UVAROV'S FORMULA FOR RATIONAL DEFORMATIONS OF ORTHOGONAL POLYNOMIALS AND ITS ASSOCIATED IDENTITY — A THOROUGH DISCUSSION

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ABSTRACT. Let $p_n(x)$, $n = 0, 1, \dots$, be the orthogonal polynomials with respect to a given density $d\mu(x)$. Furthermore, let $d\nu(x)$ be a density which arises from $d\mu(x)$ by multiplication by a rational function in x. We prove a formula that expresses the Hankel determinants of moments of $d\nu(x)$ in terms of a determinant involving the orthogonal polynomials $p_n(x)$ and associated functions $q_n(x) = \int p_n(u)\, d\mu(u)/(x-u)$. Uvarov's formula for the orthogonal polynomials with respect to $d\nu(x)$ is a corollary of our theorem. Our result generalises a Hankel determinant formula for the case where the rational function is a polynomial that existed somehow hidden in the folklore of the theory of orthogonal polynomials but has been stated explicitly only relatively recently (see [arχiv:2101.04225]). Our theorem can be interpreted in a two-fold way: analytically or in the sense of formal series. We apply our theorem to derive several curious Hankel determinant evaluations, and to show linear recursiveness for a certain class of such Hankel determinants.

1. Introduction

Recently, in [9] this author discovered a formula that expresses the Hankel determinant of linear combinations of moments of orthogonal polynomials in terms of a determinant involving these orthogonal polynomials. A literature search revealed that this formula existed in a hidden form behind a theorem (cf. [11, Theorem 2.5] or [7, Theorem 2.7.1]) that is commonly attributed to Christoffel [3] (although he had only proved it in a very special case); only recently it had been stated explicitly, by Lascoux in [10, Prop. 8.4.1] (although incorrectly), by Elouafi [5, Theorem 1] (however with an incomplete proof), and by Brézin and Hikami [2]. Four fundamentally different proofs are given in [9]: one due to this author, one following Lascoux's arguments, one completing Elouafi's arguments, and one presenting the random matrix-inspired proof of Brézin and Hikami.

The purpose of this article is to present and prove a generalisation of the aforementioned formula that is inspired by Uvarov's formula [12, 13] for the orthogonal polynomials with respect to a density that is related to another given density by the multiplication by a rational function, see Theorem 1 below.

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Let $\big(p_n(x)\big)_{n\ge0}$ be a sequence of monic polynomials over a field K of characteristic zero[1] with $\deg p_n(x)=n$, and assume that they are orthogonal with respect to the linear functional L, i.e., they satisfy $L(p_m(x)p_n(x))=\omega_n\delta_{m,n}$ with $\omega_n\ne0$ for all n, where $\delta_{m,n}$ is the Kronecker delta. Furthermore, we write μ_n for the n-th moment $L(x^n)$ of the functional L. For convenience (and in order to keep the usual analytic meaning in mind), we shall write $\int f(u)\,d\mu(u)$ instead of $L(f(x))$. This can be either read in a purely formal way, or the analyst may think of it as a concrete integral with respect to the measure given by the density $d\mu(u)$.

For the statement of our theorem, we need the "functions"[2]

$$q_n(y)=\int\frac{p_n(u)}{y-u}\,d\mu(u). \tag{1.1}$$

These can be understood in the "ordinary" analytic sense if $\mu(u)$ is a concrete measure, or, alternatively, these can be understood in the sense of formal power series in $1/y$, see Lemma 6, Equation (2.7).

Here is the main result of this article.

Theorem 1. *Let k, m and n be non-negative integers and $x_1,x_2,\dots,x_m$ and $y_1,y_2,\dots,y_k$ be variables. Then, with the above notations, for $n\ge k$ we have*

$$\frac{\det\limits_{0\le i,j\le n-1}\left(\displaystyle\int u^{i+j}\frac{\prod_{\ell=1}^m(u-x_\ell)}{\prod_{\ell=1}^k(u-y_\ell)}\,d\mu(u)\right)}{\det\limits_{0\le i,j\le n-k-1}\left(\mu_{i+j}\right)}\\
=(-1)^{n(m-k)+km}\frac{\det M_{k,m,n}(x_1,\dots,x_m,y_1,\dots,y_k)}{\left(\prod\limits_{1\le i<j\le m}(x_j-x_i)\right)\left(\prod\limits_{1\le i<j\le k}(y_i-y_j)\right)}, \tag{1.2}$$

where

$$M_{k,m,n}(x_1,\dots,x_m,y_1,\dots,y_k)=\begin{pmatrix}p_{n-k}(x_1)&p_{n-k+1}(x_1)&\dots&p_{n+m-1}(x_1)\\ \hdotsfor{4}\\ p_{n-k}(x_m)&p_{n-k+1}(x_m)&\dots&p_{n+m-1}(x_m)\\ q_{n-k}(y_1)&q_{n-k+1}(y_1)&\dots&q_{n+m-1}(y_1)\\ \hdotsfor{4}\\ q_{n-k}(y_k)&q_{n-k+1}(y_k)&\dots&q_{n+m-1}(y_k)\end{pmatrix}.$$

[1]For the analyst, (usually) this field is the field of real numbers, and a further restriction is that the linear functional L is defined by a measure with non-negative density. However, the formulae in this paper do not need these restrictions and are valid in this wider context of "formal orthogonality".

[2]We put "functions" in quotation marks since they are functions in the literal sense only if we "put on the analytic glasses" and regard $d\mu(u)$ as the density of a concrete measure, but they are not if we read this in the formal sense. In [7, Sec. 3.6], up to normalisation, the $q_n(y)$'s are called "functions of the second kind".

If $n < k$, then

$$\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}\,d\mu(u)\right) = (-1)^{n(m-k)+km}\frac{\det N_{k,m,n}(x_1,\dots,x_m,y_1,\dots,y_k)}{\left(\prod_{1\le i<j\le m}(x_j-x_i)\right)\left(\prod_{1\le i<j\le k}(y_i-y_j)\right)}, \tag{1.3}$$

where

$$N_{k,m,n}(x_1,\dots,x_m,y_1,\dots,y_k) = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & p_0(x_1) & p_1(x_1) & \dots & p_{n+m-1}(x_1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & p_0(x_m) & p_1(x_m) & \dots & p_{n+m-1}(x_m) \\ y_1^{k-n-1} & \dots & y_1^2 & y_1 & 1 & q_0(y_1) & q_1(y_1) & \dots & q_{n+m-1}(y_1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_k^{k-n-1} & \dots & y_k^2 & y_k & 1 & q_0(y_k) & q_1(y_k) & \dots & q_{n+m-1}(y_k) \end{pmatrix}.$$

Here, determinants of empty matrices and empty products are understood to equal 1.

Remarks. (1) The numerator determinant on the left-hand sides of (1.2) and (1.3) is the Hankel determinant $\det_{0\le i,j\le n-1}(\rho_{i+j})$, where the ρ_s's are the moments of the linear functional

$$p(x)\mapsto \int p(u)\frac{\prod_{\ell=1}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}\,d\mu(u).$$

(2) The theory of orthogonal polynomials guarantees that in our setting (namely due to the condition $\omega_n\ne 0$ in the orthogonality) the Hankel determinant of moments in the denominator on the left-hand side of (1.2) is non-zero.

(3) The main theorem in [9] is equivalent to the special case of Theorem 1 where $k=0$. It is worth stating this special case separately.

Corollary 2. *Let m and n be non-negative integers and $x_1,x_2,\dots,x_m$ be variables. Then we have*

$$\frac{\det_{0\le i,j\le n-1}\left(\int u^{i+j}\prod_{\ell=1}^{m}(u-x_\ell)\,d\mu(u)\right)}{\det_{0\le i,j\le n-1}(\mu_{i+j})} = (-1)^{nm}\frac{\det_{1\le i,j\le m}\left(p_{n+j-1}(x_i)\right)}{\prod_{1\le i<j\le m}(x_j-x_i)}. \tag{1.4}$$

(4) Similarly, it is worth stating the special case of Theorem 1 where $m=0$ separately.

Corollary 3. *Let k and n be non-negative integers and $y_1,y_2,\dots,y_k$ be variables. Then, for $n\ge k$ we have*

$$\frac{\det_{0\le i,j\le n-1}\left(\int \frac{u^{i+j}\,d\mu(u)}{\prod_{\ell=1}^{k}(u-y_\ell)}\right)}{\det_{0\le i,j\le n-k-1}(\mu_{i+j})} = (-1)^{nk}\frac{\det_{1\le i,j\le k}\left(q_{n-k+j-1}(y_i)\right)}{\prod_{1\le i<j\le k}(y_i-y_j)}. \tag{1.5}$$

If $n<k$, then

$$\det_{0\le i,j\le n-1}\left(\int\frac{u^{i+j}\,d\mu(u)}{\prod_{\ell=1}^k(u-y_\ell)}\right)=(-1)^{nk}\frac{\det N_{k,n}(y_1,\dots,y_k)}{\prod\limits_{1\le i<j\le k}(y_i-y_j)},\tag{1.6}$$

where

$$N_{k,n}(y_1,\dots,y_k)=\begin{pmatrix}y_1^{k-n-1} & \dots & y_1^2 & y_1 & 1 & q_0(y_1) & q_1(y_1) & \dots & q_{n-1}(y_1)\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ y_k^{k-n-1} & \dots & y_k^2 & y_k & 1 & q_0(y_k) & q_1(y_k) & \dots & q_{n-1}(y_k)\end{pmatrix}.$$

(5) Let $d\mu(u)$ be a given density. We will explain in Section 6 how Theorem 1 is related to Uvarov's formula [12, 13] for the orthogonal polynomials with respect to the linear functional defined by the density

$$\frac{\prod_{\ell=2}^m(u-x_\ell)}{\prod_{\ell=1}^k(u-y_\ell)}\,d\mu(u).$$

(6) In [9], four proofs of the special case of Theorem 1 where $k=0$ — that is, of Corollary 2 — are given, one using the method of condensation, one using classical results from the theory of orthogonal polynomials, one using a vanishing argument, and one using a random matrix-inspired argument. It is interesting to note that neither the second nor the third proof seem to extend to a proof of Theorem 1, only the condensation argument does. This is indeed the argument that we apply here in Section 4.

The rest of this article is organised as follows. In the next section, we review some classical facts from the theory of orthogonal polynomials that will be relevant for the proof of Theorem 1. Furthermore, in Lemma 6, we provide some information on the "functions" $q_n(y)$ that are so central in Theorem 1. Our proof of Theorem 1 requires several determinant identities. These are presented in Section 3, among which Jacobi's condensation formula (see Lemma 8). The actual proof of Theorem 1 is then the subject of Section 4. In Proposition 13 in Section 5 it is explained how to "read" Theorem 1 in cases where two (or more) of the x_i's or the y_i's are equal. We come back to the title of this article in Section 6 by outlining how Theorem 1 relates to Uvarov's formula [12, 13]. By considering the special case of Theorem 1 where $d\mu(u)=\sqrt{1-u^2}\,du$ — corresponding to Chebyshev polynomials of the second kind —, we derive evaluations of several curious Hankel determinants in Section 7. Finally, in Section 8, we prove that, if the coefficients of the three-term recurrence (2.1) satisfied by the orthogonal polynomials $p_n(x)$ are constants, the (scaled) Hankel determinants of moments that appear on the left-hand side of (1.2) satisfy a linear recurrence with constant coefficients.

2. Preliminaries on orthogonal polynomials

In this section we survey classical facts about orthogonal polynomials that we shall need in the sequel. We also prove some properties of the "functions" $q_n(y)$.

As in the introduction, let $\big(p_n(x)\big)_{n\ge0}$ be a sequence of monic polynomials over a field K of characteristic zero that is orthogonal with respect to the linear function L given by

$$L:p(x)\mapsto\int p(u)\,d\mu(u),$$

where this may be read formally, or — when we want to interpret this analytically — where $d\mu(u)$ is some given density. Such a sequence of orthogonal polynomials exists if and only if all Hankel determinants $\det_{0\le i,j\le n-1}(\mu_{i+j})$ of moments $\mu_s = \int u^s\, d\mu(u)$ do not vanish. For explicit formulae for the orthogonal polynomials $p_n(x)$ in terms of the moments see Lemmas 4 and 5 below.

By Favard's theorem (see e.g. [8, Theorems 11–13]), the sequence $\big(p_n(x)\big)_{n\ge0}$ is orthogonal if and only if it satisfies a *three-term recurrence*

$$p_n(x) = (x - s_{n-1})p_{n-1}(x) - t_{n-2}p_{n-2}(x), \quad \text{for } n \ge 1, \tag{2.1}$$

with initial values $p_{-1}(x) = 0$ and $p_0(x) = 1$, for some sequences $(s_n)_{n\ge0}$ and $(t_n)_{n\ge0}$ of elements of K with $t_n \ne 0$ for all n.

The t_n's are connected with the Hankel determinants of moments by the formula (see e.g. [14, Ch. IV, Cor. 6])

$$\det_{0\le i,j\le n-1}(\mu_{i+j}) = \prod_{i=0}^{n-1} t_i^{n-i-1}. \tag{2.2}$$

Conversely, we may use this formula to express t_{n-1} in terms of the Hankel determinants of moments. For convenience, we introduce the abbreviation

$$H(n) := \det_{0\le i,j\le n-1}(\mu_{i+j}). \tag{2.3}$$

Then, from (2.2) we infer

$$t_{n-1} = \frac{H(n+1)/H(n)}{H(n)/H(n-1)}. \tag{2.4}$$

Next we quote two formulae that express orthogonal polynomials in terms of their associated moments. The first can be found in [11, Eq. (2.2.6)] or [7, Eq. (2.1.11)], and the second in [11, Eq. (2.2.9)]) or [7, Eq. (2.1.10)].

Lemma 4. *Let M be a linear functional on polynomials in x with moments ν_n, $n = 0, 1, \dots$, such that all Hankel determinants $\det_{0\le i,j\le n}(\nu_{i+j})$, $n = 0, 1, \dots$, are non-zero. Then the determinants*

$$\frac{1}{\det_{0\le i,j\le n-1}(\nu_{i+j})} \det \begin{pmatrix} \nu_0 & \nu_1 & \nu_2 & \dots & \nu_n \\ \nu_1 & \nu_2 & \nu_3 & \dots & \nu_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ \nu_{n-1} & \nu_n & \nu_{n+1} & \dots & \nu_{2n-1} \\ 1 & x & x^2 & \dots & x^n \end{pmatrix}$$

are a sequence of monic orthogonal polynomials with respect to M.

Lemma 5. *Let M be a linear functional on polynomials in x with moments ν_n, $n = 0, 1, \dots$, such that all Hankel determinants $\det_{0\le i,j\le n}(\nu_{i+j})$, $n = 0, 1, \dots$, are non-zero. Then the determinants*

$$\det_{0\le i,j\le n-1}(\nu_{i+j+1} - \nu_{i+j}x) \tag{2.5}$$

are a sequence of orthogonal polynomials with respect to M.

We now turn to the "functions" $q_n(y)$ defined in (1.1). They satisfy the same three-term recurrence relation as the original orthogonal polynomials $p_n(x)$. Moreover, it follows from Lemma 4 that $q_n(y)$ can be seen as (formal) power series in $1/y$ of degree $-n-1$.[3]

Lemma 6. *We have*

$$q_n(y) = (y - s_{n-1})q_{n-1}(y) - t_{n-2}q_{n-2}(y), \quad \textit{for } n \geq 2, \tag{2.6}$$

with initial values $q_0(y) = \int \frac{d\mu(u)}{y-u}$ *and* $q_1(y) = (y - s_0)\int \frac{d\mu(u)}{y-u} - \mu_0$, *with the sequences* $(s_n)_{n\geq 0}$ *and* $(t_n)_{n\geq 0}$ *of elements of* K *that feature in the three-term recurrence* (2.1) *for the underlying orthogonal polynomials* $p_n(x)$.

Moreover, for all non-negative integers n, *as a formal power series in* $1/y$ *the "function"* $q_n(y)$ *starts as*

$$q_n(y) = \frac{H(n+1)}{H(n)} y^{-n-1} + O\left(y^{-n-2}\right), \tag{2.7}$$

where $H(n)$ *and* $H(n+1)$ *is the short notation introduced in* (2.3).

Proof. Let $n \geq 2$. From (2.1), we get

$$\int \frac{p_n(u)}{y-u}\, d\mu(u) = \int \frac{(u - s_{n-1})p_{n-1}(u)}{y-u}\, d\mu(u) - t_{n-2}\int \frac{p_{n-2}(u)}{y-u}\, d\mu(u). \tag{2.8}$$

The first term on the right-hand side can be simplified as follows:

$$\begin{aligned}\int \frac{(u - s_{n-1})p_{n-1}(u)}{y-u}\, d\mu(u) &= \int \frac{(y - s_{n-1})p_{n-1}(u)}{y-u}\, d\mu(u) - \int \frac{(y-u)p_{n-1}(u)}{y-u}\, d\mu(u)\\ &= (y - s_{n-1})q_{n-1}(y).\end{aligned}$$

If this is used in (2.8) together with the definition (1.1) of $q_n(y)$, the recurrence (2.6) results immediately. The initial values for $q_0(y)$ and $q_1(y)$ are straightforward to derive from $p_0(x) = 1$ and $p_1(x) = x - s_0$.

In order to show the second assertion, we note that, by definition of $q_n(y)$, we have

$$q_n(y) = \int \frac{p_n(u)}{y-u}\, d\mu(u) = \sum_{i=0}^{\infty} \int p_n(u) u^i y^{-i-1}\, d\mu(u).$$

Because of the orthogonality of $p_n(u)$ with respect to the density $d\mu(u)$, all terms of the above sum with $i < n$ vanish. Thus,

$$q_n(y) = \sum_{i=n}^{\infty} \int p_n(u) u^i y^{-i-1}\, d\mu(u) = y^{-n-1}\int p_n(u) u^n\, d\mu(u) + O\left(y^{-n-2}\right). \tag{2.9}$$

[3]By "degree" of a formal power series $f(y)$ in $1/y$ we mean the maximal exponent e such that y^e appears in $f(y)$ with non-zero coefficient. We warn the reader that the recurrence (2.6) is deceiving in regard of the degree of $q_n(y)$: it seems to suggest that — similar to the situation for the underlying orthogonal polynomials $p_n(x)$ — the degree would rise by 1 when going from $q_{n-1}(y)$ to $q_n(y)$. However, as (2.7) shows, on the contrary the degree *drops* by 1, caused by a cancellation of leading terms in (2.6).

Now we use the formula of Lemma 4 with $\nu_s = \mu_s$ for all s, to obtain

$$\int p_n(u)u^n\,d\mu(u) = \frac{1}{H(n)}\int \det\begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \dots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \dots & \mu_{2n-1} \\ 1 & u & u^2 & \dots & u^n \end{pmatrix} u^n\,d\mu(u)$$

$$= \frac{1}{H(n)}\begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \dots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \dots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \mu_{n+2} & \dots & \mu_{2n} \end{pmatrix} = \frac{H(n+1)}{H(n)}.$$

If this is substituted back in (2.9), the assertion of the lemma follows immediately. □

3. Auxiliary determinant identities

The purpose of this section is to collect three determinant formulae that will turn out to be crucial in our proof of Theorem 1.

The proof method of our proof in Section 4 is the *method of condensation* (frequently referred to as "Dodgson condensation"; see [8, Sec. 2.3]). This method provides inductive proofs that are based on a determinant identity due to Jacobi, which we recall in the following proposition.

Proposition 7. *Let A be an $N\times N$ matrix. Denote the submatrix of A in which rows $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_k$ are omitted by $A^{j_1,j_2,\dots,j_k}_{i_1,i_2,\dots,i_k}$. Then we have*

$$\det A\cdot\det A^{j_1,j_2}_{i_1,i_2} = \det A^{j_1}_{i_1}\cdot\det A^{j_2}_{i_2} - \det A^{j_2}_{i_1}\cdot\det A^{j_1}_{i_2} \tag{3.1}$$

for all integers i_1, i_2, j_1, j_2 with $1\le i_1 < i_2 \le N$ and $1 \le j_1 < j_2 \le N$.

We need two further determinantal formulae, which involve Hankel determinants of linear combinations of sequence elements.

The following is [9, Lemma 3 with n replaced by $n-1$].

Lemma 8. *Let $(c_n)_{n\ge 0}$ be a given sequence, and α and β be variables. Then, for all positive integers n, we have*

$$\begin{aligned}(\beta-\alpha)\det_{0\le i,j\le n-2}\big(\alpha\beta c_{i+j} + (\alpha+\beta)c_{i+j+1} + c_{i+j+2}\big)\det_{0\le i,j\le n-1}\big(c_{i+j}\big)\\ = \det_{0\le i,j\le n-2}\big(\alpha c_{i+j} + c_{i+j+1}\big)\det_{0\le i,j\le n-1}\big(\beta c_{i+j} + c_{i+j+1}\big)\\ - \det_{0\le i,j\le n-2}\big(\beta c_{i+j} + c_{i+j+1}\big)\det_{0\le i,j\le n-1}\big(\alpha c_{i+j} + c_{i+j+1}\big). \end{aligned}\tag{3.2}$$

We require another, similarly looking determinant identity.

Lemma 9. *Let $(c_n)_{n\ge0}$ be a given sequence, and α and β be variables. Then, for all positive integers n, we have*

$$\begin{aligned}\det_{0\le i,j\le n-1}\big(\alpha c_{i+j}+c_{i+j+1}\big)\det_{0\le i,j\le n-1}\big(\beta c_{i+j}+c_{i+j+1}\big)\\ =-\det_{0\le i,j\le n}\big(c_{i+j}\big)\det_{0\le i,j\le n-2}\big(\alpha\beta c_{i+j}+(\alpha+\beta)c_{i+j+1}+c_{i+j+2}\big)\\ +\det_{0\le i,j\le n-1}\big(c_{i+j}\big)\det_{0\le i,j\le n-1}\big(\alpha\beta c_{i+j}+(\alpha+\beta)c_{i+j+1}+c_{i+j+2}\big).\end{aligned}\tag{3.3}$$

Proof. In principle, it should be possible to prove this directly in the style of the proof of Lemma 8 given in [9], that is, by extracting the coefficient of $\alpha^s\beta^t$ on both sides of (3.3), and by then reducing everything to some known determinant identity. Embarrassingly, I failed to carry this through. Therefore, instead I take recourse to a dirty trick that is based on the fact that Theorem 1 had been proved earlier for the special case in which $k=0$ (cf. [9, Theorem 1] and Proposition 11 below). Namely, we interpret the sequence $(c_n)_{n\ge0}$ as the sequence of moments of some linear functional, and we let $\big(r_n(x)\big)_{n\ge0}$ be the corresponding sequence of monic orthogonal polynomials with respect to that functional. As was said earlier, for the orthogonal polynomials to exist we must assume that all Hankel determinants $\det_{0\le i,j\le n-1}(c_{i+j})$ are non-zero, which we do for the moment. Since both sides of (3.3) are polynomials in the c_i's, α and β, this restriction can be removed in the end.

Now, in the above setting, by Theorem 1 with $k=0$, $m=1$, and $x_1=-\alpha$, we have

$$\det_{0\le i,j\le n-1}\big(\alpha c_{i+j}+c_{i+j+1}\big)=H(n)r_n(-\alpha),$$

and, by Theorem 1 with $k=0$, $m=2$, $x_1=-\alpha$ and $x_2=-\beta$, we have

$$\det_{0\le i,j\le n-1}\big(\alpha\beta c_{i+j}+(\alpha+\beta)c_{i+j+1}+c_{i+j+2}\big)=H(n)\frac{r_n(-\alpha)r_{n+1}(-\beta)-r_{n+1}(-\alpha)r_n(-\beta)}{\alpha-\beta}.$$

Consequently, Equation (3.3) turns out to be equivalent with

$$\begin{aligned}\big(\tilde H(n)\big)^2r_n(-\alpha)r_n(-\beta)=\tilde H(n+1)\tilde H(n-1)\frac{r_{n-1}(-\alpha)r_n(-\beta)-r_n(-\alpha)r_{n-1}(-\beta)}{\alpha-\beta}\\ +\big(\tilde H(n)\big)^2\frac{r_n(-\alpha)r_{n+1}(-\beta)-r_{n+1}(-\alpha)r_n(-\beta)}{\alpha-\beta}.\end{aligned}$$

Here we used the short notation $\tilde H(n)=\det_{0\le i,j\le n-1}(c_{i+j})$. After cancellation of common factors using (2.2) (with $\tilde H(n)$ in place of $H(n)$ and the appropriate sequence $(\tilde t_n)_{n\ge0}$ instead of $(t_n)_{n\ge0}$), this becomes

$$\begin{aligned}(\alpha-\beta)r_n(-\alpha)r_n(-\beta)=\tilde t_{n-1}\big(r_{n-1}(-\alpha)r_n(-\beta)-r_n(-\alpha)r_{n-1}(-\beta)\big)\\ +\big(r_n(-\alpha)r_{n+1}(-\beta)-r_{n+1}(-\alpha)r_n(-\beta)\big).\end{aligned}$$

Indeed, this is trivially true because of (2.1) with n replaced by $n+1$, $p_n(x)$ replaced by $r_n(x)$, s_n replaced by $\tilde s_n$, t_n replaced by $\tilde t_n$, and $x=-\beta$, respectively $x=-\alpha$. ☐

4. Proof of Theorem 1

Here we prove Theorem 1. As announced, we shall apply the method of condensation (see Proposition 7) in order to set up an inductive proof. Propositions 11 and 12 will provide the start of the induction. In order to "keep the induction running", we need a non-vanishing result that we present first.

Lemma 10. *Under the assumption that the Hankel determinants $H(n)$ in* (2.3) *do not vanish, the numerator determinant on the left-hand sides of* (1.2) *and* (1.3)*,*

$$\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}\,d\mu(u)\right), \tag{4.1}$$

does not vanish identically.

Proof. Adopting the viewpoint of formal series in the variables $x_1,x_2,\dots,x_m$ and $1/y_1,1/y_2,\dots,1/y_k$, the highest degree term in (4.1) is

$$\begin{aligned}\det_{0\le i,j\le n-1}\left((-1)^{m-k}\mu_{i+j}\frac{\prod_{\ell=1}^{m}x_\ell}{\prod_{\ell=1}^{k}y_\ell}\right)&=(-1)^{n(m-k)}\frac{\prod_{j=1}^{m}x_j^n}{\prod_{j=1}^{k}y_j^n}\det_{0\le i,j\le n-1}\left(\mu_{i+j}\right)\\&=(-1)^{n(m-k)}H(n)\frac{\prod_{j=1}^{m}x_j^n}{\prod_{j=1}^{k}y_j^n},\end{aligned}$$

which is non-zero by our assumption of non-vanishing of $H(n)$.

If we adopt the analytic point of view, then one would multiply the determinant (4.1) by $\prod_{j=1}^{m}x_j^{-n}/\prod_{j=1}^{k}y_j^{-n}$ and then compute the limit[4] as $x_i\to\infty$ and $y_j\to\infty$ for all i and j. The result would be the determinant

$$\det_{0\le i,j\le n-1}\left((-1)^{m-k}\mu_{i+j}\right)=(-1)^{n(m-k)}H(n),$$

with the same conclusion. □

Proposition 11. *Theorem 1 holds for $k=0$.*

This is the main theorem (namely Theorem 1) in [9], for which four fundamentally different proofs are provided there.

Proposition 12. *Theorem 1 holds for $m=0$.*

Proof. We prove the claim by induction on k.

For the start of the induction we need the validity of (1.2) for $k=m=0$ — this is obvious since both left-hand and right-hand side equal 1 in that case —, of (1.3) for $m=0$ and $k=1$ — also this is easy to see since the only case to consider is $n=0$, which makes the left-hand side reduce to 1, and also the right-hand side, due to the evaluation of the Vandermonde determinant —, and of (1.2) for $m=0$ and $k=1$. The latter needs an argument, which we provide next.

For $m=0$ and $k=1$, the numerator of the left-hand side of (1.2) reads

$$\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{d\mu(u)}{u-y_1}\right). \tag{4.2}$$

We have

$$\int u^{i+j}\frac{d\mu(u)}{u-y_1}=\int u^{i+j-1}\frac{u-y_1+y_1}{u-y_1}\,d\mu(u)=\mu_{i+j-1}+y_1\int u^{i+j-1}\frac{d\mu(u)}{u-y_1}.$$

[4] This limit is unproblematic if $d\mu(u)$ is a measure with finite support.

We use this relation in the last row, that is, for $i = n-1$. Subsequently, we subtract the $(n-2)$-nd row multiplied by y_1 from the last row (the $(n-1)$-st row). Thereby the entry in the j-th column of the last row becomes μ_{n+j-2}. We repeat this operation with the $(n-2)$-nd and the $(n-3)$-rd row, with the $(n-3)$-rd and the $(n-4)$-th row, ..., and with the first row and the 0-th row. As a result, we have converted the determinant in (4.2) into the determinant

$$\det\begin{pmatrix} \int u^j \dfrac{d\mu(u)}{u-y_1} & \text{for } i=0,\ 0\le j\le n-1 \\ \mu_{i+j-1} & \text{for } 1\le i\le n-1,\ 0\le j\le n-1 \end{pmatrix}$$
$$= \int \det\begin{pmatrix} 1 & u & u^2 & \dots & u^{n-1} \\ \mu_0 & \mu_1 & \mu_2 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_n \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{n-2} & \mu_{n-1} & \mu_{n+1} & \dots & \mu_{2n-3} \end{pmatrix} \frac{d\mu(u)}{u-y_1}$$
$$= (-1)^{n-1} \det_{0\le i,j\le n-2}(\mu_{i+j}) \int \frac{p_{n-1}(u)}{u-y_1}\, d\mu(u)$$
$$= (-1)^{n} \det_{0\le i,j\le n-2}(\mu_{i+j})\, q_{n-1}(y_1),$$

where the next-to-last line is due to Lemma 4. This confirms (1.2) for $m = 0$ and $k = 1$.

We now turn to the induction step. We have to distinguish between two cases, depending on whether $n \ge k$ or not. For convenience, we rewrite (1.2) (with $m = 0$) in the form

$$(-1)^{nk} \frac{\prod_{1\le i<j\le k}(y_i - y_j)}{\det_{0\le i,j\le n-k-1}(\mu_{i+j})} \det_{0\le i,j\le n-1}\left(\int u^{i+j} \frac{d\mu(u)}{\prod_{\ell=1}^k (u-y_\ell)}\right) = \det M_{k,0,n}(y_1,\dots,y_k), \tag{4.3}$$

and we apply a similar rewriting to (1.3), leading to (4.3) with denominator omitted on the left-hand side and M replaced by N on the right-hand side.

Let $k \ge 2$ and assume that (4.3), and also the analogue corresponding to (1.3), is true for "smaller k".

We are going to use the condensation formula of Proposition 7. For $n \ge k$, the identity (3.1) with $N = k$, $A = M_{k,0,n}(y_1,\dots,y_k)$, $i_1 = j_1 = 1$ and $i_k = j_k = k$ gives

$$\begin{aligned} \det M_{k,0,n}(y_1,\dots,y_k) \cdot \det M_{k-2,0,n-1}(y_2,\dots,y_{k-1}) \\ = \det M_{k-1,0,n}(y_2,\dots,y_k) \cdot \det M_{k-1,0,n-1}(y_1,\dots,y_{k-1}) \\ - \det M_{k-1,0,n-1}(y_2,\dots,y_k) \cdot \det M_{k-1,0,n}(y_1,\dots,y_{k-1}). \end{aligned} \tag{4.4}$$

For $n < k$, with the same choices of N, i_1, i_2, j_1, j_2, but with $A = N_{k,0,n}(y_1,\dots,y_k)$, we obtain the same identity, but with N instead of M. A little detail is that, if $n = k-1$, we encounter the terms $N_{k-2,0,n-1}(\dots) = N_{k-2,0,k-2}(\dots)$ and $N_{k-1,0,n}(\dots) = N_{k-1,0,k-1}(\dots)$ in (4.4) (with M replaced by N). It can be seen by inspection that $N_{k-2,0,k-2}(\dots) = M_{k-2,0,k-2}(\dots)$ and $N_{k-1,0,k-1}(\dots) = M_{k-1,0,k-1}(\dots)$. This is the interpretation that we give these terms in that special case.

The identity (4.4) can be seen as a recurrence formula for $\det M_{k,0,n}(y_1,\dots,y_k)$, as one can use it to express $\det M_{k,0,n}(y_1,\dots,y_k)$ in terms of expressions of the form

$\det M_{l,0,s}(y_a,\dots,y_b)$ with l smaller than k (and similarly with M replaced by N), provided the determinants $\det M_{k-2,0,n-1}(y_2,\dots,y_{k-1})$ and $\det N_{k-2,0,n-1}(y_2,\dots,y_{k-1})$ are all non-zero. (We shall address the latter point later.) Hence, for carrying out the induction step it suffices to verify that the left-hand side of (4.3) satisfies the same recurrence. Consequently, we substitute this left-hand side in (4.4). After cancellation of factors that are common to both sides, we arrive at

$$(y_k-y_1)\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{d\mu(u)}{\prod_{\ell=1}^{k}(u-y_\ell)}\right)\cdot\det_{0\le i,j\le n-2}\left(\int u^{i+j}\frac{d\mu(u)}{\prod_{\ell=2}^{k-1}(u-y_\ell)}\right)$$
$$=\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{d\mu(u)}{\prod_{\ell=2}^{k}(u-y_\ell)}\right)\cdot\det_{0\le i,j\le n-2}\left(\int u^{i+j}\frac{d\mu(u)}{\prod_{\ell=1}^{k-1}(u-y_\ell)}\right)$$
$$-\det_{0\le i,j\le n-2}\left(\int u^{i+j}\frac{d\mu(u)}{\prod_{\ell=2}^{k}(u-y_\ell)}\right)\cdot\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{d\mu(u)}{\prod_{\ell=1}^{k-1}(u-y_\ell)}\right).$$

This is the special case of Lemma 8 where

$$c_n=\int u^{i+j}\frac{d\mu(u)}{\prod_{\ell=1}^{k}(u-y_\ell)},$$

$\alpha=-y_k$ and $\beta=-y_1$. Since Lemma 10 with $m=0$ says that all these determinants do not vanish identically, this establishes the induction step and proves (4.3) and thus the proposition. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. We apply induction with respect to $k+m$. As start of the induction we use Propositions 11 and 12. In other words, we know that Theorem 1 holds for $k=0$ and for $m=0$.

In preparation of the induction step, we again rewrite (1.2),

$$(-1)^{n(m-k)+km}\frac{\left(\prod_{1\le i<j\le m}(x_j-x_i)\right)\left(\prod_{1\le i<j\le k}(y_i-y_j)\right)}{\det_{0\le i,j\le n-k-1}(\mu_{i+j})}$$
$$\times\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}\,d\mu(u)\right)$$
$$=\det M_{k,m,n}(x_1,\dots,x_m,y_1,\dots,y_k),\qquad(4.5)$$

and similarly (1.3), leading to (4.5) with denominator omitted on the left-hand side and M replaced by N on the right-hand side.

Let now k and m be positive integers, and assume that (4.5), and also its analogue corresponding to (1.3), hold for "smaller $k+m$".

Here again, we are going to use the condensation formula of Proposition 7. For $n\ge k$, the identity (3.1) with $N=k$, $A=M_{k,m,n}(x_1,\dots,x_m,y_1,\dots,y_k)$, $i_1=j_1=1$

and $i_k = j_k = k+m$ gives

$$\begin{aligned}
&\det M_{k,m,n}(x_1,\dots,x_m,y_1,\dots,y_k)\cdot\det M_{k-1,m-1,n}(x_2,\dots,x_m,y_1,\dots,y_{k-1})\\
&\quad=\det M_{k,m-1,n+1}(x_2,\dots,x_m,y_1,\dots,y_k)\cdot\det M_{k-1,m,n-1}(x_1,\dots,x_m,y_1,\dots,y_{k-1})\\
&\quad-\det M_{k,m-1,n}(x_2,\dots,x_m,y_1,\dots,y_k)\cdot\det M_{k-1,m,n}(x_1,\dots,x_m,y_1,\dots,y_{k-1}). \qquad (4.6)
\end{aligned}$$

For $n<k$, with the same choices of N, i_1, i_2, j_1, j_2, but with $A = N_{k,m,n}(y_1,\dots,y_k)$, we obtain the same identity, but with N instead of M. Again we have to take notice of the little detail that, if $n = k-1$, we encounter terms such as $N_{k,m-1,n+1}(\dots) = N_{k,m-1,k}(\dots)$ and $N_{k-1,m,n}(\dots) = N_{k-1,m,k-1}(\dots)$ in (4.6) (with M replaced by N). Here also they have to be interpreted as the corresponding "M-terms".

In order to accomplish the induction step, we have to prove that the left-hand side of (4.5) satisfies the same relation. Consequently, we substitute this left-hand side in (4.6). After cancellation of factors that are common to both sides, we arrive at

$$\begin{aligned}
&\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}\,d\mu(u)\right)\cdot\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=2}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k-1}(u-y_\ell)}\,d\mu(u)\right)\\
&=-\det_{0\le i,j\le n}\left(\int u^{i+j}\frac{\prod_{\ell=2}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}\,d\mu(u)\right)\cdot\det_{0\le i,j\le n-2}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k-1}(u-y_\ell)}\,d\mu(u)\right)\\
&\quad+\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=2}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}\,d\mu(u)\right)\cdot\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k-1}(u-y_\ell)}\,d\mu(u)\right).
\end{aligned}$$

This is the special case of Lemma 9 where

$$c_n = \int u^{i+j}\frac{\prod_{\ell=2}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}\,d\mu(u),$$

$\alpha = -x_1$ and $\beta = -y_k$. Since Lemma 10 says that all these determinants do not vanish identically, this establishes the induction step and proves (4.5), and thus Theorem 1. $\square$

5. The case of equal parameters

Let

$$R(x_1,x_2,\dots,x_m,y_1,y_2,\dots,y_k)$$

denote the right-hand side of (1.2) if $n\ge k$, and the right-hand side of (1.3) if $n<k$. Since the numerator (regardless whether we are considering (1.2) or (1.3)) is skew-symmetric in the x_i's and skew-symmetric in the y_i's, it is divisible by the Vandermonde products

$$\left(\prod_{1\le i<j\le m}(x_j-x_i)\right)\left(\prod_{1\le i<j\le k}(y_i-y_j)\right)$$

in the denominator. Thus, while in its definition it seems problematic to substitute the same value for two different x_i's or for two different y_i's in $R(x_1,x_2,\dots,x_m,y_1,y_2,\dots,y_k)$, this is actually not the case. The proposition below provides an explicit expression for such substitutions of equal parameters.

Proposition 13. *Let $r, s, k_1, k_2, \dots, k_s, m_1, m_2, \dots, m_r$, and n be non-negative integers and $\xi_1, \xi_2, \dots, \xi_r$ and $\omega_1, \omega_2, \dots, \omega_s$ be variables. We write k for the sum $\sum_{i=1}^{s} k_i$, and m for the sum $\sum_{i=1}^{r} m_i$.*

Then

$$R(\xi_1, \dots, \xi_1, \xi_2, \dots, \xi_2, \dots, \xi_r, \dots, \xi_r, \omega_1, \dots, \omega_1, \omega_2, \dots, \omega_2, \dots, \omega_s, \dots, \omega_s)$$
$$= (-1)^{n(m-k)+km} \frac{\det M_{k_1,\dots,k_s,m_1,\dots,m_r,n}(\xi_1, \dots, \xi_r, \omega_1, \dots, \omega_s)}{\left(\prod_{1 \le i < j \le r} (x_j - x_i)^{m_i m_j}\right)\left(\prod_{1 \le i < j \le s} (y_i - y_j)^{k_i k_j}\right)},$$

where ξ_i is repeated m_i times and ω_i is repeated k_i times in the argument of R on the left-hand side. The matrix in the numerator on the right-hand side is defined by

$$M_{k_1,\dots,k_s,m_1,\dots,m_r,n}(\xi_1, \dots, \xi_r, \omega_1, \dots, \omega_s) = \begin{pmatrix} P_{m_1,k,n}(\xi_1) \\ \vdots \\ P_{m_r,k,n}(\xi_r) \\ Q_{k_1,k,n}(\omega_1) \\ \vdots \\ Q_{k_s,k,n}(\omega_s) \end{pmatrix},$$

with

$$P_{a,k.n}(\xi) = \left(\frac{p_{n-k+j-1}^{(i-1)}(\xi)}{(i-1)!}\right)_{1 \le i \le a,\ 1 \le j \le k+m}$$

and

$$Q_{a,k,n}(\omega) = \left(\frac{q_{n-k+j-1}^{(i-1)}(\omega)}{(i-1)!}\right)_{1 \le i \le a,\ 1 \le j \le k+m}.$$

If $b < 0$ the polynomial $p_b(\xi)$ has to be interpreted as 0, while the function $q_b(\omega)$ has to be interpreted as ω^{-b-1},

Proof. This can be proved in the same way as [9, Prop. 5]. We leave the details to the reader. $\square$

Remark. Obviously, we have

$$q_n^{(i-1)}(y) = (-1)^{i-1}(i-1)! \int \frac{p_n^{(i-1)}(u)}{(y-u)^i}\, d\mu(u).$$

6. Uvarov's formula

In this section, we show that our identity in Theorem 1 implies Uvarov's formula for the orthogonal polynomials of a density that arises from a given density by the multiplication of a rational function. The connection between Theorem 1 and Uvarov's formula is set up by Lemma 5.

Let $d\mu(u)$ be the density of a positive measure with infinite support all of whose moments exist. We let $d\nu(u)$ be the modified density

$$d\nu(u) = \frac{\prod_{\ell=2}^{m}(u - x_\ell)}{\prod_{\ell=1}^{k}(u - y_\ell)}\, d\mu(u), \tag{6.1}$$

where $x_2, x_3, \dots, x_m$ and $y_1, y_2, \dots, y_k$ are real numbers chosen so that the modification factor

$$\frac{\prod_{\ell=2}^{m}(u-x_\ell)}{\prod_{\ell=1}^{k}(u-y_\ell)}$$

is positive for all u in the support of the measure μ. In [12, 13] (cf. [7, Theorem 2.7.3]), Uvarov provides a formula for the orthogonal polynomials with respect to the density $d\nu(u)$ in (6.1), expressed in terms of the orthogonal polynomials with respect to the original density $d\mu(u)$.

Theorem 14. *Under the above assumptions, for $n \ge k$, the orthogonal polynomials with respect to the density $d\nu(u)$ given in* (6.1)*, as polynomials in x_1, can be expressed as*

$$\frac{1}{\prod_{\ell=2}^{m}(x_1-x_\ell)}\det\begin{pmatrix} p_{n-k}(x_1) & p_{n-k+1}(x_1) & \dots & p_{n+m-1}(x_1) \\ \dots & \dots & \dots & \dots \\ p_{n-k}(x_m) & p_{n-k+1}(x_m) & \dots & p_{n+m-1}(x_m) \\ q_{n-k}(y_1) & q_{n-k+1}(y_1) & \dots & q_{n+m-1}(y_1) \\ \dots & \dots & \dots & \dots \\ q_{n-k}(y_k) & q_{n-k+1}(y_k) & \dots & q_{n+m-1}(y_k) \end{pmatrix}, \tag{6.2}$$

while, for $n < k$, they can be expressed as

$$\frac{1}{\prod_{\ell=2}^{m}(x_1-x_\ell)}\det\begin{pmatrix} 0 & \dots & 0 & 0 & 0 & p_0(x_1) & p_1(x_1) & \dots & p_{n+m-1}(x_1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & p_0(x_m) & p_1(x_m) & \dots & p_{n+m-1}(x_m) \\ y_1^{k-n-1} & \dots & y_1^2 & y_1 & 1 & q_0(y_1) & q_1(y_1) & \dots & q_{n+m-1}(y_1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_k^{k-n-1} & \dots & y_k^2 & y_k & 1 & q_0(y_k) & q_1(y_k) & \dots & q_{n+m-1}(y_k) \end{pmatrix}, \tag{6.3}$$

with $q_n(y)$ the "functions" given in (1.1).

Proof. In Lemma 5 we choose for the moments ν_i the moments corresponding to the density in (6.1). Then the determinant (2.5) turns out to be exactly the determinant on the left-hand sides of (1.2) and (1.3). Hence, in view of Lemma 5, it is obvious that the left-hand sides of (1.2) and (1.3) are orthogonal polynomials with respect to the density in (6.1), as polynomials in x_1. These right-hand sides are, up to (non-zero) factors that are independent of x_1, proportional to the expressions in (6.2) and (6.3), respectively. There is one "catch" though: Lemma 5 says that this is only the case if all Hankel determinants of moments of (6.1) are non-zero. Now, the choice of the x_i's and y_i's is such that (6.1) is the density of a positive measure. It then follows from Heine's formula (cf. [11, Eq. (2.2.11)] or [7, Cor. 2.1.3]) that all Hankel determinants of moments are non-zero (in fact, positive). □

More generally, our Theorem 1 implies that, *whenever the right-hand sides of* (1.2) *and* (1.3) *are polynomials in x_1 of degree n, then they are orthogonal polynomials in x_1 for the density in* (6.1).

7. Applications: some Hankel determinant evaluations

In this final section, we apply Theorem 1 to derive some Hankel determinant evaluations featuring Catalan numbers and central binomial coefficients.

We restrict ourselves to the special case of Theorem 1 where

$$d\mu(u) = \sqrt{1 - \frac{u^2}{4}}\, du, \qquad -2 \le u \le 2. \tag{7.1}$$

The polynomials which are orthogonal with respect to this density are, essentially, the *Chebyshev polynomials of the second kind* $U_n(x)$, given by the generating function

$$\sum_{n\ge 0} U_n(x)\, z^n = \frac{1}{1 - 2xz + z^2}.$$

More precisely, we have

$$\frac{1}{\pi}\int_{-2}^{2} U_m(u/2)U_n(u/2)\sqrt{1 - \tfrac{u^2}{4}}\, du = \delta_{n,m},$$

with, again, $\delta_{n,m}$ denoting the Kronecker delta. The polynomials $U_n(x/2)$ are monic, as can be seen from

$$\sum_{n=0}^{\infty} U_n(x/2) z^n = \frac{1}{1 - xz + z^2}. \tag{7.2}$$

The non-vanishing moments of the density (7.1) are given by the *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$. To be precise, we have

$$\frac{1}{\pi}\int_{-2}^{2} u^n \sqrt{1 - \frac{u^2}{4}}\, du = \begin{cases} C_{n/2}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \qquad n \ge 0.$$

We shall apply Theorem 1 with $k = 1$ and $m = 0$, and with $k = m = 1$. For carrying out the corresponding calculations, we need the well-known evaluation of the Hankel determinant of moments of the density (7.1) (see e.g. [4, Eq. (2.6) in combination with Case (xvi) in Sec. 4])

$$\det_{0\le i,j\le n-1} (C_{(i+j)/2}) = 1 \tag{7.3}$$

(where C_s is defined to be zero if s is not an integer), and the following fundamental integral evaluation:[5]

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-u^2}}{u+a}\, du = \begin{cases} a - \sqrt{a^2 - 1}, & \text{for } a \ge 1, \\ & \text{for } \Re a > 0 \text{ and } \Im a \ne 0, \\ & \text{for } \Re a = 0 \text{ and } \Im a > 0, \\ a, & \text{for } -1 \le a \le 1, \\ a + \sqrt{a^2 - 1}, & \text{for } a \le -1, \\ & \text{for } \Re a < 0 \text{ and } \Im a \ne 0, \\ & \text{for } \Re a = 0 \text{ and } \Im a < 0. \end{cases} \tag{7.4}$$

[5]The substitution $u = 2t/(1 + t^2)$ transforms the integral into an integral of a rational function. Subsequently, a thorough case analysis has to be performed.

For a "test", we may use this to confirm the orthogonality of the $U_n(x/2)$'s. We have

$$\begin{aligned}\sum_{n\ge0} z^n \frac{1}{\pi}\int_{-2}^{2} U_n(u/2)\sqrt{1-\tfrac{u^2}{4}}\,du &= \frac{1}{\pi}\int_{-2}^{2}\frac{\sqrt{1-\frac{u^2}{4}}}{1-uz+z^2}\,du\\ &= \frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-u^2}}{1-2uz+z^2}\,du\\ &= -\frac{1}{z\pi}\int_{-1}^{1}\frac{\sqrt{1-u^2}}{u-\frac{1+z^2}{2z}}\,du. \end{aligned}\tag{7.5}$$

This calculation is meant as a calculation for (formal) power series in z. Thus, we must think of z as "small", which implies that $\frac{1+z^2}{2z}$ is "large". Hence, when we apply (7.4) with $a = -\frac{1+z^2}{2z}$, we find ourselves in the third case of the case distinction. This gives

$$\sum_{n\ge0} z^n \frac{1}{\pi}\int_{-2}^{2} U_n(u/2)\sqrt{1-\tfrac{u^2}{4}}\,du = -\frac{1}{z}\left(-\frac{1+z^2}{2z}+\sqrt{\frac{(1+z^2)^2}{4z^2}-1}\right)=1,\tag{7.6}$$

as expected. Another "test" that we may perform is the calculation of moments (of the measure) of the Chebyshev polynomials. We have

$$\begin{aligned}\sum_{n\ge0} z^n \frac{1}{\pi}\int_{-2}^{2} u^n\sqrt{1-\tfrac{u^2}{4}}\,du &= \frac{1}{\pi}\int_{-2}^{2}\frac{\sqrt{1-\frac{u^2}{4}}}{1-uz}\,du = \frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-u^2}}{1-2uz}\,du\\ &= -\frac{1}{z\pi}\int_{-1}^{1}\frac{\sqrt{1-u^2}}{u-\frac{1}{2z}}\,du = \frac{1}{2z^2}\left(1-\sqrt{1-4z^2}\right)\\ &= \sum_{n\ge0} C_n z^{2n}, \end{aligned}\tag{7.7}$$

again as expected.

We intend to apply Theorem 1 with $k=1$ and $y_1=-2a$. For convenience, let us write $\omega(a)$ for the quantity in (7.4). Then, by (7.4) and (7.7), for the moments of the density $\frac{1}{u-y_1}\sqrt{1-\frac{u^2}{4}}\,du$ we have

$$\begin{aligned}\sum_{n\ge0} z^n \frac{1}{\pi}\int_{-2}^{2}\frac{u^n\sqrt{1-\frac{u^2}{4}}}{u+2a}\,du &= \frac{1}{\pi}\int_{-2}^{2}\frac{\sqrt{1-\frac{u^2}{4}}}{(u+2a)(1-uz)}\,du\\ &= \frac{1}{\pi(1+2az)}\int_{-2}^{2}\frac{\sqrt{1-\frac{u^2}{4}}}{u+2a}\,du + \frac{z}{\pi(1+2az)}\int_{-2}^{2}\frac{\sqrt{1-\frac{u^2}{4}}}{1-uz}\,du\\ &= \frac{\omega(a)}{1+2az}+\frac{z}{1+2az}\sum_{n\ge0}C_n z^{2n}\\ &= \sum_{n\ge0}\omega(a)(-2a)^n z^n + \sum_{n\ge0} z^n \sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-2a)^{n-2k-1}C_k. \end{aligned}$$

Hence,

$$\frac{1}{\pi}\int_{-2}^{2}\frac{u^n\sqrt{1-\frac{u^2}{4}}}{u+2a}\,du=\omega(a)(-2a)^n+\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-2a)^{n-2k-1}C_k. \tag{7.8}$$

We next compute the functions $q_n(-2a)$ associated with the Chebyshev polynomials $U_n(x/2)$ (cf. (1.1)),

$$\begin{aligned}\sum_{n\ge 0}z^n q_n(-2a)&=\sum_{n\ge 0}z^n\frac{1}{\pi}\int_{-2}^{2}\frac{U_n(u/2)\sqrt{1-\frac{u^2}{4}}}{-2a-u}\,du=-\frac{1}{\pi}\int_{-2}^{2}\frac{\sqrt{1-\frac{u^2}{4}}}{(u+2a)(1-uz+z^2)}\,du\\
&=-\frac{1}{\pi(1+2az+z^2)}\int_{-2}^{2}\frac{\sqrt{1-\frac{u^2}{4}}}{u+2a}\,du\\
&\qquad-\frac{z}{\pi(1+2az+z^2)}\int_{-2}^{2}\frac{\sqrt{1-\frac{u^2}{4}}}{1-uz+z^2}\,du\\
&=-\frac{\omega(a)+z}{1+2az+z^2}=-\sum_{n\ge 0}\big(\omega(a)U_n(-a)+U_{n-1}(-a)\big)z^n,\end{aligned}$$

where we used (7.4), (7.5)/(7.6), and finally (7.2) to obtain the last line. Hence,

$$q_n(-2a)=-\big(\omega(a)U_n(-a)+U_{n-1}(-a)\big). \tag{7.9}$$

If we put these findings together, we get the following result from Theorem 1 with $k=1$ and $m=0$.

Theorem 15. *Let X and a be variables. For all positive integers n, we have*

$$\det_{0\le i,j\le n-1}\left(X(-2a)^{i+j}+\sum_{k=0}^{\lfloor (i+j-1)/2\rfloor}(-2a)^{i+j-2k-1}C_k\right)\\
=(-1)^{n-1}\big(XU_{n-1}(-a)+U_{n-2}(-a)\big). \tag{7.10}$$

Proof. Using (7.3), (7.8) and (7.9), we see that Theorem 1 with $k=1$, $m=0$, $y_1=-2a$, and $d\mu(u)=\sqrt{1-\frac{u^2}{4}}\,du$ reads

$$\det_{0\le i,j\le n-1}\left(\omega(a)(-2a)^{i+j}+\sum_{k=0}^{\lfloor (i+j-1)/2\rfloor}(-2a)^{i+j-2k-1}C_k\right)\\
=(-1)^{n-1}\big(\omega(a)U_{n-1}(-a)+U_{n-2}(-a)\big).$$

This holds for all complex numbers a. Now, if we regard the above determinant (formally) as a polynomial in $\omega(a)$, then it is not difficult to see that the degree in $\omega(a)$ is at most 1. On the other hand, the right-hand side of the above equation is visibly a polynomial of degree 1 in $\omega(a)$. If we now choose $a=-2$ (say), so that $a+\sqrt{a^2-1}=-2+\sqrt{3}$ and 1 are linearly independent over the rational numbers, then we may conclude that $\omega(a)$ can be replaced by a variable, X say. This establishes the assertion of the theorem. $\square$

If we choose $a=-1$ in the above theorem, the sum in the determinant can be simplified. Namely, we have

$$(-1)2^n+\sum_{k=0}^{\lfloor (n-1)/2\rfloor} 2^{n-2k-1}C_k=\begin{cases}-\binom{n}{n/2}, & \text{if } n \text{ is even,}\\ -\frac{1}{2}\binom{n+1}{(n+1)/2}, & \text{if } n \text{ is odd,}\end{cases} \tag{7.11}$$

as is straightforward to verify by an induction on n. Since $U_n(1)=n+1$ for all n, the choice of $X=-1$ in Theorem 15 leads to the Hankel determinant evaluation

$$\det_{0\le i,j\le n-1}\left(2^{-2\lceil (i+j)/2\rceil}\binom{2\lceil (i+j)/2\rceil}{\lceil (i+j)/2\rceil}\right)=2^{-n(n-1)}. \tag{7.12}$$

More generally, if we choose $X=-Y-1$, then we get

$$\det_{0\le i,j\le n-1}\left(Y+2^{-2\lceil (i+j)/2\rceil}\binom{2\lceil (i+j)/2\rceil}{\lceil (i+j)/2\rceil}\right)=2^{-n(n-1)}(Yn+1). \tag{7.13}$$

In a similar fashion, Theorem 1 with $k=m=1$, $x_1=b$, $y_1=-2a$, and $d\mu(u)=\sqrt{1-\frac{u^2}{4}}\,du$ implies the following result.

Theorem 16. *Let X, a and b be variables. For all positive integers n, we have*

$$\begin{aligned}\det_{0\le i,j\le n-1}\Bigg(&X(-2a)^{i+j+1}+\sum_{k=0}^{\lfloor (i+j)/2\rfloor}(-2a)^{i+j-2k}C_k\\ &-b\bigg(X(-2a)^{i+j}+\sum_{k=0}^{\lfloor (i+j-1)/2\rfloor}(-2a)^{i+j-2k-1}C_k\bigg)\Bigg)\\ &=U_{n-1}(b/2)\big(XU_n(-a)+U_{n-1}(-a)\big)-U_n(b/2)\big(XU_{n-1}(-a)+U_{n-2}(-a)\big).\end{aligned} \tag{7.14}$$

Again, it is worth stating the identities that one obtains in the special case where $a=-1$ explicitly. Namely, in view of the simplification (7.11) and of $U_n(1)=n+1$, in that case Equation (7.14) reduces to

$$\begin{aligned}\det_{0\le i,j\le n-1}\Bigg(&Y+2^{-2\lceil (i+j+1)/2\rceil}\binom{2\lceil (i+j+1)/2\rceil}{\lceil (i+j+1)/2\rceil}\\ &-b\bigg(\tfrac{1}{2}Y+2^{-2\lceil (i+j)/2\rceil-1}\binom{2\lceil (i+j)/2\rceil}{\lceil (i+j)/2\rceil}\bigg)\Bigg)\\ &=(-1)^{n-1}2^{-n^2}\Big(U_{n-1}(b/2)(Y(n+1)+1)-U_n(b/2)(Yn+1)\Big).\end{aligned} \tag{7.15}$$

Since $U_n(0)=(-1)^{n/2}$ if n is even and $U_n(0)=0$ otherwise, for $b=0$ Equation (7.15) becomes

$$\det_{0\le i,j\le n-1}\left(Y+2^{-2\lceil (i+j+1)/2\rceil}\binom{2\lceil (i+j+1)/2\rceil}{\lceil (i+j+1)/2\rceil}\right)=(-1)^{\binom{n}{2}}2^{-n^2}\big(2\lceil n/2\rceil Y+1\big). \tag{7.16}$$

Finally, since

$$U_n(1/2)=\begin{cases}1, & \text{if } n\equiv 0,1 \bmod 6,\\ 0, & \text{if } n\equiv 2 \bmod 3,\\ -1, & \text{if } n\equiv 3,4 \bmod 6,\end{cases}$$

for $b=1$ Equation (7.15) becomes

$$\det_{0\le i,j\le n-1}\left(Y+2^{-2\lceil (i+j)/2\rceil}\binom{2\lceil (i+j)/2\rceil}{\lceil (i+j+1)/2\rceil}\right)\\
=\begin{cases}2^{-n(n-1)}Y, & \text{if } n\equiv 0 \pmod 3,\\ -2^{-n(n-1)}(Y(n+1)+1), & \text{if } n\equiv 1 \pmod 3,\\ 2^{-n(n-1)}(Yn+1), & \text{if } n\equiv 2 \pmod 3.\end{cases} \tag{7.17}$$

When testing (7.13) and (7.16) with the help of *Mathematica*, initially I mistyped the exponent of 2. This led to the discovery of two more — conjectural — Hankel determinant evaluations.

Conjecture 17. *Let Y be a variable. For all positive integers n, we have*

$$\det_{0\le i,j\le n-1}\left(Y+2^{-i-j}\binom{2\lceil (i+j)/2\rceil}{\lceil (i+j)/2\rceil}\right)\\
=2^{-(n-1)^2}\times\begin{cases}2(-1)^s(2sY+1), & \text{if } n=3s,\\ (-1)^s((2s+1)Y+1), & \text{if } n=3s+1,\\ (-1)^{s+1}((2s+1)Y+1), & \text{if } n=3s+2,\end{cases} \tag{7.18}$$

and

$$\det_{0\le i,j\le n-1}\left(Y+2^{-i-j-1}\binom{2\lceil (i+j+1)/2\rceil}{\lceil (i+j+1)/2\rceil}\right)\\
=\begin{cases}(-1)^{\lfloor n/6\rfloor}2^{-n(n-1)}\left(\frac{1}{2}\left\lfloor\frac{4n+2}{3}\right\rfloor Y+1\right), & \text{if } n \text{ is even},\\ (-1)^{\lfloor n/6\rfloor}2^{-n(n-1)}\left(\frac{1}{2}\left\lfloor\frac{4n+4}{3}\right\rfloor Y+1\right), & \text{if } n \text{ is odd}.\end{cases} \tag{7.19}$$

8. Recursiveness of Hankel determinants of moments

In [9, Cor. 8], it was shown that, if the coefficients of the three-term recurrence (2.1) satisfied by the orthogonal polynomials $p_n(x)$ are constants, the scaled Hankel determinants of moments on the left-hand side of (1.4) (i.e., Equation (1.2) with $k=0$) satisfy a linear recurrence with constant coefficients. Since, by (2.6), the "functions" $q_n(y)$ satisfy the same type of three-term recurrence, this result — and its proof — extend straightforwardly to the scaled Hankel determinants of moments on the left-hand side of (1.2). It should be noted that the form of the scaling in (8.1) below is based on (2.2).

Corollary 18. *Within the setup in Sections 1 and 2, let $s_i\equiv s$ and $t_i\equiv t$ for $i\ge 1$. Furthermore, let*

$$H_n=t_0^{-(n-k-1)}t^{-\binom{n-k-1}{2}}\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^m(u-x_\ell)}{\prod_{\ell=1}^k(u-y_\ell)}\,d\mu(u)\right). \tag{8.1}$$

Then the sequence $(H_n)_{n\ge 0}$ of (scaled) Hankel determinants of moments satisfies a linear recurrence of the form

$$\sum_{i=0}^{2^{k+m}}c_iH_{n-i}=0,\quad \textit{for } n>2^{k+m}, \tag{8.2}$$

for some constants $c_i \in K$, normalised by $c_0 = 1$. Explicitly, these constants can be computed as the coefficients of the characteristic polynomial (in x) $\sum_{i=0}^{2^{k+m}} c_i x^{2^{k+m}-i}$ of the tensor product of 2×2 matrices

$$\begin{pmatrix} s-x_1 & t \\ -1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} s-x_m & t \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} s-y_1 & t \\ -1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} s-y_k & t \\ -1 & 0 \end{pmatrix}. \tag{8.3}$$

In particular, we have

$$c_1 = -\left(\prod_{\ell=1}^{m}(s-x_\ell)\right)\left(\prod_{\ell=1}^{k}(s-y_\ell)\right) \tag{8.4}$$

and

$$c_{2^{k+m}-i} = t^{(k+m)(2^{k+m-1}-i)} c_i, \quad \text{for } i = 0, 1, \dots, 2^{k+m}. \tag{8.5}$$

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References

[1] J. Baik, P. Deift and E. Strahov, *Products and ratios of characteristic polynomials of random Hermitian matrices*, J. Math. Phys. **44** (2003), 3657–3670.

[2] E. Brézin and S. Hikami, *Characteristic polynomials of random matrices*, Comm. Math. Phys. **214** (2000), 111–135.

[3] E. B. Christoffel, *Über die Gaußische Quadratur und eine Verallgemeinerung derselben*, J. reine angew. Math. **55** (1858), 61–82.

[4] J. Cigler and C. Krattenthaler, *Hankel determinants of linear combinations of moments of orthogonal polynomials*, Int. J. Number Theory **17** (2021), 341–369.

[5] M. Elouafi, *A unified approach for the Hankel determinants of classical combinatorial numbers*, J. Math. Anal. Appl. **431** (2015), 1253–1274.

[6] M. E. H. Ismail and R. W. Ruedemann, *Relation between polynomials orthogonal on the unit circle with respect to different weights*, J. Approx. Theory **71** (1992), 39–60.

[7] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge University Press, Cambridge, 2009.

[8] C. Krattenthaler, *Advanced determinant calculus*, Séminaire Lotharingien Combin. **42** (1999) ("The Andrews Festschrift"), Article B42q, 67 pp.

[9] C. Krattenthaler, *Hankel determinants of linear combinations of moments of orthogonal polynomials, II*, preprint, `ar`χ`iv:2101.04225`.

[10] A. Lascoux, *Symmetric functions and combinatorial operators on polynomials*, CBMS Regional Conference Series in Mathematics, vol. 99, Amer. Math. Soc., Providence, RI, 2003.

[11] G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc., Providence, RI, 1939.

[12] V. B. Uvarov, *Relation between polynomials orthogonal with different weights* (Russian), Dokl. Akad. Nauk SSSR **126** (1959), 33–36.

[13] V. B. Uvarov, *The connection between systems of polynomials that are orthogonal with respect to different distribution functions* (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **9** (1969), 1253–1262.

[14] X. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux*, UQAM, Montréal, Québec, 1983; available at `http://www.xavierviennot.org/xavier/polynomes_orthogonaux.html`.

[15] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math. **85** (1997), 67–86.

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