On the Arithmetic and Shift Complexities of Inverting of Difference Operator Matrices

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Let K be a difference field of characteristic 0 with an automorphism (a shift) σ.
A scalar linear difference operator f is an element of $K[\sigma, \sigma^{-1}]$. For a non-zero scalar operator $f = \sum a_i \sigma^i$ we define its *upper and lower orders*:

$$\overline{\mathrm{ord}}\, f = \max\{i \mid a_i \neq 0\}, \quad \underline{\mathrm{ord}}\, f = \min\{i \mid a_i \neq 0\}$$

and its *order*

$$\mathrm{ord}\, f = \overline{\mathrm{ord}}\, f - \underline{\mathrm{ord}}\, f.$$

Conventionally, $\overline{\mathrm{ord}}\, 0 = -\infty$, $\underline{\mathrm{ord}}\, 0 = \infty$, $\mathrm{ord}\, 0 = -\infty$.

For a finite set F of scalar operators (e.g., for a vector, a matrix, a row of a matrix) we define $\overline{\mathrm{ord}}\, F$ as the maximum of the upper orders of its components), $\underline{\mathrm{ord}}\, F$ — as the minimum of the lower orders of the components and $\mathrm{ord}\, F$ as $\overline{\mathrm{ord}}\, F - \underline{\mathrm{ord}}\, F$.

A difference operator $n \times n$-matrix is a matrix belonging to $\mathrm{Mat}_n(K[\sigma, \sigma^{-1}])$. In the sequel we will connect with a difference operator matrix some matrices belonging to $\mathrm{Mat}_n(K)$. To avoid a terminological confuses we will call operator matrices just *operators*. The case of scalar operators will be discussed separately.
An operator is of *full rank* if its rows are linearly independent over $K[\sigma, \sigma^{-1}]$.

Any non-zero operator $L \in \mathrm{Mat}_n(K[\sigma, \sigma^{-1}])$ can be represented as

$$L = A_l \sigma^l + A_{l-1} \sigma^{l-1} + \cdots + A_t \sigma^t, \tag{1}$$

$A_l, A_{l-1}, \dots, A_t \in \mathrm{Mat}_n(K)$,
the matrices A_l, A_t (the *leading* and the *trailing* matrices of L) are non-zero.

$$L \in \mathrm{Mat}_n(K[\sigma, \sigma^{-1}]), \tag{2}$$

An operator L is *invertible* in $\mathrm{Mat}_n(K[\sigma, \sigma^{-1}])$ and $M \in \mathrm{Mat}_n(K[\sigma, \sigma^{-1}])$ is its *inverse*, if $LM = ML = I_n$ where I_n is the unit $n \times n$-matrix.
We write L^{-1} for the matrix M.

Invertible operators are also called *unimodular* operators.

Example 1

$(K = \mathbb{Q}(x),\ \sigma f(x) = f(x+1))$:

$$\begin{pmatrix} \sigma^{-1} & -\frac{1}{x-1} \\ \frac{x^2}{2} & -\frac{x}{2}\sigma + 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{(x+1)^2}{2x}\sigma^2 + \sigma & \frac{1}{x}\sigma \\ -\frac{x^2}{2}\sigma & 1 \end{pmatrix}. \tag{3}$$

In the sequel, we will give some complexity analysis of the unimodularity testing for an operator matrix and of constructing the inverse matrix if it exists.

Let K be a difference field with an automorphism σ.
We will say that a ring Λ which is a difference extension of the field K, is an *adequate* difference extension of K, if $\mathrm{Const}\,(\Lambda)$ is a field and for an arbitrary sytstem $\sigma y = Ay,\ y = (y_1, \dots, y_n)^T$, having a non-singular matrix $A \in \mathrm{Mat}_n(K)$ the dimension of the linear over $\mathrm{Const}\,(\Lambda)$ space of solutions belonging Λ^n is equal to n.
For an arbitrary Λ the equality $\mathrm{Const}\,(\Lambda) = \mathrm{Const}\,(K)$ is not guaranteed. In the general case $\mathrm{Const}\,(K)$ is a proper subfield of $\mathrm{Const}\,(\Lambda)$.
Below, we denote by Λ a fixed adequate difference extension of a difference field K.

Besides the complexity as the number of arithmetic operations (the *arithmetic complexity*) one can consider the number of shifts in the worst case (the *shift complexity*).

Thus we will consider two complexities.
This is similar to the situation with sorting algorithms, when we consider separately the complexity as
the number of comparisons and, resp.,
the number of swaps
in the worst case.

We introduce also the *full algebraic complexity* as the total number of all operations in the worst case.

The set of unimodular $n \times n$-operators will be denoted by Υ_n.
We use the notation V_L for the space of solutions of the system $L(y) = 0$, belonging to Λ^n. We will also consider V_L as the space of solutions of operator L.

Proposition 1

Let an operator $L \in \mathrm{Mat}_n(K[\sigma, \sigma^{-1}])$ be of full rank. Then $L \in \Upsilon_n \iff V_L = 0$.

New algorithm

Theorem 1

(i) For an operator $L \in \mathrm{Mat}_n(K[\sigma, \sigma^{-1}])$ *of full rank, there exists such* $U \in \Upsilon_n$*, that the operator* $\tilde{L} = UL$ *has the following properties:*

(a) *lower orders of all rows of* $\tilde{L}$ *are equal to zero;*

(b) $\dim V_{\tilde{L}}$ *is equal to sum of the upper orders of the rows of the rows of the operator* $\tilde{L}$*;*

(c) $\operatorname{ord} U = O(nd)$.

(ii) There exists an algorithm which checks whether a given operator L *is of full rank or not, and if it is, then constructs an operator* U *described in (i). The arithmetic complexity of the algorithm is equal to* $O(n^4 d^2)$*, the shift comlexity is equal to* $O(n^2 d^2)$.

As we know, $L \in \Upsilon_n$ iff $\dim V_{\tilde{L}} = 0$ (Proposition 1), and, by (a), (b), — iff $\tilde{L} \in \mathrm{Mat}_n(K)$. To check whether an operator L is of full rank and $\tilde{L} \in \mathrm{Mat}_n(K)$ (without constructing operators U and $\tilde{L}$) we have an algorithm whose arithmetic complexity is $O(n^3 d^2)$, the shift complexity is $O(n^2 d^2)$. This gives an algorithm for the unimodularity checking. If U and $\tilde{L}$ are constructed by the algorithm mentioned in Theorem 1 (ii), and if $\tilde{L} \in \mathrm{Mat}_n(K)$, then we compute $L^{-1} = \tilde{L}^{-1} U$. By (c), the computing of the product $\tilde{L}^{-1} U$ does not change the estimates

$$O(n^4 d^2), \quad O(n^2 d^2)$$

for the arithmetic and shift complexities.

To construct the operators U, $\tilde{L}$ an extended version of EG-eliminations is used. Our previous algorithm was organized similarly, but the constructing of the operators U, $\tilde{L}$ was produced by another algorithms having the bigger shift complexity.

In addition, there is a question related to complexity of the matrix inverting for the case when matrix entries are scalar differential operators over a differential field K:

It is not clear, whether this problem can be reduced to the matrix multiplication problem as in the "commutative" case?

One more question: whether there exists an algorithm for such $n \times n$-matrices inverting with complexity $O(n^a d^b)$, where d is the maximal order of differential operators which are the matrix entries, with $a < 3$?

I think that it is possible to prove by the usual way that the matrix multiplication can be reduced to the problem of the matrix inverting (I have in mind the difference matrices). However, it is not so easy to prove that the problem of the matrix inverting can be reduced to the problem of the the matrix multiplication (I think even that this is not correct, but cannot prove this).