \frac{2}{d} \right\rfloor. \end{aligned}$$

Therefore, h_d vanishes if and only if $\Delta = 1$ and either $K = 1 - \frac{2}{d}$ or $K = 1 - \frac{1}{d}$. In the former case, we have $2N = K + 1 - \frac{2}{d} = 2 - \frac{4}{d}$, that is, $N = 1 - \frac{2}{d}$. By definition of N and K, this means that $d \mid n$ and $d \mid k$; this is the third case of (7.8). In the latter case, we have $2N = 2 - \frac{3}{d}$, that is, $N = 1 - \frac{3}{2d}$, which is impossible since dN must be an integer.

We have found all cases for which the left-hand side of (7.8) is non-zero, and thus proved the fourth case of (7.8). It is then an easy task to evaluate the left-hand side of (7.8) in the second and third case, using the simple fact (5.4). □

8. Trees with given vertex degrees under ordinary rotation

Let $\mathbf{n} = (n_1, n_2, \dots)$ be a sequence of non-negative integers satisfying (2.5). In particular, only finitely many of the n_i's are non-zero. In this section we focus on the set $\mathrm{T}(\mathbf{n})$ of rooted plane trees with degree distribution $\mathbf{n}$. The cardinality of $\mathrm{T}(\mathbf{n})$ is given by (2.8), where $n+1 = \sum_i n_i$ is the total number of nodes. Note that $n_m = 0$ for $m > n$. We consider the action on $\mathrm{T}(\mathbf{n})$ of the cyclic group of order $2n$ generated by the ordinary rotation R.

In the statement of the theorem below and as well in the rest of the article, we need the *q-multinomial coefficient* defined by

$$\begin{bmatrix} M \\ N_1, N_2, \dots, N_\ell \end{bmatrix}_q = \frac{[M]_q!}{[N_1]_q!\,[N_2]_q! \cdots [N_\ell]_q!},$$

given that $N_1 + N_2 + \cdots + N_\ell = M$. Of course this specialises to the usual multinomial coefficient at $q = 1$. If $(N_1, N_2, \dots)$ is a sequence of non-negative integers having a finite sum M, so that only finitely many of the N_i's are non-zero, we write

$$\begin{bmatrix} M \\ N_1, N_2, \dots \end{bmatrix}_q = \frac{[M]_q!}{\prod_{i \ge 1} [N_i]_q!}.$$

Theorem 16. *Let* $\mathbf{n} = (n_1, n_2, \dots)$ *be a sequence of non-negative integers satisfying* (2.5). *Write* $n+1 = \sum_i n_i$. *Then the cyclic group of order* $2n$ *acts by the rotation* R *on* $\mathrm{T}(\mathbf{n})$, *and the triple*

$$\left(\mathrm{T}(\mathbf{n}), \langle R\rangle, \frac{1+q^n}{[n+1]_q}\begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Example 6. For $\mathbf{n} = (3, 0, 1, 0, \dots)$ and $\mathbf{n} = (2, 2, 0, \dots)$, corresponding to the trees of Figure 2, we obtain the same polynomials as those given by Theorem 7, namely $1+q^3$ and $1+q^2+q^4$, respectively.

Proof of Theorem 16. Polynomiality and non-negativity of coefficients of the cyclic sieving polynomial are proved in Lemma 17 below.

Let ω be a primitive $2n$-th root of unity. We have to prove

$$\mathrm{Fix}_{R^e}(\mathrm{T}(\mathbf{n})) = \frac{1+q^n}{[n+1]_q}\begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q\Bigg|_{q=\omega^e} \tag{8.1}$$

for all integers e. Again, it suffices to prove this for values of $e \in [\![0, 2n-1]\!]$ equal to 0 or dividing $2n$. Lemma 18 below gives the corresponding values of the right-hand side of (8.1).

For $e = 0$, all trees of $\mathrm{T}(\mathbf{n})$ are fixed by R^e, and their number is given by (2.8). This coincides with the value at $q = 1$ of the right-hand side of (8.1).

We now assume that $0 < e < 2n$, with e a divisor of $2n$. We revisit the proof of Theorem 5 while keeping track of vertex degrees.

If e is odd and equals n, the map Φ sends trees of $\mathrm{T}(\mathbf{n})$ fixed by R^e bijectively to rooted trees having $\frac{n+1}{2}+1$ nodes, among which $\frac{n_1}{2}+1$ leaves and $\frac{n_i}{2}$ nodes of degree i for $i \geq 2$, and in which one of the *non-root* leaves is marked. Observe that each n_i must be even. By (2.8) with n replaced by $\frac{n+1}{2}$, n_1 replaced by $\frac{n_1}{2}+1$, and n_i replaced by $\frac{n_i}{2}$ for $i \geq 2$, there are

$$\left(\tfrac{n_1}{2}+1\right)\cdot\frac{2}{\frac{n+1}{2}+1}\binom{\frac{n+1}{2}+1}{\frac{n_1}{2}+1, \frac{n_2}{2}, \dots}$$

ways for choosing such a tree and marking one of the $\frac{n_1}{2}+1$ leaves. We must however subtract the number of those trees where the marked leaf is the root. By (2.6) with n replaced by $\frac{n+1}{2}$, n_1 replaced by $\frac{n_1}{2}+1$, and n_i replaced by $\frac{n_i}{2}$ for $i \geq 2$, this number equals

$$\frac{1}{\frac{n+1}{2}}\binom{\frac{n+1}{2}}{\frac{n_1}{2}, \frac{n_2}{2}, \dots}.$$

The difference of these two numbers simplifies to

$$\frac{2\frac{n+1}{2}-1}{\frac{n+1}{2}}\binom{\frac{n+1}{2}}{\frac{n_1}{2}, \frac{n_2}{2}, \dots},$$

which is equivalent to the expression in the second alternative of (8.6). This proves (8.1) in the present case.

Now let e be even with $e = 2n/d$. Recall from the proof of Proposition 3 that a tree of $\mathrm{T}(\mathbf{n})$ fixed by R^e has a central node of degree ℓ divisible by d, and is obtained by

gluing d copies of the same tree around the central vertex. This implies in particular that $n_\ell \equiv 1 \pmod d$, while each n_i is divisible by d for $i \neq \ell$. This observation constrains the degree distribution $\mathbf{n}$ as in the third case of Lemma 18 below, and it implies that, given $\mathbf{n}$ and e, there is at most one possible value for the degree ℓ of the central node. Moreover, these conditions on the n_i's, combined with (2.5), imply that $d \mid \ell$, which is therefore redundant. Analogously, these conditions imply that d divides $n = -1 + \sum n_i$, meaning that $e = 2n/d$ is even, which is also redundant, given the conditions on the n_i's. By Proposition 3, the map Ψ_d sends trees of $\mathrm{T}(\mathbf{n})$ fixed by R^e bijectively to rooted trees with a total of $\frac{e}{2}+1$ nodes, including

- $\bar n_i := \frac{n_i}{d}$ nodes of degree i for $i \neq \{\ell/d, \ell\}$,
- $\bar n_\ell := \frac{n_\ell - 1}{d}$ nodes of degree ℓ,
- $\bar n_{\ell/d} := \frac{n_{\ell/d}}{d} + 1$ nodes of degree ℓ/d, one of which (corresponding to c) is marked.

According to (2.8), the number of such trees is equal to

$$\bar n_{\ell/d} \cdot \frac{2}{\frac{e}{2}+1} \binom{\frac{e}{2}+1}{\bar n_1, \bar n_2, \dots} = 2 \binom{\frac{e}{2}}{\frac{n_1}{d}, \dots, \frac{n_{\ell-1}}{d}, \frac{n_\ell - 1}{d}, \frac{n_{\ell+1}}{d}, \dots}.$$

This number coincides with the expression in the third alternative of (8.6), and is thus proving (8.1) also in this case.

For all other values of $\mathbf{n}$ and e, there are no trees fixed by R^e in $\mathrm{T}(\mathbf{n})$. This completes the proof of the theorem. □

Lemma 17. *Let $\mathbf{n} = (n_1, n_2, \dots)$ be a sequence of non-negative integers satisfying (2.5). Write $n+1 := \sum_i n_i$. Then the expression*

$$\frac{1+q^n}{[n+1]_q} \begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q \tag{8.2}$$

is a polynomial in q with non-negative coefficients.

Proof. To see that the expression in (8.2) is a polynomial, we write

$$\frac{1+q^n}{[n+1]_q} \begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q = \frac{[2n]_q}{[n]_q[n+1]_q} \begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q = \prod_{d \geq 2} \Phi_d(q)^{i_d}, \tag{8.3}$$

using again the notation $\Phi_d(q)$ for the d-th cyclotomic polynomial, where

$$i_d = \chi\big(d \mid 2n\big) - \chi\big(d \mid n\big) + \left\lfloor \frac{n}{d} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{n_i}{d} \right\rfloor. \tag{8.4}$$

We recall that $n = -1 + \sum_i n_i$.

From now on, let $d \geq 2$. We write $N_i = \{n_i/d\}$ for $i \geq 1$, where $\{\alpha\} := \alpha - \lfloor \alpha \rfloor$ denotes the fractional part of α, as before. Using this notation, we may rewrite Equation (8.4) as

$$i_d = \chi\big(d \mid 2n\big) - \chi\big(d \mid n\big) + \left\lfloor \sum_{i \geq 1} N_i - \frac{1}{d} \right\rfloor. \tag{8.5}$$

This is non-negative except possibly if $\sum_{i \geq 1} N_i = 0$, in which case the third term on the right-hand side above equals -1. This last condition is equivalent to all N_i's being equal to zero, which in its turn is equivalent to all n_i's being divisible by d. It then follows from (2.5) that $d = 2$. Moreover, $n = -1 + \sum n_i$ is then odd. Therefore, the

first term on the right-hand side of (8.5) equals 1, while the second is 0. This proves non-negativity of i_d, and establishes the polynomiality part of the lemma.

We will now show that the expression in (8.2) is a polynomial with non-negative coefficients. In order to do this, we proceed in complete analogy to the corresponding part in the proof of Lemma 8. Namely, since it is well-known that q-binomial coefficients are reciprocal and unimodal polynomials, since the product of reciprocal and unimodal polynomials is also reciprocal and unimodal, and since a q-multinomial coefficient can be written as a product of q-binomial coefficients, q-multinomial coefficients are reciprocal and unimodal. In particular,

$$\begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q$$

is a reciprocal and unimodal polynomial in q. It is of degree C, say, where an explicit formula for C could be given (namely the second elementary symmetric polynomial in the n_i's), which however is irrelevant. Hence, the coefficient of q^i in

$$(1-q)\begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q$$

is non-negative for $0 \le i \le \lceil C/2 \rceil$. Therefore the same must be true for the series

$$\frac{1+q^n}{1-q^{n+1}} \times (1-q)\begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q = \frac{1+q^n}{[n+1]_q}\begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q.$$

However, we know from above that the last expression is a polynomial. Since it is reciprocal and has degree C, we infer that also the other coefficients of the polynomial must be non-negative.

This completes the proof of the lemma. □

Lemma 18. *Let* $\mathbf{n} = (n_1, n_2, \dots)$ *be a sequence of non-negative integers satisfying* (2.5). *Write* $n+1 = \sum_i n_i$. *Let* ω *be a primitive* $2n$*-th root of unity, and let* e *be a non-negative integer* $< 2n$ *being equal to* 0 *or dividing* $2n$. *In the latter case, write* $d := 2n/e$, *so that* $d \ge 2$. *Then we have*

$$\frac{1+q^n}{[n+1]_q}\begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q\Bigg|_{q=\omega^e} = \begin{cases} \frac{2}{n+1}\binom{n+1}{n_1, n_2, \dots}, & \text{if } e = 0, \\ \frac{n}{\frac{n+1}{2}}\binom{\frac{n+1}{2}}{\frac{n_1}{2}, \frac{n_2}{2}, \dots}, & \text{if } d = 2 \text{ and each } n_i \text{ is even}, \\ 2\binom{\frac{e}{2}}{\frac{n_1}{d}, \dots, \frac{n_{\ell-1}}{d}, \frac{n_\ell - 1}{d}, \frac{n_{\ell+1}}{d}, \dots}, & \text{if } d \mid (n_\ell - 1) \text{ for some } \ell, \\ & \text{while } d \mid n_i \text{ for } i \ne \ell, \\ 0, & \text{otherwise.} \end{cases} \tag{8.6}$$

Note that, in the third case, d divides n since $n = -1 + \sum n_i$, which means that e is even. Moreover, as already observed, Condition (2.5) implies that $d \mid \ell$ in this case.

Proof of Lemma 18. The case $e = 0$ is obvious, so we assume that e is a positive divisor of $2n$ with $e < 2n$. Let ω be a primitive $2n$-th root of unity. Then ω^e is a primitive d-th root of unity. According to (8.3), in order to obtain a non-zero value of the specialisation

on the left-hand side of (8.6), the number i_d given by (8.4) or (8.5) must be zero. Note that, by assumption, we have $d \mid 2n$, so that

$$i_d = 1 - \chi\big(d \mid n\big) + \left\lfloor \sum_{i\geq 1} N_i - \frac{1}{d} \right\rfloor, \tag{8.7}$$

where, as in the proof of Lemma 17, we denote by N_i the fractional part of n_i/d.

As discussed earlier in the proof of Lemma 17, the third summand in (8.7) is negative (and then equals -1) if and only if $\sum_{i\geq 1} N_i = 0$. Then $d = 2$, all N_i's are even and n is odd, so that $i_d = 0$ indeed. This is the second case in (8.6). It is then an easy task, using the simple fact (5.4), to determine the value on the left-hand side of (8.6) in this case.

If, on the other hand, we have $\sum N_i > 0$, then the last summand on the right-hand side of (8.7) is non-negative. Then $i_d = 0$ if and only if d divides n and $\sum N_i \leq 1$ (recall that each N_i is of the form r/d, for $r \in [\![0, d-1]\!]$). However, the fact that d divides $n = \sum n_i - 1$ means that $\sum N_i - \frac{1}{d}$ is an integer: combined with $\sum N_i \leq 1$, this forces $\sum N_i = \frac{1}{d}$. In other words, all n_i's but one are divisible by d, and one of them, say n_ℓ, equals 1 modulo d. We are thus in the third case of (8.6). It is then not too difficult to obtain the evaluation on the left-hand side of (8.6) in this case, using the simple fact (5.4). The key is to write

$$\begin{aligned}\frac{1+q^n}{[n+1]_q} \begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q &= \frac{1-q^{2n}}{1-q^n} \cdot \frac{1-q}{1-q^{n_\ell}} \\ &\quad \times \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^2)(1-q)}{(1-q^{n_\ell-1})(1-q^{n_\ell-2})\cdots(1-q)\prod_{i\neq \ell}\big((1-q^{n_i})(1-q^{n_i-1})\cdots(1-q)\big)}.\end{aligned}$$

Since $d \mid n$, it follows from (5.4) that the first ratio on the right-hand side tends to 2 as $q \to \omega^e$ (a primitive d-th root of unity). Similarly, since $d \mid (n_\ell - 1)$, by (5.4) the second ratio tends to 1. The numerator and the denominator of the third ratio are both products of n terms of the form $1 - q^a$, where the exponents of q, when reduced modulo d, equal $0, d-1, d-2, \dots, 1, 0, d-1, \dots, 1, \dots, 0, d-1, d-2, \dots, 1$, in this order, both in the numerator and denominator. We can then split the third ratio as a product of n elementary ratios to which (5.4) applies. □

9. Trees with given vertex degrees under δ-rotation

In this section we fix $\delta \geq 1$ and a sequence $\mathbf{n} = (n_1, n_2, \dots)$ satisfying (2.5), and we consider the set $\mathrm{T}_\delta(\mathbf{n})$ of rooted plane trees whose root corner has degree δ (assuming $n_\delta > 0$). The cardinality of this set is given by (2.9). The δ-rotation R_δ defined in Section 4 acts on these trees by moving the root corner to the next corner of degree δ, in counterclockwise order.

Theorem 19. *Let $\delta \geq 1$ be an integer. Let $\mathbf{n} = (n_1, n_2, \dots)$ be a sequence of non-negative integers satisfying (2.5) and $n_\delta > 0$. Write $n + 1 = \sum_i n_i$. The cyclic group of order δn_δ acts by the rotation R_δ on $\mathrm{T}_\delta(\mathbf{n})$, and the triple*

$$\left(\mathrm{T}_\delta(\mathbf{n}), \langle R_\delta\rangle, \frac{[\delta n_\delta]_q}{[n_\delta]_q\,[n]_q} \begin{bmatrix} n \\ n_1, \dots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \dots \end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Example 7. For $\delta = 1$ and $\mathbf{n} = (3, 0, 1, 0, \ldots)$ or $\mathbf{n} = (2, 2, 0, \ldots)$, corresponding to the leaf-rooted trees of Figure 2, we obtain the same polynomials as those given by Theorem 10, namely 1 in both cases. For $\delta = 3$ and $\mathbf{n} = (3, 0, 1, 0, \ldots)$, we obtain 1 as well, as in Theorem 13. For $\delta = 2$ and $\mathbf{n} = (2, 2, 0, \ldots)$, we obtain $1 + q^2$, again as in Theorem 13.

Proof of Theorem 19. Polynomiality and non-negativity of coefficients of the cyclic sieving polynomial are proved in Lemma 20 below.

Let ω be a primitive δn_δ-th root of unity. We have to prove

$$\mathrm{Fix}_{R_\delta^e}(\mathrm{T}_\delta(\mathbf{n})) = \frac{[\delta n_\delta]_q}{[n_\delta]_q\,[n]_q} \left[\begin{matrix} n \\ n_1, \ldots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \ldots \end{matrix} \right]_q \Bigg|_{q=\omega^e} \tag{9.1}$$

for all integers e. It suffices to prove this for values of e in $[\![0, \delta n_\delta - 1]\!]$ that equal 0 or divide δn_δ. Lemma 21 below gives the corresponding values of the right-hand side of (9.1).

We begin with $e = 0$. Clearly, all trees in $\mathrm{T}_\delta(\mathbf{n})$ are invariant under $R_\delta^0 = \mathrm{id}$. By (2.9) we know the cardinality of $\mathrm{T}_\delta(\mathbf{n})$, which agrees with the value of the right-hand side of (9.1) at $q = 1$.

We now assume $0 < e < \delta n_\delta$, with e a divisor of δn_δ. We rely in what follows on Proposition 4, which relates the rotations R_δ and R. Recall in particular that the only possible values of e are $e = \delta n_\delta/2$, with n odd (corresponding to $f = n$ in Proposition 4) and those for which $\delta n_\delta/e$ divides n (corresponding to even values of f). We also use Proposition 3, as usual, to characterise trees fixed by a given power of R.

If $e = \delta n_\delta/2$, with $n + 1$ and δn_δ even, Proposition 4 tells us that we just need to count trees of $\mathrm{T}_\delta(\mathbf{n})$ fixed by R^n. By Proposition 3, the map Φ sends them bijectively to trees rooted at a corner of degree δ, having a total of $1 + \frac{n+1}{2}$ nodes, among which

- $\frac{n_i}{2}$ nodes of degree i, for $i \geq 2$,
- $1 + \frac{n_1}{2}$ leaves, one of which is marked, and is not the root node.

The latter condition indicates that the case $\delta = 1$ has to be studied separately. Note that each n_i must be even.

Assume first that $\delta \geq 2$. By (2.9) with n replaced by $\frac{n+1}{2}$, n_1 replaced by $\frac{n_1}{2} + 1$, and n_i replaced by $\frac{n_i}{2}$ for $i \geq 2$, the number of such trees is

$$\mathrm{Fix}_{R_\delta^e(\mathrm{T}_\delta(\mathbf{n}))} = \left(\tfrac{n_1}{2} + 1\right) \cdot \frac{\delta}{\frac{n+1}{2}} \binom{\frac{n+1}{2}}{\frac{n_1}{2} + 1, \frac{n_2}{2}, \ldots, \frac{n_{\delta-1}}{2}, \frac{n_\delta}{2} - 1, \frac{n_{\delta+1}}{2}, \ldots}.$$

If $\delta = 1$ we may use (2.9) again, with the same replacements as above, to express the relevant number of trees as

$$\frac{n_1}{2} \cdot \frac{1}{\frac{n+1}{2}} \binom{\frac{n+1}{2}}{\frac{n_1}{2}, \frac{n_2}{2}, \ldots}.$$

We happily observe that, in both cases ($\delta \geq 2$ or $\delta = 1$), the expression that we obtain is equivalent to the expression in the second alternative of (9.7). This proves (9.1) in the present case.

Now, if $d := \delta n_\delta/e$ divides n, let us write $f = 2n/d$. Proposition 4 tells us that we have to count trees of $\mathrm{T}_\delta(\mathbf{n})$ fixed by R^f. As in the proof of Theorem 16, these trees are centred at a node of degree ℓ, where $n_\ell - 1$ is divisible by d, while for $i \neq \ell$ the number n_i is divisible by d. This constrains the degree distribution $\mathbf{n}$ as in the third case of Lemma 21 below. By Proposition 3, the map Ψ_d sends trees of $\mathrm{T}_\delta(\mathbf{n})$ fixed by R^f bijectively to trees with a total of $\frac{f}{2}+1$ nodes, among which

- $\bar n_i := \frac{n_i}{d}$ nodes of degree i for $i \notin \{\ell/d, \ell\}$,
- $\bar n_\ell := \frac{n_\ell - 1}{d}$ nodes of degree ℓ,
- $\bar n_{\ell/d} := \frac{n_{\ell/d}}{d} + 1$ nodes of degree ℓ/d, one of which is marked.

Moreover, these trees are rooted

- either at a corner of degree δ, not incident to the marked node,
- or possibly at a corner incident to the marked node if $\ell = \delta$.

We will consider the cases $\delta = \ell/d$, $\delta = \ell$ and $\delta \notin \{\ell/d, \ell\}$ separately, but then happily observe that they reduce to the same final counting formula.

Assuming first that $\delta \notin \{\ell/d, \ell\}$, we derive from the next-to-last expression in (2.9) that

$$\mathrm{Fix}_{R_\delta^e}(\mathrm{T}_\delta(\mathbf{n})) = \bar n_{\ell/d} \cdot |\,\mathrm{T}_\delta(\bar n_1, \bar n_2, \ldots)| = \bar n_{\ell/d} \cdot \delta \bar n_\delta \cdot \frac{(\frac{f}{2}-1)!}{\bar n_1!\,\bar n_2! \cdots}. \tag{9.2}$$

If $\delta = \ell/d$, we forbid the marked node to be the root node, even though it has degree δ. Thus the above formula is modified into

$$\mathrm{Fix}_{R_\delta^e}(\mathrm{T}_\delta(\mathbf{n})) = (\bar n_\delta - 1) \cdot |\,\mathrm{T}_\delta(\bar n_1, \bar n_2, \ldots)| = (\bar n_\delta - 1) \cdot \delta \bar n_\delta \cdot \frac{(\frac{f}{2}-1)!}{\bar n_1!\,\bar n_2! \cdots}.$$

Finally, if $\delta = \ell$, we allow the marked node to be the root node, even though its degree is $\ell/d = \delta/d$ rather than δ. That is, to the generic contribution (9.2) corresponding to trees in which the root has degree δ, we have to add the number of trees of $\mathrm{T}(\bar n_1, \bar n_2, \ldots)$ rooted at a corner of degree δ/d. It then follows from the next-to-last expression in (2.9) that

$$\mathrm{Fix}_{R_\delta^e}(\mathrm{T}_\delta(\mathbf{n})) = \bar n_{\ell/d} \cdot \delta \bar n_\delta \cdot \frac{(\frac{f}{2}-1)!}{\bar n_1!\,\bar n_2! \cdots} + \frac{\delta}{d}\, \bar n_{\delta/d} \cdot \frac{(\frac{f}{2}-1)!}{\bar n_1!\,\bar n_2! \cdots}.$$

In all three cases, the formula that we obtain coincides with the expression in the third alternative of (9.7). This proves (9.1) also in this case.

For all other values of $\mathbf{n}$ and e, there are no trees fixed by R_δ^e in $\mathrm{T}_\delta(\mathbf{n})$. This completes the proof of the theorem. □

Lemma 20. *Let $\delta \geq 1$, and let $\mathbf{n} = (n_1, n_2, \ldots)$ be a sequence of non-negative integers satisfying (2.5). Write $n+1 = \sum_i n_i$. Then the expression*

$$\frac{[\delta n_\delta]_q}{[n_\delta]_q\,[n]_q} \begin{bmatrix} n \\ n_1, \ldots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \ldots \end{bmatrix}_q$$

is a polynomial in q with non-negative coefficients.

Proof. We write

$$\frac{[\delta n_\delta]_q}{[n_\delta]_q\,[n]_q} \begin{bmatrix} n \\ n_1, \ldots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \ldots \end{bmatrix}_q = \prod_{d \geq 2} \Phi_d(q)^{j_d}, \tag{9.3}$$

using again the notation $\Phi_d(q)$ for the d-th cyclotomic polynomial in q, where

$$j_d = \chi\big(d \mid \delta n_\delta\big) - \chi\big(d \mid n_\delta\big) + \left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{n_\delta - 1}{d} \right\rfloor - \sum_{\substack{i \geq 1 \\ i \neq \delta}} \left\lfloor \frac{n_i}{d} \right\rfloor . \tag{9.4}$$

Recall that $n+1 = \sum_i n_i$. Fix $d \geq 2$. We introduce the fractional parts $N_i = \{n_i/d\}$ for $i \neq \delta$ and $N_\delta = \{(n_\delta - 1)/d\}$. Using this notation, we rewrite Equation (9.4) as

$$j_d = \chi\big(d \mid \delta n_\delta\big) - \chi\big(d \mid n_\delta\big) + \left\lfloor \sum_{i \geq 1} N_i - \frac{1}{d} \right\rfloor . \tag{9.5}$$

This is non-negative except possibly if $\sum_{i \geq 1} N_i = 0$. By definition of the N_i's, the last condition is equivalent to all N_i's being equal to zero, which in its turn is equivalent to all n_i's being divisible by d except for n_δ, which satisfies the congruence $n_\delta \equiv 1 \pmod{d}$. It then follows from (2.5) that d divides δ. Hence the right-hand side of (9.5) reduces to $1 - 0 + (-1) = 0$. Thus, in all cases the conclusion is that j_d is non-negative. This proves the polynomiality part of the lemma.

For proving non-negativity of the coefficients, we start again by recalling that q-multinomial coefficients are reciprocal and unimodal polynomials, and hence

$$\begin{bmatrix} n \\ n_1, \dots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \dots \end{bmatrix}_q$$

is a reciprocal and unimodal polynomial in q. It is of degree E, say, where an explicit formula for E could be given, which however is irrelevant. It follows that the coefficient of q^i in

$$(1-q) \begin{bmatrix} n \\ n_1, \dots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \dots \end{bmatrix}_q$$

is non-negative for $0 \leq i \leq \lceil E/2 \rceil$. Therefore the same must be true for the series

$$\frac{[\delta n_\delta]_q}{[n_\delta]_q} \frac{1}{1-q^n} \times (1-q) \begin{bmatrix} n \\ n_1, \dots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \dots \end{bmatrix}_q , \tag{9.6}$$

because $[\delta n_\delta]_q / [n_\delta]_q = 1 + q^{n_\delta} + \cdots + q^{(\delta-1)n_\delta}$ is a polynomial with non-negative coefficients. However, we know from above that the last expression is a polynomial. The degree of this polynomial is $E - n + 1 + n_\delta(\delta - 1) \leq E$, because $(\delta - 1)n_\delta \leq n - 1$. (Indeed, this is obvious if $\delta = 1$, and if $\delta \geq 2$, it follows from (2.5) that $(\delta - 2)n_\delta \leq -2 + n_1 \leq -2 + (n + 1 - n_\delta)$, which gives the desired result.) We know that at least the first half of the coefficients of (9.6) are non-negative. It is obviously a reciprocal polynomial. Hence the other coefficients of the polynomial must be non-negative as well.

This completes the proof of the lemma. □

Lemma 21. *Let $\delta \geq 1$, and let $\mathbf{n} = (n_1, n_2, \dots)$ be a sequence of non-negative integers satisfying (2.5). Write $n + 1 = \sum_i n_i$. Furthermore, let ω be a primitive δn_δ-th root of unity, and let e be a non-negative integer $< \delta n_\delta$ being equal to 0 or dividing δn_δ. In the*

latter case, write $d := \delta n_\delta/e$, *so that* $d \geq 2$. *Then we have*

$$\frac{[\delta n_\delta]_q}{[n_\delta]_q\,[n]_q} \begin{bmatrix} n \\ n_1, \dots, n_{\delta-1}, n_\delta - 1, n_{\delta+1}, \dots \end{bmatrix}_q \Bigg|_{q=\omega^e}$$

$$= \begin{cases} \frac{\delta}{n} \binom{n}{n_1,\dots,n_{\delta-1},n_\delta-1,n_{\delta+1},\dots}, & \text{if } e = 0, \\ \delta \cdot \binom{\frac{n+1}{2}-1}{\frac{n_1}{2},\dots,\frac{n_{\delta-1}}{2},\frac{n_\delta}{2}-1,\frac{n_{\delta+1}}{2},\dots}, & \text{if } d = 2 \text{ and each } n_i \text{ is even}, \\ \frac{\delta n_\delta}{n} \cdot \binom{\frac{n}{d}}{\frac{n_1}{d},\dots,\frac{n_{\ell-1}}{d},\frac{n_\ell-1}{d},\frac{n_{\ell+1}}{d},\dots}, & \text{if } d \mid (n_\ell - 1) \text{ for some } \ell \\ & \text{while } d \mid n_i \text{ for } i \neq \ell, \\ 0, & \text{otherwise.} \end{cases} \tag{9.7}$$

Note that in the third case we have $d \mid n$, so that if $d = 2$ then n is even; the second and third case are thus disjoint because n is odd in the second case.

Proof of Lemma 21. The case where $e = 0$ is trivial. Therefore we assume from now on that e is a positive divisor of δn_δ with $e < \delta n_\delta$. Then ω^e is a primitive d-th root of unity. According to (9.3), in order to obtain a non-zero value of the specialisation on the left-hand side of (9.7), the number j_d given by (9.4) or (9.5) must be zero. Note that, by assumption, we have $d \mid \delta n_\delta$, so that, with the notation N_i used in the proof of Lemma 20,

$$j_d = 1 - \chi\big(d \mid n_\delta\big) + \left\lfloor \sum_{i \geq 1} N_i - \frac{1}{d} \right\rfloor. \tag{9.8}$$

As already discussed in the proof of Lemma 20, the third summand on the right-hand side is negative if and only if $\sum N_i = 0$, and in this case it equals -1. We have seen that $j_d = 0$ in this case, that d divides $n_\delta - 1$ and all n_i's with $i \neq d$. We are then in the third case of (9.7) with $\ell = \delta$. The left-hand side of (9.7) is then evaluated using the simple fact (5.4).

If, on the other hand $\sum N_i > 0$, then the third summand in (9.8) is non-negative. The equation $j_d = 0$ holds if and only if $d \mid n_\delta$ (which means that $N_\delta = 1 - \frac{1}{d}$) and $\sum N_i \leq 1$. Given the value of N_δ, this means that $\sum_{i \neq \delta} N_i \leq 1/d$, or equivalently $\sum_{i \neq \delta} N_i \in \{0, 1/d\}$.

- If $\sum_{i \neq \delta} N_i = 0$, then each n_i is divisible by d (including n_δ, as we have seen above). Then (2.5) forces $d = 2$, and we are in the second case of (9.7). The left-hand side of (9.7) is then evaluated using the simple fact (5.4).
- If $\sum_{i \neq \delta} N_i = 1/d$, then there exists $\ell \neq \delta$ such that $N_\ell = 1/d$ (that is, $d \mid (n_\ell - 1)$), while $d \mid N_i$ for $i \neq \ell$ (including $i = \delta$). We are then in the third case of (9.7), with $\ell \neq \delta$. The left-hand side of (9.7) is then evaluated using the simple fact (5.4).

□

10. Trees with given vertex degrees under internal rotation

In this section we fix a sequence $\mathbf{n} = (n_1, n_2, \dots)$ of non-negative integers satisfying (2.5), and consider the set $\mathrm{T}_i(\mathbf{n})$ of rooted plane trees with degree distribution $\mathbf{n}$

whose root corner is incident to a non-leaf. The cardinality of this set is given by (2.7), where $n+1=\sum_i n_i$ is the total number of nodes. The internal rotation R_i acts on these trees by moving the root corner to the next corner not incident to a leaf, in counterclockwise order.

Theorem 22. *Let* $\mathbf{n}=(n_1,n_2,\dots)$ *be a sequence of non-negative integers satisfying* (2.5). *Write* $n+1=\sum_i n_i$. *The cyclic group of order* $2n-n_1$ *acts on* $\mathrm{T}_i(\mathbf{n})$ *by the internal rotation* R_i, *and the triple*

$$\left(\mathrm{T}_i(\mathbf{n}),\langle R_i\rangle,\frac{[2n-n_1]_q}{[n+1]_q\,[n]_q}\begin{bmatrix}n+1\\ n_1,n_2,\dots\end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Example 8. For $\mathbf{n}=(3,0,1,0,\dots)$ and $\mathbf{n}=(2,2,0,\dots)$, corresponding to the trees of Figure 2, we obtain the same polynomials as those given by Theorem 13, namely 1 and $1+q^2$, respectively.

Proof of Theorem 22. Polynomiality and non-negativity of coefficients of the cyclic sieving polynomial are proved in Lemma 23 below.

Let ω be a primitive $(2n-n_1)$-th root of unity. We have to prove

$$\mathrm{Fix}_{R_i^e}(\mathrm{T}_i(\mathbf{n}))=\frac{[2n-n_1]_q}{[n+1]_q\,[n]_q}\begin{bmatrix}n+1\\ n_1,n_2,\dots\end{bmatrix}_q\Bigg|_{q=\omega^e} \tag{10.1}$$

for all integers e. It suffices to prove this for values $e\in[\![0,2n-1-n_1]\!]$ equal to 0 or dividing $2n-n_1$. Lemma 24 below gives the corresponding values of the right-hand side of (10.1).

When $e=0$, all trees in $\mathrm{T}_i(\mathbf{n})$ are fixed by $R_i^0=\mathrm{id}$, and their number is given by (2.7). This coincides with the value at $q=1$ of the right-hand side of (10.1).

We now assume that $0<e<2n-n_1$, with e a divisor of $2n-n_1$. We revisit the proof of Theorem 13 while keeping track of all vertex degrees. In particular, we use Proposition 4 to relate the rotations R_i and R. Recall that the only possible values of e are $e=n-\frac{n_1}{2}$, with n odd (corresponding to $f=n$ in the notation of Proposition 4), and those for which $(2n-n_1)/e$ divides n (corresponding to even values of f).

If $e=n-\frac{n_1}{2}$, with $n+1$ and n_1 even, Proposition 4 tells us that we simply have to count trees of $\mathrm{T}_i(\mathbf{n})$ fixed by R^n. By Proposition 3, the map Φ sends them bijectively to trees of $\mathrm{T}_i(1+\frac{n_1}{2},\frac{n_2}{2},\frac{n_3}{2},\dots)$ having a marked leaf. Such trees have $(n+1)/2$ edges. Hence, using (2.7), we conclude that

$$\mathrm{Fix}_{R_i^e}(\mathrm{T}_i(\mathbf{n}))=\left(1+\tfrac{n_1}{2}\right)\cdot\frac{2^{\frac{n+1}{2}}-1-\frac{n_1}{2}}{\frac{n+1}{2}\left(1+\frac{n+1}{2}\right)}\binom{1+\frac{n+1}{2}}{1+\frac{n_1}{2},\frac{n_2}{2},\dots}.$$

This expression is equivalent to the expression in the second alternative of (10.6). This proves (10.1) in this case.

Now, if $d:=(2n-n_1)/e$ divides n, let us write $f:=2n/d$. By Proposition 4, we need to count trees of $\mathrm{T}_i(\mathbf{n})$ fixed by R^f. As in the proof of Theorem 16, these trees are centred at a node of degree ℓ (divisible by d), where $n_\ell-1$ is divisible by d, while for $i\neq\ell$ the number n_i is divisible by d. This constrains the degree distribution $\mathbf{n}$ as in

the third case of Lemma 24 below. By Proposition 4, the map Ψ_d sends trees of $\mathrm{T}_i(\mathbf{n})$ fixed by R^f bijectively to trees with a total of $\frac{f}{2}+1$ nodes, among which

- $\bar n_i := \frac{n_i}{d}$ nodes of degree i for $i \notin \{\ell/d, \ell\}$,
- $\bar n_\ell := \frac{n_\ell - 1}{d}$ nodes of degree ℓ,
- $\bar n_{\ell/d} := \frac{n_{\ell/d}}{d} + 1$ nodes of degree ℓ/d, one of which is marked.

Moreover, these trees are rooted

- either at a corner incident to a non-leaf,
- or possibly at the marked node when it is a leaf, that is, when $\ell = d$.

Assuming first that $\ell \neq d$, we conclude from (2.7) that

$$\mathrm{Fix}_{R_i^e}(\mathrm{T}_i(\mathbf{n})) = \bar n_{\ell/d} \cdot \frac{f-\bar n_1}{\frac{f}{2}(1+\frac{f}{2})} \binom{1+\frac{f}{2}}{\bar n_1, \bar n_2, \dots}.$$

If $\ell = d$, we must add to the above term the number of leaf-rooted trees having degree distribution $(\bar n_1, \bar n_2, \dots)$. Using (2.6), we conclude that

$$\begin{aligned}\mathrm{Fix}_{R_i^e}(\mathrm{T}_i(\mathbf{n})) &= \bar n_1 \cdot \frac{f-\bar n_1}{\frac{f}{2}(1+\frac{f}{2})} \binom{1+\frac{f}{2}}{\bar n_1, \bar n_2, \dots} + \frac{1}{\frac{f}{2}} \binom{\frac{f}{2}}{\bar n_1 - 1, \bar n_2, \dots} \\ &= \frac{f-\bar n_1 + 1}{\frac{f}{2}} \binom{\frac{f}{2}}{\bar n_1 - 1, \bar n_2, \dots}.\end{aligned}$$

We happily observe that, in both cases ($\ell \neq d$ or $\ell = d$), the expression that we obtain is equivalent to the expression in the third alternative of (10.6). This proves (10.1) also in this case.

For all other values of e and $\mathbf{n}$, there are no trees fixed by R_i^e in $\mathrm{T}_i(\mathbf{n})$. This completes the proof of the theorem. □

Lemma 23. *Let $\mathbf{n} = (n_1, n_2, \dots)$ be a sequence of non-negative integers satisfying (2.5). Write $n+1 = \sum_i n_i$. Then the expression*

$$\frac{[2n-n_1]_q}{[n+1]_q\,[n]_q} \begin{bmatrix} n+1 \\ n_1, n_2, \dots \end{bmatrix}_q \tag{10.2}$$

is a polynomial in q with non-negative coefficients.

Proof. Taking inspiration from the decomposition (7.3), we decompose (10.2) as

$$\frac{1}{[n]_q} \begin{bmatrix} n \\ n_1 - 1, n_2, \dots \end{bmatrix}_q + q^{n_1} \frac{1+q^{n-n_1}}{[n+1-n_1]_q} \begin{bmatrix} n-1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n+1-n_1 \\ n_2, n_3, \dots \end{bmatrix}_q.$$

By Lemma 20 with $\delta = 1$, we know that the first term in this expression is a polynomial in q with non-negative coefficients.

To see that the second term is a polynomial, we write

$$\frac{1+q^{n-n_1}}{[n+1-n_1]_q} \begin{bmatrix} n-1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n+1-n_1 \\ n_2, n_3, \dots \end{bmatrix}_q = [2n-2n_1]_q \frac{[n-1]_q!}{\prod_{i\geq 1}[n_i]_q!} = \prod_{d\geq 2} \Phi_d(q)^{k_d}, \tag{10.3}$$

using again the notation $\Phi_d(q)$ for the d-th cyclotomic polynomial in q, where

$$k_d = \chi\big(d \mid (2n-2n_1)\big) + \left\lfloor \frac{n-1}{d} \right\rfloor - \sum_{i\geq 1} \left\lfloor \frac{n_i}{d} \right\rfloor. \tag{10.4}$$

From now on, let $d \geq 2$. We introduce the fractional parts $N_i = \{n_i/d\}$ for $i \geq 1$. Using this notation, we may rewrite Equation (10.4) as

$$k_d = \chi\big(d \mid (2n - 2n_1)\big) + \left\lfloor \sum_{i\geq 1} N_i - \frac{2}{d} \right\rfloor. \tag{10.5}$$

This is non-negative except possibly if $\sum_{i\geq 1} N_i = 0$ or $\sum_{i\geq 1} N_i = \frac{1}{d}$. In that case the second term on the right-hand side is -1.

If $\sum_{i\geq 1} N_i = 0$, this is equivalent to all N_i's being equal to zero, which in its turn is equivalent to all n_i's being divisible by d. It then follows from (2.5) that $d = 2$, so that $k_d = 0$.

If $\sum_{i\geq 1} N_i = \frac{1}{d}$, this is equivalent to all N_i's being equal to zero except for one that equals $\frac{1}{d}$, say $N_\ell = \frac{1}{d}$ for a specific ℓ. In its turn, this is equivalent to all n_i's being divisible by d except for n_ℓ which satisfies $n_\ell \equiv 1 \pmod d$. In particular, d divides $n = -1 + \sum n_i$. Hence, if $\ell \geq 2$, then d divides $n - n_1$, so that the first term on the right-hand side of (10.5) equals $+1$ which guarantees non-negativity of the right-hand side. However, we cannot have $\ell = 1$, that is, $n_1 \equiv 1 \pmod d$, because (2.5) would then give $-1 \equiv -2 \pmod d$, a contradiction since $d \geq 2$.

This proves the polynomiality part of the lemma.

We will now show that the expression in (10.3) is in fact a polynomial with non-negative coefficients. In order to do this, we proceed in complete analogy to the corresponding part in the proof of Lemma 8. Namely, since it is well-known that q-binomial and q-multinomial coefficients are reciprocal and unimodal polynomials, and since the product of reciprocal and unimodal polynomials is also reciprocal and unimodal, we conclude that the product

$$\begin{bmatrix} n-1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n+1-n_1 \\ n_2, n_3, \dots \end{bmatrix}_q$$

is a reciprocal and unimodal polynomial in q. It is of degree F, say, where an explicit formula for F could be given, which however is irrelevant. Hence, the coefficient of q^i in

$$(1-q) \begin{bmatrix} n-1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n+1-n_1 \\ n_2, n_3, \dots \end{bmatrix}_q$$

is non-negative for $0 \leq i \leq \lceil F/2 \rceil$. Therefore the same must be true for the series

$$\frac{1+q^{n-n_1}}{1-q^{n+1-n_1}} \times (1-q) \begin{bmatrix} n-1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n+1-n_1 \\ n_2, n_3, \dots \end{bmatrix}_q = \frac{1+q^{n-n_1}}{[n+1-n_1]_q} \begin{bmatrix} n-1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n+1-n_1 \\ n_2, n_3, \dots \end{bmatrix}_q.$$

However, we know from above that the last expression is a polynomial. Since it is reciprocal and has degree F, we infer that also the other coefficients of this polynomial must be non-negative.

This completes the proof of the lemma. □

Lemma 24. *Let* $\mathbf{n} = (n_1, n_2, \dots)$ *be a sequence of non-negative integers satisfying* (2.5). *Write* $n + 1 = \sum_i n_i$. *Furthermore, let* ω *be a primitive* $(2n - n_1)$*-th root of unity, and let* e *be a non-negative integer* $< 2n - n_1$ *being equal to* 0 *or dividing* $2n - n_1$. *In the*

latter case, write $d=(2n-n_1)/e$, *so that* $d\ge 2$. *Then we have*

$$\frac{[2n-n_1]_q}{[n+1]_q\,[n]_q}\begin{bmatrix} n+1 \\ n_1,n_2,\dots\end{bmatrix}_q\Bigg|_{q=\omega^e} = \begin{cases} \frac{2n-n_1}{n(n+1)}\binom{n+1}{n_1,n_2,\dots}, & \text{if } e=0,\\ \frac{2n-n_1}{n+1}\binom{\frac{n+1}{2}}{\frac{n_1}{2},\frac{n_2}{2},\dots}, & \text{if } d=2 \text{ and each } n_i \text{ is even},\\ \frac{2n-n_1}{n}\binom{\frac{n}{d}}{\frac{n_1}{d},\dots,\frac{n_{\ell-1}}{d},\frac{n_\ell-1}{d},\frac{n_{\ell+1}}{d},\dots}, & \text{if } d\mid (n_\ell-1) \text{ for some } \ell,\\ & \text{while } d\mid n_i \text{ for } i\neq\ell,\\ 0, & \text{otherwise.}\end{cases} \tag{10.6}$$

Note that in the third case we have $d\mid n$, so that if $d=2$ then n is even; the second and third case are thus disjoint because n is odd in the second case.

Proof of Lemma 24. The case where $e=0$ is trivial. Therefore we assume from now on that e is a positive divisor of $2n-n_1$ with $e<2n-n_1$. Then ω^e is a primitive d-th root of unity. According to (5.3), in order to obtain something non-zero upon the specialisation $q=\omega^e$ on the left-hand side of (10.6), there must be an equal number of factors of the form $1-q^\alpha$ with $d\mid\alpha$ in the numerator and in the denominator. In other words, we must have

$$1+\left\lfloor\frac{n-1}{d}\right\rfloor-\sum_{i\ge 1}\left\lfloor\frac{n_i}{d}\right\rfloor=0. \tag{10.7}$$

As in the previous proof, we introduce the fractional parts $N_i=\{\frac{n_i}{d}\}$ for $i\ge 1$. With this notation, Equation (10.7) turns into

$$1+\left\lfloor\sum_{i\ge 1}N_i-\frac{2}{d}\right\rfloor=0.$$

This equation holds if and only if $\sum N_i\in\{0,1/d\}$.

If $\sum N_i=0$, then each N_i is 0, so that d divides each n_i. Condition (2.5) then forces $d=2$, and we are in the second case of (10.6). One then evaluates the left-hand side of (10.6) in this case using the simple fact (5.4).

If $\sum N_i=1/d$, then each N_i is 0 except one, say N_ℓ, which equals $1/d$. This means d divides $n_\ell-1$ and each n_i with $i\neq\ell$. We are thus in the third case of (10.6). One then evaluates the left-hand side of (10.6) in this case using the simple fact (5.4). It is useful to recall that, due to (2.5), d divides ℓ in this case, so that in particular $\ell\neq 1$. □

11. Connections with earlier work

So far, we have stated all our cyclic sieving results in the uniform language of rooted plane trees. However, it is well known that these trees are in bijection with many other families of Catalan objects. When we consider a class of trees ($\mathrm{T}(n,k)$, $\mathrm{T}(\mathbf{n})$, ...), a bijection to other objects, and a rotation on trees (R, R_e, R_i, or R_δ) that translates nicely through the bijection, we obtain a cyclic sieving phenomenon for the image of the class of trees under the bijection. Some of these results are new, and some are not. We briefly discuss some examples in this section. We describe the bijections we use

rather informally, with the help of figures. We hope that this will be sufficient for the readers.

11.1. Non-crossing perfect matchings

We start with the classical bijection between non-crossing (perfect) matchings of $2n$ points placed on a circle and rooted plane trees with n edges, illustrated in Figure 6. Then the (ordinary) rotation R on trees, of order $2n$, rotates the matching by $(2\pi)/(2n)$ in clockwise order, that is, replaces the point i by $i+1$, modulo $2n$.

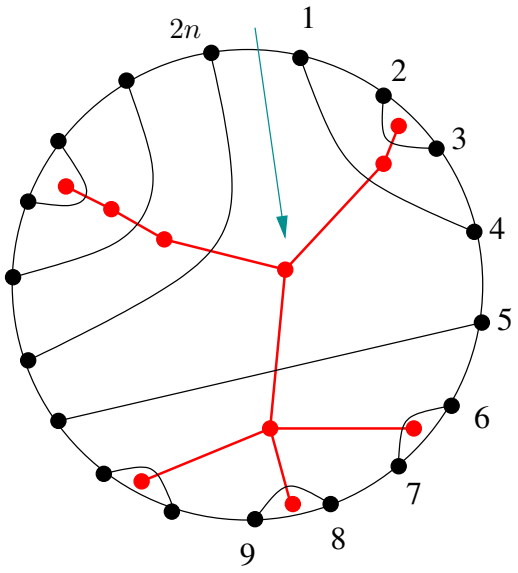


FIGURE 6. A non-crossing matching on 2×9 elements and the corresponding rooted plane tree with $n = 9$ edges. The number of short edges in the matching is $k = 5$.

Our first theorem, Theorem 5, thus translates into a cyclic sieving phenomenon for non-crossing perfect matchings, for this natural rotation. This result is not new: it is stated in [29, Thm. 8.1], and follows for instance by combining a bijection between non-crossing matchings and two-row rectangular Young tableaux with a cyclic sieving phenomenon for these tableaux under *promotion*; see Theorems 1.4 and 1.5 in [25] as well as [27, Thm. 1.3]. Let us rephrase this perfect matching version of Theorem 5 in a way that will be convenient in Section 12. We denote by $\mathrm{NCM}(n)$ the set of non-crossing matchings of $2n$ points.

Corollary 25. *Let ω be a primitive $2n$-th root of unity, and let e be a non-negative integer less than $2n$, equal to 0 or dividing $2n$. Let $d := 2n/e$, with $d = \infty$ if $e = 0$. The number of non-crossing matchings on $2n$ points left invariant by a rotation of*

$(2\pi e)/(2n) = 2\pi/d$ *is given by*

$$\mathrm{Fix}_{2\pi/d}(\mathrm{NCM}(n)) = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}\Bigg|_{q=\omega^e} = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}\Bigg|_{q=\mathrm{e}^{2\pi i/d}}$$
$$= \begin{cases} \frac{1}{n+1}\binom{2n}{n}, & \text{if } e=0,\\ \binom{n}{\lceil n/2\rceil}, & \text{if } d=2 \text{ with } n \text{ odd},\\ \binom{2n/d}{n/d}, & \text{if } d \mid n,\\ 0, & \text{otherwise.}\end{cases}$$

Through the bijection of Figure 6, the number of leaves of the tree, k say, corresponds to the number of *short edges* of the matching, that is, edges of the form $(i, i+1)$, modulo $2n$. This means that our second theorem, Theorem 7, is equivalent to the refinement of the cyclic sieving phenomenon for non-crossing matchings obtained in [2, Thm. 4.8]. The equivalence is not immediately obvious because of an additional factor $q^{k(k-2)}$ in the cyclic sieving polynomial of the latter theorem: however, we note that this factor takes the value 1 at ω^e for each value of e for which at least one tree of $\mathrm{T}(n,k)$ is fixed by R^e (cf. Lemma 9). One advantage of this additional factor is that now the sum of the refined cyclic sieving polynomials coincides with the q-Catalan polynomial [2, Prop. 4.5]:

$$\sum_k q^{k(k-2)} \cdot \frac{1+q^n}{[n-1]_q}\begin{bmatrix}n-1\\ k-2\end{bmatrix}_q\begin{bmatrix}n\\ k\end{bmatrix}_q = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q.$$

Using this summation (which is a special case of Heine's q-analogue of Gauß' hypergeometric ${}_2F_1[1]$-summation; see [14, Eq. (1.5.2)]), we could thus have derived Theorem 5 from Theorem 7.

Conversely, we can translate some of our results in terms of non-crossing matchings. One example is Theorem 10, which translates into a nice cyclic sieving phenomenon for non-crossing matchings containing the pair $(2n, 1)$.

11.2. Non-crossing partitions

We now consider another classical bijection, this time between non-crossing partitions of n points on a circle [22] and rooted plane trees with n edges (Figure 7, left). It is not immediately obvious to understand how the ordinary rotation R on trees, of order $2n$, translates in terms of partitions. However, the double rotation R^2, of order n, simply becomes the rotation by $2\pi/n$ on non-crossing partitions; that is, i is replaced by $i+1$ modulo n.

Hence our first theorem, Theorem 5, gives by restriction to even values of e a (known) cyclic sieving phenomenon for rotation on non-crossing partitions of n points, with cyclic sieving polynomial $\begin{bmatrix}2n\\ n\end{bmatrix}_q/[n+1]_q$; see [29, Thm. 10.1], where this result is derived from a refined cyclic sieving phenomenon where the number of blocks of the partition is also fixed [26, Thm. 7.2]. See [8] and Subsection 11.4 for generalisations to non-crossing partitions associated with well-generated complex reflection groups (the above classical case corresponding to the symmetric group).

However, it is also possible to describe the action of the rotation R in terms of non-crossing partitions. Indeed, it corresponds to taking the so-called *Kreweras complement* of the partition. This complement is obtained by inserting n new vertices $1', 2', \ldots, n'$

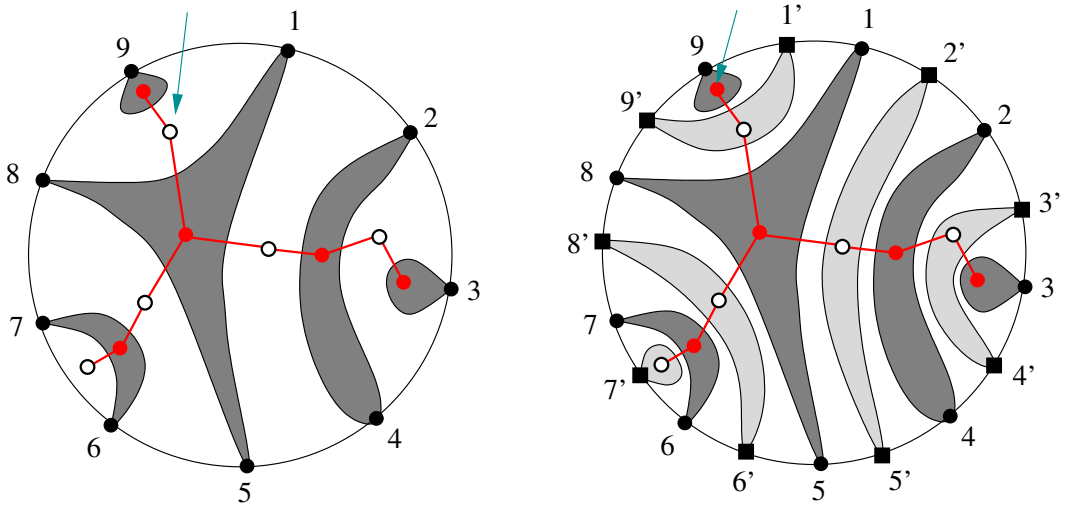


FIGURE 7. *Left*: A non-crossing partition on $n = 9$ elements and the corresponding rooted plane tree with n edges. *Right*: Taking the Kreweras complement (shown in light grey) of the partition amounts to the ordinary rotation R on the tree.

between the original ones and organising them in blocks that correspond to the empty areas of the original partition (see Figure 7, right). Thus, our first theorem, Theorem 5, also translates into a (known) cyclic sieving phenomenon for complementation of non-crossing partitions, proved in [6, Thm. 1.5] for arbitrary finite Weyl groups, and in [18, 20] for well-generated complex reflection groups; see Subsection 11.4. According to Bessis and Reiner, who had conjectured this result [8, Conj. 6.5] in the more general setting proved in [18, 20], the classical case was proved earlier by D. White in a personal communication. Later it was also established independently by C. Heitsch [16].

11.3. Triangulations and dissections of the disk

Another classical family in the Catalan zoo for which cyclic sieving results have been established is the set of triangulations of a polygon. There exists a simple bijection between leaf-rooted *ternary trees* with k leaves and $k-2$ inner nodes of degree 3 — that is, in our notation, trees of $\mathrm{T}_l(\mathbf{n})$, with $\mathbf{n} = (k, 0, k-2, 0, \ldots)$ — and triangulations of a k-gon; see Figure 8.

Through this bijection, the external rotation R_l, of order k when considered on trees with k leaves, becomes the rotation of triangulations by $2\pi/k$, which sends each point i to $i+1$. Hence, a special case of Theorem 19 (for $\delta = 1$, and $\mathbf{n}$ as above) gives a cyclic sieving phenomenon for triangulations of the disk, and this phenomenon is not new [29, Thm. 12.2].

More generally, let us call any collection of non-intersecting diagonals (i, j) with $|i - j| > 1$ a *dissection* of the k-gon. Equivalently, a dissection can be seen as an *outerplanar map* with faces of degree at least 3 (see Figure 9). A simple bijection sends these dissections to leaf-rooted trees having no vertex of degree 2. The degree of a face is the degree of the node that lies in it. Again, the external rotation on such trees corresponds to the clockwise rotation by $2\pi/k$ of the dissection. Hence the case $\delta = 1$, $n_2 = 0$ of Theorem 19 gives a cyclic sieving phenomenon for dissections of the k-gon

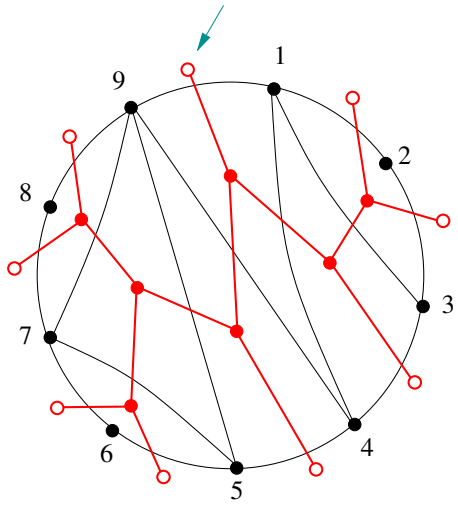


FIGURE 8. A triangulation of the k-gon, with $k = 9$, and the corresponding leaf-rooted ternary tree with 9 leaves and 7 inner nodes.

with prescribed face degrees, which we believed to be new... until we found it in a very recent independent preprint [1, Thm. 1]. In particular, we obtain a cyclic sieving phenomenon for m-angulations of the k-gon, for m fixed.

Conversely, the known cyclic sieving phenomenon for dissections of the k-gon with d edges [26, Thm. 7.1] gives a cyclic sieving phenomenon for the external rotation on trees with $d+1$ inner nodes and k leaves, in which vertices of degree 2 are forbidden.

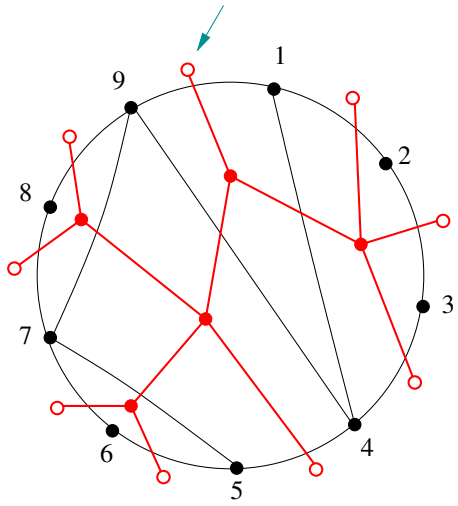


FIGURE 9. A dissection of the k-gon, with $k = 9$, having $d = 4$ diagonals, and the corresponding leaf-rooted tree with $k = 9$ leaves and $d + 1 = 5$ inner nodes. The degree of the faces of the dissection, or equivalently the degrees of the inner nodes of the tree, are $3, 3, 3, 4$ and 4.

11.4. Non-crossing partitions associated with reflection groups

The purpose of this subsection is to describe the setting of generalised non-crossing partitions associated with well-generated complex reflection groups as introduced by

Armstrong [5], and to explain how the general cyclic sieving results in [18, 20] cover the cyclic sieving results worked out in Subsections 11.1 and 11.2 for non-crossing matchings and non-crossing partitions. Phrased differently, here we make the indications on results for reflection groups made in these subsections explicit.

In order to do this, we need to recall the definition of *m-divisible non-crossing partitions associated with reflection groups*. Let W be a *well-generated complex reflection group* and c a *Coxeter element* in W. While, for our purposes, it is entirely sufficient to assume that $W = S_n$, the group of permutations of $\{1, 2, \ldots, n\}$, and $c = (1, 2, \ldots, n)$ (that is, the long cycle), we refer the interested reader to [23] for an excellent introduction into the theory of complex reflection groups. The *reflection length* $\ell_T(w)$ of an element $w \in W$ is defined as the smallest ℓ such that $w = t_1 t_2 \cdots t_\ell$ with all t_i's being reflections in W. For example, for the case of $W = S_n$ and $c = (1, 2, \ldots, n) = (1, 2)(2, 3) \cdots (n-1, n)$, we have $\ell_T(c) = n - 1$ since the reflections in S_n are the transpositions. Given these ingredients, Armstrong's [5, Sec. 3.3] definition of m-divisible non-crossing partitions reads as follows.

Definition 26. Let W be a well-generated complex reflection group and c a fixed Coxeter element in W. For a positive integer m, the set of m-divisible non-crossing partitions for W, denoted by $NC^{(m)}(W)$, is defined by

$$NC^{(m)}(W) = \big\{(w_0; w_1, \ldots, w_m) : w_0 w_1 \cdots w_m = c \text{ and } \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)\big\}.$$

Remarks. (1) In the case where $m = 1$, we write $NC(W)$ instead of $NC^{(1)}(W)$ for short and call this the set of (ordinary) non-crossing partitions associated with the reflection group W. Moreover, an element $(w_0; w_1)$ in $NC(W) = NC^{(1)}(W)$ (whose product equals the Coxeter element c) is usually identified with w_0.

(2) We have suppressed the dependence on the Coxeter element c in the notation, since here we are only interested in enumeration. Indeed, it can be shown that any two Coxeter elements are related to each other by conjugation and (possibly) an automorphism on the field of complex numbers (see [30, Thm. 4.2] or [23, Cor. 11.25]). It follows that any two sets $NC^{(m)}(W)$ corresponding to different Coxeter elements can be bijectively mapped to each other by these transformations.

In the case of $W = S_n$, Armstrong [5, Thm. 4.3.8] described a translation of the above abstract definition of m-divisible non-crossing partitions into the combinatorial language of non-crossing partitions according to Kreweras (those of Subsection 11.2). The resulting partitions are the m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ of Edelman [11], which are defined by the property that all block sizes are divisible by m. Given an element $(w_0; w_1, \ldots, w_m) \in NC^{(m)}(S_n)$, the bijection, $\bigtriangledown_{S_n}$ say, from [5, Thm. 4.3.8] works as follows: for $i = 1, 2, \ldots, m$, let $\tau_{m,i}$ be the transformation that maps a permutation $w \in S_n$ to a permutation $\tau_{m,i}(w) \in S_{mn}$ by letting

$$(\tau_{m,i}(w))(mk + i - m) = mw(k) + i - m, \quad k = 1, 2, \ldots, n,$$

and $(\tau_{m,i}(w))(l) = l$ for all $l \not\equiv i \pmod m$. With the choice of Coxeter element $c = (1, 2, \ldots, n)$, the announced bijection maps $(w_0; w_1, \ldots, w_m) \in NC^{(m)}(S_n)$ to

$$\bigtriangledown_{S_n}(w_0; w_1, \ldots, w_m) = (1, 2, \ldots, mn)\,(\tau_{m,1}(w_1))^{-1}\,(\tau_{m,2}(w_2))^{-1} \cdots (\tau_{m,m}(w_m))^{-1},$$

where the cycles in the disjoint cycle decomposition of the above permutation correspond to the blocks in the non-crossing partition of $\{1, 2, \dots, mn\}$. We refer the reader to [5, Sec. 4.3.2] for the details.

Example 9. Let $n = 7$, $m = 3$, $w_0 = (4, 5, 6)$, $w_1 = (3, 6)$, $w_2 = (1, 7)$, and $w_3 = (1, 2, 6)$. Then $(w_0; w_1, w_2, w_3)$ is mapped to

$$\begin{aligned}\bigtriangledown_{S_7}(w_0; w_1, w_2, w_3) &= (1, 2, \dots, 21)\,(7, 16)\,(2, 20)\,(18, 6, 3)\\ &= (1, 2, 21)\,(3, 19, 20)\,(4, 5, 6)\,(7, 17, 18)\,(8, 9, \dots, 16),\end{aligned}$$

which gives a partition of $\{1, \dots, 21\}$ with one block of size 9 and four blocks of size 3.

If $m = 1$, then $\bigtriangledown_{S_n}$ takes a pair $(w_0; w_1) \in NC^{(1)}(S_n) = NC(S_n)$, ignores w_1, and interprets w_0 as a partition of $\{1, 2, \dots, n\}$ by converting each cycle in the disjoint cycle decomposition of w_0 into a block containing the numbers involved in that cycle. It was already shown by Biane [9] that this produces indeed a non-crossing partition. Conversely, one recovers w_0 from the partition by associating to each block a cycle in which the elements of the block are ordered clockwise; in other words, each cycle of w_0 is *increasing*, meaning that it can be written $(i_1, i_2, \dots, i_k)$ for an increasing sequence $i_1 < i_2 < \cdots < i_k$.

Example 10. Let $m = 1$, $W = S_9$,

$$w_0 = (1, 5, 8)(2, 4)(6, 7) \quad \text{and} \quad w_1 = (1, 4)(2, 3)(5, 7)(8, 9).$$

Then we have $w_0 w_1 = (1, 2, \dots, 9) = c$, and thus the above described identification produces the non-crossing partition $\{\{1, 5, 8\}, \{2, 4\}, \{3\}, \{6, 7\}, \{9\}\}$, that is, the non-crossing partition shown on the left of Figure 7.

Armstrong [5, Thm. 4.3.13] has shown that the cycle structure of the first component of an m-divisible non-crossing partition determines the block structure of its image under his bijection.

Proposition 27. *Let $(w_0; w_1, \dots, w_m) \in NC^{(m)}(S_n)$. The non-crossing partition $\bigtriangledown_{S_n}(w_0; w_1, \dots, w_m)$ of $\{1, 2, \dots, mn\}$ has b_i blocks of size mi if and only if w_0 has b_i cycles of length i in its disjoint cycle decomposition.*

In Example 9, we have $w_0 = (4, 5, 6) = (1)(2)(3)(7)(4, 5, 6)$ which has one cycle of length 3 and four cycles of length 1. Indeed, the image of $(w_0; w_1, w_2, w_3)$ in the example has one block of size $3 \cdot 3 = 9$ and four blocks of size $3 \cdot 1 = 3$.

We introduced the concept of Kreweras complement in Subsection 11.2 (by way of an example; see Figure 7). As shown by Kreweras [22, p. 339], this may be equivalently realised in the language of permutations. Namely, given a non-crossing partition π, and the associated permutation w_0 constructed as above, the Kreweras complement of π is the partition associated with the permutation cw_0^{-1}, where $c = (1, 2, \dots, n)$ is again the long cycle. It is useful to note that the cycles of cw_0^{-1} are also increasing; see Example 11 below.

More generally, the *Kreweras complement for well-generated complex reflection groups* W is defined as the map

$$w \mapsto cw^{-1}, \quad \text{for } w \in NC(W), \tag{11.1}$$

under the identification of $NC(W)$ with the first component of the tuples $(w_0; w_1)$ lying in $NC(W) = NC^{(1)}(W)$.

Example 11. Consider the non-crossing partition

$$\{\{1,5,8\},\{2,4\},\{3\},\{6,7\},\{9\}\}$$

on the left of Figure 7. We have

$$\begin{aligned}(1,2,\ldots,9)\big((1,5,8)(2,4)(6,7)\big)^{-1} &= (1,2,\ldots,9)(1,8,5)(2,4)(6,7)\\ &= (1,9)(2,5)(3,4)(6,8).\end{aligned}$$

We see that this is indeed the Kreweras complement indicated by the light-grey blocks on the right of Figure 7.

Algebraically, the two-fold application of the Kreweras complement is

$$w \mapsto c\left(cw^{-1}\right)^{-1} = cwc^{-1},$$

that is, conjugation by the Coxeter element c. Hence, in the special case where $W = S_n$, the two-fold application of the Kreweras complement acts by mapping each number i to $i+1$ modulo n, as we already observed in Subsection 11.2.

We are now able to discuss the known cyclic sieving phenomena for Armstrong's m-divisible non-crossing partitions for reflection groups. Namely, Armstrong defined an action on m-divisible non-crossing partitions in [5, Def. 3.4.15] by

$$A : (w_0; w_1, \ldots, w_m) \mapsto (v; cw_mc^{-1}, w_1, \ldots, w_{m-1}),$$

where $v = (cw_mc^{-1})w_0(cw_mc^{-1})^{-1}$. Armstrong shows that this action generates a cyclic group of order mh, where h is the *Coxeter number* of W, that is, the largest of the *invariant degrees* of W. For $W = S_n$ the Coxeter number is n. Therefore, in the case of the symmetric group S_n, the above action has order mn. Using the bijection ∇_{S_n}, this can also be seen from the fact that the action A in terms of Edelman's m-divisible non-crossing partitions is given by *rotation*, i.e., every element i in a partition is replaced by $i+1$, taken modulo mn. In particular, when $m = 1$ we recover the rotation on non-crossing partitions that was identified with R^2 in Subsection 11.2.

Putting results from [18] and [20] together, we have the following cyclic sieving phenomenon.

Theorem 28. *The triple* $(NC^{(m)}(W), \langle A\rangle, \mathrm{Cat}^{(m)}(W;q))$ *exhibits the cyclic sieving phenomenon, where*

$$\mathrm{Cat}^{(m)}(W;q) := \prod_{i=1}^{n} \frac{[mh+d_i]_q}{[d_i]_q},$$

n is the rank of W, $(d_1, d_2, \ldots, d_n)$ the list of invariant degrees of W, and $h = d_n$ the Coxeter number of W.

Remark. We refer the reader again to [23] for the definitions of the rank and the invariant degrees. For our discussion, the only fact that is relevant is that

$$\mathrm{Cat}^{(m)}(S_n;q) = \frac{1}{[n]_q}\begin{bmatrix}(m+1)n\\ n-1\end{bmatrix}_q. \tag{11.2}$$

For $m = 1$, this is MacMahon's q-Catalan polynomial, and one recovers the cyclic sieving phenomenon for non-crossing partitions under rotation mentioned in Subsection 11.2.

In the case where $w_0 = \mathrm{id}$, the action A reduces to

$$A : (\mathrm{id}; w_1, \dots, w_m) \mapsto (\mathrm{id}; cw_mc^{-1}, w_1, \dots, w_{m-1}). \tag{11.3}$$

Combining again results from [18] and [20], we obtain a second cyclic sieving phenomenon.

Theorem 29. *Let $NC^{(m;0)}(W)$ denote the subset of $NC^{(m)}(W)$ consisting of those elements for which $w_0 = \mathrm{id}$. Then the triple $(NC^{(m;0)}(W), \langle A\rangle, \mathrm{Cat}^{(m-1)}(W;q))$ exhibits the cyclic sieving phenomenon.*

Let us consider the case where $m = 2$ and $W = S_n$, that is, the 2-divisible non-crossing partitions $(\mathrm{id}; w_1, w_2)$ for S_n. By Proposition 27, the map ∇_{S_n} sends them to partitions of $\{1, 2, \dots, 2n\}$ in which all blocks have cardinality 2, hence to non-crossing *matchings*. We will come back to this at the end of this section. For the moment, observe that there is an alternative point of view on these objects: forgetting $w_0 = \mathrm{id}$, these are the pairs (w_1, w_2) with $w_1w_2 = c$ and $\ell_T(w_1) + \ell_T(w_2) = \ell_T(c) = n - 1$. So we simply recover non-crossing partitions of $NC(S_n)$.

Obviously, w_1 determines w_2. The action (11.3) reduces to

$$(w_1, w_2) \mapsto (cw_2c^{-1}, w_1),$$

or, if we focus on the first component,

$$w_1 \mapsto cw_2c^{-1} = c(w_1^{-1}c)c^{-1} = cw_1^{-1}.$$

As we can see, this is exactly the Kreweras complement defined in (11.1). Hence, the special case of Theorem 29 where $m = 2$ and $W = S_n$ is equivalent to the cyclic sieving result for non-crossing partitions under the action of the Kreweras complement mentioned in Subsection 11.2 (and, thus, also to Theorem 5).

Finally, in order to understand the connection with the non-crossing (perfect) matchings of Subsection 11.1, we still consider the 2-divisible non-crossing partitions $(\mathrm{id}; w_1, w_2)$ for S_n, but now we map them using ∇_{S_n} to non-crossing partitions in which all blocks have size 2; in other words, to non-crossing matchings of $2n$ points. More precisely, the blocks of the matching associated with $(\mathrm{id}; w_1, w_2)$ can be seen to be the pairs $\{2k, 2w_1(k) - 1\}$, for $1 \le k \le n$, or equivalently the pairs $\{2w_2(k), 2k + 1\}$. Now apply Armstrong's action:

$$(\mathrm{id}; w_1, w_2) \mapsto (\mathrm{id}; cw_2c^{-1}, w_1).$$

The matching corresponding to the new 2-divisible partition has blocks $\{2w_1(k), 2k+1\}$, which means that the action A translates into standard rotation on matchings (that is, shifting everything by one unit). Hence, we see that the special case of Theorem 29 where $m = 2$ and $W = S_n$ is also equivalent to Corollary 25 (and, thus again, to Theorem 5).

Example 12. Let $n = 9$, $w_1 = (1, 5, 8)(2, 4)(6, 7)$ and $w_2 = (1, 4)(2, 3)(5, 7)(8, 9)$. Then the matching associated with $(\mathrm{id}; w_1, w_2)$ by ∇_{S_9} has blocks

$$\{1, 16\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}, \{10, 15\}, \{11, 14\}, \{12, 13\}, \{17, 18\}.$$

One can check that the blocks of the matching associated with $(\mathrm{id}; cw_2c^{-1}, w_1)$ are

$$\{2, 17\}, \{3, 10\}, \{4, 9\}, \{5, 8\}, \{6, 7\}, \{11, 16\}, \{12, 15\}, \{13, 14\}, \{18, 1\}.$$

12. Tree-rooted planar maps and their cyclic sieving

The aim of this section is to establish cyclic sieving phenomena for *tree-rooted maps*, that is, planar maps equipped with a distinguished spanning tree. A rotation that generalises the ordinary rotation R acts on these objects.

We begin with some definitions, and then prove three cyclic sieving results; see Theorems 30, 33, and 34 below. The existence of such phenomena was actually the original motivation of this paper. To finish, we translate these results in terms of cubic maps equipped with a Hamiltonian cycle, and in terms of lattice walks confined to a quadrant.

12.1. Definitions

A (*planar*) *map* is a proper embedding of a connected planar graph in the oriented sphere, considered up to orientation preserving homeomorphism. Loops and multiple edges are allowed (see Figure 10, left). The *faces* of a map are the connected components of its complement. The numbers of vertices, edges, and faces of a planar map $\mathfrak{m}$, denoted by $\mathrm{v}(\mathfrak{m})$, $\mathrm{e}(\mathfrak{m})$, and $\mathrm{f}(\mathfrak{m})$, are related by Euler's relation

$$v(\mathfrak{m}) + \mathrm{f}(\mathfrak{m}) = \mathrm{e}(\mathfrak{m}) + 2.$$

The *degree* of a vertex is the number of half-edges incident to it (a loop thus contributes 2 to the degree of the vertex where it is attached). A *corner* is a sector delimited by two consecutive half-edges around a vertex; hence a vertex of degree k is incident to k corners. A *rooted* map is a map with a distinguished corner. It is equivalent to distinguish and orient one edge (the one that follows the root corner in counterclockwise order, oriented away from the root vertex). All our maps are planar and rooted. We often draw them in the plane by putting the root corner in the infinite face.

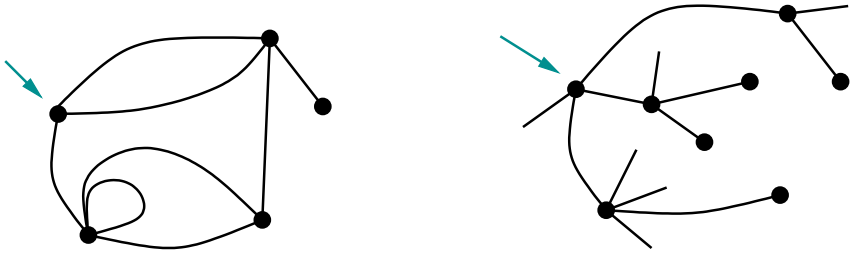


Figure 10. *Left*: a rooted planar map. *Right*: a rooted b-tree with 8 vertices and 6 buds.

Observe that a rooted tree, as defined in Section 2, is just a rooted one-face map. A *b-tree* is a tree with additional half-edges, called *buds*, attached to the vertices; see the right of Figure 10 for an example. A bud contributes 1 to the degree of the vertex to which it is attached. Adding a bud to a vertex also increases the number of corners by 1. Note that there is no corner at the open (vertex-less) endpoint of a bud. A *rooted* b-tree is a b-tree with a distinguished corner.

A *tree-rooted map*, in Mullin's terminology [24], is a rooted map with a distinguished spanning tree. A tree-rooted map with $i+1$ vertices and $j+1$ faces (and, thus, $n = i+j$ edges) can be decomposed into a rooted b-tree having $i+1$ vertices and $2j$ buds, coupled with a non-crossing matching of the $2j$ buds (see Figure 11). This correspondence is bijective. Note that all corners of the map are incident to the spanning tree.

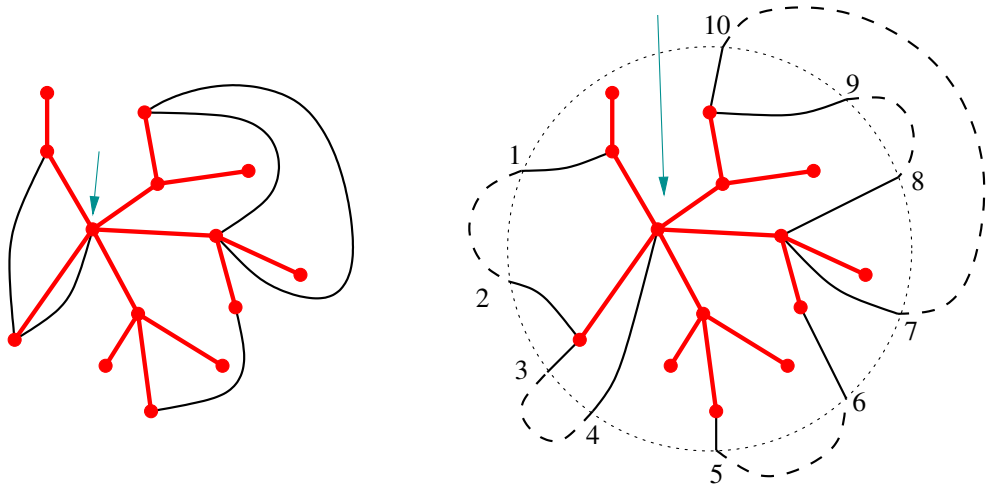


FIGURE 11. A tree-rooted map with $i+1=14$ vertices and $j+1=6$ faces (left). It can be seen as a rooted b-tree with 14 vertices and $2j=10$ buds, coupled with a non-crossing matching of the 10 buds (shown in dashed lines on the right).

On tree-rooted maps, we introduce an extension of the ordinary rotation R that we defined on rooted trees in Section 4. By definition, it acts by moving the root corner to the next corner of the map around the spanning tree, in counterclockwise order; see an example in Figure 12. (In order to understand why the third orbit has cardinality 2 only, we have to remember that maps are drawn *on the sphere*.) On tree-rooted maps with n edges, this extended rotation, which we still denote by R, realises an action of the cyclic group of order $2n$. On the other hand, the rotation R acts similarly on b-trees. Given that b-trees with $n+1$ nodes and b buds have $2n+b$ corners, this rotation gives an action of the cyclic group of order $2n+b$ on such trees. We shall make use of this separate consideration as a step towards the proofs of Theorems 30 and 34, see Propositions 31 and 35.

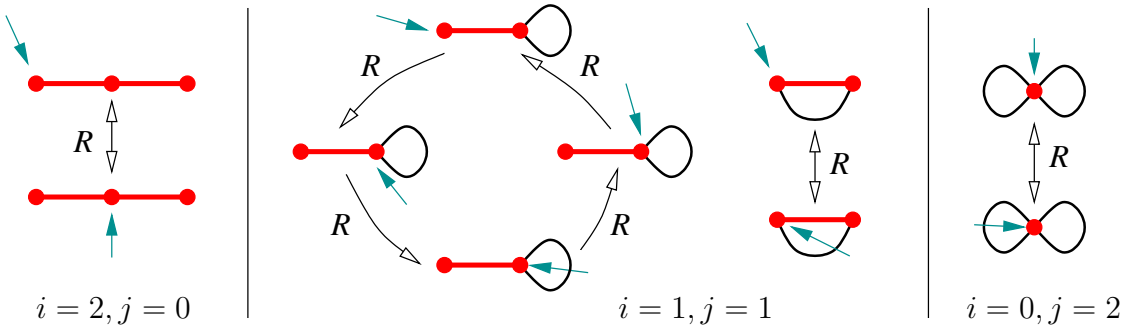


FIGURE 12. The 10 tree-rooted maps with $n=2$ edges ($i+1$ vertices, $j+1$ faces), and the action of the rotation R on these maps. The spanning trees are shown in thick red lines.

12.2. Tree-rooted maps with prescribed number of vertices and faces

For non-negative integers i and j, we denote by $\mathrm{TM}(i,j)$ the set of tree-rooted maps having $i+1$ vertices and $j+1$ faces. (Note the shift in the variables i and j; this is analogous to $\mathrm{T}(n)$ being the set of rooted trees with n edges, hence $n+1$ vertices.)

Theorem 30. *Let i and j be non-negative integers. Then the cyclic group of order $2i+2j$ acts on $\mathrm{TM}(i,j)$ by the rotation R, and the triple*

$$\left(\mathrm{TM}(i,j), \langle R\rangle, \frac{1}{[i+1]_q[j+1]_q}\begin{bmatrix}2i+2j\\ i,i,j,j\end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

The case $q=1$ is due to Mullin [24, Eq. (4.6)]. The above theorem extends Theorem 5, which corresponds to the case $j=0$.

Example 13. For $i=2, j=0$, one finds the polynomial $1+q^2$, in agreement with the first orbit in Figure 12. We obtain the same polynomial (which is expected, by a duality argument) for $i=0, j=2$. In the more interesting case where $i=j=1$, we obtain $P(q)=(1+q^2)(1+q+q^2)$, and one can check that cyclic sieving holds indeed with the help of Figure 12.

Recall that a tree-rooted map is obtained by coupling a b-tree with a perfect matching (cf. Figure 11). Accordingly, we will show that the above cyclic sieving phenomenon results from the combination of two cyclic sieving results, one for b-trees (established in the proposition below), and one for non-crossing matchings (Corollary 25).

Proposition 31. *Let $\mathrm{BT}(b,n)$ be the set of b-trees with b buds and $n+1$ nodes. The cyclic group of order $m := 2n+b$ acts on $\mathrm{BT}(b,n)$ by the rotation R, and the triple*

$$\left(\mathrm{BT}(b,n), \langle R\rangle, \frac{1}{[n+1]_q}\begin{bmatrix}2n+b\\ b,n,n\end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Note that this result specialises to Theorem 5 when $b=0$.

Proof. Polynomiality and non-negativity of coefficients of the cyclic sieving polynomial follow from the rewriting

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n+b\\ b,n,n\end{bmatrix}_q = \begin{bmatrix}2n+b\\ b\end{bmatrix}_q \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q$$

and the well-known facts that q-binomial coefficients and q-Catalan numbers are polynomials in q with non-negative coefficients; see [3, Thm. 3.1] and the first paragraph of the proof of Theorem 5.

Let ω be a primitive m-th root of unity, with $m=2n+b$. We have to prove

$$\mathrm{Fix}_{R^e}(\mathrm{BT}(b,n)) = \frac{1}{[n+1]_q}\begin{bmatrix}2n+b\\ b,n,n\end{bmatrix}_q\Bigg|_{q=\omega^e} \tag{12.1}$$

for all integers e. It suffices to prove this for values of $e \in [\![0, m-1]\!]$ equal to 0 or dividing m. Lemma 32 below gives the corresponding values of the right-hand side of (12.1).

We begin with $e = 0$. To construct a b-tree of $\mathrm{BT}(b, n)$, we start from a rooted tree with $n+1$ nodes (counted by the Catalan number C_n) and then spread b buds around this tree. When walking around the b-tree, starting from the root corner, one sees a sequence of $2n$ edge-sides and b buds. This sequence is counted by the binomial coefficient $\binom{2n+b}{b}$. Hence

$$|\,\mathrm{BT}(b,n)| = \binom{2n+b}{b}\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n+b}{b,n,n}, \tag{12.2}$$

which is indeed the value at $q = 1$ of (12.1).

Now let e be a positive divisor of $m = 2n + b$, and let $d = m/e$. Let τ^+ be a tree of $\mathrm{BT}(b, n)$, and let τ be the rooted tree obtained by removing all buds of τ^+. Then τ has n edges. By adapting the proof of Proposition 4, we obtain that

$$R^e(\tau^+) = \tau^+ \quad \text{implies} \quad R^f(\tau) = \tau,$$

with $f = 2n/d$. In particular, there can exist b-trees fixed by R^e only if $e = n + \frac{b}{2}$ with n odd and b even (that is, in particular, $f = n$ and $d = 2$), or if m/e divides n (that is, f is even).

So let us first take $e = n + \frac{b}{2}$, with b and $n+1$ even, and let $\tau^+ \in \mathrm{BT}(b, n)$ be fixed by R^e. The associated tree τ is fixed by R^n, and thus has a central edge c by Proposition 3. We now proceed as in Proposition 3. We return to τ^+, cut the edge c in the middle, retain the component containing the root, and mark the bud that comes from the edge c; this gives a tree τ_1^+ in $\mathrm{BT}\left(1 + \frac{b}{2}, \frac{n-1}{2}\right)$, with a marked bud, and the correspondence $\tau^+ \mapsto \tau_1^+$ is one-to-one. Hence in this case,

$$\mathrm{Fix}_{R^e}(\mathrm{BT}(b,n)) = \left(1+\frac{b}{2}\right)\left|\mathrm{BT}\left(1+\frac{b}{2}, \frac{n-1}{2}\right)\right|,$$

which, according to (12.2), is equivalent to the expression in the second alternative of (12.3). This proves (12.1) in this case.

Now assume that $d = m/e$ divides n, and let $f = 2n/d$. Let $\tau^+ \in \mathrm{BT}(b, n)$ be fixed by R^e. The associated tree τ is fixed by R^f, where f is even, and thus has a central node c by Proposition 3. We now proceed as in Proposition 3. Let ℓ be the degree of c in τ^+. As in Figure 4, we denote the b-trees attached at the centre by $\tau_1, \dots, \tau_\ell$ — some of them may consist of a single bud. Our convention here is that τ_1 contains the edge *or the bud* that follows the root corner, in counterclockwise order. Then $d := m/e$ divides ℓ, and the trees τ_a and $\tau_{a+\ell/d}$ are copies of one another. The map Ψ_d, now extended to trees with buds, then sends trees of $\mathrm{BT}(b, n)$ fixed by R^e bijectively to trees of $\mathrm{BT}(b/d, n/d)$ having a marked node. Hence

$$\mathrm{Fix}_{R^e}(\mathrm{BT}(b,n)) = \left(1+\frac{n}{d}\right)\left|\mathrm{BT}\left(\frac{b}{d}, \frac{n}{d}\right)\right|,$$

which, according to (12.2), is equivalent to the expression in the third alternative of (12.3). (Note that the right-hand side is zero if b is not a multiple of d.) This proves (12.1) in this case, and concludes the proof of the proposition. □

The following lemma extends Lemma 6, which corresponds to the case $b = 0$.

Lemma 32. *Let $n \geq 1$, $b \geq 0$, and write $m = 2n+b$. Furthermore, let ω be a primitive m-th root of unity, and let e be a non-negative integer $< m$ equal to 0 or dividing m. In the latter case, write $d := m/e$. Then we have*

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n+b\\ b,n,n\end{bmatrix}_q\Bigg|_{q=\omega^e} = \begin{cases} \frac{1}{n+1}\binom{2n+b}{b,n,n}, & \text{if } e=0,\\ \binom{n+\frac{b}{2}}{\frac{b}{2},\frac{n-1}{2},\frac{n+1}{2}}, & \text{if } d=2 \text{ with } n+1 \text{ and } b \text{ even},\\ \binom{e}{\frac{b}{d},\frac{n}{d},\frac{n}{d}}, & \text{if } d \text{ divides } n \text{ and } b,\\ 0, & \text{otherwise.}\end{cases} \tag{12.3}$$

Proof. The case $e = 0$ being obvious, we assume that e is a positive divisor of $m = 2n+b$ with $e < m$. According to (5.3), in order to obtain a non-zero value of the specialisation on the left-hand side of (12.3), there must be an equal number of factors of the form $1-q^\alpha$ with $d \mid \alpha$ in the numerator and in the denominator, with $d \geq 2$. In other words, we must have

$$\left\lfloor \frac{2n+b}{d} \right\rfloor - \left\lfloor \frac{b}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n+1}{d} \right\rfloor = 0.$$

Introducing the fractional parts $N = \{\frac{n}{d}\}$ and $B = \{\frac{b}{d}\}$, this can be rewritten as

$$2N + B = \left\lfloor N + \frac{1}{d} \right\rfloor.$$

Here, the left-hand side does not need taking an integer part since, by assumption, we have $d \mid (2n+b)$. Clearly, the only possible values of the right-hand side in the above equation are 0 or 1. In the former case, we infer $N = B = 0$, that is, we have that d divides n and b, and we are in the third case of (12.3). If $2N + B = 1 = \lfloor N + \frac{1}{d} \rfloor$, we have $N = 1 - \frac{1}{d}$, so that $B = \frac{2}{d} - 1$, which forces $d = 2$, with b and $n+1$ even. We are then in the second case of (12.3). Consequently, in the fourth case of (12.3), the value is 0.

It is then an easy task to evaluate the left-hand side of (12.3) in the second and third case, using the simple fact (5.4). □

We can now prove the cyclic sieving phenomenon for tree-rooted maps with $i+1$ vertices and $j+1$ faces.

Proof of Theorem 30. We first note that the expression of the theorem can be written as a product:

$$\frac{1}{[i+1]_q[j+1]_q}\begin{bmatrix}2i+2j\\ i,i,j,j\end{bmatrix}_q = \begin{bmatrix}2i+2j\\ 2i,2j\end{bmatrix}_q \times \frac{1}{[i+1]_q}\begin{bmatrix}2i\\ i\end{bmatrix}_q \times \frac{1}{[j+1]_q}\begin{bmatrix}2j\\ j\end{bmatrix}_q.$$

This is a polynomial with non-negative coefficients because this is the case for each of the three factors.

Let $n = i + j$ be the number of edges in a tree-rooted map of $\mathrm{TM}(i,j)$. We have to prove that

$$\mathrm{Fix}_{R^e}(\mathrm{TM}(i,j)) = \frac{1}{[i+1]_q[j+1]_q}\begin{bmatrix}2i+2j\\ i,i,j,j\end{bmatrix}_q\Bigg|_{q=\omega^e}$$

for all integers e. It suffices to prove this for values of $e \in [\![0, 2n-1]\!]$ equal to 0 or dividing $2n$.

For $e = 0$, all elements of $\mathrm{TM}(i,j)$ are fixed by R^e. As recalled in Subsection 12.1, such tree-rooted maps are obtained by combining a non-crossing perfect matching of $2j$ points, counted by the Catalan number C_j, with a b-tree having $i+1$ vertices and $b = 2j$ buds; see Figure 11. According to (12.2), this gives

$$|\,\mathrm{TM}(i,j)| = \frac{1}{j+1}\binom{2j}{j} \cdot \frac{1}{i+1}\binom{2i+2j}{i,i,2j} = \frac{1}{(i+1)(j+1)}\binom{2i+2j}{i,i,j,j},$$

as desired.

Now let $e \in [\![1, 2n-1]\!]$ be a divisor of $2n$, and write $d := 2n/e$. Consider a tree-rooted map $(\mathfrak{m}, \tau)$ in $\mathrm{TM}(i,j)$ that is fixed by R^e, and let τ^+ be the associated b-tree. Since τ^+ is simply obtained by opening the edges of $\mathfrak{m}$ that are not in τ, then τ^+ itself is fixed by R^e. Conversely, take a b-tree τ^+ with $i+1$ vertices and $b = 2j$ buds, invariant by R^e, and match its buds with a non-crossing perfect matching $\mathfrak{p}$; then the resulting tree-rooted map is invariant by R^e if and only if $\mathfrak{p}$ is invariant by a rotation of $(2\pi)/d$. Hence, using the notation of Corollary 25,

$$\begin{aligned}
\mathrm{Fix}_{R^e}(\mathrm{TM}(i,j)) &= \mathrm{Fix}_{R^e}(\mathrm{BT}(2j,i)) \cdot \mathrm{Fix}_{2\pi/d}(\mathrm{NCM}(j)) \\
&= \frac{1}{[i+1]_q}\begin{bmatrix}2i+2j\\ i,i,2j\end{bmatrix}_q\Bigg|_{q=\mathsf{e}^{2\pi i/d}} \times \frac{1}{[j+1]_q}\begin{bmatrix}2j\\ j\end{bmatrix}_q\Bigg|_{q=\mathsf{e}^{2\pi i/d}} \qquad (12.4)\\
&= \frac{1}{[i+1]_q[j+1]_q}\begin{bmatrix}2i+2j\\ i,i,j,j\end{bmatrix}_q\Bigg|_{q=\mathsf{e}^{2\pi i/d}}, \qquad (12.5)
\end{aligned}$$

where we have used Proposition 31 (with b replaced by $2j$ and n replaced by i) and Corollary 25 (with n replaced by j) to obtain the second line. By Lemma 2, this concludes the proof. □

12.3. Tree-rooted maps with prescribed number of edges

We now derive a cyclic sieving result for tree-rooted maps with n edges from the previous theorem on $\mathrm{TM}(i,j)$, using simply summation, in the spirit of [2]. We denote the set of tree-rooted maps with n edges by $\mathrm{TM}(n)$.

Theorem 33. *Let n be a non-negative integer. Then the cyclic group of order $2n$ acts on $\mathrm{TM}(n)$ by the rotation R, and the triple*

$$\left(\mathrm{TM}(n), \langle R\rangle, \frac{1}{[n+1]_q[n+2]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q\begin{bmatrix}2n+2\\ n+1\end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

The case $q = 1$ is due to Mullin [24, Eq. (4.7)].

Example 14. For $n = 2$, as in Figure 12, we obtain the polynomial

$$P(q) = \left(q^2+1\right)\left(q^4+q^3+q^2+q+1\right)\left(q^2-q+1\right) = q^8+2q^6+q^5+2q^4+q^3+2q^2+1,$$

and we can check, by evaluating it at $1, i$ and -1, that cyclic sieving indeed holds.

Remark. Below we prove this theorem by summation of (a slight modification of) the cyclic sieving polynomials of Theorem 30. In the same fashion, we could have derived our first theorem, Theorem 5, from its refinement, Theorem 7, but we gave instead a direct, sum-free proof of Theorem 5. In contrast, we have no direct proof of the above theorem. Bernardi [7] gave a direct combinatorial proof of the case $q = 1$, by constructing a bijection between tree-rooted maps with n edges and pairs consisting of a tree with n edges and a non-crossing partition of $n + 1$ elements. However, this bijection does not seem to behave nicely when applying the rotation R to tree-rooted maps.

Proof of Theorem 33. A tree-rooted map with n edges has $i + 1$ vertices and $j + 1$ edges, for some i and j that sum to n. Theorem 30 gives a cyclic sieving polynomial for these tree-rooted maps under the rotation R. Recall that we have constructed it as the product of the cyclic sieving polynomials obtained for b-trees of $\mathrm{BT}(2j, i)$ and for non-crossing perfect matchings of $2j$ elements; cf. (12.4) and (12.5), and the paragraph above these lines.

Now let $e \in [\![0, 2n - 1]\!]$ be either 0 or a divisor of $2n$. Let $d = 2n/e$. Then $d \in \{2, 3, \dots, 2n\} \cup \{\infty\}$. For ω a primitive $2n$-th root of unity, ω^e is a primitive d-th root of unity (or 1, if $e = 0$). By Lemma 32 (with b replaced by $2j$ and n replaced by i) and Corollary 25 (with n replaced by j), the cyclic sieving polynomial in the statement of Theorem 30 vanishes at ω^e unless

- $e = 0$,
- or $d = 2$,
- or d divides both i and j.

This implies that

$$\frac{1}{[i+1]_q[j+1]_q}\begin{bmatrix}2i+2j\\ i,i,j,j\end{bmatrix}_q \cdot q^{(n+1-i)j}$$

is another cyclic sieving polynomial for $\mathrm{TM}(i, j)$, since the power of q that we have multiplied with takes the value 1 at ω^e in all the above cases. (Note that $(n+1-i)j = (j + 1)j$ is always even.)

Hence a cyclic sieving polynomial for $\mathrm{TM}(n)$ is

$$\sum_{i+j=n} \frac{1}{[i+1]_q[j+1]_q}\begin{bmatrix}2i+2j\\ i,i,j,j\end{bmatrix}_q q^{(n+1-i)j} = \frac{[2n]_q!}{[n+1]_q!^2}\sum_{i+j=n}\begin{bmatrix}n+1\\ i\end{bmatrix}\begin{bmatrix}n+1\\ j\end{bmatrix} q^{(n+1-i)j}$$
$$= \frac{[2n]_q!}{[n+1]_q!^2}\begin{bmatrix}2n+2\\ n\end{bmatrix},$$

by the q-analogue of the Chu–Vandermonde identity [3, Eq. (3.3.10)]. This gives the polynomial announced in Theorem 33. Since it is obtained as a sum of cyclic sieving polynomials, its coefficients are clearly non-negative. □

12.4. Tree-rooted maps with prescribed vertex degrees

Let $\mathbf{n} = (n_1, n_2, \dots)$ be a sequence of non-negative numbers containing finitely many non-zero entries. Let $\mathrm{TM}(j, \mathbf{n})$ be the set of tree-rooted maps having n_k vertices of degree k for all k, and $j+1$ faces. The b-trees associated with such maps have n_k vertices of degree k for all k, and $b = 2j$ buds. Adding a node of degree 1 at the end of each

bud gives a tree with $b+n_1$ leaves and n_k nodes of degree k for $k \geq 2$. By (2.5), such a tree can only exist if

$$-b+\sum_{k\geq 1}(k-2)n_k = -2. \tag{12.6}$$

Assuming this holds (with $b = 2j$), a tree-rooted map of $\mathrm{TM}(j,\mathbf{n})$ has $i+1$ vertices and n edges, with

$$i+1 := \sum_k n_k \qquad \text{and} \qquad n := i+j = j-1+\sum_k n_k.$$

The following theorem extends Theorem 16, which corresponds to the case $j = 0$.

Theorem 34. *Let $j \geq 0$ and $\mathbf{n} = (n_1, n_2, \ldots)$ satisfy (12.6) with $b = 2j$. Define n as above. The cyclic group of order $2n$ acts on $\mathrm{TM}(j,\mathbf{n})$ by the rotation R, and the triple*

$$\left(\mathrm{TM}(j,\mathbf{n}), \langle R\rangle, \frac{[2n]_q}{[j+1]_q[n+j+1]_q\,[n+j]_q} \begin{bmatrix} n+j+1 \\ j, j, n_1, n_2, \ldots \end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Example 15. Let us take $j = 1$, $n_1 = n_3 = 1$, the other n_k's being zero. Then $i = 1$, $n = 2$, and we obtain the polynomial $P(q) = [4]_q = 1+q+q^2+q^3$. We can check that cyclic sieving holds, using the second orbit in Figure 12.

Again, this theorem decouples into the cyclic sieving phenomenon for non-crossing matchings of $2j$ points stated in Corollary 25 (which we apply with n replaced by j) and the following cyclic sieving result dealing with b-trees with fixed vertex degrees (which we apply with n replaced by i and b replaced by $2j$). The proof of Theorem 34 then mimics that of Theorem 30, and we do not repeat the argument.

Proposition 35. *Let $b \geq 0$ and $\mathbf{n} = (n_1, n_2, \ldots)$ satisfy (12.6). Let $\mathrm{BT}(b,\mathbf{n})$ denote the set of all b-trees with b buds and n_k nodes of degree k, for $k \geq 1$. Furthermore, write $n+1 := \sum_i n_i$. Then the cyclic group of order $m := 2n+b$ acts on $\mathrm{BT}(b,\mathbf{n})$ by the rotation R, and the triple*

$$\left(\mathrm{BT}(b,\mathbf{n}), \langle R\rangle, \frac{[2n+b]_q}{[n+b+1]_q\,[n+b]_q} \begin{bmatrix} n+b+1 \\ b, n_1, n_2, \ldots \end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Proof. Polynomiality and non-negativity of coefficients of the cyclic sieving polynomial are proved in Lemma 36 below.

Let ω be a primitive m-th root of unity. We have to prove

$$\mathrm{Fix}_{R^e}(\mathrm{BT}(b,\mathbf{n})) = \frac{[2n+b]_q}{[n+b+1]_q\,[n+b]_q} \begin{bmatrix} n+b+1 \\ b, n_1, n_2, \ldots \end{bmatrix}_q \Bigg|_{q=\omega^e} \tag{12.7}$$

for all integers e. It suffices to prove this for values of $e \in [\![0, m-1]\!]$ equal to 0 or dividing m. Lemma 37 below gives the corresponding values of the right-hand side of (12.7).

We begin with $e = 0$. To construct a b-tree of $\mathrm{BT}(b,\mathbf{n})$, we start from a rooted tree of $\mathrm{T}(b+n_1, n_2, n_3, \ldots)$. We choose b of its leaves, which we convert into buds. This

yields a tree with buds, rooted either at a corner (and there are $2n+b$ of them) or at one of the b buds. A double rooting argument then gives

$$\begin{aligned} |\operatorname{BT}(b,\mathbf{n})| &= \frac{2n+b}{2n+2b}\binom{b+n_1}{b}\,|\operatorname{T}(b+n_1,n_2,\ldots)| \\ &= \frac{2n+b}{(n+b)(n+b+1)}\binom{b+n+1}{b,n_1,n_2,\ldots} \end{aligned} \tag{12.8}$$

by (2.8). This proves the case $e=0$ of (12.7).

Next we assume that $e>0$, and we refine the argument of the proof of Proposition 31, taking into account the number of vertices of each degree. We write as usual $d := m/e = (2n+b)/e$.

If $d=2$ and the number $n+1$ of vertices is even, then we need to count b-trees with $1+\frac{b}{2}$ buds, $n_i/2$ nodes of degree i for $i\geq 1$, and a marked bud. In particular, b must be even. This gives

$$\operatorname{Fix}_{R^e}(\operatorname{BT}(b,\mathbf{n})) = \left(1+\frac{b}{2}\right)\left|\operatorname{BT}\left(1+\frac{b}{2},\frac{\mathbf{n}}{2}\right)\right| = \frac{2n+b}{n+b+1}\binom{\frac{b+n+1}{2}}{\frac{b}{2},\frac{n_1}{2},\ldots}$$

by (12.8). This coincides with the expression in the second alternative of (12.10). This proves (12.7) in this case.

Now we assume that $d=m/e$ divides n. Then the b-trees of $\operatorname{BT}(b,\mathbf{n})$ fixed by R^e have a central node, of degree ℓ divisible by d. Moreover, ℓ must be the (necessarily unique) integer such that $n_\ell \equiv 1 \pmod d$ while d divides n_i for $i\neq\ell$. By extending the map Ψ_d to b-trees, we obtain a bijection between b-trees of $\operatorname{BT}(b,\mathbf{n})$ fixed by R^e and b-trees having

- $\frac{b}{d}$ buds,
- $\bar n_i := \frac{n_i}{d}$ nodes of degree i for $i\notin\{\ell/d,\ell\}$,
- $\bar n_\ell := \frac{n_\ell-1}{d}$ nodes of degree ℓ,
- $\bar n_{\ell/d} := \frac{n_{\ell/d}}{d}+1$ nodes of degree ℓ/d, one of which is marked.

This gives

$$\begin{aligned} \operatorname{Fix}_{R^e}(\operatorname{BT}(b,\mathbf{n})) &= \bar n_{\ell/d}\left|\operatorname{BT}\left(\frac{b}{d},\bar n_1,\bar n_2,\ldots\right)\right| \\ &= \frac{2n+b}{n+b}\binom{\frac{n+b}{d}}{\frac{b}{d},\frac{n_1}{d},\ldots,\frac{n_{\ell-1}}{d},\frac{n_\ell-1}{d},\frac{n_{\ell+1}}{d},\ldots}, \end{aligned}$$

by (12.8). This coincides with the expression in the third alternative of (12.10), and concludes the proof of the proposition. □

Lemma 36. *Let $b\geq 0$ and $\mathbf{n}=(n_1,n_2,\ldots)$ satisfy (12.6). Let $n+1 := \sum_i n_i$. Then the expression*

$$\frac{[2n+b]_q}{[n+b+1]_q\,[n+b]_q}\begin{bmatrix} n+b+1 \\ b,n_1,n_2,\ldots\end{bmatrix}_q \tag{12.9}$$

is a polynomial in q with non-negative coefficients.

Proof. Let us introduce numbers $\tilde n_i$ defined by

$$\tilde n_1 = b \quad\text{and}\quad \tilde n_{i+1} = n_i \text{ for } i\geq 1.$$

Define $\tilde{n} = -1 + \sum_i \tilde{n}_i$. Then

$$n = -1 + \sum n_i = -1 - b + \sum \tilde{n}_i = \tilde{n} - \tilde{n}_1.$$

This allows us to rewrite (12.9) as

$$\frac{[2\tilde{n} - \tilde{n}_1]_q}{[\tilde{n}+1]_q[\tilde{n}]_q} \begin{bmatrix} \tilde{n}+1 \\ \tilde{n}_1, \tilde{n}_2, \dots \end{bmatrix}.$$

We recognise the expression of Lemma 23, with n_i replaced by $\tilde{n}_i$. However, we cannot apply directly Lemma 23, because the $\tilde{n}_i$'s do not satisfy (2.5). Indeed, in view of (12.6),

$$\sum_{i\geq 1}(i-2)\tilde{n}_i = -b + \sum_{i\geq 1}(i-1)n_i = -b + \sum_{i\geq 1}(i-2)n_i + \sum_{i\geq 1} n_i = -2 + (n+1) \neq -2.$$

However, we can reproduce the first steps of the proof of Lemma 23, as long as we do not use (2.5). This leads us to the following point: the expression (12.9) is a polynomial if and only if for all $d \geq 2$, the following quantity is non-negative:

$$k'_d := \chi\left(d \mid (2n+b)\right) + \left\lfloor B + \sum_{i\geq 1} N_i - \frac{2}{d} \right\rfloor,$$

where we have used the fractional parts $B = \{b/d\}$ and $N_i = \{n_i/d\}$. This is the counterpart of (10.4). The above quantity is non-negative except possibly if $B + \sum N_i$ equals 0 or $\frac{1}{d}$. If this holds, the final term above equals -1.

If $B + \sum N_i = 0$, that is, if d divides b and each n_i, then it follows from (12.6) that $d = 2$, so that $d \mid (2n+b)$ and $k'_d = 0$.

If $B + \sum N_i = \frac{1}{d}$, then either $B = \frac{1}{d}$ and all N_i's vanish, or $N_\ell = \frac{1}{d}$ for some ℓ, while $B = N_i = 0$ for $i \neq \ell$. In the first case, d divides $b-1$ and all n_i's, and it follows from (12.6) that $d = 1$, which is impossible. In the second case, d divides b, $n_\ell - 1$ and all n_i's for $i \neq \ell$. Since $n = -1 + \sum n_i$, we conclude that d divides n, and thus $2n+b$. Hence k'_d vanishes in this case.

This proves the polynomiality part of the lemma. Now we observe that the non-negativity part in the proof of Lemma 23 does not use any assumption on the n_i's, and conclude that all coefficients of (12.9) are non-negative. □

Lemma 37. *Let $b \geq 0$ and $\mathbf{n} = (n_1, n_2, \dots)$ satisfy (12.6). Let $n+1 := \sum_i n_i$. Write $m := 2n+b$. Furthermore, let ω be a primitive m-th root of unity, and let e be a non-negative integer $< m$ being equal to 0 or dividing m. In this case, write $d := m/e$. Then we have*

$$\frac{[2n+b]_q}{[n+b+1]_q\,[n+b]_q} \begin{bmatrix} n+b+1 \\ b, n_1, n_2, \dots \end{bmatrix}_q \Bigg|_{q=\omega^e}$$

$$= \begin{cases} \frac{2n+b}{(n+b)(n+b+1)} \binom{b+n+1}{b,n_1,n_2,\dots}, & \text{if } e = 0, \\ \frac{2n+b}{n+b+1} \binom{\frac{b+n+1}{2}}{\frac{b}{2}, \frac{n_1}{2}, \dots}, & \text{if } d = 2 \text{ with } b \text{ and each } n_i \text{ even}, \\ \frac{2n+b}{n+b} \binom{\frac{n+b}{d}}{\frac{b}{d}, \frac{n_1}{d}, \dots, \frac{n_{\ell-1}}{d}, \frac{n_\ell - 1}{d}, \frac{n_{\ell+1}}{d}, \dots}, & \text{if } d \mid b,\ d \mid (n_\ell - 1) \text{ for some } \ell \\ & \text{and } d \mid n_i \text{ for } i \neq \ell, \\ 0, & \text{otherwise.} \end{cases} \tag{12.10}$$

Proof. The case where $e = 0$ is trivial. Therefore we assume from now on that e is a positive divisor of m with $e < m$. Then ω^e is a primitive d-th root of unity. According to (5.3), in order to obtain a non-zero value of the specialisation on the left-hand side of (12.10), there must be an equal number of factors of the form $1 - q^\alpha$ with $d \mid \alpha$ in the numerator and in the denominator. In other words, we must have

$$1 + \left\lfloor \frac{n+b-1}{d} \right\rfloor - \left\lfloor \frac{b}{d} \right\rfloor - \sum_i \left\lfloor \frac{n_i}{d} \right\rfloor = 0.$$

Introducing the fractional parts $B := \{b/d\}$ and $N_i = \{n_i/d\}$, this may be rewritten as

$$1 + \left\lfloor B + \sum N_i - \frac{2}{d} \right\rfloor = 0.$$

This forces $B + \sum N_i$ to be contained in $\{0, \frac{1}{d}\}$.

If $B + \sum N_i = 0$, that is, $B = N_i = 0$ for all i, then d divides b and each n_i. However, by assumption d divides $2n + b$, hence d divides $2n = 2\sum n_i - 2$. This forces $d = 2$, and we are in the second case of (12.10).

If $B + \sum N_i = 1/d$ with $B = 1/d$ and $N_i = 0$ for all i, then d divides $b - 1$ and each n_i, so it divides $n + 1 = \sum n_i$ as well. However, by assumption d divides $2n + b = 2(n + 1) + (b - 1) - 1$, which means that $d = 1$, which is impossible.

Finally, if $B + \sum N_i = 1/d$ with $N_\ell = 1/d$, $B = 0$ and $N_i = 0$ for $i \neq \ell$, then d divides b, $n_\ell - 1$ and each n_i for $i \neq \ell$. Hence, we are in the third case of (12.10).

It is then an easy task to evaluate the left-hand side of (12.10) in the second and third case, using the simple fact (5.4). □

12.5. Two reformulations

12.5.1. **Cubic maps with a Hamiltonian cycle.** There exists a simple bijection between tree-rooted maps and *cubic* maps (maps in which all vertices have degree 3) equipped with a *Hamiltonian cycle* containing the root edge. (By definition, a Hamiltonian cycle visits exactly once each vertex.) This bijection is illustrated in Figure 13 and works as follows: starting from a tree-rooted map $(\mathfrak{m}, \tau)$, say with $i+1$ vertices and $j + 1$ faces, and drawn with a circle surrounding τ as we did in Figure 11 (right), one forms a non-crossing matching dual to the i edges of τ, using the classical construction of Figure 6. This matching is shown in dashed red lines in Figure 13. Note that its edges must intersect only one edge of $\mathfrak{m}$ (and this edge is in τ). Now one has $2i + 2j = 2n$ points on the surrounding circle. The cubic map then consists of these $2n$ vertices, together with the edges of the surrounding circle (which forms the distinguished Hamiltonian cycle), plus the i edges of the matching, and finally the j edges that lie outside the circle. The root edge of the cubic map is chosen to be the unique edge of the cycle that faces the root corner of $\mathfrak{m}$, oriented in counterclockwise direction.

Note that a tree-rooted map with $i + 1$ vertices, $j + 1$ faces, n edges, n_k nodes of degree k is sent to a cubic map (with a distinguished Hamiltonian cycle) having $2n$ vertices, i edges inside the cycle, j edges outside the cycle, and n_k *inner* faces (those lying inside the distinguished cycle) that contain k edges of the cycle.

It is not hard to see that the rotation R on tree-rooted maps just moves the root-edge of the cubic map one step in counterclockwise direction around the cycle. Our three cyclic sieving theorems then translate naturally into cyclic sieving phenomena for cubic maps with a distinguished Hamiltonian cycle.

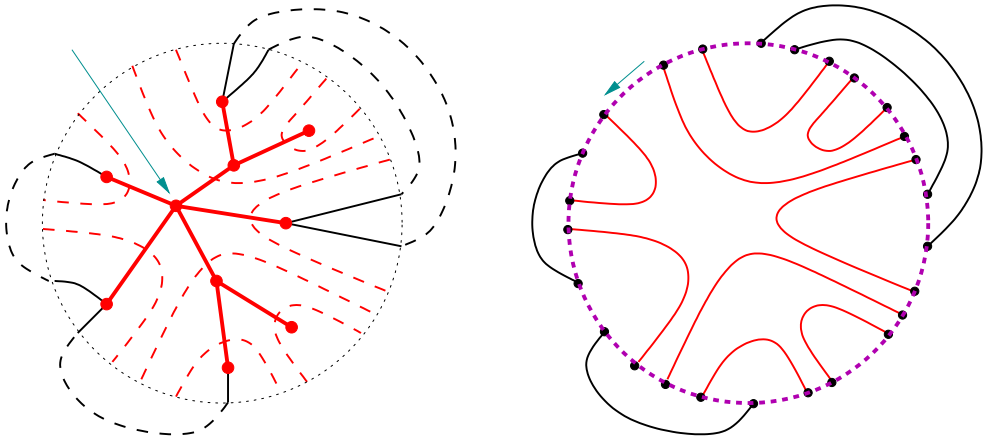


FIGURE 13. A tree-rooted map with $i+1 = 10$ vertices and $j+1 = 5$ faces (left), and the corresponding cubic map with a distinguished Hamiltonian cycle (containing the root edge). The rotation R on the tree-rooted map moves the root edge of the cubic map along the distinguished cycle.

12.5.2. **Square lattice walks in the first quadrant.** Les us finally recall the classical bijection between tree-rooted maps with n edges and walks with north, east, south and west steps (N, E, S, W) that start and end at $(0, 0)$, and never leave the non-negative quadrant $\mathbb{N}^2$, with $\mathbb{N} = \{0, 1, 2, \ldots\}$. This bijection works as follows: starting from the root corner of the tree-rooted map, and walking around the tree in counterclockwise direction, one encodes each edge of the tree by E (respectively W) when it is visited for the first (respectively second) time, and each edge not in the tree by N (respectively S) when it is visited for the first (respectively second) time, until one is back to the root corner. For instance, the encoding of the tree-rooted map of Figure 13 starts $ENWESNWE\ldots$ If the map has $i + 1$ vertices and $j + 1$ faces, then the walk has i east steps and j north steps.

The effect of the rotation R on walks is not hard to describe: if a tree-rooted map is encoded by the walk (or word) $w = aw_1\bar{a}w_2$, where $a \in \{N, E\}$ and $\bar{a} \in \{S, W\}$ is the step that *matches* the first letter of w, then applying R gives the walk $w_1aw_2\bar{a}$. To be more precise: if $a = N$, then $aw_1\bar{a}$ is the shortest non-empty prefix of w that ends on the x-axis; if $a = E$, then $aw_1\bar{a}$ is the shortest non-empty prefix of w that ends on the y-axis. Theorem 33 (respectively Theorem 30) thus translates into a cyclic sieving phenomenon for quadrant walks with $2n$ steps (respectively with i east steps and j north steps).

Note that this cyclic action on $NSEW$ quadrant walks is similar to another cyclic action defined on quadrant walks with S, W, and NE steps, known as *Kreweras walks*, for which a cyclic sieving phenomenon has been conjectured; see [17, Conj. 6.2] and [15].

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